GASPER'S DETERMINANT THEOREM, REVISITED

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Abstract. Let $n \geq 2$ be a natural number, M a real $n \times n$ matrix, s the sum of the entries of M and q the sum of their squares. With $\alpha := s/n$ and $\beta := q/n$, Gasper's determinant bound says that $|\det M| \leq \beta^{n/2}$, and in case of $\alpha^2 \geq \beta$:

$$|\det M| \leq |\alpha| \left(\frac{n\beta - \alpha^2}{n-1} \right)^{\frac{n-1}{2}}$$

This article gives a corrected proof of Gasper's theorem and lists some more applications.

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1. Introduction

The present article is primarily a revised version of [6], ironing out a flaw in the proof of [6], Theorem 1, adding statements about complex matrices and about infinite determinants and mentioning a few more applications. We do not repeat the numerical results concerning determinants of matrices whose entries are a permutation of the numbers $1, \ldots, n^2$. See [13] for these.

Throughout, let $n > 1$ be a natural number and $N := \{1, \ldots, n\}$. Whenever not stated otherwise, *matrix* means a real $n \times n$ matrix, the set of which we denote by $\mathbb{M}$.

For $M \in \mathbb{M}$ and $i, j \in N$ we denote by M_i the i-th row of M, by M^j the j-th column of M, and by $M_{i,j}$ the entry of M at position (i, j). If M is a matrix or a row or column of a matrix, then by $s(M)$ we denote the sum of the entries of M and by $q(M)$ the sum of their squares.

The identity matrix is denoted by I. By J we name the matrix which has 1 as all of its entries, while e is the column vector in $\mathbb{R}^n$ with all entries being 1. Matrices of the structure $xI + yJ$ will play an important role, so we state some of their properties:

Lemma 1. *Let $x, y \in \mathbb{R}$ and $M := xI + yJ$. Then we have:*

(1) $\det M = x^{n-1}(x + ny)$
(2) *M is invertible if and only if $x \notin \{0, -ny\}$.*
(3) *If M is invertible, then $M^{-1} = \frac{1}{x} I - \frac{y}{x(x+ny)} J$.*

Proof. Because $J = ee^T$, it holds that

$$Me = (xI + yee^T)e = (x + ye^T e)e = (x + ny)e \quad \text{and} \quad Mv = (xI + yee^T)v = xv$$

for all $v \in \mathbb{R}^n$ with $v \perp e$. Hence M has the eigenvalue x with multiplicity $n - 1$ and the simple eigenvalue $x + ny$. This shows (1). (2) is an immediate consequence of (1). (3) can be verified by a straight calculation. □

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2. Matrices with given entry sum and square sum

Let $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$. We inspect the following set of matrices:

$$\mathbb{M}_{\alpha,\beta} := \{M \in \mathbb{M} : s(M) = n\alpha,\ q(M) = n\beta\}$$

Lemma 2.

(1) *If* $\alpha^2 > n\beta$, *then* $\mathbb{M}_{\alpha,\beta} = \emptyset$.
(2) *If* $\alpha^2 = n\beta$, *then* $\det M = 0$ *for all* $M \in \mathbb{M}_{\alpha,\beta}$.
(3) *If* $\alpha^2 \le n\beta$, *then there exists an* $M \in \mathbb{M}_{\alpha,\beta}$ *with*

$$\det M = \alpha \left(\frac{n\beta - \alpha^2}{n-1}\right)^{\frac{n-1}{2}}.$$

(4) *If* $\alpha^2 \le \beta$, *then there exists an* $M \in \mathbb{M}_{\alpha,\beta}$ *with* $\det M = \beta^{\frac{n}{2}}$.

