CURIOUS CONJECTURES ON THE DISTRIBUTION OF PRIMES AMONG THE SUMS OF THE FIRST $2n$ PRIMES

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ABSTRACT. Let p_n be nth prime, and let $(S_n)_{n=1}^{\infty} := (S_n)$ be the sequence of the sums of the first $2n$ consecutive primes, that is, $S_n = \sum_{k=1}^{2n} p_k$ with $n = 1, 2, \ldots$. Heuristic arguments supported by the corresponding computational results suggest that the primes are distributed among sequence (S_n) in the same way that they are distributed among positive integers. In other words, taking into account the Prime Number Theorem, this assertion is equivalent to

$$\#\{p : p \text{ is a prime and } p = S_k \text{ for some } k \text{ with } 1 \le k \le n\}$$

$$\sim \#\{p : p \text{ is a prime and } p = k \text{ for some } k \text{ with } 1 \le k \le n\} \sim \frac{\log n}{n} \text{ as } n \to \infty,$$

where $|S|$ denotes the cardinality of a set S. Under the assumption that this assertion is true (Conjecture 3.3), we say that (S_n) satisfies the Restricted Prime Number Theorem. Motivated by this, in Sections 1 and 2 we give some definitions, results and examples concerning the generalization of the prime counting function $\pi(x)$ to increasing positive integer sequences.

The remainder of the paper (Sections 3-7) is devoted to the study of mentioned sequence (S_n). Namely, we propose several conjectures and we prove their consequences concerning the distribution of primes in the sequence (S_n). These conjectures are mainly motivated by the Prime Number Theorem, some heuristic arguments and related computational results. Several consequences of these conjectures are also established.

1. Introduction, Motivation and Preliminaries

An extremely difficult problem in number theory is the *distribution of the primes* among the natural numbers. This problem involves the study of the asymptotic behavior of the counting function $\pi(x)$ which is one of the more intriguing functions in number theory. The function $\pi(x)$ is defined as the number of primes $\le x$. For elementary methods in the study of the distribution of prime numbers, see [12].

Although questions in number theory were not always mathematically *en vogue*, by the middle of the nineteenth century the problem of counting primes had attracted the attention of well-respected mathematicians such as Legendre, Tchébychev, and the prodigious Gauss.

A query about the frequency with which primes occur elicited the following response: *I pondered this problem as a boy, and determined that, at around x, the primes occur with density* $1/\log x$–C. F. Gauss (letter to Encke, 24 December 1849). Gauss wrote:

This remark of Gauss can be interpreted as predicting that

$$\#\{\text{primes } \le x\} \approx \sum_{n=2}^{\lfloor x\rfloor} \frac{1}{\log n} \approx \int_2^x \frac{dt}{\log t} = \mathrm{Li}(x).$$

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Studying tables of primes, C. F. Gauss in the late 1700s and A.-M. Legendre in the early 1800s conjectured the celebrated *Prime Number Theorem*:

$$\pi(x)=|\{p\le x:\, p \text{ prime}\}|\sim\frac{x}{\log x}$$

(here, as always in the sequel, $|S|$ denotes the cardinality of a set S).

This theorem was proved much later ([8, p. 10, Theorem 1.1.4]; for its simple analytic proof see [31] and [46], and for its history see [3], [21], [22] and [28, p. 21]. Briefly, $\pi(x)\sim x/\log x$ as $x\to\infty$, or in other words, the density of primes $p\le x$ is $1/\log x$; that is, the ratio $\pi(x)\,:\,(x/\log x)$ converges to 1 as x grows without bound. Using L'Hôpital's rule, Gauss showed that the logarithmic integral $\int_2^x dt/\log t$, denoted by $\mathrm{Li}(x)$, is asymptotically equivalent to $x/\log x$. Recall that Gauss felt that $\mathrm{Li}(x)$ gave better approximations to $\pi(x)$ than $x/\log x$ for large values of x.

Though unable to prove the Prime Number Theorem, several significant contributions to the proof of Prime Number Theorem were given by P. L. Chebyshev in his two important 1851–1852 papers ([6] and [7]). Chebyshev proved that there exist positive constants c_1 and c_2 and a real number x_0 such that $c_1x/\log x\le\pi(x)\le c_1x/\log x$ for $x>x_0$. In other words, $\pi(x)$ increases as $x\log x$. Using methods of complex analysis and the ingenious ideas of Riemann (forty years prior), this theorem was first proved in 1896, independently by J. Hadamard and C. de la Vallée-Poussin (see e.g., [33, Section 4.1]).

A *generalized prime system* (or *g-prime system*) $\mathcal{G}$ is a sequence of positive real numbers $q_1,q_2,q_3,\ldots$ satisfying $1<q_1\le q_2\le\cdots\le q_n\le q_{n+1}\le\cdots$ and $q_n\to\infty$ as $n\to\infty$. From these can be formed the system $\mathcal{N}$ of *generalized integers* or *Beurling integers*; that is, the numbers of the form $q_1^{k_1}q_2^{k_l}\cdots q_l^{k_l}$, where $l\in\mathbb{N}$ and $k_1,k_2,\ldots,k_l\in\mathbb{N}_0:=\mathbb{N}\bigcup\{0\}$. Notice that $\mathcal{N}$ denotes the *multiplicative semigroup* generated by $\mathcal{G}$, and it consists of the unit 1 together with all finite power-products of g-primes, arranged in increasing order and counted with multiplicity.

Clearly, this system generalizes the notion of primes and positive integers obtained from them. Such systems were first introduced by A. Beurling [5] and have been studied by many authors since then (see in particular [4], [2], [11], [13], [25], [32] and [47]). In particular, Nyman [32] and Malliavin [25] sharpened Beurling's results in various ways.

Much of the theory concerns connecting the asymptotic behaviour of *g-prime counting function* and *g- counting function* $\pi_{\mathcal{G}}(x)$ and $N_{\mathcal{G}}(x)$, defined on $[1,\infty)$ respectively by

$$\pi_{\mathcal{G}}(x)=\sum_{q\in\mathcal{G},q\le x}1\quad\text{and}\quad N_{\mathcal{G}}(x)=\sum_{n\in\mathcal{N},n\le x}1,$$

where in the first sum the summation is taken over all g-primes, counting multiplicities. Similarly, for the second sum $\sum_{n\in\mathcal{N},n\le x}1$. Accordingly, we have

$$\pi_{\mathcal{G}}(x)=\#\{i:\,q_i\in\mathcal{G}\text{ and }g_i\le x\}\quad\text{and}\quad N_{\mathcal{G}}(x)=\#\{i:\,n_i\in\mathcal{N}\text{ and }n_i\le x\}.$$

If $\mathcal{G}=\{a_1,a_2,\ldots,a_n,a_{n+1},\ldots\}=(a_n)_{n=1}^{\infty}$ is a sequence such that $a_1\le a_2\le\cdots\le a_n\le a_{n+1}\le\cdots$, then obviously, we have $\pi_{\mathcal{G}}(a_n)=n$ for each $n\in\mathbb{N}$.

In 1937 Beurling proved [5, Théorème IV] that if $N_{\mathcal{G}}$ satisfies the asymptotic relation $N_{\mathcal{G}}(x)=Ax+O(x/\log^{\gamma}x)$ with some constants $A>0$ and $\gamma>3/2$, then the number of q_n's such that $q_n\le x$ is equal to $x/\log x+o(x/\log x)$, i.e.,

$$\pi_{\mathcal{G}}(x)=\frac{x}{\log x}+o\left(\frac{x}{\log x}\right).$$

In other words, the conclusion of the Prime Number Theorem (in the sequel, shortly written as PNT) is valid for such a system $\mathcal{G}$ (from this reason often called a *Beurling prime number system*). Beurling also gave an example in which $N_{\mathcal{G}}(x) = Ax + O(x/\log^{3/2} x)$ but PNT is not valid. This result was refined in 1969 by H. G. Diamond [9, Theorem (B)]. In 1970 Diamond [10] also proved that Beurling's condition is sharp, namely, the PNT does not necessarily hold if $\gamma = 3/2$.

In particular, if $\mathcal{G}$ is a set $\mathcal{P} := \{p_1, p_2, p_3, \ldots\}$ of all primes $2 = p_1 < p_2 < p_3 < \cdots$ with the associated multiplicative semigroup $\mathcal{N} = \mathbb{N} = \{1, 2, 3, \ldots\}$, then PNT states that

$$\pi(x) \sim \frac{x}{\log x}, \quad \text{as } x \to \infty,$$

where $\pi(x)$ is the usual *prime counting function*, that is,

$$\pi(x) = \sum_{p \text{ prime},\ p<x} 1.$$

As observed in [4, Introduction], the additive structure of the positive integers is not particularly relevant to the distribution of primes. Therefore, for a given g-prime system $\mathcal{G}$ defined above, it can be of interest to consider the distribution of g-primes (the elements in $\mathcal{G}$) with respect to certain associated system of generalized integers without any algebraic (multiplicative) structure. This means that the associated system $\mathcal{N}$ to $\mathcal{G}$ defined above may be some subset of $[1, +\infty)$ which is not a multiplicative semigroup (generated by $\mathcal{G}$).

In particular, here we mainly consider the case when $\mathcal{G}$ is an infinite set of primes and the associated system $\mathcal{N}$ to $\mathcal{G}$ is an increasing integer sequence $(a_n)_{n=1}^{\infty}$. We focus our attention when $\mathcal{G}$ is a set of all primes whose associated system $\mathcal{N}$ is the sequence $(a_n)_{n=1}^{\infty} := (\sum_{i=1}^{2n} p_i)_{n=1}^{\infty}$ where $2 = p_1 < p_2 < \cdots < p_n < \cdots$ are all the primes.

Let $(\mathcal{G}, \mathcal{N} := (a_k)_{k=1}^{\infty})$ be a pair defined above. Then we define its *counting function* $N_{\mathcal{G},(a_k)}(x)$ as

$$N_{\mathcal{G},(a_k)}(x) = \#\{i : \ i \in \mathbb{N} \text{ and } a_i \leq x\}.$$

Furthermore, the *prime counting function* for $(\mathcal{G}, \mathcal{N} := (a_k)_{k=1}^{\infty})$ is the function $x \mapsto \pi_{\mathcal{G},(a_k)}(x)$ defined on $[1, \infty)$ as

$$\pi_{\mathcal{G},(a_k)}(x) = \#\{q : \ q \in \mathcal{G} \text{ and } q = a_i \text{ for some } i \text{ with } a_i \leq x\}. \tag{1}$$

Heuristic and computational results show that for many "natural pairs" $(\mathcal{G}, \mathcal{N} := (a_k)_{k=1}^{\infty})$ the associated counting function $N_{\mathcal{G},(a_k)}(x)$ has certain asymptotic growth as $x \to \infty$. Notice that for each $n \in \mathbb{N}$ we have

$$\pi_{\mathcal{G},(a_k)}(a_n) = \#\{q : \ q \in \mathcal{G} \text{ and } q = a_i \text{ for some } i \text{ with } 1 \leq i \leq n\}. \tag{2}$$

The *normalizable prime counting function* for $(\mathcal{G}, \mathcal{N} = (a_k)_{k=1}^{\infty})$ is the function $(n, x) \mapsto p_{\mathcal{G},(a_k)}(n, x)$ defined for $(n, x) \in \mathbb{N} \times [1, +\infty)$ as

$$p_{\mathcal{G},(a_k)}(n, x) = \frac{\log a_n}{a_n} \pi_{\mathcal{G},(a_k)}(x). \tag{3}$$

The above expression induces the companion sequence $(b_n)_{n=1}^{\infty}$ of $(a_k)_{k=1}^{\infty}$ defined as

$$b_n = p_{\mathcal{G},(a_k)}(a_n) = \frac{\log a_n}{a_n} \pi_{\mathcal{G},(a_k)}(a_n), \ n = 1, 2, \ldots. \tag{4}$$

We also define another "companion" sequence $(c_n)_{n=1}^{\infty}$ of $(a_k)_{k=1}^{\infty}$ defined as

$$c_n = \frac{a_n}{(\log n)(\log a_n)} = \frac{\pi_{\mathcal{G},(a_k)}(a_n)}{b_n \log n},\ n = 1, 2, \ldots . \tag{5}$$

Here as always in the sequel, we will suppose that $\mathcal{G}$ is a set of all primes whose associated system $\mathcal{N}$ is a sequence $(a_k)_{k=1}^{\infty}$. For brevity, in the sequel the functions defined by (1), (2), (3) and (4) will be denoted by $p_{(a_k)}(x)$, $\pi_{(a_k)}(a_n)$, $p_{(a_k)}(n, x)$ and $p_{(a_k)}(a_n)$, respectively.

Definition 1.1. Let Ω be a set of all nonnegative continuous real functions defined on $(1, +\infty)$ and let $(a_k)_{k=1}^{\infty} := (a_k)$ be an increasing sequence of positive integers. We say that (a_k) is a *prime-like sequence* if there exists the function $\omega_{(a_k)} = \omega \in \Omega$ such that the function $n \mapsto \pi_{(a_k)}(a_n)$ defined by (2) is asymptotically equivalent to $\omega(n)$ as $n \to \infty$. Then we say that a sequence (a_k) satisfies the *ω-Restricted Prime Number Theorem*.

In particular, if $\omega(x) \sim x/\log x$ as $x \to \infty$, then we say that a sequence (a_k) satisfies the *Restricted Prime Number Theorem* (RPNT).

