The Euler and Springer numbers as moment sequences

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Abstract

I study the sequences of Euler and Springer numbers from the point of view of the classical moment problem.

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1 Introduction

The representation of combinatorial sequences as moment sequences is a fascinating subject that lies at the interface between combinatorics and analysis. For instance, the Apéry numbers

$$A_n \;=\; \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \tag{1.1}$$

play a key role in Apéry's celebrated proof [6] of the irrationality of $\zeta(3)$ [31,47,73,80]; they also arise in Ramanujan-like series for $1/\pi$ [20] and $1/\pi^2$ [79]. As such, they have elicited much interest, both combinatorial [21, 62, 72] and number-theoretic [1, 14, 23, 35,58]. A few years ago I conjectured [25], based on extensive numerical computations, that the Apéry numbers are a Stieltjes moment sequence, i.e. $A_n = \int x^n \, d\mu(x)$ for some positive measure μ on $[0, \infty)$. Very recently this conjecture has been proven by Edgar [26], in a *tour de force* of special-functions work; he gives an explicit formula, in terms of Heun functions, for the (unique) representing measure μ. The more general conjecture [66] that the Apéry polynomials

$$A_n(x) \;=\; \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 x^k \tag{1.2}$$

are a Stieltjes moment sequence for all $x \ge 1$ remains open.

In this paper I propose to study the moment problem for two less recondite combinatorial sequences: the Euler numbers and the Springer numbers. Many of the results given here are well known; others are known but perhaps not as well known as they ought to be; a few seem to be new. This paper is intended as a leisurely survey that presents the relevant results in a unified fashion and employs methods that are as elementary as possible.

The *Euler numbers* E_n are defined by the exponential generating function

$$\sec t + \tan t \;=\; \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \,. \tag{1.3}$$

The E_{2n} are also called *secant numbers*, and the E_{2n+1} are called *tangent numbers*. The Euler numbers are positive integers that satisfy the recurrence

$$E_{n+1} \;=\; \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} E_{n-k} E_k \qquad \text{for } n \ge 1 \tag{1.4}$$

with initial condition $E_0 = E_1 = 1$; this recurrence follows easily from the differential equation $\mathcal{E}'(t) = \frac{1}{2}[1 + \mathcal{E}(t)^2]$ for the generating function $\mathcal{E}(t) = \sec t + \tan t$. André [3, 4] showed in 1879 that E_n enumerates the alternating (down-up) permutations of $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$, i.e. the permutations $\sigma \in \mathfrak{S}_n$ that satisfy $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \ldots$.[1]

[1] As Josuat-Vergès *et al.* point out [50, p. 1613], André's work "is perhaps the first example of an inverse problem in the theory of generating functions: given a function whose Taylor series has nonnegative integer coefficients, find a family of combinatorial objects counted by those coefficients."

More recently, other combinatorial objects have been found to be enumerated by the Euler numbers: complete increasing plane binary trees, increasing 0-1-2 trees, André permutations, simsun permutations, and many others; see [33, 52, 68, 74] for surveys. The sequence of Euler numbers starts as

$$(E_n)_{n\geq 0} \;=\; 1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, \ldots \tag{1.5}$$

and can be found in the On-Line Encyclopedia of Integer Sequences [57] as sequence A000111. It follows from (1.3) that E_n has the asymptotic behavior

$$E_n \;=\; \frac{4}{\pi}\left(\frac{2}{\pi}\right)^n n! \;+\; O\Big(\Big(\frac{2}{3\pi}\Big)^n n!\Big) \tag{1.6}$$

as $n \to \infty$.[2]

The *Springer numbers* S_n are defined by the exponential generating function [40, 41, 67]

$$\frac{1}{\cos t - \sin t} \;=\; \sum_{n=0}^{\infty} S_n \frac{t^n}{n!}\;. \tag{1.7}$$

Arnol'd [7] showed in 1992 that S_n enumerates a signed-permutation analogue of the alternating permutations. More precisely, recall that a *signed permutation* of $[n]$ is a sequence $\pi = (\pi_1, \ldots, \pi_n)$ of elements of $[\pm n] \stackrel{\text{def}}{=} \{-n, \ldots, -1\} \cup \{1, \ldots, n\}$ such that $|\pi| \stackrel{\text{def}}{=} (|\pi_1|, \ldots, |\pi_n|)$ is a permutation of $[n]$. In other words, a signed permutation π is simply a permutation $|\pi|$ together with a sign sequence $\operatorname{sgn}(\pi)$. We write $\mathfrak{B}_n$ for the set of signed permutations of $[n]$; obviously $|\mathfrak{B}_n| = 2^n n!$. Then a *snake of type* B_n is a signed permutation $\pi \in \mathfrak{B}_n$ that satisfies $0 < \pi_1 > \pi_2 < \pi_3 > \pi_4 < \ldots$. Arnol'd [7] showed that S_n enumerates the snakes of type B_n. Several other combinatorial objects are also enumerated by the Springer numbers: Weyl chambers in the principal Springer cone of the Coxeter group B_n [67], topological types of odd functions with $2n$ critical values [7], and certain classes of complete binary trees and plane rooted forests [49]. The sequence of Springer numbers starts as