Proof. (1) Suppose $\mathbb{M}_{\alpha,\beta} \neq \emptyset$, say $M \in \mathbb{M}_{\alpha,\beta}$. Reading M and J as elements of $\mathbb{R}^{n^2}$, Cauchy's inequality gives:

$$\alpha^2 = \frac{1}{n^2}\Big(\sum_{i,j=1}^{n} M_{i,j}\Big)^2 = \frac{1}{n^2}\langle M, J\rangle^2 \le \frac{1}{n^2}\|M\|_2^2\,\|J\|_2^2 = \sum_{i,j=1}^{n} M_{i,j}^2 = n\beta$$

(2) For $\alpha^2 = n\beta$ and $M \in \mathbb{M}_{\alpha,\beta}$, the calculation in (1) shows $|\langle M, J\rangle| = \|M\|_2\,\|J\|_2$. But this holds only if M is a scalar multiple of J, so we have $\det M = 0$ because of $\det J = 0$.

(3) Suppose $\alpha^2 \le n\beta$. With $\gamma := \left(\frac{n\beta-\alpha^2}{n-1}\right)^{\frac{1}{2}}$ set $M := \gamma I + \frac{1}{n}(\alpha - \gamma)J$. Then $M \in \mathbb{M}_{\alpha,\beta}$, and by Lemma 1:

$$\det M = \gamma^{n-1}\left(\gamma + n\tfrac{1}{n}(\alpha - \gamma)\right) = \alpha\gamma^{n-1}$$

(4) Suppose $\alpha^2 \le \beta$. In case of $\alpha \ge 0$, set $\gamma := \frac{1}{2}\big(\frac{3\alpha}{\sqrt{\beta}} - 1\big)$, so $\gamma^2 \le 1$. Set

$$A := \begin{pmatrix} \alpha & \sqrt{\beta - \alpha^2} \\ -\sqrt{\beta - \alpha^2} & \alpha \end{pmatrix} \quad \text{and} \quad B := \sqrt{\beta}\begin{pmatrix} \gamma & \sqrt{1-\gamma^2} & 0 \\ -\sqrt{1-\gamma^2} & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $s(A) = 2\alpha$, $q(A) = 2\beta$, $\det A = \beta$, $s(B) = 3\alpha$, $q(B) = 3\beta$, $\det B = \beta^{\frac{3}{2}}$. In case of $n = 2k$ with $k \in \mathbb{N}$, use k copies of A to build the block matrix

$$M := \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix},$$

which has the required properties. In case of $n = 2k+1$ with $k \in \mathbb{N}$, use $k-1$ copies of A to build the block matrix

$$M := \begin{pmatrix} A & & & \\ & \ddots & & \\ & & A & \\ & & & B \end{pmatrix},$$

which again satisfies the requirements.

In case of $\alpha < 0$, an $M' \in \mathbb{M}_{-\alpha,\beta}$ with $\det M' = \beta^{\frac{n}{2}}$ exists. For even n, $M := -M' \in \mathbb{M}_{\alpha,\beta}$ has the requested determinant, while for odd n swapping two rows of $-M'$ gives the desired matrix M. ∎

In the proofs of (3) and (4) of Lemma 2 we have specified matrices whose determinants will below turn out to be the greatest possible. The determinant values relate like following:

Lemma 3. *For $\alpha^2 \leq n\beta$ the inequality*

$$|\alpha| \left(\frac{n\beta - \alpha^2}{n-1} \right)^{\frac{n-1}{2}} \leq \beta^{\frac{n}{2}}$$

holds, with equality if and only if $\alpha^2 = \beta$.

Proof. With $f(x) := x\left(\frac{n-x}{n-1}\right)^{n-1}$ for $x \in [0, n]$ we have

$$|\alpha| \left(\frac{n\beta - \alpha^2}{n-1} \right)^{\frac{n-1}{2}} \beta^{-\frac{n}{2}} = \sqrt{f\left(\tfrac{\alpha^2}{\beta}\right)}.$$

The proof is completed by applying the AM-GM inequality to $f(x)^{1/n}$:

$$f(x)^{\frac{1}{n}} = \left(x \left(\tfrac{n-x}{n-1} \right)^{n-1} \right)^{\frac{1}{n}} \leq \tfrac{1}{n} \left(x + (n-1) \tfrac{n-x}{n-1} \right) = 1$$

with equality if and only if $x = \frac{n-x}{n-1}$, i. e. if and only if $x = 1$. □

