

# COUNTING GRADED LATTICES OF RANK THREE THAT HAVE FEW COATOMS

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ABSTRACT. We consider the problem of computing  $R(c, a)$ , the number of unlabeled graded lattices of rank 3 that contain  $c$  coatoms and  $a$  atoms. More specifically we do this when  $c$  is fairly small, but  $a$  may be large. We describe a computational method that, for a fixed  $c$ , combines direct enumeration of the connection graphs of  $c$  coatoms, and Redfield–Pólya counting for distributing atoms between the coatoms. Using this method we compute  $R(c, a)$  for  $c \leq 9$  and  $a \leq 1000$ . With the help of these computations we also derive  $R(c, a)$  in closed form for  $c \leq 7$ .

## 1. INTRODUCTION

Let  $R(c, a)$  denote the number of unlabeled graded lattices of rank 3 that contain  $c$  coatoms and  $a$  atoms, with  $c, a \geq 1$ . If  $c$  and  $a$  are very small,  $R(c, a)$  can be determined by actually generating the lattices, for example by the `nauty` one-liner `genbg -Z1 -d1 -u {c} {a}`. This takes time at least linear in  $R(c, a)$ , so this approach is limited to very small instances. By this method the author has computed  $R(c, a)$  for  $c + a \leq 21$  (OEIS sequence A300260 [10]).

In this work we seek exact values of  $R(c, a)$  when  $c$  is small, but  $a$  may be large. The converse case is handled by duality since  $R(c, a) = R(a, c)$ . Throughout this paper we treat  $c$  as a small constant. As a warm-up, and to illustrate the underlying ideas, we manually derive closed form expressions for  $R(c, a)$  when  $c \leq 3$ . We then present a computational method that, for a given value of  $c$ , uses `nauty` [17] to enumerate the ways how  $c$  coatoms can be connected, and then employs the cycle index theorem to count distributions of atoms among the  $c$  coatoms. The method has been implemented using the computer algebra system GAP [7]. With this program we compute  $R(c, a)$  for  $c \leq 9$  and  $a \leq 1000$ . Then, by doing an exact polynomial fit on residue classes of the results, we derive  $R(c, a)$  in closed form for also  $4 \leq c \leq 7$ .

## 2. RELATED WORK

Although lattices have their algebraic aspect as commutative, idempotent algebras, in the context of counting and enumeration a lattice is usually seen as a kind of a directed graph, defined by its covering relation (or “Hasse graph”). This places the problem in the realm of *graph enumeration*.

Exact counts of lattices are often determined by complete enumeration, that is, by generating all lattices [8, 9, 11, 15] or those in some specific class [6, 11]. Counting of rank-3 lattices does not seem to have been addressed directly, although some asymptotics are known. The total number of rank-3 graded lattices of  $n + 2$

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*Key words and phrases.* Graded lattices; isomorphism; cycle index theorem; computer algebra.

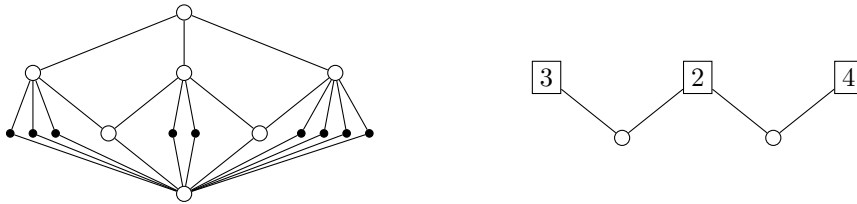


FIGURE 1. A rank-3 lattice (loners shown as small dots) and its connection graph (coatoms shown as boxes with numbers of loners).

elements is  $R(n) = \sum_{c=1}^{n-1} R(c, n-c)$ . From the results of Klotz and Lucht [13] and Kleitman and Winston [12] we have

$$1.2777n^{3/2+o(n^{3/2})} \leq R(n) \leq 1.6994n^{3/2+o(n^{3/2})}.$$

Of course, any algorithm that generates all lattices can be adapted to the special case of rank-3 lattices. However, the structure of rank-3 lattices makes them amenable to non-generative methods. The present work combines both approaches: it *generates* a large number of basic cases, and for each case *counts* further lattices without generating them. Both are done by computation, but with quite different tools. Thus the work has connections both to graph generating methods and to counting theory. Such connections have been seen in graph counting (for example, when labeled graphs are treated as unlabeled objects decorated by labels), but not so much with lattices.

### 3. CONNECTION GRAPHS

A graded lattice of rank 3, or a *rank-3 lattice* has elements on four levels: top, coatoms, atoms, and bottom. From now on we ignore the top and the bottom, since the structure of a rank-3 lattice is determined by its two central levels. These levels form a bipartite graph with two distinct color classes (coatoms and atoms). Furthermore, there are no isolated elements (as the lattice is graded), and any two elements have at most one common neighbor. This characterization (cf. [12]) gives rise to the **nauty** one-liner mentioned in the introduction, which generates the bipartite graphs one by one, and outputs the count.