$$(S_n)_{n\geq 0} \;=\; 1, 1, 3, 11, 57, 361, 2763, 24611, 250737, 2873041, 36581523, \ldots \tag{1.8}$$

and can be found in [57] as sequence A001586. It follows from (1.7) that S_n has the asymptotic behavior

$$S_n \;=\; \frac{2\sqrt{2}}{\pi}\left(\frac{4}{\pi}\right)^n n! \;+\; O\Big(\Big(\frac{4}{3\pi}\Big)^n n!\Big) \tag{1.9}$$

as $n \to \infty$.

[2] **Warning:** The Euler numbers found in classical books of analysis are somewhat different from these: $E_{2n-1}^{\text{classical}} = 0$ and $E_{2n}^{\text{classical}} = (-1)^n E_{2n}$. Moreover, our tangent numbers E_{2n+1} are classically written as a complicated expression in terms of Bernoulli numbers:

$$E_{2n-1} \;=\; \frac{(-1)^{n-1} 2^{2n} (2^{2n}-1) B_{2n}}{2n} \qquad \text{for } n \geq 1\;.$$

The definition given here is the one nowadays universally used by combinatorialists, since it makes E_n a positive integer that has a uniform combinatorial interpretation for n even and n odd.

In this paper I propose to study the sequences of Euler and Springer numbers from the point of view of the classical moment problem [2, 63–65, 71]. Let us recall that a sequence $\boldsymbol{a} = (a_n)_{n\ge 0}$ of real numbers is called a *Hamburger* (resp. *Stieltjes*) *moment sequence* if there exists a positive measure μ on $\mathbb{R}$ (resp. on $[0,\infty)$) such that $a_n = \int x^n \, d\mu(x)$ for all $n \ge 0$. A Hamburger (resp. Stieltjes) moment sequence is called *H-determinate* (resp. *S-determinate*) if there is a unique such measure μ; otherwise it is called *H-indeterminate* (resp. *S-indeterminate*). Please note that a Stieltjes moment sequence can be S-determinate but H-indeterminate [2, p. 240] [65, p. 96]. The Hamburger and Stieltjes moment properties are also connected with the representation of the ordinary generating function $A(t) = \sum_{n=0}^{\infty} a_n t^n$ as a Jacobi-type or Stieltjes-type continued fraction; this connection will be reviewed in Section 2 below.

Many combinatorial sequences turn out to be Hamburger or Stieltjes moment sequences, and it is obviously of interest to find explicit expressions for the representing measure(s) μ and/or the continued-fraction expansions of the ordinary generating function. In this paper we will address both aspects for the Euler and Springer numbers and some sequences related to them.

2 Preliminaries on the moment problem

In this section we review some basic facts about the moment problem [2, 63–65, 71, 76] that will be used repeatedly in the sequel.

In the Introduction we defined Hamburger and Stieltjes moment sequences. We begin by noting some elementary consequences of these definitions:

1) If $\boldsymbol{a} = (a_n)_{n\ge 0}$ is a Stieltjes moment sequence, then every arithmetic-progression subsequence $(a_{n_0+jN})_{N\ge 0}$ with $n_0 \ge 0$ and $j \ge 1$ is again a Stieltjes moment sequence.

2) If $\boldsymbol{a} = (a_n)_{n\ge 0}$ is a Hamburger moment sequence, then every arithmetic-progression subsequence $(a_{n_0+jN})_{N\ge 0}$ with $n_0 \ge 0$ *even* and $j \ge 1$ is again a Hamburger moment sequence; and if also j is even, then it is a Stieltjes moment sequence.

3) For a sequence $\boldsymbol{a} = (a_n)_{n\ge 0}$, the following are equivalent:

(a) $\boldsymbol{a}$ is a Stieltjes moment sequence.

(b) The "aerated" sequence $\widehat{\boldsymbol{a}} = (a_0, 0, a_1, 0, a_2, 0, \ldots)$ is a Hamburger moment sequence.

(c) There exist numbers $a_0', a_1', a_2', \ldots$ such that the "modified aerated" sequence $\widehat{\boldsymbol{a}}' = (a_0, a_0', a_1, a_1', a_2, a_2', \ldots)$ is a Hamburger moment sequence.