Proposition 1.2. *Let $(a_k)_{k=1}^{\infty}$ be a positive integer sequence, and let $(b_n)_{n=1}^{\infty}$ be its companion sequence defined by* (4)*. Then*

$$\limsup_{n\to\infty} b_n \le 1. \tag{6}$$

Proof. Taking the obvious inequality $\pi_{(a_k)}(a_n) \le \pi(a_n)$ with $n = 1, 2, \ldots$ into (4) we get

$$b_n \le \frac{\pi(a_n)\log a_n}{a_n},\ n = 1, 2, \ldots,$$

which by the Prime Number Theorem immediately yields

$$\limsup_{n\to\infty} b_n \le \lim_{n\to\infty} \frac{\pi(a_n)\log a_n}{a_n} = 1,$$

as desired. □

Proposition 1.3. *Let (a_n) be a prime-like sequence with the associated function $\omega(x)$. Then*

$$\limsup_{n\to\infty} \frac{\omega(n)}{\pi(a_n)} \le 1. \tag{7}$$

This means that $\omega(n)$ grows slowly than $\pi(a_n)$ as $n \to \infty$.

Proof. Notice that the inequality (7) is equivalent to

$$\limsup_{n\to\infty} \frac{\log a_n}{a_n} \pi_{(a_k)}(a_n) \le 1. \tag{8}$$

Since by the assumption, $\omega(n) \sim \pi_{(a_k)}(a_n)$, the inequality (8) yields

$$\limsup_{n\to\infty} \frac{\log a_n}{a_n} \omega(n) \le 1,$$

whence, in view of the fact that $\log a_n/a_n \sim 1/\pi(a_n)$, immediately follows (7). □

Remark **1.4.** The inequality (6) is sharp since by the Prime Number Theorem (see Example 2.1), equality in (6) holds for the sequences $a_k = k$ with $k = 1, 2, \ldots$.

Remark **1.5.** If a sequence (a_k) satisfies the *ω-Restricted Prime Number Theorem*, then by (4) we have

$$\omega(n) \sim \pi_{(a_k)}(a_n) = \frac{a_n b_n}{\log a_n} \quad \text{as } n \to \infty. \tag{9}$$

Remark **1.6.** Let (a_k) be a sequence satisfying the ω- Restricted Prime Number Theorem. Then clearly, $\omega_{(a_k)}(n) \le n$ for all sufficiently large n. Moreover, $\omega_{(a_k)}(n) \sim n$ as $n \to \infty$ if and only if the density of primes in a sequence (a_k) is equal to 1.

Notice also that by the Prime Number Theorem, $\omega_{\mathbb{N}}(x) = x/\log x$ for the sequence of all positive integers $\mathbb{N} = \{1, 2, \dots n, \dots\}$, that is, $\mathbb{N}$ satisfies the Restricted Prime Number Theorem (cf. Conjecture 3.3).

Here, as always in the sequel, $\mathcal{P} = (p_n) := \{p_1, p_2, \dots p_n, \dots\}$ will denote the set of all primes, where $2 = p_1 < 3 = p_2 < p_3 < \cdots < p_n < \cdots$. Moreover, (a_n) will always denotes an infinite strictly increasing sequence of positive integers. Hence, for such a sequence must be $a_n \ge n$ for each $n \in \mathbb{N}$.

The remainder of the paper is organized as follows. In Section 2 we present five examples concerning the determination of the function $\omega_{(a_k)}(x)$ and a sequence (b_n) associated to a given sequence (a_k). In particular, we consider the sequence $(a_k)_{k=1}^{\infty}$ with $a_k = a + (k-1)d$, where $a \ge 1$ and $d > 1$ are relatively prime integers.

In Section 3 we consider the distribution of primes in the sequence $(S_n)_{n=1}^{\infty}$ whose terms are given by $S_n = \sum_{i=1}^{2n} p_i$, where p_i is the ith prime. Heuristic arguments supported by related computational results suggest the curious conjecture that the sequence (S_n) satisfies the Restricted Prime Number Theorem (Conjecture 3.3). In other words, this means that the primes are distributed amongst all the terms of the sequence (S_n) in the same way that they are distributed amongst all the positive integers. Under this conjecture, we prove that if q_k is the kth prime in $(S_n)_{n=1}^{\infty}$, then $q_k \sim 2k^2 \log^3 k \sim 2p_k^2 \log k$ as $k \to \infty$ (Corollaries 3.6 and 3.7).

Assuming that Conjecture 3.3 is true, in Section 4 we give the asymptotic expression for the kth prime in the sequence (S_n) (Corollary 4.2); namely, $q_k \sim 2k^2 \log^3 k$ as $k \to \infty$. This result is refined by Theorem 4.4. We also conjecture that $\lfloor k \log k \rfloor + 1 \le m$ for each pair (k, m) of positive integers with $k \ge 1$ and $q_k = S_m$ (Conjecture 4.6). Some consequences of Conjectures 3.3 and 4.6 are also presented.

Section 5 is devoted to the estimations of the values M_k $(k = 1, 2, \dots)$ involving in the expression for q_k from Theorem 4.4. We also propose some other conjectures concerning the sequences (S_k) and (M_k). Related consequences are also established.

The conjectures presented in this paper, as well as some their consequences, are mainly supported by some computational results given in Section 6. In particular, the number $\pi_n := k$ of primes in the set $\mathcal{S}_n := \{S_1, S_2, \dots, S_n\}$ for 38 values of n up to $10^9 + 5 \cdot 10^8$ are presented in Table 1. For such values k and the associated indices m such that $q_k = S_m$, the corresponding approximate values of q_k, M_k (together with lower and upper bounds of M_k), $(k \log k)/m$ and $S_m \sqrt{k \log k}/(2m^{5/2} \log m)$ are also given in this table. Under the previous notations, related numerical results for $q_k/(2k^2 \log^3 k)$, $q_k/(2m^2 \log m)$ and two estimates involving q_k which are discussed in Section 4, are given in Table 3. Some additional computational results, the conjectures and their consequences are also given in Section 6.

In the last Section 7 we propose the stroner (asymptotic) version of Conjecture 3.3 which coincides with well known form of Prime Number Theorem involving the function $\mathrm{li}(x)$.

Notice that similar considerations to those for the sequence (S_n) concerning alternating sums of consecutive primes are given in [29].

2. Examples

Example 2.1. For the sequence (a_k) with $a_k = k$ $(k = 1, 2, \ldots)$, we clearly have $\pi_{(k)}(n) = \pi(n)$, and hence

$$b_n = \frac{\pi(n)\log n}{n}, \quad n = 1, 2, \ldots. \tag{10}$$

By the Prime Number Theorem, from (10) we find that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\pi(n)\log n}{n} = 1. \tag{11}$$

Example 2.2. Let (a_k) be a sequence of all primes, that is, $a_k = p_k$ with $k \in \mathbb{N} := \{1, 2, \ldots\}$, where p_k is the kth prime. Since by (1), $\pi_{(p_k)}(p_n) = \pi(p_n) = n$, substituting this into (4) yields

$$b_n = \frac{n\log p_n}{p_n}, \quad n = 1, 2, \ldots. \tag{12}$$

Now applying to (12) the well known fact that $p_n \sim n\log n$ as $n \to \infty$ (see, e.g., [30]), we find that

$$\lim_{n\to\infty} b_n = 1. \tag{13}$$

Notice also that the known inequality $p_n > n\log n$ with $n \geq 1$ (see, e.g., [38, (3.10) in Theorem 3]) implies that $b_n < 1$ for all $n \geq 1$.

Example 2.3. Suppose that a and d are relatively prime positive integers. Then concerning Dirichlet's theorem de la Vallée Poussin established (see, e.g., [35, p. 205]) that the number of primes $p < x$ with $p \equiv a (\mathrm{mod}\, d)$ is approximately

$$\frac{\pi(x)}{\varphi(d)} \sim \frac{1}{\varphi(d)} \cdot \frac{x}{\log x}. \tag{14}$$

Here $\varphi(n)$ is the Euler totient function defined as the number of positive integers not exceeding n and relatively prime to n. Note that the right hand side of (14) is the same for any a such that $\gcd(a, d) = 1$. This shows that primes are in a certain sense uniformly distributed in reduced residue classes with respect to a fixed modulus. Notice that for a sequence $(a_k)_{k=1}^{\infty}$ given by $a_k = a + (k-1)d$, (14) can be written as

$$\pi_{(a_k)}(a_n) \sim \frac{\pi(a_n)}{\varphi(d)} \text{ as } n \to \infty. \tag{15}$$

Inserting (15) together with $\pi(a_k) \sim a_k/\log a_k$ into (4) immediately gives

$$\lim_{n\to\infty} b_n = \frac{1}{\varphi(d)} \lim_{n\to\infty} \frac{\pi(a_n)\log a_n}{a_n} = \frac{1}{\varphi(d)}. \tag{16}$$

Then substituting (16) into (9), we obtain that for the associated function $\omega_{a,d}(x) := \omega_{(a_k}$ of the sequence (a_k) there holds
(17)
$$\omega_{a,d}(n) \sim \pi_{(a_k)}(a_n)\frac{a_n b_n}{\varphi(d)\log a_n} = \frac{a+(n-1)d}{\varphi(d)\log(a+(n-1)d)} \sim \frac{dn}{\varphi(d)\log n} \text{ as } n\to\infty.$$
It follows that $\omega_{a,d}(x) = dx/(\varphi(d)\log x)$ for $x\in(1,+\infty)$.

Example 2.4. Let (a_n) be a sequence defined as $a_n = 2^{p_n}-1$, where p_n is the nth prime. The numbers a_n are called *Mersene numbers*. A prime that appears in the sequence (a_n) is called *Mersenne prime*. Namely, it is easy to show (see, e.g., [36, p. 28]) that if 2^n-1 is prime, then so is n. The greatest known Mersenne prime is $2^{43112609}-1$ with the exponent 43112609 (12978169 digit number), and it is discovered in August 2008. This is in fact one between 45 known Mersenne primes, and so $a_{45}\le 2^{43112609}-1$.

In 1980 H. Lenstra and C. Pomerance, working independently, came the conclusion that the probability that a Mersenne number 2^p-1 is prime is $e^\gamma\log(ap)/(p\log 2)$ with $\gamma = 0.577216\ldots$ (the Euler-Mascheroni constant), where $a=2$ if $p\equiv 3(\bmod 4)$ and $a=6$ if $p\equiv 1(\bmod 4)$. Recall that the constant $e^\gamma = 1.781072\ldots$ is important in number theory; namely, $e^\gamma = \lim_{n\to\infty}\frac{1}{\log p_n}\prod_{k=1}^n\frac{p_k}{p_k-1}$ which restates the third of Mertens' theorems ([27], also see [23, pp. 351–353, Theorem 428]). Then notice that the distribution of the log of the Mersenne primes is a *Poisson Process* (see [45]).

Accordingly to the above assumption given by Lenstra and Pomerance, if $a_k = 2^{q_k}-1$, where $(q_k)_{k=1}^\infty$ is a sequence of all primes $\equiv 3(\bmod 4)$ ($q_1=3, q_2=7, q_3=11,\ldots$), for the associated function $\omega^{(3,4)}(x)$ to (a_k) we have that "the expected number" of primes between the first n terms of the sequence (q_k) is

$$\sim \omega^{(3,4)}(n) \sim \sum_{k=1}^n \frac{e^\gamma\log(2q_k)}{q_k\log 2} \text{ as } n\to\infty. \tag{18}$$

Since $q_k\sim p_{2k}\sim 2k\log k$, substituting this into (18) and using the well known asymptotic formula $\sum_{k=1}^n 1/k\sim\gamma+\log n$ as $n\to\infty$, we get

$$\begin{aligned}\omega^{(3,4)}(n) &\sim \frac{e^\gamma}{2\log 2}\sum_{k=2}^n\frac{\log(4k\log k)}{k\log k} = \frac{e^\gamma}{2\log 2}\sum_{k=2}^n\frac{\log k+\log 4+\log\log k}{k\log k}\\ &\sim \frac{e^\gamma}{2\log 2}\left(\sum_{k=2}^n\frac{1}{k}+\sum_{k=2}^n\frac{\log 4}{k\log k}+\sum_{k=2}^n\frac{\log\log k}{k\log k}\right)\\ &\sim \frac{e^\gamma}{2\log 2}\left((\gamma+\log n)+\log 4\int_2^n\frac{dx}{x\log x}+\int_2^n\frac{\log\log x}{x\log x}\,dx\right)\\ &\quad\text{(the changes } \log x = s \text{ and } \log\log x = t\\ &= \frac{e^\gamma}{2\log 2}\left(\log n+\int_{\log 2}^{\log n}\frac{ds}{s}+\int_{\log\log 2}^{\log\log n}t\,dt\right)\\ &\sim \frac{e^\gamma}{2\log 2}\left(\log n+\log\log n+\frac{(\log\log n)^2}{2}\right) \text{ as } n\to\infty.\end{aligned} \tag{19}$$

This shows that $\omega^{(3,4)}(x) = e^\gamma\left(\log x+\log\log x+(\log\log x)^2/2\right)/(2\log 2)$, and hence $\pi_{(a_k)}(a_n)\sim e^\gamma/(2\log 2)\left(\log n+\log\log n+(\log\log n)^2/2\right)$. Substituting this in (4), where (b_n) is the companion' sequence of (a_k), and using the fact that $q_n \sim 2n\log n$, we find that

$$b_n \sim \frac{e^{\gamma} n(\log n)^2}{4^{n\log n}} \text{ as } n\to\infty. \tag{20}$$

Similarly, under the above assumptions attributed by Lenstra and Pomerance, if $a'_k = 2^{r_k}-1$, where $(r_k)_{k=1}^{\infty}$ is a sequence of all primes $\equiv 1(\bmod 4)$ $(r_1=5, r_2=13, r_3=17,\ldots)$, then for the associated function $\omega^{(1,4)}(x)$ to (a'_k) and the companion sequence (b'_n) of (a'_k) the same relations (18)–(20) are satisfied.

