Enumerating All Subgraphs without Forbidden Induced Subgraphs via Multivalued Decision Diagrams

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Abstract

We propose a general method performed over multivalued decision diagrams that enumerates all subgraphs of an input graph that are characterized by input forbidden induced subgraphs. Our method combines elaborations of classical set operations and the developing construction technique, called the frontier based search, for multivalued decision diagrams. Using the algorithm, we enumerated all the chordal graphs of size at most 10 on multivalued decision diagrams.

1 Introduction

Enumeration is a fundamental topic in computer science. Especially, *subgraph enumeration problem* is a well-studied topic. Given a graph and constraints, the problem is to output all the subgraphs satisfying the constraints in the graph. Several well-known techniques for enumeration have been proposed [9, 1] and applied to several graph classes. For example, [6] enumerates all subgraphs belonging to the class of *chordal graphs* based on reverse search [1]. These traditional algorithms enumerate subgraphs one by one explicitly and take time depending on the number of output subgraphs. Unfortunately, the number of output subgraphs is exponentially huge in the size of the input graph.

On the other hand, to approach the subgraph enumeration problem by implicit enumeration, techniques constructing a compressed representation such as *Zero-suppressed Binary Decision Diagram* (ZDD) [8] are well-studied. Computation time of such techniques does not depend on the number of subgraphs but the size of the constructed ZDDs. A classical technique such as the one proposed in [2] is to use *apply operation* and *family algebra* that are useful function of ZDD. Moreover, a novel algorithm named

frontier-based search (FBS) [5] has been developed recently. FBS has been applied for enumerating various classes of subgraphs such as paths, cycles, forests, partitions, and so on. However, graph classes handled by ZDD based techniques are limited to rather simple ones only.

In this paper, we propose a general technique that enumerates subgraphs belonging to graph classes characterized by *forbidden induced subgraphs*. Several graph classes such as chordal, interval, split, and threshold graphs are characterized by rather simple forbidden induced subgraphs, like cycles, paths, and their complements. For example, a graph is called chordal if and only if it has no cycles of size at least 4 as a vertex induced subgraph. The proposed method needs to be given a ZDD representing forbidden induced subgraphs, which we assume to be computed by an existing method or some way. Our technique consists of the following three steps, which involve FBS and family algebra over *multivalued decision diagrams (MDD)* [4]:

1. Enumerating forbidden induced subgraphs on a ZDD (in some way);

2. Adding edges induced by the forbidden induced subgraphs as an MDD by FBS;

3. Constructing a ZDD enumerating subgraphs that avoid forbidden subgraphs as induced ones by a novel operation of family algebra.

The frist step depends on the target graph class but the last two steps do not. This paper describes those two steps.

As a demonstration of our method, we enumerated chordal graphs by experiments. We succeeded in enumerating all the 215,488,096,587 chordal graphs of size 10 as a ZDD.

2   Preliminaries

2.1   Forbidden induced subgraphs

Let $G = (V, E)$ be a graph with a vertex set V and an edge set $E \subseteq \{\{u, v\} \mid u, v \in V\}$. For any vertex subset $U \subseteq V$, $E[U]$ denotes the set of edges whose end points are both included in U, i.e., $E[U] = \{\, e \in E \mid e \subseteq U \,\}$, called *induced edges (by U)*. For any edge subset $D \subseteq E$, $\bigcup D$ denotes the set of the end points of each edge in D, i.e., $\bigcup D = \bigcup_{\{u,v\} \in D} \{u, v\}$, called *induced vertices (by D)*. We call $(U, E[U])$ the *(vertex) induced subgraph (by U)*. Let $G[D] = (\bigcup D, D)$, called the *edge induced subgraph (by D)*. This paper often identifies an edge induced subgraph $G' = (\bigcup D, D)$ and the edge set D.

Some graph classes are characterized by forbidden subgraphs. We say that a graph class $\mathcal{G}$ is *FIS-characterized* by a graph class $\mathcal{F}$ if $\mathcal{G}$ consists of graphs $G = (V, E)$ such that none of the vertex subsets of V induces a graph belonging to $\mathcal{F}$, i.e.,

$$(V, E) \in \mathcal{G} \iff \forall U \subseteq V, (U, E[U]) \notin \mathcal{F}.$$

For example, the class of chordal graphs is FIS-characterized by the class of cycles of size at least 4.