Indeed, (b) $\Longrightarrow$ (c) is trivial, and (c) $\Longrightarrow$ (a) follows from property #2: concretely, if $\widehat{\boldsymbol{a}}'$ is represented by a measure $\widehat{\mu}'$ on $\mathbb{R}$, then $\boldsymbol{a}$ is represented by the measure μ on $[0,\infty)$ that is the image of $\widehat{\mu}'$ under the map $x \mapsto x^2$ [namely, $\mu(A) = \widehat{\mu}'(\{x\colon\, x^2 \in A\})$]. And for (a) $\Longrightarrow$ (b), if $\boldsymbol{a}$ is represented by a measure μ supported on $[0,\infty)$, then $\widehat{\boldsymbol{a}}$ is represented by the even measure $\widehat{\mu} = (\tau^+ + \tau^-)/2$ on $\mathbb{R}$, where $\tau^\pm$ is the image of μ under the map $x \mapsto \pm\sqrt{x}$.

4) For a sequence $\boldsymbol{a} = (a_n)_{n\ge 0}$, the following are equivalent:

(a) $\boldsymbol{a}$ is a Stieltjes moment sequence.

(b) Both $\boldsymbol{a}$ and the once-shifted sequence $\widetilde{\boldsymbol{a}} = (a_{n+1})_{n\geq 0}$ are Stieltjes moment sequences.

(c) Both $\boldsymbol{a}$ and $\widetilde{\boldsymbol{a}}$ are Hamburger moment sequences.

Here (a) $\iff$ (b) $\implies$ (c) is easy (using property #1); unfortunately I do not know any completely elementary proof of (c) $\implies$ (a), but it is anyway an immediate consequence of Theorems 2.1 and 2.2 below (see also [12, p. 187]).

5) If $\boldsymbol{a} = (a_n)_{n\geq 0}$ and $\boldsymbol{b} = (b_n)_{n\geq 0}$ are Hamburger (resp. Stieltjes) moment sequences, then any linear combination $\alpha\boldsymbol{a} + \beta\boldsymbol{b}$ with $\alpha, \beta \geq 0$ is also a Hamburger (resp. Stieltjes) moment sequence: if $\boldsymbol{a}$ (resp. $\boldsymbol{b}$) has representing measure μ (resp. ν), then $\alpha\boldsymbol{a} + \beta\boldsymbol{b}$ has representing measure $\alpha\mu + \beta\nu$.

6) If $\boldsymbol{a} = (a_n)_{n\geq 0}$ and $\boldsymbol{b} = (b_n)_{n\geq 0}$ are Hamburger (resp. Stieltjes) moment sequences, then their entrywise product $\boldsymbol{ab} \stackrel{\text{def}}{=} (a_n b_n)_{n\geq 0}$ is also a Hamburger (resp. Stieltjes) moment sequence: if $\boldsymbol{a}$ (resp. $\boldsymbol{b}$) has representing measure μ (resp. ν), then $\boldsymbol{ab}$ has representing measure given by the product convolution $\mu \diamond \nu$:

$$(\mu \diamond \nu)(A) \;=\; (\mu \times \nu)\big(\{(x,y) \in \mathbb{R}^2\colon\; xy \in A\}\big) \qquad \text{for } A \subseteq \mathbb{R} \tag{2.1}$$

[that is, $\mu \diamond \nu$ is the image of $\mu \times \nu$ under the map $(x, y) \mapsto xy$]. We will often use this fact in the contrapositive: if $\boldsymbol{b}$ is a Hamburger (resp. Stieltjes) moment sequence and $\boldsymbol{ab}$ is *not* a Hamburger (resp. Stieltjes) moment sequence, then $\boldsymbol{a}$ is *not* a Hamburger (resp. Stieltjes) moment sequence. Indeed, the non-Hamburger (resp. non-Stieltjes) property of $\boldsymbol{ab}$ can be viewed as a strengthened form of the non-Hamburger (resp. non-Stieltjes) property of $\boldsymbol{a}$.

We now recall the well-known [2, 34, 63–65, 71, 76] necessary and sufficient conditions for a sequence $\boldsymbol{a} = (a_n)_{n\geq 0}$ to be a Hamburger or Stieltjes moment sequence. To any infinite sequence $\boldsymbol{a} = (a_n)_{n\geq 0}$ of real numbers, we associate for each $m \geq 0$ the m-shifted infinite Hankel matrix

$$H_\infty^{(m)}(\boldsymbol{a}) \;=\; (a_{i+j+m})_{i,j\geq 0} \;=\; \begin{bmatrix} a_m & a_{m+1} & a_{m+2} & \cdots \\ a_{m+1} & a_{m+2} & a_{m+3} & \cdots \\ a_{m+2} & a_{m+3} & a_{m+4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.2}$$

and the m-shifted $n \times n$ Hankel matrix

$$H_n^{(m)}(\boldsymbol{a}) \;=\; (a_{i+j+m})_{0\leq i,j\leq n-1} \;=\; \begin{bmatrix} a_m & a_{m+1} & \cdots & a_{m+n-1} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+n-1} & a_{m+n} & \cdots & a_{m+2n-2} \end{bmatrix} . \tag{2.3}$$