3. Main results

Let $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$. By Lemma 2 there exists an $M \in \mathbb{M}_{\alpha,\beta}$ with $\det M \neq 0$ if and only if $\alpha^2 < n\beta$. By possibly swapping two rows of M, $\det M > 0$ can be achieved. As $\mathbb{M}_{\alpha,\beta}$ is compact, the determinant function assumes a maximum value on $\mathbb{M}_{\alpha,\beta}$. Gasper's theorem provides insight into the properties of the matrices with maximal determinant:

Theorem 1 (O. Gasper, 2009). *Let $\alpha^2 < n\beta$ and $M \in \mathbb{M}_{\alpha,\beta}$ with maximal determinant. Then*

$$\text{if } \alpha^2 \leq \beta \ : \ \begin{cases} (A) & MM^T = \beta I \\ (B) & \det M = \beta^{\frac{n}{2}} \end{cases}$$

$$\text{if } \alpha^2 \geq \beta \ : \ \begin{cases} (C) & s(M_i) = \alpha \quad \text{and} \quad s(M^j) = \alpha \quad \text{for all } i, j \in N \\ (D) & MM^T = (\beta - \delta)I + \delta J \quad \text{with } \delta := \frac{\alpha^2 - \beta}{n-1}, \text{ so } \beta - \delta = \frac{n\beta - \alpha^2}{n-1}. \\ (E) & \det M = |\alpha| \, (\beta - \delta)^{\frac{n-1}{2}} \end{cases}$$

Proof. From Lemma 2 we know that $\det M > 0$. The matrix M solves an extremum problem with equality contraints

$$\text{(P)} \qquad \begin{cases} \det X \longrightarrow \max \\ s(X) = n\alpha \\ q(X) = n\beta \end{cases} \qquad (X \in \mathbb{M}^*),$$

where $\mathbb{M}^*$ is the set of invertible matrices. The Lagrange function of (P) is given by

$$L(X, \lambda, \mu) = \det X - \lambda(s(X) - n\alpha) - \mu(q(X) - n\beta),$$

so there exist $\lambda, \mu \in \mathbb{R}$ with $\frac{d}{dM_{i,j}} L(M, \lambda, \mu) = 0$ for all $i, j \in N$. By Jacobi's formula

$$\left(\frac{d}{dM_{i,j}} \det M \right)_{i,j} = (\det M) \left(M^T\right)^{-1}$$

we get[1] $(\det M)\,(M^T)^{-1} - \lambda J - 2\mu M = 0$, i. e.

$$(\det M)I = \lambda J M^T + 2\mu M M^T. \tag{1}$$

Suppose $\mu = 0$. Then applying the determinant function to (1) and using $\det J = 0$ would give $(\det M)^n = \det(\lambda J M^T) = \det(J)\det(\lambda M) = 0$, a contradiction to $\det M > 0$. Hence

$$\mu \neq 0. \tag{2}$$

As JM^T has the diagonal elements $s(M_1), \ldots, s(M_n)$, and MM^T has the diagonal elements $q(M_1), \ldots, q(M_n)$, we get $n \det M = \lambda s(M) + 2\mu q(M) = \lambda n\alpha + 2\mu n\beta$ by applying the trace function to (1), consequently

$$\det M = \lambda\alpha + 2\mu\beta. \tag{3}$$

The symmetry of $(\det M)I$ and the symmetry of $2\mu MM^T$ in (1) show that λJM^T is symmetric. As all rows of JM^T are identical, namely equal to $(s(M_1), \ldots, s(M_n))$, we obtain

$$\lambda s(M_1) = \cdots = \lambda s(M_n). \tag{4}$$

In the following, we inspect the cases $\lambda = 0$ and $\lambda \neq 0$ and prove:

$$\begin{cases} \lambda = 0 & \Longrightarrow \quad \alpha^2 \leq \beta \;\wedge\; (A) \;\wedge\; (B) \\ \lambda \neq 0 & \Longrightarrow \quad \alpha^2 \geq \beta \;\wedge\; (C) \;\wedge\; (D) \;\wedge\; (E) \end{cases} \tag{5}$$