Example 2.5. Let $(a_k)_{k=1}^{\infty}$ be an increasing sequence of positive integers satisfying

$$\frac{\log a_k}{a_k} = o(k^{-1}). \tag{21}$$

Then from (4) and the obvious fact that $\pi_{(a_k)}(a_n)\le n$ for each $n\in\mathbb{N}$, we find that

$$\lim_{n\to\infty} b_n = 0. \tag{22}$$

In particular, (22) holds for any sequence (a_k) satisfying one of the following asymptotics: $a_n\sim a^n$ with a fixed $a>1$; $a_n\sim n\log^{\alpha} n$ with $\alpha>1$; $a_n\sim n^{\alpha}$ with $\alpha>1$; or $a_n\sim n^{\alpha}\log^{\beta} n$ with $\alpha\ge 1$ and $\beta>1$.

Accordingly, we ask the following question.

Question 2.6. *For what real numbers $\alpha\in(0,1)$ there exists a sequence (a_k) whose companion sequence (b_n) defined by (4) satisfies the limit relation*

$$\limsup_{n\to\infty} b_n = \alpha?$$

3. Distribution of primes in the sequence (S_n) with $S_n=\sum_{i=1}^{2n} p_i$

Here, as always in the sequel, we consider the distribution of primes in the sequence $(S_n)_{n=1}^{\infty}$ whose terms are given by $S_n=\sum_{i=1}^{2n} p_i$, where p_i is the ith prime. Recall that the prime counting function $\pi(x)$ is defined as the number of primes $\le x$.

Proposition 3.1. *Let (S_n) be the sequence defined as $S_n=\sum_{i=1}^{2n} p_i$. Then as $n\to\infty$,*

$$S_n\sim 2n^2\log n \tag{23}$$

and

$$\pi(S_n)\sim n^2. \tag{24}$$

Furthermore, if x is a real number such that $S_n\le x<S_{n+1}$, then

$$n\sim\sqrt{\frac{x}{\log x}} \text{ as } n\to\infty. \tag{25}$$

Proof. Let (S'_n) be the sequence defined as $S'_n=\sum_{i=1}^{n} p_i$ (this is Sloane's sequence A007504 in [42]). By the Prime Number Theorem, we have (see, e.g., [43, page 5]),

$$\begin{aligned} S'_n &:= \sum_{i=1}^{n} p_n \sim \sum_{k=1}^{n} k\log k \sim \int_1^n x\log x\,dx = \frac{x^2}{2}\log x\Big|_1^n - \int_1^n \frac{x^2}{2}(\log x)'\,dx \\ &\sim \frac{n^2\log n}{2} \text{ as } n\to\infty. \end{aligned} \tag{26}$$

It follows from (26) that

$$S_n = S'_{2n} \sim 2n^2 \log n, \tag{27}$$

which implies (23). By the Prime Number Theorem, from (27) we have

$$\pi(S_n) \sim \frac{2n^2 \log n}{\log(2n^2 \log n)} \sim \frac{2n^2 \log n}{\log 2 + 2\log n + \log\log n} \sim n^2. \tag{28}$$

Finally, (25) immediately follows from (23). □

Remark **3.2.** For refinements of the estimate (23), see [15], [39] and [41, Theorem 2.3]). We see from (23) that there are $\sim n^2$ primes less than S_n. Using this fact, Z.-W. Sun [43, Remark 1.6] conjectured that the number of primes in the interval $(\sum_{i=1}^n p_i, \sum_{i=1}^{n+1} p_i)$ is asymptotically equivalent to $n/2$ as $n \to \infty$. Under the validity of this conjecture, in particular it follows that the number of primes in the interval (S_n, S_{n+1}) is asymptotically equivalent to n as $n \to \infty$. Moreover, we also believe that the "probability" that $\sum_{i=1}^{2n} p_i$ is a prime is $2n/p_{2n}$, which is $\sim 1/\log n$ because of $p_{2n} \sim 2n \log 2n$. Notice that the "probability" of a large integer n being a prime is also asymptotically equal to $1/\log n$.

Furthermore, some computational results and heuristic arguments show that between these $\sim n^2$ primes which are less than S_n there are $\sim 2n/\log S_n \sim n/\log n$ primes that belong to the set $\mathcal{S}_n := \{S_1, S_2, \ldots, S_n\}$. For example, if $n = 10^8$ then $n/\log n = 10^8/\log 10^8 = 5428681.02$, while from the second column of Table 1 of Section 6 we see that there are 5212720 primes in the set $\mathcal{S}_{10^8}$ (cf. Table 2 of Section 6). Accordingly, we propose the following curious conjecture which is basic in this paper.

Conjecture 3.3. *The sequence* (S_n) *with* $S_n = \sum_{i=1}^{2n} p_i$ *satisfies the Restricted Prime Number Theorem. In other words,*

(29)
$$\begin{aligned}\pi_n : = \pi_{(S_k)}(S_n) &= \#\{p : p \text{ is a prime and } p = S_i \text{ for some } i \text{ with } 1 \le i \le n\}\\ &\sim \frac{n}{\log n} \quad \text{as } n \to \infty.\end{aligned}$$

Let us recall that in all results of this section (Corollaries 3.4, 3.6, 3.7, 3.8 and 3.13) we assume the truth of Conjecture 3.3. In particular, Conjecture 3.3 implies Euclid's theorem (on the infinitude of primes) for (S_n) as follows.

Corollary 3.4 (Euclid's theorem for the sequence (S_n))**.** *The sequence* (S_n) *contains infinitely many primes.*

Remark **3.5.** Notice that the sequence (S_n) is closely related to the Sloane's sequence A013918 [42] containing all primes (in increasing order) equal to the sum of the first m primes for some $m \in \mathbb{N}$ (A013918 is in fact the intersection of A000040-the sequence of all primes and A007504-sum of first n primes). The first few terms of the sequence A013918 are: $2, 5, 17, 41, 197, 281, 7699, 8893, 22039$; see the related link by T. D. Noe [42, A013918] which gives the table of the first 10000 terms of this sequence (10000th term is 402638678093). Notice also that the Sloane's sequence A013916 in [42] associated to the sequence A013918 gives numbers n such that the sum of the first n primes is prime. The first few terms of this sequence are: $1, 2, 4, 6, 12, 14, 60, 64, 96$ (see the related link by D. W. Wilson [42, A013918] which gives table of the first 10000 terms of this sequence (10000th term is 244906). Similarly, the second Sloane's sequence A013917 ((a_n)) associated to A013918, is defined as: a_n is prime and sum of all primes $\le a_n$ is prime. The first few terms of this sequence are: $2, 3, 7, 13, 37, 43, 281$.

As a further application of Conjecture 3.3, here we obtain the asymptotic expression for the kth prime in the sequence (S_n).

Corollary 3.6 (The asymptotic expression for the kth prime in the sequence (S_n))**.** *Let q_k $(k=1,2,\ldots)$ be the kth prime in the sequence (S_n). Then*

$$q_k \sim 2k^2\log^3 k \text{ as } k\to\infty. \tag{30}$$

Proof. If for a pair (k,n) there holds $q_k=S_n$, then by Conjecture 3.3, we have

$$k\sim \frac{n}{\log n}\quad \text{as } n\to\infty, \tag{31}$$

so that $n\sim k\log n$, and hence $\log n\sim\log k$ as $n\to\infty$. Inserting this into (23), we find that

$$q_k=S_n\sim 2n^2\log n\sim 2(k\log n)^2\log n = 2k^2\log^3 n\sim 2k^2\log^3 k,$$

as desired. □

Corollary 3.7. *Let q_k $(k=1,2,\ldots)$ be the kth prime in the sequence (S_n). Then*

$$q_k\sim 2p_k^2\log k \text{ as } k\to\infty \tag{32}$$

and

$$q_k\sim p_{k^2}\log^2 k \text{ as } k\to\infty. \tag{33}$$

Proof. From (30) and the fact that $p_k\sim k\log k$ we find that

$$q_k\sim 2(k\log k)^2\log k\sim 2p_k^2\log k,$$

which proves (32).

Similarly, from (30) and $p_{k^2}\sim k^2\log k^2=2k^2\log k$ we find that

$$q_k\sim (k^2\log k^2)\log^2 k\sim p_{k^2}\log^2 k,$$

which implies (33). □

Furthermore, we have the following result.

Corollary 3.8. *Let q_k be the kth prime in the sequence (S_n) with $q_k=S_n$. Then*

$$\lim_{k\to\infty}\frac{k\log k}{n}=1. \tag{34}$$

Proof. The asymptotic relation (31) implies that $\log n/\log k\sim 1$, which substituting in (31) immediately gives (34). □

Motivated by some heuristic arguments and computations for some small integer values d, we propose the following generalization of Conjecture 3.3.

Conjecture 3.9. *For any fixed nonnegative integer d the sequence $(S_n^{(d)})_{n=1}^{\infty}$ defined as*

$$S_n^{(d)}=2d+S_n=2d+\sum_{i=1}^{2n}p_i,\quad n=1,2,\ldots$$

satisfies the Restricted Prime Number Theorem. In other words, as $n\to\infty$,

$$\begin{aligned}\pi_n^{(d)}:=&\pi_{(2d+S_k)}(2d+S_n)=\#\{p:\ p \text{ is a prime and } p=2d+S_i\\ &\text{for some } i \text{ with } 1\le i\le n\}\sim\frac{n}{\log n}.\end{aligned} \tag{35}$$

For $d = 0$ this conjecture is in fact Conjecture 3.3 (*cf. Sloane's sequence* A013918 *mentioned above*).

Remark **3.10.** Conjecture 3.3 and the fact that by (23) $S_n \sim 2n^2 \log n$ imply that the average difference between consecutive primes in the sequence (S_n) near to $2n^2$ is approximately $\log(2n^2) \sim 2\log n$.

Remark **3.11.** Numerous computational results concerning the sums of the first n primes (partial sums of consecutive primes) given by the Sloane's sequence A007504 (here denoted as S'_n), and certain their curious arithmetical properties are presented in the following Sloane's sequences in OEIS [42]: A051838, A116536, A067110, A067111, A045345, A114216, A024011, A077023, A033997, A071089, A083186, A166448, A196527, A065595, A165906, A061568, A066039, A077022, A110997, A112997, A156778, A167214, A038346, A038347, A054972, A072476, A076570, A076873, A077354, A110996, A123119, A189072, A196528, A022094, A024447, A121756, A143121, A117842, A118219, A131740, A143215, A161436, A161490, A013918 etc.

Since the sequence (S_n) is a subsequence of the sequence (S'_n) with $S'_n = \sum_{k=1}^n p_k$ whose all terms with odd indices n are even integers, it follows that in accordance to Definition 1.1, Conjecture 3.3 is equivalent to

$$\omega_{(S'_k)}(n) \sim \frac{n}{2\log n}.$$

Therefore, Conjecture 3.3 is equivalent with the following one.