We also define the Hankel determinants

$$\Delta_n^{(m)}(\boldsymbol{a}) \;=\; \det H_n^{(m)}(\boldsymbol{a}) \,. \tag{2.4}$$

Theorem 2.1 (Necessary and sufficient conditions for Hamburger moment sequence)**.** *For a sequence* $\boldsymbol{a} = (a_n)_{n\ge 0}$ *of real numbers, the following are equivalent:*

(a) $\boldsymbol{a}$ *is a Hamburger moment sequence.*

(b) $H_\infty^{(0)}(\boldsymbol{a})$ *is positive-semidefinite. [That is, all the principal minors of* $H_\infty^{(0)}(\boldsymbol{a})$ *are nonnegative.]*

(c) There exist numbers $\alpha_0 \ge 0$, $\beta_1, \beta_2, \ldots \ge 0$ *and* $\gamma_0, \gamma_1, \ldots \in \mathbb{R}$ *such that*

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{\alpha_0}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \cdots}}} \tag{2.5}$$

in the sense of formal power series. [That is, the ordinary generating function $f(t) = \sum\limits_{n=0}^{\infty} a_n t^n$ *can be represented as a Jacobi-type continued fraction with nonnegative coefficients* $\boldsymbol{\beta}$ *and* α_0*.]*

There is also a refinement that is often useful: $\boldsymbol{a}$ is a Hamburger moment sequence with a representing measure μ having infinite support $\iff$ $H_\infty^{(0)}(\boldsymbol{a})$ is positive-definite (i.e. all the principal minors are strictly positive) $\iff$ all the leading principal minors $\Delta_n^{(0)}$ are strictly positive $\iff$ all the β_i are strictly positive.

Theorem 2.2 (Necessary and sufficient conditions for Stieltjes moment sequence)**.** *For a sequence* $\boldsymbol{a} = (a_n)_{n\ge 0}$ *of real numbers, the following are equivalent:*

(a) $\boldsymbol{a}$ *is a Stieltjes moment sequence.*

(b) Both $H_\infty^{(0)}(\boldsymbol{a})$ *and* $H_\infty^{(1)}(\boldsymbol{a})$ *are positive-semidefinite. [That is, all the principal minors of* $H_\infty^{(0)}(\boldsymbol{a})$ *and* $H_\infty^{(1)}(\boldsymbol{a})$ *are nonnegative.]*

(c) $H_\infty^{(0)}(\boldsymbol{a})$ *is totally positive. [That is,* all *the minors of* $H_\infty^{(0)}(\boldsymbol{a})$ *are nonnegative.]*

(d) There exist numbers $\alpha_0, \alpha_1, \ldots \ge 0$ *such that*

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}} \tag{2.6}$$

in the sense of formal power series. [That is, the ordinary generating function $f(t) = \sum\limits_{n=0}^{\infty} a_n t^n$ *can be represented as a Stieltjes-type continued fraction with nonnegative coefficients.]*

(e) *There exist numbers* $\alpha_0 \ge 0$, $\beta_1, \beta_2, \ldots \ge 0$ *and* $\gamma_0, \gamma_1, \ldots \ge 0$ *such that the infinite tridiagonal matrix*

$$A(\boldsymbol{\beta}, \boldsymbol{\gamma}) \;=\; \begin{bmatrix} \gamma_0 & 1 & & & \\ \beta_1 & \gamma_1 & 1 & & \\ & \beta_2 & \gamma_2 & 1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{2.7}$$

is totally positive and

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{\alpha_0}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \cdots}}} \tag{2.8}$$

in the sense of formal power series. [That is, the ordinary generating function $f(t) = \sum\limits_{n=0}^{\infty} a_n t^n$ *can be represented as a Jacobi-type continued fraction with a totally positive production matrix.]*

Once again, there is a refinement: $\boldsymbol{a}$ is a Stieltjes moment sequence with a representing measure μ having infinite support $\iff$ $H_\infty^{(0)}(\boldsymbol{a})$ and $H_\infty^{(1)}(\boldsymbol{a})$ are positive-definite (i.e. all the principal minors are strictly positive) $\iff$ all the leading principal minors $\Delta_n^{(0)}$ and $\Delta_n^{(1)}$ are strictly positive $\iff$ $H_\infty^{(0)}(\boldsymbol{a})$ is strictly totally positive (i.e. all the minors are strictly positive) $\iff$ all the α_i are strictly positive $\iff$ all the β_i are strictly positive.