Case $\lambda = 0$: Then (3) reads $\det M = 2\mu\beta$, so taking (2) into account and dividing (1) by 2μ gives $\beta I = MM^T$, i. e. (A). From this, (B) follows by applying the determinant function. Using the inequality between arithmetic mean and root mean square and the fact that the matrix $(1/\sqrt{\beta})M$ is orthogonal and thus an isometry w.r.t. the euclidean norm $\|\;\|_2$, we get

$$\alpha^2 = \Big(\frac{1}{n}\sum_{i=1}^{n} s(M_i)\Big)^2 \leq \frac{1}{n}\sum_{i=1}^{n} s(M_i)^2 = \frac{1}{n}\,\|Me\|_2^2 = \frac{1}{n}\,\beta\|e\|_2^2 = \frac{1}{n}\,\beta n = \beta. \tag{6}$$

Case $\lambda \neq 0$: Then $s(M_1) = \cdots = s(M_n)$ by (4). With $s(M_1) + \cdots + s(M_n) = s(M) = n\alpha$ this shows $s(M_i) = \alpha$ for all $i \in N$. Using $(\det M)I = \lambda M^T J + 2\mu M^T M$ instead of (1) yields $s(M^j) = \alpha$ for all $j \in N$, so (C) is done. Furthermore, $JM^T = \alpha J$, and (1) becomes

$$2\mu MM^T = (\det M)I - \lambda\alpha J, \tag{7}$$

hence $q(M_i) = (MM^T)_{i,i} = (\det M - \lambda\alpha)/(2\mu)$ for all $i \in N$, so $q(M_1) = \cdots = q(M_n)$. Using $q(M_1) + \cdots + q(M_n) = q(M) = n\beta$ shows

$$(MM^T)_{i,i} = q(M_i) = \beta \quad \text{for all } i \in N. \tag{8}$$

Let $i, j \in N$ with $i \neq j$. Then (7) gives $(MM^T)_{i,k} = -\lambda\alpha/(2\mu)$ for all $k \in N \setminus \{i\}$. With

$$\sum_{k=1}^{n} (MM^T)_{i,k} = \sum_{k=1}^{n}\sum_{p=1}^{n} M_{i,p}\, M_{k,p} = \sum_{p=1}^{n} M_{i,p}\, s(M^p) = \sum_{p=1}^{n} M_{i,p}\,\alpha = s(M_i)\,\alpha = \alpha^2$$

and (8) we get

$$(MM^T)_{i,j} = \frac{1}{n-1}\sum_{k\neq i} (MM^T)_{i,k} = \frac{1}{n-1}\left(\sum_{k=1}^{n} (MM^T)_{i,k} - (MM^T)_{i,i}\right) = \frac{\alpha^2 - \beta}{n-1} = \delta,$$

which, again with (8), proves (D). With Lemma 1 this yields

$$(\det M)^2 = \det(MM^T) = (\beta - \delta)^{n-1}(\beta - \delta + n\delta) = (\beta - \delta)^{n-1}\alpha^2,$$

[1]This is where a clerical mistake happened in [6]. Here we have corrected $-\lambda M - 2\mu J$ by $-\lambda J - 2\mu M$ and adapted the remainder of the proof accordingly.

and taking the square root gives (E). Suppose $\alpha^2 < \beta$. Then by Lemma 2 there would exist an $M' \in \mathbb{M}_{\alpha,\beta}$ with $\det M' = \beta^{\frac{n}{2}}$, and by Lemma 3

$$\det M = |\alpha|(\beta-\delta)^{\frac{n-1}{2}} < \beta^{\frac{n}{2}} = \det M',$$

which contradicts the maximality of $\det M$. Hence $\alpha^2 \geq \beta$.