Conjecture 3.3'. *Let (S'_n) be a sequence defined as $S'_n = \sum_{k=1}^n p_k$, $n = 1, 2, \ldots$. Then*

$$\omega_{(S'_k)}(x) = \frac{x}{2\log x} \quad for \quad x \in (1, \infty). \tag{36}$$

Proposition 3.12. *For each $n \geq 3$ we have*

$$1 \leq \frac{S_n}{2n^2 \log n} < 1 + \frac{\log 2}{\log n} + \frac{\log\log(2n)}{\log n}. \tag{37}$$

Proof. By *Mandl's inequality* (see, e.g., [39], [15]), for each $n \geq 9$ there holds

$$S'_n < \frac{n}{2} p_n \tag{38}$$

(for a refinement of (38), see [17, the inequality 2.4]). Mandl's inequality (38) with $2n$ instead of n becomes $S_n < np_{2n}$ with $n \geq 5$. This inequality together with the known inequality (see, e.g., [38, p. 69])

$$p_{2n} < 2n(\log n + \log 2 + \log\log(2n)) \quad \text{for all } n \geq 3$$

immediately yields

$$S_n < 2n^2(\log n + \log 2 + \log\log(2n)) \quad \text{for all } n \geq 5. \tag{39}$$

On the other hand, a lower bound for S'_n can be obtained by using *Robin's inequality* (see, e.g., [15, p. 51]) which asserts that for every $n \geq 2$

$$np_{[n/2]} \leq S'_n. \tag{40}$$

The inequality (40) with $2n$ instead of n and the inequality $n \log n \leq p_n$ with $n \geq 3$ (see, e.g., [38, p. 69]) yield

$$2n^2 \log n \leq S_n \quad \text{for } n \geq 3. \tag{41}$$

The inequalities (39) and (41) immediately yield

$$\log n \le \frac{S_n}{2n^2} < \log n + \log 2 + \log\log(2n) \quad \text{for all } n \ge 5,$$

or equivalently,

$$1 \le \frac{S_n}{2n^2 \log n} < 1 + \frac{\log 2}{\log n} + \frac{\log\log(2n)}{\log n} \quad \text{for all } n \ge 5. \tag{42}$$

The inequalities given by (42) coincide with these of (37) for $n \ge 5$. A direct calculation chows that (37) is also satisified for $n = 3$ and $n = 4$. This completes the proof. □

Corollary 3.13. *Let $q_k = S_m$ be the kth prime in the sequence $(S_n)_{n=1}^{\infty}$. Then for all $k \ge 3$ there holds*

$$2m^2 \log m < q_k < 2m^2(\log m + \log(\log(2m) + \log 2). \tag{43}$$

Proof. The above inequalities coincide with (37) of Proposition 3.12 with $n = m$ and $q_k = S_m$. □

Remark **3.14.** Z.-W. Sun [44, the case $\alpha = 1$ in Lemma 3.1] showed that for all $n \ge 2$

$$S_n' > 2 + \frac{n^2 \log n}{2}\left(1 - \frac{1}{2\log n}\right),$$

which with $2n$ instead of n becomes

$$S_n > 2 + 2n^2\left(\log n + \log\frac{2}{\sqrt{e}}\right) \approx 2 + 2n^2(\log n + 0.193147),$$

whence it follows that

$$\frac{S_n}{2n^2 \log n} > 1 + \frac{0.193147}{n}.$$

The above inequality is stronger that the left hand side of the inequality (37). Accordingly, if $q_k = S_m$, then the first inequality of (43) can be refined in the form

$$q_k > 2 + 2m^2(\log m + 0.193147) \quad \text{for all } k \ge 3.$$

On the other hand, combining the inequalities (46) and (47) from the next section with the inequalities $S_n > 2np_n$ and $S_n < np_{2n}$ (given in proof of Proposition 3.12), respectively, we immediately obtain the following refinement of Proposition 3.12.

Proposition 3.15. *For each $n \ge 3$ there holds*

$$\frac{S_n}{2n^2 \log n} \ge 1 + \frac{\log\log n - 1}{\log n} + \frac{\log\log n - 2.2}{\log^2 n},$$

and for each $n \ge 344192$, we have

$$\frac{S_n}{2n^2 \log n} \le 1 + \frac{\log\log(2n) + \log 2 - 1}{\log n} + \frac{\log\log(2n) - 2}{(\log n)\log(2n)}.$$

Remark **3.16.** If $q_k = S_m$, then in view of the first inequality of Proposition 3.15, the first inequality of (43) may be replaced by the following one:

$$q_k > 2m^2\left(\log m + \log\log m - 1 + \frac{\log\log m - 2.2}{\log m}\right) \quad \text{for all } k \ge 3.$$

Remark **3.17.** The inequalities (38) and (40) and the asymptotic expression $p_n \sim n \log n$ show that the average of the first n primes is asymptotically equal to $(n \log n)/2$ (cf. Sloane's sequence A060620 in [42]), that is,

$$\frac{S'_n}{n} \sim \frac{n \log n}{2} \quad \text{as} \quad n \to \infty.$$

Conjecture 3.3 suggests the fact that for the sequence (S_n) would be valid the analogues of some other classical results and conjectures closely related to the Prime Number Theorem and Riemann Hypothesis. In particular, if $Q = \{q_1, q_2, \dots, q_k, \dots\}$ is a set of all primes $q_1 < q_2 < \dots < q_k < \cdots$ in the sequence (S_n), it can be of interest to establish the asymptotic expression for q_k as $k \to \infty$.

Finally, heuristic arguments, some computational results and Conjecture 3.3 lead to the follwing its generalization (cf. Sloane's sequence A143121 - triangle read by rows, $T(n,k) = \sum_{j=k}^{n} p_j$, $1 \le k \le n$; see the columns in Example of this sequence).

Conjecture 3.18. *For any fixed positive integer k, let $(S_n^{(k)}) := (S_n^{(k)})_{n=1}^{\infty}$ be the sequence whose nth term is defined as*

$$S_n^{(k)} = \sum_{i=1}^{2n+1} p_{i+k}, \quad n \in \mathbb{N}.$$

Then the sequence $(S_n^{(k)})$ satisfies the Restricted Prime Number Theorem.

For example, there are 78498 (resp. 664579) primes less than 10^6 (resp. 10^7), while the computations show that among the first 10^6 (resp. 10^7) terms of the sequences (S_n), $(S_n^{(k)})$ with $k = 1, 2, \dots, 12$ there are 69251 (resp. 594851), 69581 (resp. 594377), 68844 (resp. 593632), 68883 (resp. 595733), 69602 (resp. 596609), 69540 (resp. 596558), 69414 (resp. 595539), 69317 (resp. 594626), 69455 (resp. 595474), 69268 (resp. 594542), 68891 (resp. 593807), 69251 (resp. 594383), 69564 (resp. 595270) primes, respectively.

4. THE ASYMPTOTIC EXPRESSION FOR THE kTH PRIME IN THE SEQUENCE (S_n)

As an easy consequence of the Prime Number Theorem, it can be deduced that $p_n \sim n \log n$ as $n \to \infty$ (see, e.g., [26]). Furthermore, a particular asymptotic expansion for p_n (see [26] or [34, the equality (66) of Section 6]; also see Sloane's sequence A200265) yields

$$p_n = n\left(\log n + \log\log n + O\left(\frac{\log\log n}{\log n}\right)\right). \tag{44}$$

It is also known that (see [16] and [38, p. 69])

$$n(\log n + \log\log n - 1) < p_n < n(\log n + \log\log n). \tag{45}$$

A more precise work about this can be found in [37] and [40] where related results are as follows:

$$n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2.2}{\log n}\right) \le p_n \quad \text{for } n \ge 3 \tag{46}$$

and

$$p_n \leq n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n}\right) \quad \text{for } n \geq 688383. \tag{47}$$

The inequalities (46) and (47) immediately yield the following result.

Corollary 4.1. *For each $n \geq 688383$ the interval*

$$\left[n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2.2}{\log n}\right), n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n}\right)\right] \tag{48}$$

contains at least one prime. Furthermore, the lenght l_n of this interval is

$$l_n = \frac{0.2n}{\log n} \sim \frac{0.2p_n}{\log^2 n}. \tag{49}$$

As an application of Conjecture 3.3, we obtain the following result.

Corollary 4.2. *Let q_k be the kth prime in the sequence (S_n) with $S_n = \sum_{k=1}^{2n} p_k$. If $q_k = S_m$, then under Conjecture 3.3 there holds*

$$q_k \sim 2k^2 \log^3 k \sim 2m^2 \log m \sim p_{\lfloor k^2 \log^2 k \rfloor} \quad \text{as } k \to \infty, \tag{50}$$

and

$$\lim_{k\to\infty} \frac{k \log k}{m} = 1. \tag{51}$$

Proof. The first asymptotic relation of (50) coincides with (30) of Corollary 3.6. Further, by (37) of Proposition 3.12, we have

$$q_k = S_m \sim 2m^2 \log m. \tag{52}$$

Moreover, we have

$$p_{[k^2 \log^2 k]} \sim k^2 (\log^2 k) \log(k^2 \log^2 k) \sim 2k^2 \log^3 k \quad \text{as } k \to \infty. \tag{53}$$

The last two asymptotic expressions of (50) follow from (52) and (53).

It remains to prove (51). If we suppose that (51) is not satisfied, then there exists $\varepsilon > 0$ and an infinite subsequence $(k_j, m_j)_{j=1}^{\infty}$ of the sequence $(k, m)_{k=1}^{\infty}$ such that $k_j \log k_j \geq (1+\varepsilon)m_j$ for all $j \in \mathbb{N}$ or $k_j \log k_j \leq (1-\varepsilon)m_j$ for all $j \in \mathbb{N}$. In the first case, using (50) for all sufficiently large j, we find that

$$m_j^2 \log m_j \sim k_j^2 \log^3 k_j \geq (1+\varepsilon)^2 m_j^2 \log k_j,$$

whence we immediately get $\log m_j \geq (1+\varepsilon)^2 \log k_j$, or equivalently, $m_j \geq k_j^t$ with $t = (1+\varepsilon)^2 > 1$. From the previous inequality and the fact that $t > 2$ we have

$$m_j^2 \log m_j > m_j^2 \geq k_j^{2t} \gg k_j^2 \log^3 k_j,$$

which contradicts the fact that by (50) $2k^2 \log^3 k \sim 2m^2 \log m$. In a similar way as in the first case, in the second case we find that $m_j \leq k_j^s$ for a constant $s = (1-\varepsilon)^2 < 1$. Then choosing a sufficiently large j_0 such that $\log m_j < m_j^{1/s-1}$ for all $j > j_0$, in view of the fact that $s + 1 < 2$ we get

$$m_j^2 \log m_j < m_j^{1+1/s} \leq k_j^{s+1} \ll k_j^2 \log^3 k_j.$$

A contradiction, and therefore, (51) is true. □

Remark **4.3.** From (50) and $p_k \sim k \log k$ we see that

$$\frac{q_k}{2k \log^2 k} \sim k \log k \sim p_k \quad \text{as} \quad k \to \infty. \tag{54}$$

The above asymptotic expression together with the assumption that Conjecture 3.3 is true suggests the fact that for the sequence $\left(q_k/(k \log^2 k)\right)$ would be satisfied the asymptotic expansion similar to (44), i.e.,

$$\frac{q_k}{2k \log^2 k} = k(\log k + \log \log k + Q_k), \tag{55}$$

with some sequence (Q_k). Motivated by (55), we establish the following asymptotic expression for the kth prime in the sequence (S_n).

Theorem 4.4 (The asymptotic expression for the kth prime in the sequence (S_n))**.** *Let q_k be the kth prime in the sequence (S_n) $(k = 2, 3, \dots)$. Then under Conjecture* 3.3 *there exists a sequence (M_k) of positive rael numbers such that $\lim_{k \to \infty} M_k = 1$ and*

$$q_k = 2M_k^5 k^2 \log^2 k(\log k + \log \log k + 2 \log M_k). \tag{56}$$

For the proof of Theorem 4.4 we will need the following result.

Lemma 4.5. *Let $S_m = q_k$ be the kth prime in the sequence (S_n). Then under Conjecture $(3, 3)$ we have*

$$q_k \sim \frac{2m^2 \sqrt{m} \log m}{\sqrt{k \log k}} \quad as\ k \to \infty. \tag{57}$$

Proof. First notice that, under notations of Lemma 4.5, Conjecture 3.3 yields (cf. (51) of Corollary 4.2)

$$k \sim \frac{m}{\log m} \quad as\ k \to \infty. \tag{58}$$

Using (58), we find that

$$\frac{2m^2 \sqrt{m} \log m}{\sqrt{k \log k}} \sim \frac{2m^2 \sqrt{m} \log m}{\sqrt{m \left(1 - \frac{\log \log m}{\log m}\right)}} \sim 2m^2 \log m \quad as\ k \to \infty. \tag{59}$$

The asymptotic relation (59) and the fact that by (50) of Corollary 4.2, $q_k = S_m \sim 2m^2 \log m$ immediately yield (57). ◻

Proof of Theorem 4.4. Let $(C_k)_{k=2}^{\infty}$ be a sequence of positive real numbers such that $m(k) = m = C_k k \log k$ with $k \geq 2$ and $q_k = S_m$. Then by (51) of Corollary 4.2, we have $C_k \to 1$ as $k \to \infty$. Taking $m = C_k k \log k$ into (57) of Lemma 4.5, as $k \to \infty$ we obtain that

$$\begin{aligned} q_k &\sim \frac{2\sqrt{C_k^5 k^5 \log^5 k}(\log k + \log \log k + \log C_k)}{\sqrt{k \log k}} \\ &= 2C_k^2 \sqrt{C_k} k^2 \log^2 k(\log k + \log \log k + \log C_k) =: f(k, C_k). \end{aligned} \tag{60}$$

Let (δ_k) be a positive real sequence such that

$$q_k = \delta_k f(k, C_k) \quad \text{for each } k \geq 2. \tag{61}$$

Then from (60) we see that $\delta_k \to 1$ as $k \to \infty$. For a fixed $k \geq 2$ consider the equation $f_k(x) = \delta_k f(k, C_k)$ which can be written in the form

$$x^2\sqrt{x}(\log k + \log\log k + \log x) = \delta_k C_k^2\sqrt{C_k}(\log k + \log\log k + \log C_k). \tag{62}$$

Notice that for any fixed integer $k \geq 2$, the real function $f_k(x)$ defined as

$$f_k(x) = 2x^2\sqrt{x}(\log k + \log\log k + \log x), \quad x > 0,$$

satisfies the limit relations $\lim_{x\to+\infty} f_k(x) = +\infty$ and $\lim_{x\to+0} f_k(x) = 0$. From this it can be easily shown that for each integer $k \geq 2$ the equation (62) has a positive real solution x_k. Using the facts that $\lim_{x\to+\infty} C_k = \lim_{x\to+\infty} \delta_k$=1, it can be easily show that $\lim_{k\to\infty} x_k = 1$. Then taking $x_k = M_k^2$ ($k = 2, 3, \dots$), then $\lim_{k\to\infty} M_k = 1$ and by (62) we find that $f_k(M_k^2) = \delta_k f(k, C_k) = q_k$, whence it follows that

$$q_k = 2M_k^5 k^2 \log^2 k(\log k + \log\log k + 2\log M_k).$$

This proves (56) and the proof is completed. □

Computational results (cf. the eighth column of Table 1 of Section 6) suggest the additional relationship between k's and m's as follows.