From the 2×2 minors of $H_\infty^{(0)}(\boldsymbol{a})$ and $H_\infty^{(1)}(\boldsymbol{a})$, we see that a Stieltjes moment sequence is log-convex: $a_n a_{n+2} - a_{n+1}^2 \ge 0$. (This is also easy to prove directly.) But it goes without saying that the Stieltjes moment property is *much* stronger than log-convexity.

For future reference, let us also recall the formula [76, p. 21] [75, p. V-31] for the contraction of an S-fraction to a J-fraction: (2.6) and (2.8) are equal if

$$\gamma_0 \;=\; \alpha_1 \tag{2.9a}$$

$$\gamma_n \;=\; \alpha_{2n} + \alpha_{2n+1} \qquad \text{for } n \ge 1 \tag{2.9b}$$

$$\beta_n \;=\; \alpha_{2n-1}\alpha_{2n} \tag{2.9c}$$

Concerning H-determinacy and S-determinacy, we limit ourselves to quoting the following sufficient condition [64, Theorems 1.10 and 1.11] due to Carleman in 1922:

Theorem 2.3 (Sufficient condition for determinacy of moment problem)**.**

(a) *A Hamburger moment sequence* $\boldsymbol{a} = (a_n)_{n\ge 0}$ *satisfying* $\sum\limits_{n=1}^{\infty} a_{2n}^{-1/2n} = \infty$ *is H-determinate.*

(b) *A Stieltjes moment sequence* $\boldsymbol{a} = (a_n)_{n\ge 0}$ *satisfying* $\sum\limits_{n=1}^{\infty} a_n^{-1/2n} = \infty$ *is S-determinate.*

In Corollary 2.9 below, we will prove, by elementary methods, a slightly weakened version of Theorem 2.3. It should be stressed that the conditions of Theorem 2.3 are sufficient for determinacy, but in no way necessary [45]; indeed, there are determinate Hamburger and Stieltjes moment sequences with arbitrarily rapid growth [65, pp. 89, 135]. In fact, given any H-indeterminate Hamburger (resp. Stieltjes) moment sequence $\boldsymbol{a} = (a_n)_{n \geq 0}$, there exists an H-determinate Hamburger (resp. Stieltjes) moment sequence $\boldsymbol{a}' = (a'_n)_{n \geq 0}$ that differs from $\boldsymbol{a}$ only in the zeroth entry: $0 < a'_0 < a_0$ while $a'_n = a_n$ for all $n \geq 1$.[3]

We will need one other fact about determinacy [13, p. 178]:

Proposition 2.4 (S-determinacy with H-indeterminacy)**.** *Let* $\boldsymbol{a}$ *be a Stieltjes moment sequence that is S-determinate but H-indeterminate. Then the unique measure on* $[0, \infty)$ *representing* $\boldsymbol{a}$ *is the Nevanlinna-extremal measure corresponding to the parameter value* $t = 0$*, hence is a discrete measure concentrated on the zeros of the D-function from the Nevanlinna parametrization (and in particular has an atom at 0).*

We refrain from explaining what is meant by "Nevanlinna-extremal measure" and "Nevanlinna parametrization" [2, 18, 65], but simply stress that in this situation the representing measure must be discrete.[4]

We will also make use of a generalization of the moment problem from positive measures to signed measures. So let μ be a finite signed measure on $\mathbb{R}$; it has a unique Jordan decomposition $\mu = \mu_+ - \mu_-$ where μ_+, μ_- are nonnegative and mutually singular [43]. We write $|\mu| = \mu_+ + \mu_-$. We will always assume that $|\mu|$ has finite moments of all orders, i.e. $\int\limits_{-\infty}^{\infty} |x|^n \, d|\mu|(x) < \infty$ for all $n \geq 0$. The moments $a_n = \int\limits_{-\infty}^{\infty} x^n \, d\mu(x)$ are then well-defined; we say that μ represents $\boldsymbol{a} = (a_n)_{n \geq 0}$.