We have now proved (5) and are ready to deduce the statements of the theorem: If $\alpha^2 < \beta$, then (5) shows that $\lambda = 0$ and thus (A) and (B). If $\alpha^2 > \beta$, then (5) shows that $\lambda \neq 0$ and thus (C), (D) and (E). Finally suppose $\alpha^2 = \beta$. Then $\delta = 0$, hence (A) $\Longleftrightarrow$ (D) and (B) $\Longleftrightarrow$ (E). If $\lambda \neq 0$, then (5) shows (C), (D) and (E), from which (A) and (B) follow. If $\lambda = 0$, then (5) shows (A) and (B), from which (D) and (E) follow. It remains to prove (C) in the case of $\alpha^2 = \beta$ and $\lambda = 0$. To this purpose, look at (6) again, where $\alpha^2 = \beta$ shows that $s(M_1) = \cdots = s(M_n)$, and (C) follows as in the case $\lambda \neq 0$. □

For calculating upper bounds for the determinants of given matrices, we note this handy consequence of Theorem 1:

Proposition 1. *Let* $M \in \mathbb{M}$, $\alpha := \frac{1}{n}s(M)$, $\beta := \frac{1}{n}q(M)$, $\kappa := \frac{n\beta-\alpha^2}{n-1}$. *Then*

$$\begin{aligned} &\textit{if } \alpha^2 < \beta \ :\ |\det M| \leq \beta^{\frac{n}{2}} \\ &\textit{if } \alpha^2 = \beta \ :\ |\det M| \leq |\alpha|\kappa^{\frac{n-1}{2}} = \beta^{\frac{n}{2}} \\ &\textit{if } \alpha^2 > \beta \ :\ |\det M| \leq |\alpha|\kappa^{\frac{n-1}{2}} < \beta^{\frac{n}{2}} \end{aligned}$$

Proof. This is trivial if $\det M = 0$. In case of $\det M \neq 0$ we get $\alpha^2 < n\beta$ by Lemma 2, and the stated inequalities are true by Lemma 3 and Theorem 1. □

Note that Lemma 3 says that $|\alpha|\kappa^{\frac{n-1}{2}} < \beta^{\frac{n}{2}}$ is true in case of $\alpha^2 < \beta$, too. However, as the following examples demonstrates, $|\det M|$ is not necessarily bounded by the left hand side in this situation:

$$M := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ,\ |\det M| = 1\ ,\ |\alpha|\kappa^{\frac{n-1}{2}} = 0$$

Proposition 1 can be used to derive bounds for the determinants of complex matrices also:

Corollary 1. *Let* $A, B \in \mathbb{M}$, $M := A + iB$, $\alpha := \frac{1}{n}s(A)$, $\beta := \frac{1}{n}(q(A)+q(B))$, $\kappa := \frac{2n\beta-\alpha^2}{2n-1}$. *Then*

$$\begin{aligned} &\textit{if } \alpha^2 < \beta \ :\ |\det M| \leq \beta^{\frac{n}{2}} \\ &\textit{if } \alpha^2 = \beta \ :\ |\det M| \leq |\alpha|^{\frac{1}{2}}\kappa^{\frac{2n-1}{4}} = \beta^{\frac{n}{2}} \\ &\textit{if } \alpha^2 > \beta \ :\ |\det M| \leq |\alpha|^{\frac{1}{2}}\kappa^{\frac{2n-1}{4}} < \beta^{\frac{n}{2}} \end{aligned}$$

Proof. For the real $2n \times 2n$-matrix

$$M' := \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

we have $s(M') = 2s(A)$ and $q(M') = 2q(A) + 2q(B))$, hence $\alpha' := \frac{1}{2n}s(M') = \frac{1}{n}s(A) = \alpha$ and $\beta' := \frac{1}{2n}q(M') = \frac{1}{n}(q(A)+q(B)) = \beta$, and Proposition 1 applied to M' gives

$$\begin{aligned} &\text{if } \alpha^2 < \beta \ :\ |\det M'| \leq \beta^{\frac{2n}{2}} \\ &\text{if } \alpha^2 = \beta \ :\ |\det M'| \leq |\alpha|\kappa^{\frac{2n-1}{2}} = \beta^{\frac{2n}{2}} \\ &\text{if } \alpha^2 > \beta \ :\ |\det M'| \leq |\alpha|\kappa^{\frac{2n-1}{2}} < \beta^{\frac{2n}{2}} \end{aligned}$$

The claimed inequalities follow by using $\det M' = |\det M|^2$, see [1], Fact 3.24.7 vii). □

To get the more attractive case of $\alpha^2 > \beta$ in Corollary 1, it can help to recall the equality $|\det(A+iB)| = |\det(B+iA)|$ and apply Corollary 1 to the latter matrix. As an example take $A := \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. For $A+iB$ we get $\alpha = 0$, $\beta = 2$ and the bound $|\det(A+iB)| \leq 2$. But $B + iA$ gives $\alpha = 2$, $\beta = 2$ and the bound $|\det(B + iA)| \leq 4 \cdot 27^{-\frac{1}{4}} \approx 1.75$.