Conjecture 4.6. *For each pair* (k, m) *with* $k \geq 1$ *and* $q_k = S_m$ *we have*

$$\lfloor k\log k\rfloor + 1 \leq m, \tag{63}$$

or equivalently,

$$q_k \geq S_{\lfloor k\log k\rfloor+1}. \tag{64}$$

Furthermore, for each $k \geq 10^4$,

$$m \leq \lfloor 1.4k\log k\rfloor, \tag{65}$$

or equivalently,

$$q_k \leq S_{\lfloor 1.4k\log k\rfloor}. \tag{66}$$

Corollary 4.7. *If the inequality* (63) *of Conjecture* 4.6 *is true, then for each* $k \geq 1$ *there holds*

$$q_k > 2k^2(\log^2 k)(\log k + \log\log k). \tag{67}$$

Proof. Combining the inequality (63) with the inequality on the left hand side of (37) of Proposition 3.12, we find that

$$\begin{aligned} q_k = S_m \geq S_{\lfloor k\log k\rfloor+1} &\geq 2(\lfloor k\log k\rfloor + 1)^2\log(\lfloor k\log k\rfloor + 1) \\ &> 2k^2(\log^2 k)\log(k\log k) = 2k^2(\log^2 k)(\log k + \log\log k), \end{aligned} \tag{68}$$

as desired. □

Corollary 4.8. *If the inequality* (63) *of Conjecture* 4.6 *is true, then* $M_k > 1$ *for each* $k \geq 1$, *where* (M_k) *is the sequence defined by* (56) *of Theorem* 4.4.

Proof. The assertion follows immediately from the inequality (67) and the expression (56) for q_k given by Theorem 4.4. □

Finally, in view of the data of the last column in Table 1 of Section 6 and some considerations presented above, we propose the following conjecture which is stronger than Corollary 4.7.

Conjecture 4.9. *For every $k \geq 252028$ with $q_k = S_m$ there holds*

$$q_k > \frac{2m^2\sqrt{m}\log m}{\sqrt{k\log k}}. \tag{69}$$

In view of the well known inequality $p_k > k(\log k + \log\log k)$, the following conjecture is also stronger than Corollary 4.7.

Conjecture 4.10. *There exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ there holds*

$$q_k > 2kp_k \log^2 k,$$

where p_k is the kth prime.

Remark **4.11.** The last column of Table 1 presented in Section 6 shows that

$$q_k \approx \frac{2m^2\sqrt{m}\log m}{\sqrt{k\log k}}$$

is a "good" approximation for the kth prime sum q_k. Notice that this approximation can be written as

$$q_k \approx 2m^2 \log m \cdot \sqrt{\frac{m}{k\log k}},$$

where the values $\sqrt{m/(k\log k)}$ slowly tend to 1 as k grows. In particular, from the last row of Table 1 of Section 6 we see that for $m = 10^9 - 2$ (i.e., for $k = 46388006$) we have $\sqrt{m/(k\log k)} \approx 1.105079$. Hence, in view of the above approximation, we believe that for all values m up to 10^9 there holds

$$q_k > 2.2m^2 \log m.$$

Notice that some values of the sequences (Q'_k) such that

$$Q'_k = \frac{q_k - 2k^2(\log^2 k)(\log k + \log\log k)}{2k^2\log^2 k\log\log k}$$

and the sequence (Q''_k) such that

$$Q''_k = \frac{q_k - 2(p_k)^2\log k}{2k^2\log^2 k\log\log k}$$

are given in Table 3 of Section 6. Table 3 also shows that for almost all values m up to 10^9 (i.e., for $k \leq 46388006$) there holds

$$kp_k > 1.1m^2 \log m.$$

Finally, Table 2 of Section 6 leads to the following conjecture whose both parts are obviously stronger than Conjecture 4.6.

Conjecture 4.12. *Let $\pi(x)$ be the prime counting function, and let π_n be the number of primes in the set $\{S_1, S_2, \ldots, S_n\}$. Then*

$$\pi_n < \pi(n)$$

for for each $n \geq 10^4$ and

$$\pi_n < \frac{n}{\log n}$$

for each $n \geq 10^5$.

5. ESTIMATIONS OF VALUES M_k FROM THEOREM 4.4

The computational results related to the search of primes in the sequence (S_n) given in the following section (Table 1) and some heuristic arguments suggest the fact that the sequence $(S_n/(2n^2\log n))_{n=2}^{\infty}$ plays an important role for estimating the values M_k $(k=1,2,\ldots)$ in the expression (56) for the kth prime q_k in the sequence (S_n).

Here we first consider the sequence $(S_n/(2n^2))$.

Proposition 5.1. *The sequence (v_n) defined as*

$$v_n=\frac{S_n}{2n^2},\quad n\in\mathbb{N}, \tag{70}$$

is increasing for $n\geq 2$.

Proof. Since $S_{n+1}=S_n+p_{2n+1}+p_{2n+2}$, an easy calculation shows that $r_n<r_{n+1}$ is equivalent with

$$\frac{S_n}{2n^2}<\frac{p_{2n+1}+p_{2n+2}}{2(2n+1)},$$

which can be written as

$$\frac{p_{2n+1}+p_{2n+2}}{2}>S_n\left(\frac{1}{n}+\frac{1}{2n^2}\right). \tag{71}$$

By a refinement of Mandl's inequality due to Hassani [17], for every $n\geq 10$ we have

$$\frac{n}{2}p_n-\sum_{i=1}^{n}p_i>0.01659n^2. \tag{72}$$

Replacing n by $2n$ into (72) it becomes

$$p_{2n}-\frac{S_n}{n}>0.06636n^2\quad\text{for all } n\geq 5. \tag{73}$$

Further, by the inequality (37) of Proposition 3.12 we have

$$\log(2n)+\log\log(2n)>\frac{S_n}{2n^2}\quad\text{for all } n\geq 5. \tag{74}$$

By using `Mathematica 8`, it is easy to to prove the inequality

$$0.06636n^2>\log(2n)+\log\log(2n)\quad\text{for all } n\geq 8. \tag{75}$$

Finally, combining the inequalities (73), (74), (75) and the obvious inequality $(p_{2n+1}+p_{2n+2})/2>p_{2n}$ immediately gives (71) for all $n\geq 8$. This together with a direct verification that $v_n<v_{n+1}$ for $2\leq n\leq 8$ concludes the proof. □

Remark **5.2.** Notice that the sequence (v_n) defined by (70) is a subsequence of the sequence (v_n') defined as

$$v_n'=\frac{2S_n'}{n^2}:=\frac{2\sum_{i=1}^{n}p_i}{n^2},\quad n\in\mathbb{N};$$

namely, $v_n=v_{2n}'$ for all $n=1,2,\ldots$. Similarly as in the proof of Proposition 5.1, it can be shown that the sequence (v_n') is increasing for $n\geq 4$.

Contrary to Proposition 5.1, we propose the following conjecture.

Conjecture 5.3. *The sequence* (t_n) *defined as*

$$t_n = \frac{S_n}{2n^2 \log n}, \quad n \in \mathbb{N} \setminus \{1\}, \tag{76}$$

is decreasing on the range $\{n \in \mathbb{N} : n \geq 1100\}$ ($m = 1099$ *is a maximal value between total* 40 *values up to* $n = 200000$ *for which* $t_{m+1} > t_m$).

Remark **5.4.** Notice that the sequence (t_n) defined by (76) is a subsequence of the sequence (t'_n) defined as

$$t'_n = \frac{2S'_n}{n^2 \log(n/2)} := \frac{2\sum_{i=1}^n p_i}{n^2 \log(n/2)}, \quad n \in \mathbb{N}; \tag{77}$$

namely, $t_n = t'_{2n}$ for all $n = 1, 2, \ldots$. We conjecture that the sequence (t'_n) is decreasing on the range $\{n \in \mathbb{N} : n \geq 2199\}$ ($m = 2198$ is a maximal value up to $n = 10^6$ for which $t'_{m+1} > t'_m$).

Corollary 5.5. *Let* (t_n) *be the sequence defined in Conjecture* 5.3*. Then under Conjecture* 5.3*, for each* $n \geq 1100$ *there holds*

$$t_{n+1} < t_n < t_{n+1}\left(1 + \frac{1}{n \log n}\right). \tag{78}$$

Proof. Proposition 5.1, Conjecture 5.3 and the well known inequality $(1 + 1/n)^n < e$ with $n \geq 1$ immediately imply that for all $n \geq 1100$ there holds

$$\begin{aligned} 0 < t_n - t_{n+1} &= \frac{S_n}{2n^2 \log n} - \frac{S_{n+1}}{2(n+1)^2 \log(n+1)} \\ &< \frac{S_{n+1}}{2(n+1)^2 \log n} - \frac{S_{n+1}}{2(n+1)^2 \log(n+1)} \\ &= \frac{S_{n+1}}{2(n+1)^2} \cdot \frac{\log\left(1 + \frac{1}{n}\right)^n}{n(\log n)(\log(n+1))} \\ &< \frac{S_{n+1}}{2(n+1)^2 \log(n+1)} \cdot \frac{1}{n \log n} \\ &= \frac{t_{n+1}}{n \log n}. \end{aligned} \tag{79}$$

From (79) we immediately get (78). □

Corollary 5.6. *Let* (t_n) *be the sequence defined in Conjecture* 5.3*. Then under Conjecture* 5.3*, for each* $n \geq 1101$ *we have*

$$t_n > \frac{17}{8 \log 2}\left(\left(1 + \frac{1}{2 \log 2}\right)\left(1 + \frac{1}{3 \log 3}\right) \cdots \left(1 + \frac{1}{(n-1)\log(n-1)}\right)\right)^{-1} \tag{80}$$

and

$$S_n > \frac{17n^2 \log n}{4 \log 2}\left(\left(1 + \frac{1}{2 \log 2}\right)\left(1 + \frac{1}{3 \log 3}\right) \cdots \left(1 + \frac{1}{(n-1)\log(n-1)}\right)\right)^{-1}. \tag{81}$$

Proof. By the right hand side of the inequality (78), we obtain that for each $n \geq 1101$

$$t_n > t_{n-1}\left(1 + \frac{1}{(n-1)\log(n-1)}\right)^{-1}. \tag{82}$$

By iterating the inequality (82) $(n-2)$ times and taking $S_2 = 17$ in $(n-2)$th step, we immediately obtain the inequality (80). Substituting $t_n = S_n/(2n^2 \log n)$ into (80) gives the inequality (81). □

Notice that under Conjecture 5.3, the sequence (t_n) defined by (76) as

$$t_n = \frac{S_n}{2n^2 \log n}, \quad n \in \mathbb{N} \setminus \{1\},$$

is decreasing on the range $\{n \in \mathbb{N} : n \geq 1100\}$. As noticed above, the computational results for "prime sums" given in Table 1 of Section 6 suggest the fact that the sequence $(t_n)_{n=2}^{\infty}$ plays an important role for estimating the values M_k $(k = 1, 2, \ldots)$ in the expression (56) for the kth prime q_k in the sequence (S_n). Accordingly, we propose the following two conjectures concerning the upper and lower bounds of the sequence (M_k).

Conjecture 5.7 (The upper bound of the sequence (M_k))**.** *Let* $(M_k)_{k=1}^{\infty}$ *be the sequence defined by by the expression* (56) *of Theorem* 4.4. *Then*

$$M_k \leq t_k = \frac{S_k}{2k^2 \log k} := M_k^{(u)} \quad \text{for all } k \geq 2. \tag{83}$$

Corollary 5.8. *Let* $(M_k)_{k=1}^{\infty}$ *be the sequence defined by the expression* (56) *of Theorem* 4.4. *Then under Conjecture* 5.7 *there holds*

$$M_k \leq 1 + \frac{\log 2 + \log\log(2k)}{\log k} \quad \text{for all } k \geq 2. \tag{84}$$

Proof. Combining the inequality on the right hand side of (37) from Proposition 3.12 with the inequality (83), we immediately obtain (84). □

Corollary 5.9. *Let* q_k *be the* k*th prime in the sequence* (S_n) $(k = 1, 2, \ldots)$. *Then under Conjectures* 3.3 *and* 5.7 *for all* $k \geq 2$ *there holds*

(85)
$$q_k < 2k^2 \log^3 k \left(1 + \frac{\log 2 + \log\log(2k)}{\log k}\right)^5 \left(1 + \frac{\log\log k}{\log k} + \frac{2\log 2 + 2\log\log(2k)}{\log^2 k}\right).$$

Proof. Applying the inequality $\log(1+x) < x$ with $x > 0$ to (84), we find that

$$\log M_k \leq \frac{\log 2 + \log\log(2k)}{\log k} \quad \text{for all } k \geq 2. \tag{86}$$

Inserting the inequalities (84) and (86) into the expression (56) of Theorem 4.4 for q_k, we immediately obtain (85). □

Corollary 5.10. *Let* q_k *be the* k*th prime in the sequence* (S_n) $(k = 1, 2, \ldots)$. *Then under Conjectures* 3.3 *and* 5.7 *there holds*

$$q_k = 2k^2 \log^2 k (\log k + O(\log\log k)), \tag{87}$$

or equivalently,

$$\frac{q_k}{2k^2 \log^3 k} = 1 + O\left(\frac{\log\log k}{\log k}\right). \tag{88}$$

Proof. The inequality (85) immediately yields the asymptotic expression (87). □

Corollary 5.10 can be refined as follows.