2.2 Multi-valued Decision Diagrams

A k-*colored subset* of a finite set E is a k-tuple $\vec{D} = (D_1, \ldots, D_k)$ of subsets $D_i \subseteq E$ such that $D_i \cap D_j = \emptyset$ for any distinct i and j. To represent and manipulate sets of k-colored subsets, we use k-*valued decision diagrams (k-DDs)*, which are special types of *multi-valued decision diagram* with lack of a reduction rule.

A k-DD over a finite set $E = \{e_1, \ldots, e_m\}$ is a labeled rooted directed acyclic graph $\mathbf{Z} = (N, A, \ell)$ with a node set N, an arc set A and a labeling function ℓ. The node set N has exactly one root node ρ and exactly two terminal nodes $\bot$ and $\top$. Each non-terminal node $\alpha \in N \setminus \{\top, \bot\}$ has a label $\ell(\alpha) \in \{1, \ldots, m\}$ and has exactly $k+1$ outgoing arcs called 0-arc, 1-arc, ..., and k-arc. The node pointed at by the j-arc of α is called the j-child and denoted by α_j for each $j \in \{0, 1, \ldots, k\}$. It is satisfied that $\ell(\alpha_j) = \ell(\alpha) + 1$ if α_j is not a terminal.

Each path π in a k-DD represents a k-colored subset $[\![\pi]\!] = (D_1, \ldots, D_k)$ of E defined by

$$D_j = \{\, e_{\ell(\beta)} \mid \pi \text{ includes the } j\text{-arc of } \beta \,\},$$

for $j \in \{1, \ldots, k\}$. The k-DD $\mathbf{Z}$ itself represents a set of k-colored subsets

$$[\![\mathbf{Z}]\!] = \{\, [\![\pi]\!] \mid \pi \text{ is a path from the root } \rho \text{ to the terminal } \top \,\}.$$

We call a k-DD *reduced* if there are no distinct nodes α and β such that $\ell(\alpha) = \ell(\beta)$ and $\alpha_j = \beta_j$ for all $j \in \{0, \ldots, k\}$. If a k-DD has nodes that violate this condition, those can be merged repeatedly until it becomes reduced. This reduction does not change the semantics of the k-DD.

We remark that k-DDs, 2-DDs, and 3-DDs are almost identical to MDDs, binary decision diagrams as well as zero-suppressed binary decision diagrams, and ternary decision diagrams, respectively, except a reduction rule that eliminates nodes so that the obtained data structure will be more compact. It is possible for our algorithm with slight modification to handle "zero-suppress" k-DDs, where a node can be eliminated if all the j-children for $1 \leq j \leq k$ point at the terminal $\bot$. However, for simplicity, we have defined k-DDs without employing such a reduction rule, where the label of a child node is always bigger than the parent's by one.

3 Proposed Algorithm

Suppose that a graph class $\mathcal{G}$ is FIS-characterized by $\mathcal{F}$. In this section, we present an algorithm that enumerates all the subgraphs of an input graph G belonging to $\mathcal{G}$ provided that all the forbidden induced subgraphs of G belonging to $\mathcal{F}$ is also given as an input. Hereafter we fix an input graph $G = (V, E)$ and restrict $\mathcal{G}$ and $\mathcal{F}$ to be the subgraphs of G. Here by a subgraph of G we mean a graph $G' = (V, D)$ for some $D \subseteq E$. Therefore, we may identify a graph and its edge set. That is, $\mathcal{G}$ and $\mathcal{F}$ are represented as sets of subsets of E. The set $\mathcal{F}$ is given as a 2-DD.