For a real matrix M, i.e. $B = 0$, Corollary 1 can yield a larger bound than Proposition 1, and it can give different bounds for M and for iM. For example the matrix $M := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ gets the bound $\frac{3}{4}\sqrt{3} \approx 1.30$ from Proposition 1 and the bound $\frac{1}{4}125^{\frac{1}{4}}\sqrt{3} \approx 1.45$ from Corollary 1, while Corollary 1 applied to iM gives the bound 1.5. This unhappy situation prompts for

Question 1. *Is there a better way to transfer Proposition 1 to complex matrices?*

4. Applications

The bounds for $\det M$ in Proposition 1 and Corollary 1 refer only to $s(M)$ and $q(M)$ and so do not use any positional information. As sections 4.1–4.6 show, they can still serve for deducing interesting inequalities. But the strength of Proposition 1 manifests better when it is applied to problems like in sections 4.7 and 4.8.

4.1. **Hadamard's inequality.** For a complex $n \times n$ matrix M with $|M_{i,j}| \leq 1$ for all $i, j \in N$, Corollary 1 shows:

$$|\det M| \leq \beta^{\frac{n}{2}} = \left(\frac{1}{n}\sum_{i,j=1}^{n} |M_{i,j}|^2\right)^{\frac{n}{2}} \leq \left(\frac{1}{n}\sum_{i,j=1}^{n} 1\right)^{\frac{n}{2}} = n^{\frac{n}{2}}$$

This is Hadamard's inequality, see [8]. For $\gamma \geq 0$ and $|M_{i,j}| \leq \gamma$ for all $i, j \in N$ we get in the same way the inequality $|\det M| \leq \gamma^n \, n^{n/2}$. However, Hadamard's more general bound

$$|\det M| \leq \prod_{i=1}^{n} \left(\sum_{j=1}^{n} |M_{i,j}|^2\right)^{\frac{1}{2}}$$

cannot be derived from the $\beta^{\frac{n}{2}}$ bound. The AM-GM inequality shows that this bound is less than or equal to the $\beta^{\frac{n}{2}}$ bound, and there are cases where it is strictly smaller. But matrices exist where the case $\alpha^2 > \beta$ in Proposition 1 applies and yields a bound that is better than Hadamard's. As an example, Hadamard's bound for the matrix $\left(\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right)$ is $\sqrt{65}$ while Proposition 1 gives the bound $\sqrt{32}$.

4.2. **Best's inequality.** If $M_{i,j} \in \{-1, 1\}$ for all $i, j \in N$ and $|\det M| = n^{n/2}$, i.e. M is a *Hadamard matrix*, then Proposition 1 shows that $\alpha^2 \leq \beta$ must hold. The value $s(M)$ is called the *excess* of M. Because $q(M) = n^2$ in case of $M_{i,j} \in \{-1, 1\}$, Proposition 1 yields an upper bound for the excess:

$$M \text{ is a Hadamard matrix} \quad \Longrightarrow \quad s(M) \leq n\sqrt{n}$$

This is known as Best's inequality, see [2].

4.3. **Inequality of determinant and trace.** For a positive integer m and positive definite matrices A and B, Theorem 2.8 in [5] says

$$(\det AB)^{\frac{m}{n}} \leq \frac{1}{n}\operatorname{tr}(A^m B^m),$$

which for $m = 1$ and $B = A^T$ reads $(\det A)^{\frac{2}{n}} \leq \frac{1}{n}\operatorname{tr}(AA^T)$. As $\operatorname{tr}(AA^T) = q(A)$, Proposition 1 shows that, for the latter inequality, A does not need to be positive definite.