In sharp contrast to Theorems 2.1 and 2.2, the moment problem for signed measures has a trivial existence condition and an extraordinary nonuniqueness:

Theorem 2.5 (Pólya [59, 60])**.** *Let* $\boldsymbol{a} = (a_n)_{n \geq 0}$ *be* any *sequence of real numbers, and let* S *be* any *closed unbounded subset of* $\mathbb{R}$*. Then there exists a signed measure*

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[3] Proof (for experts): If $\boldsymbol{a}$ is an indeterminate Hamburger moment sequence, then the Nevanlinna-extremal measure corresponding to the parameter value $t = 0$ (call it μ_0) is a discrete measure concentrated on the zeros of the Nevanlinna D-function (which are all real and simple, and one of which is 0). If, in addition, $\boldsymbol{a}$ is a Stieltjes moment sequence, then the orthonormal polynomials $P_n(x)$ have all their zeros in $(0, \infty)$, so $P_n(0) P_n(x) > 0$ for all $x \leq 0$; it follows that $D(x) = x \sum_{n=0}^{\infty} P_n(0) P_n(x)$ has all its zeros in $[0, \infty)$, so that μ_0 is supported on $[0, \infty)$. Now consider the measure $\mu' = \mu_0 - \mu_0(\{0\}) \delta_0$: it is H-determinate [2, p. 115] [11, p. 111] and its moment sequence $\boldsymbol{a}'$ differs from $\boldsymbol{a}$ only in the zeroth entry. I thank Christian Berg for drawing my attention to this result and its proof.

[4] Proof of Proposition 2.4 (for experts): Let $\boldsymbol{a}$ be a Stieltjes moment sequence that is H-indeterminate. Then it was shown in footnote 3 that the N-extremal measure μ_0 is a discrete measure on $[0, \infty)$ representing $\boldsymbol{a}$. If $\boldsymbol{a}$ is also S-determinate, then μ_0 is the unique measure on $[0, \infty)$ representing $\boldsymbol{a}$. I again thank Christian Berg for drawing my attention to this result and its proof.

μ with support in S that represents $\boldsymbol{a}$ [that is, $\int_{-\infty}^{\infty} |x|^n \, d|\mu|(x) < \infty$ for all $n \ge 0$ and $a_n = \int_{-\infty}^{\infty} x^n \, d\mu(x)$ for all $n \ge 0$].

So for any sequence $\boldsymbol{a}$ (even the zero sequence!) there are continuum many distinct signed measures μ, with disjoint supports, that represent $\boldsymbol{a}$. (For instance, we can take $S = \mathbb{Z} + \lambda$ for any $\lambda \in [0, 1)$.) See also Bloom [15] for a slight refinement; and see Boas [16] for a different proof of a weaker result.

The requirement here that S be unbounded is essential; among signed measures with bounded support, uniqueness holds. More generally, uniqueness holds among signed measures that have exponential decay. To show this, we begin with some elementary lemmas:

Lemma 2.6 (Bounded support)**.** *Let $\boldsymbol{a} = (a_n)_{n \ge 0}$ be a sequence of real numbers, let μ be a signed measure on $\mathbb{R}$ that represents $\boldsymbol{a}$, and let $R \in [0, \infty)$.*

- *(a) If μ is supported in $[-R, R]$, then $|a_n| \le \|\mu\| \, R^n$, where $\|\mu\| = |\mu|(\mathbb{R})$.*
- *(b) Conversely, if μ is a* positive *measure and $|a_n| \le CR^n$ for some $C < \infty$, then μ is supported in $[-R, R]$.*

PROOF. (a) is trivial.

(b) Suppose that μ is a positive measure such that $\mu\big((-\infty, -R-\epsilon] \cup [R+\epsilon, \infty)\big) = K > 0$ for some $\epsilon > 0$. Then $a_{2n} \ge K(R+\epsilon)^{2n}$ for all $n \ge 0$, which contradicts the hypothesis $|a_n| \le CR^n$. $\square$

Remark. This proof shows that (b) holds under the weaker hypothesis $\liminf_{n\to\infty} |a_{2n}|^{1/2n} \le R$. $\blacksquare$

Lemma 2.7 (Exponential decay)**.** *Let $\boldsymbol{a} = (a_n)_{n \ge 0}$ be a sequence of real numbers, let μ be a signed measure on $\mathbb{R}$ that represents $\boldsymbol{a}$, and let $\epsilon > 0$.*

- *(a) If $\displaystyle\int_{-\infty}^{\infty} e^{\epsilon|x|} \, d|\mu|(x) = C < \infty$, then $|a_n| \le C\epsilon^{-n} n!$.*
- *(b) Conversely, if μ is a* positive *measure and $|a_n| \le C\epsilon^{-n} n!$ for some $C < \infty$, then $\displaystyle\int_{-\infty}^{\infty} e^{\delta|x|} \, d\mu(x) < \infty$ for all $\delta < \epsilon$.*

PROOF. (a) Since $|x^n| \le \epsilon^{-n} n! e^{\epsilon|x|}$, it follows that $|a_n| \le C\epsilon^{-n} n!$.