For counting purposes it is better to simplify the representation of a rank-3 lattice. Let us divide the atoms into *connectors*, which are covered by two or more coatoms, and *loners*, which are covered by one coatom only. In lattice theory loners are known as meet-irreducible atoms, but we want a shorter name here. The *connection graph* is the bicolored graph spanned by the coatoms and the connectors. Any rank-3 lattice is uniquely represented by (1) its connection graph and (2) for each coatom  $u$ , an integer  $\ell(u)$  indicating how many loners it covers. This is illustrated in Figure 1.

With any given value of  $c$  there is only a finite collection of connection graphs. Let  $r$  be the number of connectors. Because in a lattice the upper covers of two atoms cannot contain two common elements, we must have  $r \leq \binom{c}{2}$ ; the maximum is reached when every pair of coatoms has its own connector. With  $c = 2$  there are two graphs, one with a connector and one without. With  $c = 3$ , the number of connectors is  $r \in [0, 3]$ , and if  $r = 1$ , either two or three coatoms are connected. So

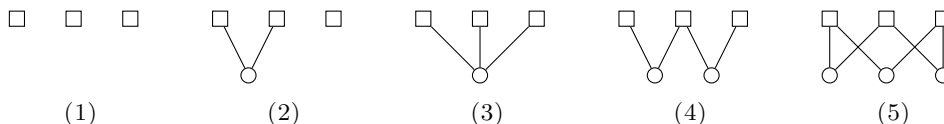


FIGURE 2. The five connection graphs of three coatoms.

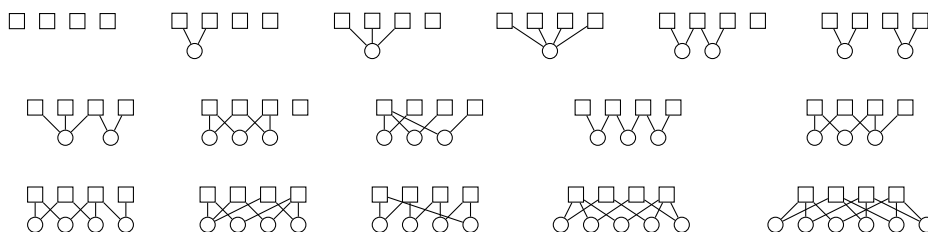


FIGURE 3. The sixteen connection graphs of four coatoms.

there are five connection graphs, which are shown in Figure 2. With  $c = 4$  there are sixteen graphs, shown in Figure 3.

Now all rank-3 lattices of  $c$  coatoms and  $a$  atoms can be counted in two phases:

- (1) *List* all nonisomorphic connection graphs of  $c$  coatoms.
- (2) For each connection graph, *calculate* the number of ways to distribute  $a - r$  loners among the  $c$  coatoms.

The first phase is straightforward using tools from `nauty`, although the list is long if  $c$  is large. It is similar to listing all rank-3 lattices, except that elements in the first color class (coatoms) can have zero neighbors, and elements in the second color class (connectors) must have at least two neighbors. Since  $r \leq \binom{c}{2}$ , iterating `genbg -Z1 -d0:2 ${c} ${r}` for  $r = 0, 1, \dots, \binom{c}{2}$  lists the connection graphs.

The second phase is like distributing  $a - r$  identical balls into  $c$  boxes. However, a couple of issues must be observed. A simple thing is to ensure that every coatom covers at least one atom. If there are  $s$  coatoms that do not cover a connector, simply allocate one loner to each, and distribute the remaining  $n = a - r - s$  balls into  $c$  boxes, with empty boxes allowed.

Somewhat more complicated is to handle the symmetries of each connection graph, so that we correctly count the nonisomorphic lattices. In Figure 2, graphs (1), (3), (5) have all coatoms in symmetric position, so we are putting  $k$  balls into three identical boxes. But in graphs (2) and (4), two coatoms are in symmetric position (two identical boxes) and the third is not (a distinct box). Dealing with symmetry comprises the bulk of this work. We do it manually for  $c \leq 3$ , but for  $c \geq 4$  we turn to computational solutions.

#### 4. LATTICES OF AT MOST THREE COATOMS

With  $c \leq 3$  there are few connection graphs, and we can treat them by hand. Clearly  $R(1, a) = 1$ . Before going further we recall some elementary results on occupancy. Let  $p_k(n)$  be the number of ways to distribute  $n$  identical balls into  $k$  identical boxes, empty boxes allowed. For all  $n \geq 0$  we have [1, Theorem 6.4]

$$(1) \quad p_2(n) = \lfloor n/2 + 1 \rfloor,$$

$$(2) \quad p_3(n) = \lfloor n^2/12 + n/2 + 1 \rfloor.$$

Also let  $p_{2,1}(n)$  be the number of ways to distribute  $n$  identical balls into two identical boxes and one distinct box, empty boxes allowed. For all  $n \geq 0$  we have

$$(3) \quad p_{2,1}(n) = \sum_{i=0}^n p_2(i) = \lfloor n^2/4 + n + 1 \rfloor.$$

This may be recognized as the quarter-squares sequence A002620 [10]. We adopt the convention that  $p_k(n) = p_{2,1}(n) = 0$  when  $n < 0$ , and note that Eqs. (1)–(3) are then valid also for  $p_2(-1)$ ,  $p_3(-2)$ ,  $p_3(-1)$ , and  $p_{2,1}(-1)$ .