4.4. **Ryser's inequality.** If $M_{i,j} \in \{0,1\}$ for all $i,j \in N$ and $t := s(M)$ is the number of 1's in M, then in Proposition 1 with $k := t/n$ we have $\alpha = \beta = k$ and get

$$\begin{aligned}
\text{if } t < n &\ :\ |\det M| \le k^{\frac{n}{2}} \\
\text{if } t = n &\ :\ |\det M| \le 1 \\
\text{if } t > n &\ :\ |\det M| \le k^{\frac{n+1}{2}} \left(\frac{n-k}{n-1}\right)^{\frac{n-1}{2}}
\end{aligned}$$

The inequality for the case $t > n$ is Ryser's determinant bound [12], Theorem 3. For the case of $t = 2n$ this was improved by Bruhn and Rautenbach, see [4], Theorem 3, and [10].

4.5. **The inequalities of Brent, Osborne and Smith.** Let $\varepsilon > 0$, $E \in \mathbb{M}$ with $|E_{i,j}| \le \varepsilon$ for all $i,j \in N$ and $M := I - E$. Then

$$\beta \le \frac{1}{n}\left(n(1+\varepsilon)^2 + (n^2-n)\varepsilon^2\right) = 1 + 2\varepsilon + n\varepsilon^2$$

in Proposition 1 gives

$$|\det M| \le \left(1 + 2\varepsilon + n\varepsilon^2\right)^{\frac{n}{2}} .$$

This is [3], Theorem 3 (8). In case of $E_{i,i} = 0$ for all $i \in N$ we have

$$\beta \le \frac{1}{n}\left(n + (n^2-n)\varepsilon^2\right) = 1 + (n-1)\varepsilon^2$$

and so

$$|\det M| \le \left(1 + (n-1)\varepsilon^2\right)^{\frac{n}{2}} ,$$

which is [3], Theorem 3 (9). While, as is demonstrated in [3], both inequalities follow from Hadamard's inequality, the above reasoning shows that these bounds do not depend on the arrangement of the dominant entries, and it can also be used to derive bounds if the number of dominant entries is different from n.

4.6. **Determinants of von Koch matrices.** Let $A := (A_{i,j} : i,j = 1 \ldots \infty)$ be an infinite matrix of real numbers $A_{i,j}$ with

$$\sum_{i=1}^{\infty} |A_{i,i}| < \infty \quad \text{and} \quad \sum_{i,j=1}^{\infty} |A_{i,j}|^2 < \infty$$

and $A(n) := (A_{i,j} : i,j = 1 \ldots n)$ for $n \in \mathbb{N}$ be the $n \times n$ finite submatrix. Then by [9] or [7], page 170, the *infinite determinant*

$$\det(I - A) := \lim_{n\to\infty} \det(I - A(n)),$$

is well-defined, where I is the suitable finite or infinite identity matrix. By Proposition 1

$$|\det(I - A(n))| \le \left(\frac{1}{n}\sum_{i,j=1}^{n}(\delta_{i,j} - A_{i,j})^2\right)^{\frac{n}{2}} = \left(1 + \frac{1}{n}\sum_{i,j=1}^{n} A_{i,j}^2 - \frac{2}{n}\sum_{i=1}^{n} A_{i,i}\right)^{\frac{n}{2}} ,$$

and as $\left(1 + \frac{x_n}{n}\right)^{\frac{n}{2}} \to e^{\frac{x}{2}}$ for $n \to \infty$ for a real sequence (x_n) with limit x:

$$|\det(I - A)| \le \exp\left(\frac{1}{2}\sum_{i,j=1}^{\infty} A_{i,j}^2 - \sum_{i=1}^{\infty} A_{i,i}\right) \tag{9}$$

One might want to apply the better bound of Proposition 1's case $\alpha^2 > \beta$, but an evaluation reveals that for $n \to \infty$ this gives the identical inequality (9).