Recall the definition of FIS-characterization:

$$D \in \mathcal{G} \iff \forall U \subseteq \bigcup D, \, D[U] \notin \mathcal{F}.$$

3

In other words, if a graph $D \in \mathcal{G}$ contains a subgraph D' belonging to $\mathcal{F}$, then D' must induce edges with which it does not belong to $\mathcal{F}$. Our proposed method can be divided into two phases. In the first phase, as its details will be described in Section 3.1, we construct a 3-DD $\mathbf{I}$ for the set of 2-colored subsets (F_1, F_2) of E such that $F_1 \in \mathcal{F}$ and $F_2 = E[\bigcup F_1] \setminus F_1$. Then we construct a 2-DD $\mathbf{Z}$ for the set of edge sets D such that

$$\forall (F_1, F_2) \in [\![\mathbf{I}]\!], \, (F_1 \subseteq D \implies F_2 \cap D \neq \emptyset),$$

as described in Section 3.2. The following lemma ensures that our method indeed gives the desired subgraphs.

Lemma 1. *Let $D \subseteq E$. The following conditions are equivalent:*

1. $\forall U \subseteq \bigcup D, \, D[U] \notin \mathcal{F}$,

2. $\forall F \in \mathcal{F}, \, (F \subseteq D \implies (E[\bigcup F] \setminus F) \cap D \neq \emptyset)$.

Proof. ($\Leftarrow$) Suppose that D does not satisfy the first condition. There is $U \subseteq \bigcup D$ such that $(U, D[U]) \in \mathcal{F}$. Let $F = D[U]$. By definition $F \subseteq D$ and $E[\bigcup F] \setminus F \subseteq E[U] \setminus D[U] \subseteq E \setminus D$. We have $E[\bigcup F] \setminus F \cap D = \emptyset$.

($\Rightarrow$) Suppose that D does not satisfy the second condition. There is $F \in \mathcal{F}$ such that $F \subseteq D$ and $(E[\bigcup F] \setminus F) \cap D = \emptyset$. We will show that $U = \bigcup F$ disproves the first statement. Since $F \subseteq D$, clearly $U \subseteq \bigcup D$. It suffices to show that $F = D[U]$, which implies $D[U] \in \mathcal{F}$. Clearly

$$F = F[\bigcup F] \subseteq D[\bigcup F] = D[U].$$

By assumption,

$$D[U] \setminus F = (E[U] \cap D) \setminus F = (E[U] \setminus F) \cap D = \emptyset$$

and thus $D[U] \subseteq F$. $\qquad\square$

3.1 Edge induction

This subsection presents an algorithm that gives a 3-DD $\mathbf{I}$ representing $[\![\mathbf{I}]\!] = \{ (F, E[\bigcup F] \setminus F) \mid F \in [\![\mathbf{F}]\!] \}$ from an input 2-DD $\mathbf{F}$ representing a set $[\![\mathbf{F}]\!]$ of edge sets. That is, we "color" the edges induced by $\bigcup F$ with the second color for each $F \in \mathcal{F}$. Our algorithm can be seen as an instance of the so-called *frontier-based search*, which is a generic framework for enumerating all the subgraphs with a specific property from an input graph. Algorithm 1 constructs a 3-DD in a top-down manner, where the initial 3-DD has only the root node $\rho_{\mathbf{I}}$ with $\ell(\rho_{\mathbf{I}}) = 1$. By giving children to already constructed nodes, we expand the diagram. Each node α of the diagram under construction has auxiliary information called *configuration*, which is a pair (n_α, f_α) of a node n_α of the input 2-DD $\mathbf{F}$ and a map f_α from a subset $E^{\langle \ell(\alpha) \rangle} = \bigcup E^{\geq \ell(\alpha)} \cap \bigcup E^{< \ell(\alpha)}$ of V to $\{-1, 0, 1, 2\}$, where $E^{\geq i} = \{e_i, \ldots, e_m\}$ and $E^{<i} = \{e_1, \ldots, e_{i-1}\}$. No distinct nonterminal nodes have the same configuration. The first component n_α satisfies the property that for any path π from the root $\rho_{\mathbf{I}}$ to α in $\mathbf{I}$, there is a path θ from the root $\rho_{\mathbf{F}}$ to n_α in $\mathbf{F}$ such that $[\![\pi]\!] = ([\![\theta]\!], F_2)$ for some $F_2 \subseteq E$ (but not vice versa). The default value of f_α is set to $f_\alpha(u) = 0$ for all $u \in E^{\langle i \rangle}$. If it has non-zero value,

- $f_\alpha(u) = -1$ means that u must not occur in $[\![\pi]\!]$,

- $f_\alpha(u) = 1$ means that there is no v' such that $\{u, v'\}$ is colored 1 in $[\![\pi']\!]$ but there must be v such that $\{u, v\}$ is colored 1 in $[\![\pi]\!]$,

- $f_\alpha(u) = 2$ means that there is v' such that $\{u, v'\} \in E^{<i}$ is colored 1 in $[\![\pi']\!]$,

for any path π from the root $\rho_\mathbf{I}$ to the terminal $\top$ passing through α and any path π' from $\rho_\mathbf{I}$ to α. The algorithm starts with the root node $\rho_\mathbf{I}$ with configuration $(\rho_\mathbf{F}, \emptyset)$.