**Theorem 1.** *For any  $a \geq 1$ , we have  $R(2, a) = a$ .*

*Proof.* Either there is a connector or not. In either case the two coatoms are in symmetric position. In the connected case we have  $r = 1$  and  $s = 0$ , and the remaining  $a - 1$  atoms can be distributed in  $p_2(a - 1)$  ways. In the disconnected case we have  $r = 0$  and  $s = 2$ , and the remaining  $a - 2$  atoms can be distributed in  $p_2(a - 2)$  ways; this is zero if  $a = 1$ .

Adding up and simplifying, we get  $R(2, a) = p_2(a - 1) + p_2(a - 2) = a$ .  $\square$

**Theorem 2.** *For any  $a \geq 1$ , we have  $R(3, a) = \lfloor (3/4)a^2 + (1/3)a + 1/4 \rfloor$ .*

*Proof.* We consider the five possible connection graphs (see Figure 2).

Graph (1): No connectors,  $r + s = 3$ . The remaining  $a - 3$  atoms can be distributed among the three symmetric coatoms in  $p_3(a - 3)$  ways.

Graph (2): One atom connects two coatoms,  $r + s = 2$ . The two connected coatoms are in symmetric position, so the remaining  $a - 2$  atoms can be distributed in  $p_{2,1}(a - 2)$  ways.

Graph (3): One atom connects three coatoms,  $r + s = 1$ . All coatoms are symmetric, so the remaining  $a - 1$  atoms can be distributed in  $p_3(a - 1)$  ways.

Graph (4): Two connectors,  $r + s = 2$ . The two coatoms at the ends are symmetric, so the remaining  $a - 2$  atoms can be distributed in  $p_{2,1}(a - 2)$  ways.

Graph (5): Three connectors,  $r + s = 3$ . All coatoms are symmetric, so the remaining  $a - 3$  atoms can be distributed in  $p_3(a - 3)$  ways.

In all five cases, if  $a < r + s$ , then  $p_3$  or  $p_{2,1}$  has a negative argument, and a zero value by our convention. Adding up we get

$$R(3, a) = 2p_3(a - 3) + p_3(a - 1) + 2p_{2,1}(a - 2).$$

Substituting (2) and (3), observing that they are valid for all  $a \geq 1$  even if the arguments become negative, and simplifying, we obtain the stated result.  $\square$

For  $c = 4$  one could continue in the same manner. But the sixteen connection graphs exhibit various kinds of symmetry, and manual case-by-case analysis becomes tedious and error-prone. We now turn to a computational method that analyzes the symmetries of the connection graphs, and also provides a numerical solution to our counting problem. In fact, the method will also help us to derive closed form expressions for  $R(4, a)$ ,  $R(5, a)$ ,  $R(6, a)$ , and  $R(7, a)$ .

## 5. BALLS INTO BOXES WITH SYMMETRY

We now address the second phase of our task: given a bicolored connection graph  $\Gamma$ , and an integer  $n$ , count the ways of distributing  $n$  identical balls (the

loners) among the vertices in the first color class (the coatoms). By saying “bicolored” we keep the connectors distinct from the coatoms. Otherwise, in graph 5 of Figure 2 there would be an automorphism that swaps them. We could as well orient the edges from atoms to coatoms.

The count is affected by the automorphism group  $\text{Aut}(\Gamma)$ , or in fact by its action *on the coatoms*, since that is where we place the balls. Indeed, although graphs (1), (3), (5) in Figure 2 are different as graphs, they have the same symmetry between the coatoms. For us they have the same kind of solution, namely  $p_3(n)$  as we saw in the previous section (but with different values of  $n$ ).

Our counting problem is solved by an application of the cycle index theorem from the Redfield–Pólya counting theory. For a general introduction to this theory see Cameron’s textbook [4]. The particular application to the balls-into-boxes problem can be found in Lisoněk’s thesis [16, Chapter 4], where a distribution of a number of balls under permutation group  $G$  is called a  $G$ -*partition* of that number. (Beware that Lisoněk’s symbols are the reverse of ours: he puts  $c$  balls into  $n$  boxes.) Other mentions of the problem include [2, 5]. For completeness of exposition, we describe the calculation here, without any claim of originality.

Consider  $G = \text{Aut}(\Gamma)$  as a permutation group on the  $c$  coatoms. For each group element  $g \in G$ , let  $m_j(g)$  be the number of cycles of length  $j$  in the disjoint cycle representation of  $g$ , and define the cycle index monomial in  $c$  indeterminates  $t_1, \dots, t_c$  as follows:

$$z_g(t_1, \dots, t_c) = t_1^{m_1(g)} t_2^{m_2(g)} \dots t_c^{m_c(g)}.$$

The cycle index of  $G$  is the average of the cycle index monomials,

$$Z_G(t_1, \dots, t_c) = \frac{1}{|G|} \sum_{g \in G} z_g(t_1, \dots, t_c).$$

A distribution of balls (loners) into  $c$  boxes (coatoms) is described by a function  $\ell : \{1, 2, \dots, c\} \rightarrow \mathbb{N}$ , with  $\ell(i)$  telling how many balls are in the  $i$ th box. Using terminology from counting theory, the nonnegative integers  $\ell(i)$  are *figures* attached to each coatom. Each figure has a weight, which is here simply the integer itself. The weight of a function  $\ell$  is  $\sum_{i=1}^c \ell(i)$ . This is the total number of balls in the distribution, so we want to count functions of weight  $n$ .