Corollary 5.11. *Let q_k be the kth prime in the sequence (S_n) $(k = 1, 2, \ldots)$. Then under Conjectures* 3.3 *and* 5.7 *there exists an absolute positive constant C with $1 \le C \le 6$ such that*

$$q_k = 2k^2 \log^2 k(\log k + C \log\log k + o(\log\log k)).$$

Proof. Using the binomial expansion, we find that

$$\left(1 + \frac{\log 2 + \log\log(2k)}{\log k}\right)^5 = 1 + \frac{5\log\log k}{\log k} + o\left(\frac{\log\log k}{\log k}\right),$$

which substituting in (85) immediately yields the estimation from Corollary 5.11. □

Remark **5.12.** The determination of a constanst C from Corollary 5.11 is closely related to the sequence (Q_k) with $Q_k = (q_k - 2k^2 \log^3 k)/(2k^2(\log^2 k)\log\log k)$, whose values are presented in Table 3 of Section 6. Related data from Table 3 and the additional computations suggest that

$$q_k < 2k^2 \log^2 k(\log k + 6\log\log k) \quad \text{for all } k \ge 5 \cdot 10^6.$$

Conjecture 5.13 (A refined upper bound of the sequence (M_k))**.** *Let $(M_k)_{k=1}^{\infty}$ be the sequence defined by the expression* (56) *of Theorem* 4.4*. Then*

$$M_k \le t_{\lfloor k\log k\rfloor} = \frac{S_{\lfloor k\log k\rfloor}}{2(\lfloor k\log k\rfloor)^2 \log\lfloor k\log k\rfloor} := M_k^{(l)} \quad \text{for all } k \ge 5\times 10^7, \tag{89}$$

where $\lfloor k\log k\rfloor$ is the greatest integer not exceeding $k\log k$.

Corollary 5.14. *Let (t_n) be the sequence defined by* (76)*. Then under the inequality* (63) *of Conjecture* 4.6 *and Conjecture* 5.13*, for all $k \ge 2$ the interval*

$$\begin{aligned}&\Big[2k^2(\log^2 k)(\log k + \log\log k), 2k^2(\log^2 k)\Big(1 + \frac{\log(2\log(2k))}{\log k}\Big)^5 \times \\ &\times (\log k + \log\log k + 2\log\Big(1 + \frac{\log(2\log(2k))}{\log k}\Big)\Big)\Big]\end{aligned} \tag{90}$$

contains at least one prime that belongs to the sequence (S_n). In particular, the prime q_k belongs to the interval given by (90)*.*

Furthermore, for all $k \ge 2$ the length l_k of the interval (90) *satisfies the inequality*

(91)

$$l_k < 62k^2(\log k)\log(k\log k)\log(2\log(2k)) + 4k^2(\log k + 31\log(2\log(2k))\log(2\log(2k))).$$

Proof. The first assertion immediately follows from the inequality (67) of Corollary 4.7 and the inequality (83) of Conjecture 5.7. Notice that by the inequality on the right hand side of (37) from Proposition 3.12, we find that

$$t_k < 1 + \frac{\log(2\log(2k))}{\log k} \quad \text{for all } k \ge 5. \tag{92}$$

Then the inequality (83) of Conjecture 5.7 and the inequality (92) immediately yield

(93)

$$\begin{aligned}q_k &= 2M_k^5 k^2(\log^2 k)(\log k + \log\log k + 2\log M_k)\\ &\le 2t_k^5 k^2(\log^2 k)(\log k + \log\log k + 2\log t_k)\\ &\le 2k^2(\log^2 k)\Big(1 + \frac{\log(2\log(2k))}{\log k}\Big)^5(\log k + \log\log k + 2\log\Big(1 + \frac{\log(2\log(2k))}{\log k}\Big)).\end{aligned}$$

The inequalities (93) and (67) of Corollary 4.7 show that the interval defined by (90) contains the prime sum q_k.

Further, using the inequality $(1+x)^5 \le 1+31x$ for $0 \le x := \log(2\log(2k))/\log k \le 1$ and the inequality $\log(1+x) < x$ for $x := \log(2\log(2k))/\log k > 0$, the length l_k of interval defined by (90) can be estimated as follows.

(94)

$$\begin{aligned} l_k \le & \left(\left(\left(1+\frac{\log(2\log(2k))}{\log k}\right)^5 - 1\right) 2k^2(\log^2 k)(\log k + \log\log k)\right. \\ & + 4\left(1+\frac{\log(2\log(2k))}{\log k}\right)^5 k^2(\log^2 k)\log\left(1+\frac{\log(2\log(2k))}{\log k}\right) \\ \le & \frac{31\log(2\log(2k))}{\log k} 2k^2(\log^2 k)(\log k+\log\log k) \\ & + 4\left(1+31\cdot\frac{\log(2\log(2k))}{\log k}\right) k^2(\log^2 k)\frac{\log(2\log(2k))}{\log k} \\ = & 62k^2(\log k)\log(k\log k)\log(2\log(2k)) + 4k^2(\log k + 31\log(2\log(2k))\log(2\log(2k)). \end{aligned}$$

This completes the proof. □

Nevertheless the fact that $M_k^{(l)}$ is probably the upper bound of M_k for all $n > 5\cdot 10^7$ (see Table 1), we propose the following conjecture.

Conjecture 5.15. *Let* (t_n) *be the sequence defined by* (76). *Then for all* $k \ge 2$ *the interval*

(95)

$$[2t^5_{\lfloor k\log k\rfloor}k^2(\log^2 k)(\log k+\log\log k+2\log t_{\lfloor k\log k\rfloor}), 2t_k^5k^2(\log^2 k)(\log k+\log\log k+2\log t_k)]$$

contains at least one prime sum q_i *from the sequence* (S_n).

As an application of Corollary 5.14, we obtain the following (S_n)-analogue of the well known fact that the series $\sum_{n=1}^{\infty} 1/p_n$ diverges.

Corollary 5.16. *The series*

$$\sum_{k=1}^{\infty}\frac{k\log^2 k}{q_k} \tag{96}$$

diverges.

Proof. It is easy to see that for each $k \ge 2$ the right bound of the interval given by (90) is less than $288k^2\log^3 k$, and hence, by Corollary 5.14, $q_k < 288k^2\log^3 k$ for each $k \ge 2$. Therefore, $k\log^2 k/q_k > 1/(288k\log k)$, and hence,

$$\begin{aligned} \sum_{k=1}^{n}\frac{k\log^2 k}{q_k} &> \frac{1}{288}\sum_{k=2}^{n}\frac{1}{k\log k} \sim \int_2^n \frac{d\,x}{x\log x} \\ &= \log\log x\Big|_2^n = \log\log n - \log\log 2 \to \infty \text{ as } n\to\infty. \end{aligned}$$

Therefore, the series (96) diverges. □

On the other hand, we have the following consequence of of Corollary 5.14.

Corollary 5.17. *For every $\varepsilon > 0$ the series*

$$\sum_{k=1}^{\infty} \frac{k \log^{2-\varepsilon} k}{q_k} \tag{97}$$

converges.

Proof. By Corollary 5.14 (see the interval (90)), $q_k > 2k^2 \log^3 k$ for each $k \geq 2$. Therefore, $k \log^{2-\varepsilon} / q_k < 1/(2k \log^{1+\varepsilon} k)$, and hence,

$$\sum_{k=1}^{n} \frac{k \log^{2-\varepsilon} k}{q_k} < \frac{1}{2} \sum_{k=2}^{n} \frac{1}{k \log^{1+\varepsilon} k} \sim \int_2^n \frac{d\,x}{x \log^{1+\varepsilon} x}$$
$$= -\frac{1}{\varepsilon \log^{\varepsilon} x}\bigg|_2^n = \frac{1}{\varepsilon \log^{\varepsilon} 2} - \frac{1}{\varepsilon \log^{\varepsilon} n} \to \frac{1}{\varepsilon \log^{\varepsilon} 2} \text{ as } n \to \infty.$$

Therefore, the series (97) converges. □

6. Computational results

By using `Mathematica 8`, here we present our computational results concerning the the number of expression "prime sums" q_k (under Conjecture 3.3) and related expression (the equality (56) of Theorem 4.4). The notion $\pi_n := k$ in the second column of Tables 1, 3 and 4 presents the number of primes in a set $\mathcal{S}_n := \{S_1, S_2, \dots, S_n\}$, where n is a related value given in the first column of this table. Hence, under notations of Section 1 and Conjecture 3.3,

$$k := \pi_n := \pi_{(S_k)}(S_n) = \#\{p : p \text{ is a prime and } p = S_i \text{ for some } i \text{ with } 1 \leq i \leq n\}.$$

Accordingly, the value k in the second column of Table 1 presents the number of primes in a set $\mathcal{S}_n$, where n is a related value given in the first column of this table. The appropriate rounded value of the greatest prime q_k in $\mathcal{S}_n$ is given in the third column (related exact values are given in Table 3), while in the next column it is written the values $n-m$, where m are related indices such that $q_k = S_m$. In the fifth column of Table 1 we present the corresponding values of M_k obtained as solutions of the equation (56) in Theorem 4.4. The refined upper bound $M_k^{(l)}$ and the upper bound $M_k^{(u)}$ of M_k given in Conjectures 5.13 and 5.7, respectively, are given in the next two columns of Table 1. Notice taht the data from the last two columns of this table are closely related to Conjecture 4.6 and Conjecture 4.9, respectively.

For example, a computation gives the following exact values: $q_{59129} = S_{849995} = 22420773979207$, $q_{62297} = S_{899999} = 25235697805141$, $q_{2707378} = 99262810294692679$ $q_{5212720} = S_{10^8} = 411680592327546713$.

Table 1. Distribution of primes in the sequence (S_n) in the range $1 \le n \le 10^9 + 5 \cdot 10^8$

n	$k := \pi_n$	$\approx q_k$	$n-m$	M_k	$M_k^{(l)}$	$M_k^{(u)}$	$(k \log k)/m$	$S_m\sqrt{k\log k}/$ $(2m^{5/2}\log m)$
100	23	107934	1	1.17894	1.22166	1.27421	0.737366	1.034203
1000	141	15501706	22	1.18281	1.18140	1.20278	0.713472	0.995174
10000	1098	$2.12 \cdot 10^{10}$	17	1.14356	1.16216	1.17714	0.770046	1.018392
100000	8350	$2.64 \cdot 10^{11}$	10	1.15163	1.14863	1.16175	0.752577	0.995122
155000	12379	$6.57 \cdot 10^{11}$	3	1.15259	1.14628	1.15912	0.752638	0.993165
185000	14482	$9.49 \cdot 10^{11}$	6	1.15241	1.14536	1.15821	0.750009	0.990628
200000	15504	$1.1 \cdot 10^{11}$	18	1.15241	1.14495	1.15768	0.750177	0.990392
220000	16954	$1.36 \cdot 10^{12}$	17	1.15090	1.14446	1.15714	0.752476	0.991502
296000	22327	$2.51 \cdot 10^{12}$	3	1.15773	1.14302	1.15565	0.741044	0.982660
300000	22595	$2.59 \cdot 10^{12}$	5	1.15774	1.14295	1.15556	0.740997	0.982557
350000	26038	$3.56 \cdot 10^{12}$	4	1.15675	1.14216	1.15469	0.742338	0.982816
400000	29495	$4.7 \cdot 10^{12}$	49	1.15398	1.14147	1.15399	0.746555	0.985037
450000	32928	$6.00 \cdot 10^{12}$	39	1.15169	1.14087	1.15336	0.750052	0.986842
500000	36302	$7.47 \cdot 10^{12}$	14	1.15027	1.14035	1.15274	0.752199	0.987819
550000	39788	$9.10 \cdot 10^{12}$	1	1.14730	1.13984	1.15221	0.756907	0.990517
600000	43119	$1.09 \cdot 10^{13}$	8	1.14633	1.13941	1.15179	0.758358	0.991099
650000	46488	$1.28 \cdot 10^{13}$	13	1.14485	1.13902	1.15136	0.760670	0.992278
700000	49834	$1.50 \cdot 10^{13}$	6	1.14361	1.13866	1.15094	0.762604	0.993234
800000	56419	$1.97 \cdot 10^{13}$	27	1.14232	1.13801	1.15025	0.764538	0.993953
850000	59602	$2.24 \cdot 10^{13}$	5	1.14239	1.13772	1.14992	0.764332	0.993576
900000	62770	$2.52 \cdot 10^{13}$	1	1.14245	1.13746	1.14964	0.764154	0.993230
950000	66064	$2.82 \cdot 10^{13}$	45	1.14140	1.37190	1.14937	0.765800	0.994084
10^6	69251	$3.13 \cdot 10^{13}$	5	1.14093	1.13692	1.14910	0.771847	0.950112
$2 \cdot 10^6$	131841	$1.31 \cdot 10^{14}$	23	1.13400	1.13373	1.14563	0.777169	0.998482
$3 \cdot 10^6$	192655	$3.03 \cdot 10^{13}$	14	1.13116	1.13193	1.14366	0.781454	0.999694
$4 \cdot 10^6$	252028	$5.49 \cdot 10^{14}$	23	1.12965	1.13069	1.14232	0.783641	1.0000075
$5 \cdot 10^6$	310756	$8.70 \cdot 10^{14}$	22	1.12809	1.12973	1.14127	0.786405	1.0007040
10^7	594851	$3.62 \cdot 10^{15}$	6	1.12473	1.12686	1.13814	0.790918	1.001331
$5 \cdot 10^7$	2707378	$9.92 \cdot 10^{16}$	10	1.11727	1.12067	1.131310	0.802006	1.002899
10^8	5212720	$4.11 \cdot 10^{17}$	13	1.11444	1.11819	1.12856	0.806231	1.003355
$10^8 + 5 \cdot 10^7$	7650550	$9.45 \cdot 10^{17}$	13	1.11295	1.116783	1.126996	0.808423	1.003481
$2 \cdot 10^8$	10047823	$1.70 \cdot 10^{18}$	8	1.11189	1.11581	1.12591	0.809999	1.003599
$3 \cdot 10^8$	14763858	$3.91 \cdot 10^{18}$	4	1.11032	1.11446	1.12441	0.812391	1.003890
$4 \cdot 10^8$	19404439	$7.05 \cdot 10^{18}$	1	1.10922	1.11353	1.12336	0.814065	1.004097
$5 \cdot 10^8$	23985388	$1.11 \cdot 10^{19}$	13	1.10848	1.11281	1.12256	0.815165	1.004142
$7 \cdot 10^8$	33031264	$2.21 \cdot 10^{19}$	18	1.10730	1.11176	1.12138	0.816956	1.004305
10^9	46388006	$4.60 \cdot 10^{19}$	2	1.10605	1.11066	1.12014	0.818867	1.004501
$10^9 + 5 \cdot 10^8$	68259534	$1.05 \cdot 10^{20}$	35	1.10473	1.10957	1.11877	0.820881	1.004650