4.7. **Matrices whose entries are a permutation of an arithmetic progression.**

Proposition 2. *Let p, q be real numbers with $q > 0$ and M a matrix whose entries are a permutation of the numbers $p, p+q, \dots, p+(n^2-1)q$. Set*

$$r := \frac{p}{q} + \frac{n^2-1}{2} \quad , \quad \varrho := \frac{n^3+n^2+n+1}{12} \quad \text{and} \quad \sigma := nq^2\left(r^2 + \frac{n^4-1}{12}\right).$$

Then

$$\begin{aligned} \text{if } r^2 < \varrho &: \ |\det M| \le \sigma^{\frac{n}{2}} \\ \text{if } r^2 = \varrho &: \ |\det M| \le n^n q^n\,|r|\varrho^{\frac{n-1}{2}} = \sigma^{\frac{n}{2}} \\ \text{if } r^2 > \varrho &: \ |\det M| \le n^n q^n\,|r|\varrho^{\frac{n-1}{2}} < \sigma^{\frac{n}{2}} \end{aligned}$$

Proof. With α and β as in Proposition 1 a calculation shows $\alpha^2 - \beta = n(n-1)q^2(r^2-\varrho)$, hence $\operatorname{sgn}(\alpha^2-\beta) = \operatorname{sgn}(r^2-\varrho)$, and the bounds noted in Proposition 1 yield the asserted inequalities for $|\det M|$. □

Corollary 2. *If M is a matrix whose entries are a permutation of $0, \dots, n^2-1$, then*

$$|\det M| \le n^n\,\frac{n^2-1}{2}\left(\frac{n^3+n^2+n+1}{12}\right)^{\frac{n-1}{2}}.$$

If M is a matrix whose entries are a permutation of $1, \dots, n^2$, then

$$|\det M| \le n^n\,\frac{n^2+1}{2}\left(\frac{n^3+n^2+n+1}{12}\right)^{\frac{n-1}{2}}.$$

Proof. Apply Proposition 2 to $(p,q) := (0,1)$ and to $(p,q) := (1,1)$, respectively. In both applications it is easy to see that $r^2 > \varrho$, which yields the stated bound. □

Actual maximal determinants for this kind of matrices suggest

Question 2. *Let $b(n)$ be the upper bound given in Corollary 2 for matrices with entries $1, \dots, n^2$. Regarding* [13], *do we have*

$$\lim_{n\to\infty} \frac{A085000(n)}{b(n)} = 1?$$

4.8. **A variation of the previous theme.**

Proposition 3. *Let p, q be real numbers with $q > 0$ and M a matrix such that each of the numbers $p, p+q, \dots, p+(n-1)q$ appears n times in M. Set*

$$r := \frac{p}{q} + \frac{n-1}{2} \quad , \quad \varrho := \frac{n+1}{12} \quad \text{and} \quad \sigma := nq^2\left(r^2 + \frac{n^2-1}{12}\right).$$

Then

$$\begin{aligned} \text{if } r^2 < \varrho &: \ |\det M| \le \sigma^{\frac{n}{2}} \\ \text{if } r^2 = \varrho &: \ |\det M| \le n^n q^n |r| \varrho^{\frac{n-1}{2}} = \sigma^{\frac{n}{2}} \\ \text{if } r^2 > \varrho &: \ |\det M| \le n^n q^n |r| \varrho^{\frac{n-1}{2}} < \sigma^{\frac{n}{2}} \end{aligned}$$

Proof. We have again $\alpha^2 - \beta = n(n-1)q^2(r^2-\varrho)$ and can apply Proposition 1. □

Corollary 3. *If M is a matrix such that each of the numbers $0, \dots, n-1$ appears n times in M, then*

$$|\det M| \leq n^n \, \frac{n-1}{2} \left(\frac{n+1}{12}\right)^{\frac{n-1}{2}} .$$

If M is a matrix such that each of the numbers $1, \dots, n$ appears n times in M, then

$$|\det M| \leq n^n \, \frac{n+1}{2} \left(\frac{n+1}{12}\right)^{\frac{n-1}{2}} .$$

Proof. Apply Proposition 3 to $(p, q) := (0, 1)$ and to $(p, q) := (1, 1)$. □

Again we would like to pose the

Question 3. *Let $b(n)$ be the upper bound given in Corollary 3 for matrices with entries $1, \dots, n$. Regarding* [11], *do we have*

$$\lim_{n \to \infty} \frac{A301371(n)}{b(n)} = 1?$$

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