We further define a figure-counting series

$$(4) \quad A(x) = 1 + x + x^2 + \dots = 1/(1 - x),$$

which says that there exists one figure of each nonnegative weight (the integer itself). We want the function-counting series

$$B(x) = \sum_{n \geq 0} b_n x^n,$$

whose coefficient  $b_n$  counts, up to symmetry, the functions  $\ell$  of weight  $n$ . By the cycle index theorem [4, Theorem 7.3] the function-counting series is

$$(5) \quad B(x) = Z_G(A(x), A(x^2), \dots, A(x^c)).$$

Thus, as soon as we know the cycle index  $Z_G$ , Eq. (5) gives a series whose coefficients are the counts we desire (for all  $n \geq 0$ ). This amounts to a very short piece of GAP code (Program 1). As an implementation detail, if we are content with

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**Program 1** Count distributions of  $0, \dots, n$  balls into  $c$  boxes, whose permutation group has cycle index  $Z_G$ . Counts returned as a vector.

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```

GroupBalls := function(ZG, c, n)
  local i,x,A,Axi,B;
  # Cycle index may use indeterminates 1..c, take next for x.
  x := Indeterminate(Rationals, c+1);
  # Figure-counting polynomial up to n, and substitute x^i.
  A := (1-x^(n+1)) / (1-x);
  Axi := [];
  for i in [1..c] do
    Axi[i] := Value(A, [x], [x^i]);
  od;
  # Function-counting polynomial
  B := Value(ZG, [1..c], Axi);
  # Pick the coefficients for x^0, ..., x^n
  return CoefficientsOfUnivariatePolynomial(B){[1..n+1]};
end;

```

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a finite sequence of results, we may replace the series  $A(x)$  with the polynomial  $1 + x + \dots + x^n$  without affecting the coefficients of  $B(x)$  up to  $x^n$ .

## 6. IMPLEMENTATION AND NUMERICAL RESULTS

The first phase, generating the connection graphs, was done for  $c = 2, 3, \dots, 9$  with `genbg` as previously described. The second phase, calculating  $R(c, a)$ , was implemented as a GAP program, here summarized as Algorithm 1, and run with  $a_{\max} = 1000$ .

Certain aspects of the computation are illustrated in Table I. For  $c = 9$  the most time-consuming parts were generating the connection graphs and analyzing their symmetries. We observe that even though there are many graphs, they have few different cycle indices. `GroupBalls` is called only once for each cycle index (we keep a table of cycle indices already seen).

The computation times should be taken as indicative only. They are the elapsed times when running on a single Intel Xeon E5-2680 core, with a nominal clock frequency of 2.4 GHz. The implementation was certainly not fully optimized and there may be plenty of room for improvement. We used the following program versions: `genbg` 1.4, GAP 4.8.10 [7], and `Digraphs` 0.11.0 [3].

Some of the numbers for  $c \leq 9$  and  $a \leq 1000$  are listed in Tables II and III. Full listings of the results and the programs are available separately [14]. Since  $R(9, 1000) \approx 1.775 \times 10^{27}$ , it would not have been practical to count the lattices by generating them.

## 7. OBTAINING FUNCTIONAL FORMS

In this section we show that for any  $c \geq 1$ , the function  $R(c, a)$  has a representation as a quasipolynomial in  $a$  (an existence result). Furthermore we find an explicit quasipolynomial when  $c \leq 7$ .

For each connection graph  $\Gamma$ , let  $G = G(\Gamma)$  be the permutation group of the  $c$  connectors defined by the action of  $\text{Aut}(\Gamma)$ . The function-counting series (5) is a

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**Algorithm 1** Given  $c, a_{\max}$ , calculate  $R(c, a)$  for  $a = 0, 1, 2, \dots, a_{\max}$ .

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```

1:  $N \leftarrow (0, \dots, 0)$  ▷ Vector of  $a_{\max}+1$  zeros
2:  $\mathcal{B} \leftarrow ()$  ▷ Lookup table for results by cycle index
3: for  $\Gamma \in$  all connection graphs do
4:    $c \leftarrow$  number of coatoms in  $\Gamma$ 
5:    $r \leftarrow$  number of connectors in  $\Gamma$ 
6:    $s \leftarrow$  number of coatoms not covering atoms in  $\Gamma$ 
7:    $G \leftarrow \text{AutomorphismGroup}(\Gamma)$  ▷ In Digraphs package
8:    $Z_G \leftarrow \text{CycleIndex}(G, \{1, \dots, c\})$  ▷ GAP builtin function
9:   if  $\mathcal{B}(Z_G)$  exists then ▷ Look up
10:     $B \leftarrow \mathcal{B}(Z_G)$  ▷ Use the memorized result
11:   else
12:     $B \leftarrow \text{GroupBalls}(Z_g, c, a_{\max})$  ▷ Compute
13:     $\mathcal{B}(Z_G) \leftarrow B$  ▷ Memorize result
14:   Shift  $P$  up by  $r + s$  positions, prepending zeros, truncating after  $a_{\max} + 1$ .
15:    $N \leftarrow N + B$ 
16: return  $N$ 