Recall that $\pi(x)$ denotes the number of primes less or equal to x. Then Table 2 present the quotients $\frac{\pi_n}{\frac{n}{\log n}}$ and $\frac{\pi_n}{\pi(n)}$.

Table 2. Distribution of primes in the sequence (S_n) in the range $1 \le n \le 10^9 + 5 \cdot 10^8$

n	$\frac{\pi_n}{\frac{n}{\log n}}$	$\frac{\pi_n}{\pi(n)}$	n	$\frac{\pi_n}{\frac{n}{\log n}}$	$\frac{\pi_n}{\pi(n)}$
10^2	1.059190	0.920000	10^7	0.958787	0.895079
10^3	0.973993	1.011300	$5 \cdot 10^7$	0.959903	0.902118
10^4	1.011300	0.893409	10^8	0.960219	0.904758
10^5	0.961329	0.870517	$5 \cdot 10^8$	0.960860	0.910059
10^6	0.956738	0.882201	10^9	0.961311	0.912296
$5 \cdot 10^6$	0.958679	0.891663	$10^9 + 5 \cdot 10^8$	0.961492	0.913458

Notice that from Table 1 we see that $M_k^{(l)}$ is probably the upper bound of M_k for $n > 5 \cdot 10^7$ (Conjecture 5.13) which is better estimate than $M_k^{(u)}$ (Conjecture 5.7).

The values of first three columns of Table 3 are defined in the same way as these of Table 1 (with exact values of q_k), and the related values of ratios $q_k/(2k^2 \log^3 k)$ are given in fourth column of this table. The asymptotic relation (87) of Corollary 5.10 shows that it can be of interest to analyze the sequence $(Q_k)_{k=2}^{\infty}$ whose kth term is defined by the equality

$$q_k = 2k^2 \log^3 k + 2Q_k k^2 (\log^2 k)(\log\log k), \quad k = 2, 3, \ldots, \tag{98}$$

or equivalently,

$$\frac{q_k}{2k^2 \log^3 k} = 1 + Q_k \cdot \frac{\log\log k}{\log k} \quad k = 2, 3, \ldots. \tag{99}$$

We also consider two similar sequences (Q'_k) and (Q''_k) which are closely related to Corollary 4.7 and Theorem 4.4, respectively (cf. Remark 4.11), and they are defined as

$$Q'_k = \frac{q_k - 2k^2(\log^2 k)(\log k + \log\log k)}{2k^2(\log^2 k)\log\log k}, \quad k = 2, 3, \ldots, \tag{100}$$

and

$$Q''_k = \frac{q_k - 2(p_k)^2 \log k}{2k^2(\log^2 k)\log\log k}, \quad k = 2, 3, \ldots. \tag{101}$$

Some values of these sequences are given in the last three columns of Table 3.

Table 3. Some "prime sums" q_k's in the sequence (S_n) with $n \le 10^9 + 5 \cdot 10^8$ and related values $q_k/(2k^2 \log^3 k)$, $q_k/(2m^2 \log m)$, Q_k, Q'_k and Q''_k

n	$k := \pi_n$	q_k	$\frac{q_k}{2k^2\log^3 k}$	$\frac{q_k}{2m^2\log m}$	Q_k	Q'_k	Q''_k
10	5	281	1.34807	1.47352	2.35436	0.17772	-1.76013
100	23	107934	3.30944	1.19829	6.33647	5.33647	5.44583
1000	141	15501706	3.21679	1.17689	6.86016	5.86016	5.77437
100000	8350	264074170741	2.58273	1.14710	6.49405	5.49405	5.28293
200000	15504	1116374522657	2.56968	1.14347	6.68238	5.68238	5.44027
300000	22595	2591079720139	2.61956	1.14145	7.03697	6.03697	5.79201
400000	29495	4704619172003	2.56741	1.14004	6.91365	5.91365	5.66455
500000	36302	7472533368077	2.51877	1.15867	6.77736	5.77736	5.51158
600000	43119	10901967324637	2.46956	1.13810	6.62021	5.62021	5.35079
700000	49834	15001269948023	2.43548	1.13737	6.51781	5.51781	5.24772
800000	56419	19776121232971	2.41801	1.13675	6.48149	5.48149	5.2025
900000	62770	25235697805141	2.41648	1.13621	6.51155	5.51155	5.22977
10^6	69251	31380813002879	2.40459	1.13572	6.30126	5.30126	5.02105
$4 \cdot 10^6$	252028	549524547523421	2.24844	1.12966	6.15989	5.15989	4.84045
$5 \cdot 10^6$	310756	870522520170287	2.22830	1.12873	6.12200	5.12200	4.79728
10^7	594851	3629567501866919	2.181921	1.12593	6.07346	5.07346	4.7374
$5 \cdot 10^7$	2707378	99262810294692679	2.083831	1.11987	5.95575	4.95575	4.58936
$7 \cdot 10^7$	3720648	198036667738658321	2.065440	1.11868	5.93361	4.93361	4.56144
$8 \cdot 10^7$	4220531	260463664887226043	2.059235	1.11821	5.93009	4.93009	4.55643
10^8	5212720	411680592327546713	2.047463	1.11744	5.91551	4.91551	4.53769
$2 \cdot 10^8$	10047823	1705122556732581169	2.014906	1.11511	5.88554	4.88554	4.49873
$3 \cdot 10^8$	14763858	3913274710820657161	1.995499	1.11379	5.86106	4.86106	4.46806
$4 \cdot 10^8$	19404439	7053651472078078383	1.982108	1.11287	5.84373	4.84373	4.44675
$5 \cdot 10^8$	23985388	11138479445180255153	1.972857	1.11217	5.83583	4.83583	4.43625
$7 \cdot 10^8$	33031264	22177401605086098829	1.958466	1.11113	5.81944	4.81944	4.41565
10^9	46388006	46007864234123508181	1.943427	1.11005	5.80097	4.80097	4.39275
$10^9 + 5 \cdot 10^8$	68259534	105428905479616558423	1.927428	1.10885	5.78377	4.78377	4.37116

It is easy to prove the folowing result.

Proposition 6.1. *Let* (Q_k), (Q'_k) *and* (Q''_k) *be the sequnces defined by* (99), (100) *and* (101), *respectively. Then*

$$\lim_{k\to\infty}(Q_k - Q'_k) = \lim_{k\to\infty}(Q'_k - Q''_k) = 0.$$

We also propose the following conjecture.

Conjecture 6.2. *The all sequences* (Q_k), (Q'_k) *and* (Q''_k) *converge to* 1.

Remark **6.3.** In view of the above conjecture, it can be of interest to consider the sequence (Q'''_k) defined as

$$Q'''_k = \frac{q_k - 2k^2(\log^2 k)(\log k + \log\log k)}{2k^2(\log^2 k)\log\log\log k}, \quad k = 2, 3, \dots,$$

The values of Q'''_{10^s} for $s = 3, 5, 6, 7, 8, 9$ are equals to 19.961, 15.329 14.524 13.809 13.621, 13.069, respectively.

Remark **6.4.** The values $V_k := q_k/(2m^2 \log m) = S_m/(2m^2 \log m)$ presented in the fourth column of Table 3 are in fact terms of the sequence $t_n := S_n/(2n^2 \log n)$ with $n = 2, 3, \dots$, which is decreasing under Conjecture 5.3 on the range $\{n \in \mathbb{N} : n \geq 1100\}$. Accordingly, under Conjectures 4.6 and 5.3 and the fact that $q_{151} = S_{1100} = 19949537$, we immedately get

$$V_k = t_m \leq t_{\lfloor k\log k\rfloor+1} < t_{\lfloor k\log k\rfloor} := M_k^{(l)} \quad \text{for all } k \geq 151,$$

that is,

$$V_k < M_k^{(l)} \quad \text{for all } k \geq 151, \tag{102}$$

where $M_k^{(l)}$ are approximative values for M_k given by (89) and presented in Table 1.

Moreover, the comparison of values of M_k with those of V_k from Tables 1 and 2, respectively, leads to the following conjecture.

Conjecture 6.5. *Let* $(M_k)_{k=2}^{\infty}$ *be the sequence defined by* (56) *of Theorem* 4.4*, and let* $m(k) = m$ *be defined as* $S_m = q_k$*. Then*

$$M_k < \frac{q_k}{2m^2 \log m} \quad for\ all\ k \geq 4 \times 10^6. \tag{103}$$

Consequently, we obtain the following "weak version" of Conjecture 6.5.

Corollary 6.6. *Let* $(M_k)_{k=2}^{\infty}$ *be the sequence defined by* (56) *of Theorem* 4.4*, and let* $m(k) = m$ *be defined as* $S_m = q_k$*. Then under Conjectures* 4.6*,* 5.3 *and* 6.5 *we have*

$$M_k < t_{\lfloor k\log k\rfloor+1} := \frac{S_{\lfloor k\log k\rfloor+1}}{2(\lfloor k\log k\rfloor+1)^2 \log(\lfloor k\log k\rfloor+1)} \quad for\ all\ k \geq 4 \times 10^6.$$

Proof. Combining Conjectures 4.6, 5.3 and 6.5, we find that for all $k \geq 4 \times 10^6$ with $q_k = S_m$

$$M_k < \frac{q_k}{2m^2 \log m} = \frac{S_m}{2m^2 \log m} = t_m \leq t_{\lfloor k\log k\rfloor+1} = \frac{S_{\lfloor k\log k\rfloor+1}}{2(\lfloor k\log k\rfloor+1)^2 \log(\lfloor k\log k\rfloor+1)},$$

as desired. □

Remark **6.7.** The ratios $L_k := q_k/(2k^2 \log^2 k(\log k + \log\log k))$ are closely related to Corollary 4.7. Of course, the values L_k are small than the related values $q_k/(2k^2 \log^3 k)$ presented in the fourth column of Table 3. For example, L_k is equal to $1.762999, 1.696920$ for $k = 2707378, 19404439$, respectively. However, the sequence (L_k) slowly tends to 1 as k grows. This is directly connected with the fact that the sequence $(k\log k/m(k))$ converges very slowly to 1 as k grows (see the eighth column of Table 1).

Remark **6.8.** A good approximation from Remark 4.11 arising from the last column of Table 1 can be written as

$$\sqrt{k\log k}\approx \frac{2m^2\sqrt{m}\log m}{S_m}, \tag{104}$$

where $q_k = S_m$. The approximation (104) allows us for given n to determine the index $k = k(n)$ such that the prime sum q_k is "very close" to S_n; especially, for each $n \geq 4\times 10^6$ (i.e., for $k \geq 252028$), assuming that Conjecture 4.9 is true, then $q_{k(n)} < S_n$. Accordingly, for given n we assume that $k_0(n) = \lfloor x_0 \rfloor$, where $x_0 = x_0(n)$ is a root of the equation

$$\sqrt{x\log x} = \frac{2n^2\sqrt{n}\log n}{S_n}. \tag{105}$$

For some values n from Table 1 Table 4 presents the exact largest values $k(n)$ such that $q_{k(n)} \leq S_n$ (these values are in fact, given in the second column of Table 1) and related differences $k(n) - k_0(n)$.