Algorithm 1: Inducing edges

input : a 2-DD $\mathbf{F}$ (representing forbidden induced subgraphs)
output: a 3-DD $\mathbf{I}$ (coloring the edges induced by the forbidden graphs)

let $N_1 \leftarrow \{(\rho_\mathbf{F}, \emptyset)\}$, $N_i \leftarrow \emptyset$ for $i = 2, \ldots, m$ and $N_{m+1} \leftarrow \{\top, \bot\}$;
for $i = 1, \ldots, m$ **do**
    for *each* $\alpha \in N_i$ **do**
        for $j = 0, 1, 2$ **do**
            let $\alpha_j \leftarrow \texttt{Child}(\alpha, j)$;
            if $\alpha_j \notin N_{i+1}$ **then** add a new node α_j with label $i + 1$ to N_{i+1};
            let α_j be the j-child of α;

return the 3-DD consisting of nodes of $N_1, \ldots, N_{m+1}$;

Algorithm 2 gives the configuration of the j-child of a node $\alpha \in N_i$, unless the child must be a terminal. Let the configuration of α be (β, f) and $e_i = \{u_1, u_2\}$.

Choosing the 0-arc of α means that we do not include the edge e_i in a 2-colored graph under consideration. Recall that if both u_1 and u_2 are used in a graph, then the edge e_i must be colored 1 or 2. Lines 6–9 reflect this restriction.

Choosing the j-arc with $j \geq 1$ means that the edge e_i is colored j in the resultant 2-colored graph. This case is handled on Lines 10–18. This clearly contradicts $f(u_k) = -1$ for any of $k \in \{1, 2\}$, which means that u_k must not be used. If $f(u_k) = 0$, this means that so far all edges $\{u_k, v\} \in E^{<i}$ are colored 0, i.e., they do not occur in 2-colored subgraphs under consideration. On the other hand, if $f(v) \geq 1$, this means that v will occur together with u. This contradicts that the edge $\{u_k, v\}$ remains colored 0. If $e_i = \{u_1, u_2\}$ is colored 1, this means that it is in a forbidden graph, so we let $f_j(u_k) = 2$. If $e_i = \{u_1, u_2\}$ is colored 2, this means that it is induced by an vertex in a forbidden graph, so we let $f_j(u_k) = 1$ unless $f_j(u_k) = 2$.

In addition, if $f(u_k) = 1$ and $u_k \notin \bigcup E^{\geq i+1}$, this means that u_k is supposed to have an edge colored with 1 but we have decided not to color any edges connecting u_k with 1. This restriction is checked on Line 19.

Since we do not need to remember the values of $f_j(v)$ for $v \notin E^{\langle i+1 \rangle}$ in the further computation, we restrict the domain of f_j to be $E^{\langle i+1 \rangle}$ on Line 20.

Algorithm 2: Child(α, j)

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input : node α with configuration (β, f) and a child number j

output: configuration of the j-th child α_j of α

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1 let $i \leftarrow \ell(\alpha)$ and $\{u_1, u_2\} \leftarrow e_i$;

2 **if** $j = 1$ **then** let $n_j \leftarrow \beta_1$;

3 **else** let $n_j \leftarrow \beta_0$;

4 **for** *all* $v \in E^{\langle i \rangle}$ **do** let $f_j(v) \leftarrow f(v)$;

5 **for** *all* $v \in E^{\langle i+1 \rangle} \setminus E^{\langle i \rangle}$ **do** let $f_j(v) \leftarrow 0$;

6 **if** $j = 0$ **then**

7   **if** $n_0 = \bot$ *or* $f(u_1)f(u_2) \geq 1$ **then** let $\alpha_0 \leftarrow \bot$;