(b) Applying the monotone convergence theorem to $\cosh \delta x = \sum_{n=0}^{\infty} (\delta x)^{2n}/(2n)!$, we conclude that

$$\int_{-\infty}^{\infty} (\cosh \delta x) \, d\mu(x) \;=\; \sum_{n=0}^{\infty} \frac{\delta^{2n} \, a_{2n}}{(2n)!} \;\le\; \frac{C}{1 - \delta^2/\epsilon^2} \;<\; \infty \,. \tag{2.10}$$

$\square$

Proposition 2.8 (Uniqueness in the presence of exponential decay)**.** *Let* $\boldsymbol{a} = (a_n)_{n\ge 0}$ *be a sequence of real numbers, and let* μ *and* ν *be signed measures on* $\mathbb{R}$ *that represent* $\boldsymbol{a}$*. Suppose that* μ *has exponential decay in the sense that* $\int\limits_{-\infty}^{\infty} e^{\epsilon|x|}\, d|\mu|(x) < \infty$ *for some* $\epsilon > 0$*; and suppose that* ν *is either a positive measure or else also has exponential decay. Then* $\mu = \nu$*.*

Proof. By Lemma 2.7(a), we conclude that $|a_n| \le C\epsilon^{-n} n!$ for some $C < \infty$. Then Lemma 2.7(b) implies that if ν is a positive measure, it has exponential decay. So we can assume that ν has exponential decay. It follows that $F(t) = \int\limits_{-\infty}^{\infty} e^{itx}\, d\mu(x)$ and $G(t) = \int\limits_{-\infty}^{\infty} e^{itx}\, d\nu(x)$ define analytic functions in the strip $|\mathrm{Im}\, t| < \epsilon$. Moreover, by the dominated convergence theorem they coincide in the disc $|t| < \epsilon$ with the absolutely convergent series $\sum\limits_{n=0}^{\infty} a_n (it)^n/n!$. It follows that $F = G$; and by the uniqueness theorem for the Fourier transform of tempered distributions [46, Theorem 7.1.10] (or by other arguments [65, proof of Proposition 1.5]) we conclude that $\mu = \nu$. $\square$

Corollary 2.9. *Let* $\boldsymbol{a} = (a_n)_{n\ge 0}$ *be a sequence of real numbers satisfying* $|a_n| \le AB^n n!$ *for some* $A, B < \infty$*. Then there is at most one positive measure representing* $\boldsymbol{a}$*.*

Proof. Apply Lemma 2.7(b) and then Proposition 2.8. $\square$

Corollary 2.10. *Let* $\boldsymbol{a} = (a_n)_{n\ge 0}$ *be a sequence of real numbers, and let* μ *be a* nonpositive *signed measure on* $\mathbb{R}$ *that represents* $\boldsymbol{a}$ *and has exponential decay in the sense that* $\int\limits_{-\infty}^{\infty} e^{\epsilon|x|}\, d\mu(x) < \infty$ *for some* $\epsilon > 0$*. Then* $\boldsymbol{a}$ *is* not *a Hamburger moment sequence.*

3 Euler numbers, part 1

We begin by studying the sequence of Euler numbers divided by $n!$. Our starting point is the partial-fraction expansions of secant and tangent [5, p. 11]:

$$\sec t \;=\; \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{(-1)^k}{(k+\frac{1}{2})\pi - t} \tag{3.1}$$

$$\tan t \;=\; \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{1}{(k+\frac{1}{2})\pi - t} \tag{3.2}$$

Inserting these formulae into the exponential generating function (1.3) of the Euler numbers and extracting coefficients of powers of t on both sides, we obtain

$$\frac{E_{2n}}{(2n)!} \;=\; \sum_{k=-\infty}^{\infty} (-1)^k \left[(k+\tfrac{1}{2})\pi\right]^{-(2n+1)} \tag{3.3}$$

(with the interpretation $\lim\limits_{N\to\infty} \sum\limits_{k=-N}^{N}$ when $n=0$) and

$$\frac{E_{2n+1}}{(2n+1)!} \;=\; \sum_{k=-\infty}^{\infty} \left[(k+\tfrac{1}{2})\pi\right]^{-(2n+2)} . \tag{3.4}$$

We can rewrite (3.4) as

$$\frac{E_{2n+1}}{(2n+1)!} \;=\; \sum_{k=0}^{\infty} \frac{2}{(k+\frac{1}{2})^2\pi^2} \left(\frac{1}{(k+\frac{1}{2})^2\pi^2}\right)^{n} , \tag{3.5}$$

which represents $(E_{2n+1}/(2n+1)!)_{n\ge 0}$ as the moments of a positive measure supported on a countably infinite subset of $[0, 4/\pi^2]$. It follows that $(E_{2n+1}/(2n+1)!)_{n\ge 0}$ is a Stieltjes moment sequence, which is both S-determinate and H-determinate. Theorem 2.2 then implies that the ordinary generating function of $(E_{2n+1}/(2n+1)!)_{n\ge 0}$ can be written as a Stieltjes-type continued fraction (2.6) with nonnegative coefficients α_n; in fact we have the beautiful explicit formula [76, p. 349]