```

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$c$	different		time/s		
	connection graphs	cycle indices	genbg	aut+cyc	count
2	2	1	0.0	0.0	0.1
3	5	2	0.0	0.0	0.7
4	16	6	0.0	0.0	4.0
5	72	11	0.0	0.1	13.0
6	592	26	0.0	0.5	49.6
7	10808	38	0.7	7.7	122.0
8	552251	87	61.4	284.5	459.6
9	82856695	142	19200.7	44211.1	1297.7

TABLE I. Some details of the computations for  $c \leq 9$  coatoms. Times are in seconds: genbg = generating the connection graphs, aut+cyc = finding their automorphism groups and cycle indices, count = calculating the counts.

generating function whose coefficients are the numbers of ways to distribute  $n \geq 0$  balls to the  $c$  coatoms. The series has a special form, a rational function whose denominator consists of factors of the form  $(1 - x^i)$ . From this observation it follows (Lisoněk's Theorem 4.3.5 [16]) that the coefficients are quasipolynomials of  $n$ .

A function  $f : \mathbb{N} \rightarrow \mathbb{C}$  is a *quasipolynomial* of quasiperiod  $N$ , if there are polynomials  $P_0, P_1, \dots, P_{N-1}$  such that for all  $n \geq 0$ ,

$$f(n) = P_k(n) \quad \text{when } n \equiv k \pmod{N}.$$

This definition is from Stanley [18]. (Note that  $N$  need not be minimal.) Lisoněk uses a slightly different definition that only requires the function to agree with the polynomials from some point  $n_0$  onwards. We shall use Stanley's definition here.

Quasipolynomials, also known as polynomials on residue classes (PORC), are extremely versatile for expressing combinatorial quantities, which often depend on

the parity of the argument, or its residue modulo some integer  $N$ . Typical examples are floors of polynomials, such as (1)–(3). We recall the following from Stanley:

**Proposition 1** (Part of Proposition 4.4.1 [18]). *A function  $f : \mathbb{N} \rightarrow \mathbb{C}$  is a quasipolynomial of quasiperiod  $N$ , if it has a rational generating function*

$$\sum_{n \geq 0} f(n)x^n = \frac{P(x)}{Q(x)},$$

where  $P, Q$  are polynomials; every root  $\alpha$  of  $Q$  satisfies  $\alpha^N = 1$ ; and  $\deg P < \deg Q$ .

Next we re-state Lisoněk's result using Stanley's definition and with explicit mention of a quasiperiod. If  $G$  is a permutation group acting on  $c$  boxes, we denote by  $P_G(n)$  the number of distributions of  $n \geq 0$  balls in the  $c$  boxes, when distributions related by an element of  $G$  are not distinguished. (Lisoněk calls them  $G$ -partitions of the integer  $n$ .)

**Proposition 2** (cf. Theorem 4.3.5 [16]). *The function  $n \mapsto P_G(n)$  is a quasipolynomial of quasiperiod  $N = \text{lcm}(1, 2, \dots, c)$ .*

*Proof.* The generating function of  $P_G(n)$  is  $B(x)$  as defined in (5). Now  $Z_G$  is a polynomial in  $c$  indeterminates  $t_1, \dots, t_c$ , and  $B(x)$  issues from it by the Pólya substitutions  $t_i = 1/(1 - x^i)$  for  $i = 1, 2, \dots, c$ . With this substitution, each cycle index monomial  $z_g$  becomes a rational function  $R_g(x) = 1/Q_g(x)$ , where (1) the denominator is a nonempty product of binomials of the form  $(1 - x^i)$ , and (2) the degree of the numerator is strictly smaller than the degree of the denominator. Clearly both conditions are retained when all  $R_g(x)$  are expanded to have a common denominator; call it  $Q(x)$ .

Taking the sum of the expanded rational functions and dividing by the constant  $|G|$ , we obtain  $B(x)$  as a rational function  $P(x)/Q(x)$  with  $\deg P < \deg Q$ , and  $Q(x)$  consisting of a product of binomials of the form  $(1 - x^i)$ , with  $i \in \{1, 2, \dots, c\}$ .

Let  $N = \text{lcm}(1, 2, \dots, c)$ . Every root  $\alpha$  of  $Q$  is a root of  $(1 - x^i)$  for some  $i \in \{1, 2, \dots, c\}$ , thus  $\alpha^i = 1$  and also  $\alpha^N = 1$ .

The conditions of Proposition 1 are satisfied, so the claim follows.  $\square$

A similar result follows for  $R(c, a)$ , the number of rank-3 lattices, seen as a function of  $a$  with  $c$  fixed. But we have to be careful with the initial terms of the sequence. We say that two functions  $f$  and  $g$  agree from  $n_0$  if  $f(n) = g(n)$  for all  $n \geq n_0$ . If  $\Gamma$  is a connection graph and  $a \geq 0$ , we write  $R(\Gamma, a)$  for the number of rank-3 lattices with the connection graph  $\Gamma$  and  $a$  atoms in total.