8   **else if** $f(u_1) \geq 1$ **then** let $f_0(u_2) \leftarrow -1$;

9   **else if** $f(u_2) \geq 1$ **then** let $f_0(u_1) \leftarrow -1$;

10 **else**

11   **if** $n_j = \bot$ *or* $f(u_1) = -1$ *or* $f(u_2) = -1$ **then** let $\alpha_j \leftarrow \bot$;

12   **else**

13   **for** $k = 1, 2$ **do**

14   **if** $f(u_k) = 0$ **then**

15   **for** *all* $v \in E^{\langle i \rangle}$ *such that* $\{u_k, v\} \in E^{<i}$ **do**

16   **if** $f(v) \geq 1$ **then** let $\alpha_j \leftarrow \bot$;

17   **else** let $f_j(v) \leftarrow -1$;

18   let $f_j(u_k) \leftarrow \max\{f(u_k), 3 - j\}$;

19 **if** $u_k \notin \bigcup E^{\geq i+1}$ *and* $f_j(u_k) = 1$ *for some* $k \in \{1, 2\}$ **then** let $\alpha_j \leftarrow \bot$;

20 **if** $\alpha_j \neq \bot$ **then** let $\alpha_j \leftarrow (n_j, f_j \upharpoonright E^{\langle i+1 \rangle})$;

21 **return** α_j;

3.2   Enumeration of subgraphs with no forbidden induced subgraphs

We now give an operation that computes a 2-DD **D** for

$$\llbracket \mathbf{D} \rrbracket = \chi(\llbracket \mathbf{I} \rrbracket) = \{\, D \subseteq E \mid \forall (F_1, F_2) \in \llbracket \mathbf{I} \rrbracket,\ (F_1 \subseteq D \implies F_2 \cap D \neq \emptyset)\,\}$$

from a 3-DD **I**. Note that the domain of χ is 2-colored subsets of E and the codomain is (1-colored) subsets of E. When **I** represents $\llbracket \mathbf{I} \rrbracket = \{\, (F, E[\bigcup F] \setminus F) \mid F \in \llbracket \mathbf{F} \rrbracket \,\}$ for a set $\llbracket \mathbf{F} \rrbracket$ of forbidden graphs, we obtain the FIS-characterized set, by Lemma 1. We compute **D** from a 3-DD **I** in a bottom-up recursive manner.

Here we give a semantics of a node α of a k-DD by

$$\llbracket \alpha \rrbracket = \{\, \llbracket \pi \rrbracket \mid \pi \text{ is a path from } \alpha \text{ to } \top \,\}.$$

Clearly $\llbracket \mathbf{D} \rrbracket = \llbracket \rho_{\mathbf{D}} \rrbracket$ for any k-DD **D** and its root $\rho_{\mathbf{D}}$. For a set $\mathcal{I}$ of 2-colored subsets of $E^{\geq i}$, define

$$\chi_i(\mathcal{I}) = \{\, D \subseteq E^{\geq i} \mid \forall (F_1, F_2) \in \mathcal{I},\ (F_1 \subseteq D \implies F_2 \cap D \neq \emptyset)\,\}.$$

According to this definition, the base of the recursion is given by

- $\chi_{m+1}(\llbracket \perp \rrbracket) = \chi_{m+1}(\emptyset) = \{\emptyset\}$,

- $\chi_{m+1}(\llbracket \top \rrbracket) = \chi_{m+1}(\{\emptyset, \emptyset\}) = \emptyset$.

For $i \leq m$, it holds that

$$\chi_i(\mathcal{I}) = (\chi_{i+1}(\mathcal{I}_0) \cap \chi_{i+1}(\mathcal{I}_2)) \cup (e_i * (\chi_{i+1}(\mathcal{I}_0) \cap \chi_{i+1}(\mathcal{I}_1))), \qquad (1)$$

where

$$e * \mathcal{D} = \{ \{e\} \cup D \mid D \in \mathcal{D} \} \text{ for any family } \mathcal{D} \text{ of (1-colored) subsets of } E,$$
$$\mathcal{I}_0 = \{ (F_1, F_2) \in \mathcal{I} \mid e_i \notin F_1 \cup F_2 \},$$
$$\mathcal{I}_1 = \{ (F_1 \setminus \{e_i\}, F_2) \mid e_i \in F_1, (F_1, F_2) \in \mathcal{I} \},$$
$$\mathcal{I}_2 = \{ (F_1, F_2 \setminus \{e_i\}) \mid e_i \in F_2, (F_1, F_2) \in \mathcal{I} \}.$$