Table 4. The values $k = k_0(n)$ and $k(n) - k_0(n) = k - k_0$ for some values $n \leq 10^9$

n	$k:=\pi_n$	$k-k_0$	n	k	$k-k_0$	n	k	$k-k_0$
10	5	3	10^2	23	1	10^3	141	-3
10^4	1098	-33	10^5	8350	-59	$2\cdot 10^5$	15504	-315
$3\cdot 10^5$	22595	-338	$4\cdot 10^5$	29495	-371	$5\cdot 10^5$	36302	-812
$6\cdot 10^5$	43119	-263	$7\cdot 10^5$	49834	-177	$8\cdot 10^5$	56419	-627
$9\cdot 10^5$	62770	-308	10^6	69251	-283	$2\cdot 10^6$	131841	-368
$3\cdot 10^6$	192655	-110	$4\cdot 10^6$	252028	**5**	$5\cdot 10^6$	310756	404
10^7	594851	1469	$2\cdot 10^7$	1141478	4638	$3\cdot 10^7$	1671839	7462
$4\cdot 10^7$	2193083	10997	$5\cdot 10^7$	2707378	14644	$6\cdot 10^7$	3216515	18621
$7\cdot 10^7$	3720648	22061	$8\cdot 10^7$	4220531	25021	$9\cdot 10^7$	4717545	28357
10^8	5212720	32696	10^8+10^7	5703356	35030	$10^8+2\cdot 10^7$	6191655	37303
$10^8+3\cdot 10^7$	6679364	41059	$10^8+4\cdot 10^7$	7165567	45196	$10^8+5\cdot 10^7$	7650550	49854
$10^8+6\cdot 10^7$	8132623	53221	$2\cdot 10^8$	10047823	67743	$3\cdot 10^8$	14763858	107704
$4\cdot 10^8$	19404439	149159	$5\cdot 10^8$	23985388	186542	$7\cdot 10^8$	33031264	267196
$8\cdot 10^8$	37508452	309262	$9\cdot 10^8$	41960355	351779	10^9	46388006	392660

In view of the above considerations and computational results given in Table 4, we propose the following conjecture.

Conjecture 6.9. *Let $n \geq 4\times 10^6$ be a positive integer, and let $x_0(n)$ be a real root of the equation*

$$\sqrt{x\log x} = \frac{2n^2\sqrt{n}\log n}{S_n}. \tag{106}$$

Then the set $\{S_1, S_2, \ldots, S_n\}$ contains at least $\lfloor x_0(n)\rfloor$ primes.

The inequality on right hand side of (37) of Proposition 3.12 immediately gives the following weak version of Conjecture 6.9.

Conjecture 6.10. *Let $n \geq 4\times 10^6$ be a positive integer, and let $y_0(n)$ be a real root of the equation*

$$\left(1+\frac{\log 2+\log\log(2n)}{\log n}\right)\sqrt{y\log y} = \sqrt{n}. \tag{107}$$

Then the set $\{S_1, S_2, \ldots, S_n\}$ contains at least $\lfloor y_0(n)\rfloor$ primes.

It can be also of interest to compare the values $k_0(n)$ and $k_1(n) := \lfloor y_0(n) \rfloor$ with the values $k_2(n) := \lfloor z_0(n) \rfloor$, where $z_0(n)$ is a real root of the equation

$$x \log x = n. \tag{108}$$

Corollary 6.11. *Let $n \geq 4 \times 10^6$ be a positive integer. Then under Conjecture* 6.10 *and its notations, the sequence $(S_i)_{i=1}^{\infty}$ contains at least $k_0(n) := \lfloor y_0(n) \rfloor$ primes which are less than $2n^2(\log n + \log\log(2n) + \log 2)$. In other words,*

$$q_{k_0(n)} < 2n^2(\log n + \log\log(2n) + \log 2). \tag{109}$$

Proof. The assertion immediately follows from Conjecture 6.10 and the right hand side of the inequalities (37) from Proposition 3.12. □

The values $k_0(n)$ (derived from Table 4 as the differences $k_0(n) = k(n) - (k(n) - k_0(n)))$, $k_1(n)$ and $k_2(n)$ concerning the values of n from Table 4, are presented in Table 5.

Table 5. The values $k_i(n)$, $i = 0, 1, 2$, the ratios $\delta_j(n) := (k(n) - k_j(n))/k(n)$ with $j = 1, 2$, the ratios $\eta(n) := k_0(n)/\sqrt{k_1(n)k_2(n)}$ and $\xi(n) := k(n)/\sqrt{k_1(n)k_2(n)}$

n	$k_0(n)$	$\delta_0(n)$	$k_1(n)$	$\delta_1(n)$	$k_2(n)$	$\delta_2(n)$	$\eta(n)$	$\xi(n)$
10	2	0.60000	2	0.60000	5	1.00000	0.63246	1.58114
10^2	22	0.04348	15	0.34783	29	-0.26087	1.05482	1.10277
10^3	144	-0.00213	109	0.22695	190	-0.347552	1.00063	0.97978
10^4	1131	-0.03005	846	0.22951	1382	-0.25865	1.04598	1.01546
10^5	8409	-0.0071	6928	0.17030	10770	-0.28982	0.97345	0.96666
$5 \cdot 10^5$	37114	-0.02237	30816	0.15112	46521	-0.28150	0.98022	0.95878
10^6	69534	-0.00409	58857	0.15009	87845	-0.26850	0.96703	0.96309
$4 \cdot 10^6$	**252023**	0.00002	216103	0.14254	315878	-0.25334	0.96461	0.96463
$5 \cdot 10^6$	310352	0.00130	266622	0.14202	388499	-0.25017	0.96430	0.96555
10^7	593382	0.00247	512630	0.13822	739955	-0.24393	0.96345	0.96584
$5 \cdot 10^7$	2692734	0.00541	2353142	0.13084	3329279	-0.22971	0.962043	0.96728
10^8	5180024	0.0063	4546674	0.12778	6382029	-0.22432	0.96162	0.96769
$5 \cdot 10^8$	23798846	0.00778	21080800	0.12110	29093410	-0.21296	0.96098	0.96851
10^9	45995346	0.00846	40886757	0.11859	56048389	-0.20825	0.96082	0.96902

The last column of Table 5 suggests that $\xi(n) < 1$ for all $n \geq 10^5$, which is obviously equivalent with the following conjecture.

Conjecture 6.12. *Let $n \geq 10^5$ be a positive integer, and let $k_1(n) = \lfloor y_0(n) \rfloor$ and $k_2(n) = \lfloor z_0(n) \rfloor$, where $y_0(n)$ and $z_0(n)$ are real roots of the equations* (107) *and* (108)*, respectively. Then the set $\{S_1, S_2, \dots, S_n\}$ contains less than $\lfloor \sqrt{k_1(n) \cdot k_2(n)} \rfloor$ primes.*

As an immediate consequence, we obtain the following statement.

Corollary 6.13. *Let $n \geq 10^5$ be a positive integer. Then under Conjecture* 6.12 *and its notations the sequence $(S_i)_{i=1}^{\infty}$ contains at most $\lfloor \sqrt{k_1(n) \cdot k_2(n)} \rfloor$ primes which are less than $2n^2 \log n$. In other words,*

$$q_{\lfloor \sqrt{k_1(n) \cdot k_2(n)} \rfloor} > 2n^2 \log n. \tag{110}$$

Proof. The assertion immediately follows from Conjecture 6.12 and left hand side of the inequalities of (37) from Proposition 3.12. □

Remark **6.14.** Conjecture 6.6 may be considered as the "prime sums analogue" of the well known fact that $p_k \geq k \log k$ for all $k \geq 3$, where p_k is the kth prime (see e.g., [38, p. 69])

Remark **6.15.** The approximation (105) can be written as

$$\sqrt{\frac{m}{k\log k}} \approx \frac{S_m}{2m^2\log m}, \tag{111}$$

which in view of Conjecture 5.3 asserts that the sequence $(m/(k\log k))_{k=k_1}^{\infty}$ is decreasing for a fixed large integer k_1.

On the other hand, if we write the estimate (111) in the form

$$\sqrt{\frac{m\log^2 m}{k\log k}} \approx \frac{S_m}{2m^2}, \tag{112}$$

then Proposition 5.1 suggests that the sequence $(m\log^2 m/(k\log k))_{k=k_2}^{\infty}$ is increasing for some fixed large integer k_2.

7. THE STRONGER FORM OF CONJECTURE 3.3

Around 1800, young C. F. Gauss conjectured that for large n the the number of primes not exceeding n is nearly

$$\mathrm{li}(n) := \int_2^n \frac{dt}{\log t}.$$

Heuristic and computational arguments give the impression that Restricted Prime Number Theorem (RPNT) for the sequence (S_n) (i.e., Conjecture 3.3) probably holds in its stronger form which in fact presents the well known form of Prime Number Theorem (PNT) for primes (see e.g., [18, Chapter 12]). Accordingly, we propose the following conjecture.

Conjecture 7.1. *Let $\pi_{(S_k)}(S_n) = \pi_n$ be the number of primes p in the sequence (S_k) such that $p = S_i$ for some i with $1 \le i \le n$. Then*

$$\pi_n = \mathrm{li(n)} + R(n), \tag{113}$$

where

$$\mathrm{li}(n) := \int_2^n \frac{dt}{\log t} = \frac{n}{\log n} + O\left(\frac{1}{\log^2 n}\right) \quad as\ n \to \infty. \tag{114}$$

is the logarithmic integral and

$$R(n) \ll ne^{(-C\delta(n))} \quad with \quad \delta(n) := (\log n)^{3/5}(\log\log n)^{-1/5}. \tag{115}$$

Assuming the above conjecture, and following related "PNT result" of A. Ivić and J.-M. De Koninck [20, Theorem 9.1] (see also [19, Theorem]), it can be proved the following result.

Corollary 7.2. *Under the truth and notations of Conjecture* 7.1*, we have*

$$\sum_{i=1}^{n} \frac{1}{\pi_i} = \frac{1}{2}\log^2 n + O(\log n) \quad as\ n \to \infty. \tag{116}$$

Similarly, using Conjecture 7.1 it can be proved the following result.

Corollary 7.3. *Let q_k be the kth prime in (S_n). Then under Conjecture* 7.1*,*

$$\sum_{k=1}^{n} \frac{k\log^2 k}{q_k} = \log\log n + o\left(\frac{1}{\log n}\right). \tag{117}$$

Finally, we propose the following conjecture.

Conjecture 7.4 (Chebyshev inequalities for (S_n))**.** *There exist positive constants c_1, c_2 and a positive positive integer n_0 such that*

$$\frac{c_1 n}{\log n} \leq \pi_{(S_n)}(S_n) \leq \frac{c_2 n}{\log n} \quad for\ all\ n > n_0. \tag{118}$$

Remark **7.5.** We also believe that for the sequence (S_n) are valid the analogues of some other classical results and conjectures closely related to the Prime Number Theorem.

Remark **7.6.** Numerous computational results involving sums of the first n primes (the Sloane's sequence A007504 sequence here denoted as S_n') and certain their curious arithmetical properties are presented in Sloane's sequences A051838 (numbers n such that sum of first n primes divides product of first n primes), A116536, A067110, A067111, A045345, A114216 (sum of first n primes divided by maximal power of 2), A024011 (numbers n such that nth prime divides sum of first n primes), A036439 ($a(n) = 2+$ the sum of the first $n-1$ primes), A014284 (partial sums of primes, if 1 is regarded as a prime; $1, 3, 611, 18, 29, \ldots$), A134125 (integral quotients of partial sums of primes divided by the number of summations; $5, 5, 7, 11, 16, 107, \ldots$), A134126 (indices k such that the $(k+1)$th partial sum of primes divided by k is integer; $1, 2, 4, 7, 10, 50, 130, \ldots$), A134127 (largest prime in the partial sums of primes in A134125 which have integer averages), A134129 (prime partial sums $A007504(k+1)$ such that $A007504(k+1)/k$ is integer; $5, 10, 28, 77, 160, \ldots$), A077023, A033997, A071089, A083186 (sum of first n primes whose indices are primes), A166448 (sum of first n primes minus next prime), A196527, A065595 (square of first n primes minus sum of squares of first n primes), A165906 (sum of first n primes divided by the nth prime), A061568 (number of primes $\leq$ sum of first n primes), A066039 (largest prime less than or equal to the sum of first n primes), A077022, A110997, A112997 (sum of first n primes minus sum of their indices), A156778 (sum of first n primes multiplied by $n/2$), A167214 (sum of first n primes multiplied by n), A038346 (sum of first n primes $\equiv 1 (\bmod 4)$, A038347 (sum of first n primes $\equiv 3 (\bmod 4)$, A054972 (product of sum of first n primes and product of first n primes), A072476, A076570 (greatest prime divisor of sum of first n primes), A076873 (smallest prime not less than sum of first n primes), A077354 (sum of second string of n primes-sum of first n primes, or $2n$th partial sum of primes; this is in fact our sequence (S_n)), A110996, A123119 (number of digits in sum of first n primes), A189072 (semiprimes in the sum of first n primes), A196528, A022094 (sum of first p_n primes, where p_n is the nth prime), A024447, A121756, A143121 (triangle read by rows, $T(n,k) = \sum_{j=k}^{n} p_j$, $1 \leq k \leq n$), A117842 (partial sum of smallest prime $\geq n$), A118219, A131740 (sum of n successive primes after nth prime), A143215 (the sequence whose nth term is $p_n \cdot S_n' = p_n \cdot \sum_{i=1}^{n} p_i$), A161436 (sum of all primes from nth prime to $(2n-1)$th prime), A161490, A013918 (numbers n such that n is prime and is equal to the sum of the first k primes for some k; $2, 5, 17, 41, 197, 281, 7699, 8893, \ldots$) etc.

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