**Proposition 3.** *For any fixed  $c \geq 1$ , the function  $a \mapsto R(c, a)$  agrees from  $\binom{c}{2}$  with a quasipolynomial in  $a$ , of quasiperiod  $N = \text{lcm}(1, 2, \dots, c)$ .*

*Proof.* Let  $N = \text{lcm}(1, 2, \dots, c)$  and  $n_0 = \binom{c}{2}$ .

Let  $\Gamma$  be any connection graph of  $c$  connectors, and let it have  $r$  connectors and  $s$  coatoms lacking a connector. It is clear that  $r + s \leq n_0$ . For  $a < r + s$ , we have  $R(\Gamma, a) = 0$ , and for  $a \geq r + s$  we have  $R(\Gamma, a) = P_G(a - r - s)$ . Clearly  $R(\Gamma, a)$  agrees with a quasipolynomial of quasiperiod  $N$  from  $r + s$ , thus also from  $n_0 \geq r + s$ .

Let then  $\mathcal{G}$  be the set of all connection graphs of  $c$  coatoms. Then

$$R(c, a) = \sum_{\Gamma \in \mathcal{G}} R(\Gamma, a).$$



Since every  $R(\Gamma, a)$  agrees with a quasipolynomial of quasiperiod  $N$  from  $n_0$ , so does their sum, as the family of quasipolynomials is closed under finite addition.  $\square$

This (like Lisoněk's Theorem 4.3.5) is an existence result, and does not tell us what the quasipolynomial is. There are many ways how one may seek the exact form. One could inspect the generating function itself, perhaps prove some recurrence relations, and so on. Here we take the low road: for each residue class  $k$ , if we know that the constituent polynomial  $P_k$  has degree at most  $d$ , fit a polynomial of degree  $d$  to at least  $d + 1$  known values. If done in rational (not floating-point) arithmetic, this identifies the polynomial coefficients exactly. But first we need to have an upper bound on  $d$ , so that we know how many points we need. Fortunately we can do this by an elementary argument.

**Proposition 4.** *For any  $c \geq 1$ , there is a constant  $K$  such that  $R(c, a) \leq Ka^{c-1}$  for all  $a \geq 1$ .*

*Proof.* Let  $\Gamma$  be any connection graph with  $c$  coatoms. Suppose first that  $\text{Aut}(\Gamma)$  fixes all coatoms, that is, we have  $c$  distinct boxes. At least one atom is used by the connection graph, so the number of balls to distribute is  $n \leq a - 1$ . There are at most  $a^{c-1}$  ways to put  $a - 1$  balls to  $c$  distinct boxes, since the numbers of balls in the first  $c - 1$  boxes are a  $(c - 1)$ -tuple of integers between 0 and  $a - 1$ . Then observe that if  $\text{Aut}(\Gamma)$  does not fix all coatoms, this can only decrease the number of distributions. So for all connection graphs we have  $R(\Gamma, a) \leq a^{c-1}$ . For a fixed value of  $c$ , there is a finite collection  $\mathcal{G}$  of connection graphs, so  $R(c, a) \leq |\mathcal{G}| a^{c-1}$ . Take  $K = |\mathcal{G}|$ .  $\square$

Now, by Propositions 3 and 4, we know in particular:

- $R(4, a)$  agrees with a quasipolynomial with  $N = \text{lcm}(1, 2, 3, 4) = 12$  and degree at most 3, from  $n_0 = \binom{4}{2} = 6$ . For the fit we need  $12 \times 4 = 48$  known values.
- $R(5, a)$  agrees with a quasipolynomial with  $N = \text{lcm}(1, \dots, 5) = 60$  and degree at most 4, from  $n_0 = \binom{5}{2} = 10$ . For the fit we need  $60 \times 5 = 300$  known values.
- $R(6, a)$  agrees with a quasipolynomial with  $N = \text{lcm}(1, \dots, 6) = 60$  and degree at most 5, from  $n_0 = \binom{6}{2} = 15$ . For the fit we need  $60 \times 6 = 360$  known values.
- $R(7, a)$  agrees with a quasipolynomial with  $N = \text{lcm}(1, \dots, 7) = 420$  and degree at most 6, from  $n_0 = \binom{7}{2} = 21$ . For the fit we need  $420 \times 7 = 2940$  known values.

The values of  $n_0$  are as guaranteed by Proposition 3. Once a quasipolynomial is available, one can check if it happens to agree with some of the initial terms, and extend the range accordingly.

As we already computed the values up to  $a = 1000$  before, we have enough values to fit the polynomials up to  $c = 6$ . For  $c = 7$  we reran Algorithm 1 with  $a_{\max} = 3000$ , which took about 1100 seconds.

From the polynomial fits we obtain the following quasipolynomials. Leading terms that are common to all residue classes are collected together. For the terms that depend on residue class, we use this shorthand notation: the quantity  $[c_0, c_1, \dots, c_{M-1}]$  takes the value  $c_k$  when  $a \equiv k \pmod{M}$ . For example,  $[3, -3]$  means 3 if  $a$  is even, and  $-3$  if  $a$  is odd. Note that  $M$  may be smaller than  $N$ .