That is, if $\mathcal{I} = \llbracket \alpha \rrbracket$, then $\mathcal{I}_0 = \llbracket \alpha_0 \rrbracket$, $\mathcal{I}_1 = \llbracket \alpha_1 \rrbracket$ and $\mathcal{I}_2 = \llbracket \alpha_2 \rrbracket$. If a 2-DD has a node β with label e_i such that $\llbracket \beta \rrbracket = \chi_i(\mathcal{I})$, then $\llbracket \beta_0 \rrbracket = \chi_{i+1}(\mathcal{I}_0) \cap \chi_{i+1}(\mathcal{I}_2)$ and $\llbracket \beta_1 \rrbracket = \chi_{i+1}(\mathcal{I}_0) \cap \chi_{i+1}(\mathcal{I}_1)$.

Equation (1) is justified by the following observation. Let us partition $\chi_i(\mathcal{I})$ into two depending on whether a set contains e_i, i.e., $\chi_i(\mathcal{I}) = \mathcal{D}_0 \cup (e_i * \mathcal{D}_1)$ where no sets in $\mathcal{D}_0 \cup \mathcal{D}_1$ contain e_i. Then by definition,

$$\mathcal{D}_0 = \{ D \subseteq E^{\geq i} \mid e_i \notin D \wedge \forall (F_1, F_2) \in \mathcal{I}, (F_1 \subseteq D \implies F_2 \cap D \neq \emptyset) \}$$
$$= \{ D \subseteq E^{\geq i+1} \mid \forall (F_1, F_2) \in \mathcal{I}_0, (F_1 \subseteq D \implies F_2 \cap D \neq \emptyset)$$
$$\wedge \forall (F_1, F_2) \in \mathcal{I}_1, (\{e_i\} \cup F_1 \subseteq D \implies F_2 \cap D \neq \emptyset)$$
$$\wedge \forall (F_1, F_2) \in \mathcal{I}_2, (F_1 \subseteq D \implies (\{e_i\} \cup F_2) \cap D \neq \emptyset) \}.$$

For $D \subseteq E^{\geq i+1}$, the condition $\{e_i\} \cup F_1 \subseteq D$ can never be true. In addition, $(\{e_i\} \cup F_2) \cap D \neq \emptyset$ if and only if $F_2 \cap D \neq \emptyset$. Hence,

$$\mathcal{D}_0 = \{ D \subseteq E^{\geq i+1} \mid \forall (F_1, F_2) \in \mathcal{I}_0 \cup \mathcal{I}_2, (F_1 \subseteq D \implies F_2 \cap D \neq \emptyset) \}$$
$$= \chi_{i+1}(\mathcal{I}_0) \cap \chi_{i+1}(\mathcal{I}_2).$$

On the other hand,

$$\mathcal{D}_1 = \{ D \subseteq E^{\geq i+1} \mid \forall (F_1, F_2) \in \mathcal{I}, (F_1 \subseteq \{e_i\} \cup D \implies F_2 \cap (\{e_i\} \cup D) \neq \emptyset) \}$$
$$= \{ D \subseteq E^{\geq i+1} \mid \forall (F_1, F_2) \in \mathcal{I}_0, (F_1 \subseteq \{e_i\} \cup D \implies F_2 \cap (\{e_i\} \cup D) \neq \emptyset)$$
$$\wedge \forall (F_1, F_2) \in \mathcal{I}_1, (\{e_i\} \cup F_1 \subseteq \{e_i\} \cup D \implies F_2 \cap (\{e_i\} \cup D) \neq \emptyset)$$
$$\wedge \forall (F_1, F_2) \in \mathcal{I}_2, (F_1 \subseteq \{e_i\} \cup D \implies (\{e_i\} \cup F_2) \cap (\{e_i\} \cup D) \neq \emptyset) \}.$$