**Theorem 3.** For any  $a \geq 0$ , we have

$$R(4, a) = (97/144)a^3 - (5/6)a^2 + [44/48, 47/48] a \\ + [0, 13, 8, -45, 40, -19, 0, -5, 8, -27, 40, -37] / 72.$$

**Theorem 4.** For any  $a \geq 3$ , we have

$$R(5, a) = (175/192)a^4 - (3079/480)a^3 + (11771/480)a^2 \\ + [-7268/160, -7273/160] a + h(a),$$

where

$$h(a) = [33600, 34019, 34072, 33627, 33152, 34915, 33624, 33947, 33472, 33507, \\ 34520, 34459, 32832, 33827, 34072, 34395, 33344, 34147, 33432, 33947, \\ 34240, 33699, 33752, 34267, 32832, 34595, 34264, 33627, 33152, 34147, \\ 34200, 34139, 33472, 33507, 33752, 35035, 33024, 33827, 34072, 33627, \\ 33920, 34339, 33432, 33947, 33472, 34275, 33944, 34267, 32832, 33827, \\ 34840, 33819, 33152, 34147, 33432, 34715, 33664, 33507, 33752, 34267] / 960.$$

It was indeed necessary to be careful with initial terms. The quasipolynomial in Theorem 4 does *not* agree with  $R(5, a)$  at  $a = 0, 1, 2$ . At those points, the quasipolynomial yields 35, 9, 6, while the true values of  $R(5, a)$  are 0, 1, 5.

There does not seem to be much structure in the “quasiconstant”  $h(a)$  term of  $R(5, a)$ . It has the full period 60. Perhaps this was to be expected: from Table I we recall that  $R(5, a)$  ensues as a sum over 72 graphs having 11 different cycle indices. If the graphs have residue-dependent contributions with different periods, their joint effect will easily end up with the full period. If desired, one may try to manipulate  $h(a)$  into a form that is more pleasant to human eyes.

It would have been quite tedious to derive  $R(5, a)$  manually, even if to find just the leading terms. By the help of computations we can readily see the three leading terms, which do not depend on residue class, and which provide a fairly precise picture of the growth rate of  $R(5, a)$ . And if we want the fine details, they are there. With the explicit formula one can easily calculate, say,  $R(5, 1000000) = 911451918774522871241702$ .

For reasons of clarity, for  $c = 6$  and  $c = 7$  we show here just the leading terms that are common to all residue classes, and hide the lower order terms behind an  $O$  notation. The full explicit quasipolynomials are available separately [14].

**Theorem 5.**

$$R(6, a) = (185521/86400)a^5 - (266581/6912)a^4 + (4268287/12960)a^3 + O(a^2).$$

**Theorem 6.**

$$R(7, a) = (35406319/3628800)a^6 - (205303771/604800)a^5 \\ + (986460817/181440)a^4 - (908874965/18144)a^3 + O(a^2).$$

As a redundancy check, for all available values of  $a$ , we compared the quantities evaluated with the quasipolynomials against the values directly obtained from Algorithm 1, and observed that they agree.

Finally, let us see what the computations have to say about simple cases. By the same polynomial-fit method as above, for all  $a \geq 0$  we obtain

$$R(2, a) = a,$$

$$R(3, a) = (3/4)a^2 + (1/3)a + [0, -1, -8, 3, -4, -5] / 12,$$

giving an alternative derivation of our first basic results in Section 4.

## 8. CLOSING REMARKS

The computations in this work were facilitated by the availability of several tools. For the first phase, where the connection graphs were generated, a tool of isomorphism-free graph generation (`genbg` from the `nauty` package) was essential. The second phase required the mathematical machinery of counting theory; but for actually computing the automorphism groups and cycle indices of thousands of graphs, it was convenient that those tools were available in GAP and Digraphs. Easy-to-use arithmetic on large integers, rationals, and polynomials was also helpful.

The method of converting the generating functions to functional forms by computing initial terms and then doing a polynomial fit has a certain “snake oil” appeal. It is easy to do, provided that one has a suitable existence result that the quasipolynomial actually is there to be found. However, it may require rather long sequences to be computed. Perhaps a computational method that inspects the structure of the generating function directly would be more efficient here.

Here we considered only one kind of lattices, the rank-3 lattices. Some of the methods used here may be applicable to other low-rank lattices as well. A natural next question would be that of rank-4 lattices. What is  $R(c, m, a)$ , the number of graded lattices of  $c$  coatoms,  $m$  elements in the middle level, and  $a$  atoms? Can one simply “glue” two rank-3 lattices on top of each other? Perhaps, but with two considerations: that of isomorphism, and that of ensuring that the glued results are indeed lattices.

## REFERENCES

- [1] R. B. J. T. Allenby and Alan Slomson. *How to Count: An Introduction to Combinatorics*. Chapman and Hall/CRC, 2nd edition, 2010.
- [2] Richard M. Beekman. *An Introduction to Number Theoretic Combinatorics*. lulu.com, 2017.
- [3] J. De Beule, J. Jonusas, J. D. Mitchell, M. Torpey, and W. A. Wilson. *Digraphs – a GAP package, version 0.11.0*, 2017.
- [4] Peter J. Cameron. *Notes on Counting: An Introduction to Enumerative Combinatorics*. Cambridge University Press, 2017.
- [5] Peter J. Cameron. A counting problem. <https://cameroncounts.wordpress.com/2018/03/16/a-counting-problem/>, March 2018. [A blog post].
- [6] Marcel Erné, Jobst Heitzig, and Jürgen Reinhold. On the number of distributive lattices. *Electron. J. Comb.*, 9, 2002. Article #R24.
- [7] The GAP Group. *GAP – Groups, Algorithms, and Programming, version 4.8.10*, 2018.
- [8] Volker Gebhardt and Stephen Tawn. Constructing unlabelled lattices. *ArXiv e-prints*, September 2016.
- [9] Jobst Heitzig and Jürgen Reinhold. Counting finite lattices. *Algebr. univ.*, 48(1):43–53, 2002.
- [10] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. <https://oeis.org>, 2018.

- [11] Peter Jipsen and Nathan Lawless. Generating all finite modular lattices of a given size. *Algebr. univ.*, 74(3):253–264, 2015.
- [12] D. J. Kleitman and K. J. Winston. The asymptotic number of lattices. *Ann. Discrete Math.*, 6:243–249, 1980.
- [13] Walter Klotz and Lutz Lucht. Endliche Verbände. *J. Reine Angew. Math.*, 247:58–68, 1971.
- [14] Jukka Kohonen. Counting rank-3 lattices. <https://bitbucket.org/jkohonen/count-rank3-lattices>, 2018.
- [15] Shoji Kyuno. An inductive algorithm to construct finite lattices. *Math. Comput.*, 145:409–421, 1979.
- [16] Petr Lisoněk. *Computer-assisted Studies in Algebraic Combinatorics*. PhD thesis, Johannes Kepler Universität, Linz, September 1994.
- [17] Brendan D. McKay and Adolfo Piperno. Nauty and Traces User’s Guide (Version 2.6). <http://pallini.di.uniroma1.it/Guide.html>, 2016.
- [18] Richard P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, 2nd edition, 2012.

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$a$	$c = 3$	4	5	6	7
1	1	1	1	1	1
2	3	4	5	6	7
3	8	13	20	29	39
4	13	34	68	121	197
5	20	68	190	441	907
6	29	121	441	1384	3736
7	39	197	907	3736	13530
8	50	299	1690	8934	42931
9	64	432	2916	19298	120892
10	78	600	4734	38268	306120
11	94	806	7310	70685	706642
12	112	1055	10836	123057	1506016
13	131	1352	15528	203764	2996398
14	151	1698	21619	323383	5618515
15	174	2100	29365	494925	10008899
16	197	2561	39045	734034	17053898
17	222	3085	50961	1059330	27950691
18	249	3675	65434	1492653	44275741
19	277	4338	82809	2059229	68059684
20	306	5074	103453	2788044	101869637
21	338	5891	127751	3712081	148898469
22	370	6790	156117	4868468	213061109
23	404	7777	188980	6298878	299097442
24	440	8854	226794	8049751	412683316
25	477	10029	270037	10172443	560547117
26	515	11300	319204	12723627	750594650
27	556	12677	374813	15765529	992040210
28	597	14160	437409	19366035	1295545409
29	640	15756	507553	23599151	1673363704
30	685	17465	585831	28545198	2139494240
...					
100	7533	665370	84971972	17929736129	6858729229937
200	30066	5355739	1407988534	627979574932	524132826147936
300	67600	18112775	7211812220	4914131994972	6330705903535897
400	120133	42978145	22926705532	21021167741959	36624782962133435
500	187666	83993514	56170430969	64731346381612	142179199873933941
600	270200	145200550	116748251030	162041086855752	429521796157985802
700	367733	230640920	216652928217	351737648034289	1092140851049830127
800	480266	344356289	370064725029	687975809274792	2448715582864593496
900	607800	490388325	593351403965	1242854550978032	4988371711653746757
1000	750333	672778695	905068227527	2108993735138119	9422962085155489652

TABLE II. Some values of  $R(c, a)$  for  $c = 3, \dots, 7$  coatoms.

$a$	$c = 8$	9
1	1	1
2	8	9
3	50	64
4	299	432
5	1690	2916
6	8934	19298
7	42931	120892
8	183303	690896
9	690896	3517049
10	2310366	15818049
11	6920971	63028260
12	18783412	224257964
13	46705657	719521493
14	107510169	2102741467
15	231227596	5650968147
16	468463678	14088437189
17	900399211	32842695865
18	1651885113	72096705250
19	2908101609	149972933224
20	4935241680	297260914919
21	8105691264	564176756133
22	12928165761	1029721046925
23	20083274851	1814279741924
24	30464974385	3096191012173
25	45228381098	5133079209599
26	65844403276	8288835750730
27	94161667324	13067204701747
28	132476193092	20153009591032
29	183609295480	30462135974619
30	250994166078	45201463018088
...		
100	5078592962561811	7626564586350129874
200	880085483053191106	3142649707966986066096
300	16609587584876364182	94045317769328410172825
400	130737521692628355615	1014377064737641167135036
500	642112898798336927353	6329853496024443260170625
600	2346516577212608845729	28059449401711567076441545
700	7000760472426076825846	98420943238637719981239097
800	18015850571650533933600	291130542533101026907632456
900	41425805120978743606026	756477905666369353284138046
1000	87178719353101913391613	1775181449515604936706800068

TABLE III. Some values of  $R(c, a)$  for  $c = 8, 9$  coatoms.