Obviously, the condition $(\{e_i\} \cup F_2) \cap (\{e_i\} \cup D) \neq \emptyset$ is always true. Recall that $e_i \notin F_1 \cup F_2$ for $(F_1, F_2) \in \mathcal{I}_0$ and that $e_i \notin F_2$ for $(F_1, F_2) \in \mathcal{I}_1$. By simplifying the formula, we obtain

$$\mathcal{D}_1 = \{ D \subseteq E^{\geq i+1} \mid \forall (F_1, F_2) \in \mathcal{I}_0, (F_1 \subseteq D \implies F_2 \cap D \neq \emptyset)$$
$$\wedge \forall (F_1, F_2) \in \mathcal{I}_1, (F_1 \subseteq D \implies F_2 \cap D \neq \emptyset) \}$$
$$= \chi_{i+1}(\mathcal{I}_0) \cap \chi_{i+1}(\mathcal{I}_1).$$

Algorithm 3: Computing a 2-DD for $\chi(\llbracket \mathbf{I} \rrbracket)$ from a 3-DD $\mathbf{I}$

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    input : node α of a 3-DD

    output: node β of a 2-DD such that $\llbracket \beta \rrbracket = \chi(\llbracket \alpha \rrbracket)$

    if $\alpha = \bot$ **then return** $\top$;

    else if $\alpha = \top$ **then return** $\bot$;

    else return a node with label $e_{\ell(\alpha)}$ whose 0-child represents $\chi(\llbracket \alpha_0 \rrbracket) \cap \chi(\llbracket \alpha_2 \rrbracket)$

      and 1-child represents $\chi(\llbracket \alpha_0 \rrbracket) \cap \chi(\llbracket \alpha_1 \rrbracket)$;

Algorithm 3 computes a 2-DD for $\chi(\llbracket \mathbf{I} \rrbracket)$ from (the root of) a 3-DD $\mathbf{I}$ based on Equation (1).

4 Experiments

In this section, we show experimental results of constructing 2-DDs and 3-DDs for chordal graphs to confirm the performance of our algorithm. For a given graph G, the 2-DD for all the cycles on G can be constructed by conventional frontier-based search [7]. The 2-DD for all the subgraphs of G that have a specified number of edges can be constructed by the method by Kawahara et al. [5]. Since both methods can be easily combined [5], we can obtain the 2-DD $\mathbf{F}_{\text{cho}}$ representing all the cycles with size at least four on G. By applying the algorithm in Sec. 3.1 with $\mathbf{F} = \mathbf{F}_{\text{cho}}$, we obtain the 3-DD, say $\mathbf{I}_{\text{cho}}$, and by applying the algorithm in Sec. 3.2 with $\mathbf{I} = \mathbf{I}_{\text{cho}}$, we have the 2-DD, say $\mathbf{Z}_{\text{cho}}$, for $\chi(\llbracket \mathbf{I}_{\text{cho}} \rrbracket)$, which represents the set of all the chordal subgraphs of G.

To see the scalability and bottleneck of our algorithm, we run it for complete (vertex-labeled) graphs with n vertices. Giving a complete graph as the input means that we obtain the set of all the chordal labeled (not necessarily connected) graphs with at most n vertices as a 2-DD. We implemented our algorithm in the C++ language using the TdZdd library [3] for the construction of DDs in a top-down manner. Our implementation was complied by $\texttt{g++}$ with the $\texttt{-O3}$ optimization option and run on a machine with Intel Xeon E5-2630 (2.30GHz) CPU and 128GB memory (Linux Centos 7.4).

Table 1 shows the running time and memory usage of algorithms. "Const. X time" in the table indicates the time (in seconds) for constructing the (2- or 3-) DD X. "Mem 1" shows the maximum memory usage (in MB) during constructing $\mathbf{F}_{\text{cho}}$ and $\mathbf{I}_{\text{cho}}$ (obtained by calling $\texttt{getmaxrss()}$ function after their construction finishes). "Mem 2" shows the maximum memory usage during constructing $\mathbf{Z}_{\text{cho}}$ measured by a program whose input is $\mathbf{I}_{\text{cho}}$ (that is, the usage does not include that of "Mem 1"). "OOM" means out of memory (i.e., the memory usage exceeds 128GB). We can confirm that our algorithm spent most of the time constructing $\mathbf{Z}_{\text{cho}}$.

Table 2 shows the number of non-terminal nodes of $\mathbf{F}_{\text{cho}}$, $\mathbf{I}_{\text{cho}}$ and $\mathbf{Z}_{\text{cho}}$ and that of graphs (i.e., the cardinality of the family of sets represented by DDs) in $\mathbf{F}_{\text{cho}}$ and $\mathbf{Z}_{\text{cho}}$. Note that the cardinality of the family represented by $\mathbf{F}_{\text{cho}}$ is the same as $\mathbf{I}_{\text{cho}}$. The cardinality of a family represented by a DD can be easily computed by a simple

Table 1: Running time (sec.) and memory usage (MB) for complete graphs with n vertices.

n	Const. $\mathbf{F}_{cho}$ time	Const. $\mathbf{I}_{cho}$ time	Const. $\mathbf{Z}_{cho}$ time	Mem 1	Mem 2
2	0.000	0.000	0.000	28	28
3	0.000	0.000	0.000	28	28
4	0.001	0.000	0.001	28	28
5	0.001	0.001	0.001	28	28
6	0.002	0.003	0.004	28	28
7	0.005	0.011	0.030	29	28
8	0.014	0.041	0.333	29	32
9	0.040	0.123	14.664	32	508
10	0.109	0.496	692.666	45	15738
11	0.323	1.574	OOM	92	OOM

Table 2: Number of nodes of 2-DDs and 3-DDs and that of graphs represented by the 2-DDs. "# cycles" means that the number of cycles with length at least four.

n	Const. $\mathbf{F}_{cho}$ # node	Const. $\mathbf{I}_{cho}$ # node	Const. $\mathbf{Z}_{cho}$ # node	# cycles	# Chordal labeled graphs
2	0	1	1	0	2
3	0	4	3	0	8
4	12	23	17	3	61
5	54	176	106	27	822
6	202	921	849	177	18154
7	717	4883	8768	1137	617675
8	2483	21959	111520	7962	30888596
9	8569	119624	1736915	62730	2192816760
10	29884	498703	32470737	555894	215488096587
11	105789	2324022	OOM	5487894	OOM

dynamic programming-based algorithm [7] in time proportional to the number of nodes in the DD. The numbers appearing in the column "# Chordal labeled Graphs" coincide those in the sequence A058862 in OEIS [10].

References

[1] David Avis and Komei Fukuda. Reverse search for enumeration. *Discrete Applied Mathematics*, 65(1-3):21–46, 1996.

[2] Olivier Coudert. Solving graph optimization problems with ZBDDs. In *Proc. of European Design and Test Conference*, pages 224–228, 1997.

[3] Hiroaki Iwashita and Shin-ichi Minato. Efficient top-down ZDD construction techniques using recursive specifications. *TCS Technical Reports*, TCS-TR-A-13-69, 2013.

[4] Timothy Kam, Tiziano Villa, and R.K. Brayton. Multi-valued decision diagrams: Theory and applications," multiple-valued logic. 4, Jan 1998.

[5] Jun Kawahara, Takeru Inoue, Hiroaki Iwashita, and Shin-ichi Minato. Frontier-based search for enumerating all constrained subgraphs with compressed representation. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, E100-A(9):1773–1784, 2017.

[6] Masashi Kiyomi and Takeaki Uno. Generating chordal graphs included in given graphs. *IEICE Transactions on Information and Systems*, E89-D(2):763–770, 2006.

[7] Donald E. Knuth. *The art of computer programming, Vol. 4A, Combinatorial algorithms, Part 1*. Addison-Wesley, 2011.

[8] Shin-ichi Minato. Zero-suppressed BDDs for set manipulation in combinatorial problems. In *Proc. of the 30th ACM/IEEE design automation conference*, pages 272–277, 1993.

[9] Robert. C. Read and Robert. E. Tarjan. Bounds on backtrack algorithms for listing cycles, paths, and spanning trees. *Networks*, 5:237–252, 1975.

[10] Neil J. A. Sloane. The encyclopedia of integer sequences, `http://oeis.org/A058862` (accessed 11 Apr. 2018).