$$\sum_{n=0}^{\infty} \frac{E_{2n+1}}{(2n+1)!}\, t^n \;=\; \frac{\tan\sqrt{t}}{\sqrt{t}} \;=\; \cfrac{1}{1 - \cfrac{\frac{1}{3}t}{1 - \cfrac{\frac{1}{15}t}{1 - \cfrac{\frac{1}{35}t}{1-\cdots}}}} \tag{3.6}$$

with coefficients $\alpha_n = 1/(4n^2-1) > 0$. This continued-fraction expansion of the tangent function was found by Lambert [54] in 1761, and used by him to prove the irrationality of π [53, 77]. But in fact, as noted by Brezinski [17, p. 110], a formula equivalent to (3.6) appears already in Euler's first paper on continued fractions [30]: see top p. 321 in the English translation.[5] The expansion (3.6) is a ${}_0F_1$ limiting case of Gauss' continued fraction for the ratio of two contiguous hypergeometric functions ${}_2F_1$ [76, Chapter XVIII].

We will come back to (3.3) in a moment.

Combining (3.3) and (3.4) and taking advantage of the evenness/oddness of the summands, we get

$$\frac{E_n}{n!} \;=\; \sum_{k=-\infty}^{\infty} \left[1+(-1)^k\right] \left[(k+\tfrac{1}{2})\pi\right]^{-(n+1)} \tag{3.7a}$$

$$=\; 2\sum_{k=-\infty}^{\infty} \left(\frac{2}{(4k+1)\pi}\right)^{n+1} \tag{3.7b}$$

[5] The paper [30], which is E71 in Eneström's [28] catalogue, was presented to the St. Petersburg Academy in 1737 and published in 1744.

(once again with the interpretation $\lim\limits_{N\to\infty} \sum\limits_{k=-N}^{N}$ when $n = 0$); see [27] for further discussion of this sum. For $n \ge 1$ this sum is absolutely convergent, so we can write

$$\frac{E_{n+1}}{(n+1)!} \;=\; \sum_{k=-\infty}^{\infty} \frac{8}{(4k+1)^2\pi^2} \left(\frac{2}{(4k+1)\pi} \right)^{n} , \tag{3.8}$$

which represents $(E_{n+1}/(n+1)!)_{n\ge 0}$ as the moments of a positive measure supported on a countably infinite subset of $[-2/3\pi, 2/\pi]$. It follows that $(E_{n+1}/(n+1)!)_{n\ge 0}$ is a Hamburger moment sequence, which is H-determinate. In fact, the ordinary generating function of $(E_{n+1}/(n+1)!)_{n\ge 0}$ can be written explicitly as a Jacobi-type continued fraction [37]

$$\sum_{n=0}^{\infty} \frac{E_{n+1}}{(n+1)!}\, t^n \;=\; \cfrac{1}{1 - \frac{1}{2}t - \cfrac{\frac{1}{12}t^2}{1 - \cfrac{\frac{1}{60}t^2}{1 - \cfrac{\frac{1}{140}t^2}{1 - \cdots}}}} \tag{3.9}$$

with coefficients $\gamma_0 = 1/2$, $\gamma_n = 0$ for $n \ge 1$, and $\beta_n = 1/(16n^2 - 4)$. This continued fraction can be obtained from Lambert's continued fraction (3.6) with t replaced by $t^2/4$, by using the identity

$$\sum_{n=0}^{\infty} \frac{E_{n+1}}{(n+1)!}\, t^n \;=\; \frac{\sec t + \tan t - 1}{t} \;=\; \frac{1}{\frac{t}{2}\cot\frac{t}{2} - \frac{t}{2}} . \tag{3.10}$$

By using the contraction formula (2.9), we can also rewrite (3.9) as a Stieltjes-type continued fraction [39]

$$\sum_{n=0}^{\infty} \frac{E_{n+1}}{(n+1)!}\, t^n \;=\; \cfrac{1}{1 - \cfrac{\frac{1}{2}t}{1 - \cfrac{\frac{1}{6}t}{1 + \cfrac{\frac{1}{6}t}{1 + \cfrac{\frac{1}{10}t}{1 - \cdots}}}}} \tag{3.11}$$

with coefficients $\alpha_{2k-1} = (-1)^{k-1}/(4k-2)$, $\alpha_{2k} = (-1)^{k-1}/(4k+2)$. Here the coefficients α_i are *not* all nonnegative; it follows by Theorem 2.2 and the uniqueness of Stieltjes-continued-fraction representations that $(E_{n+1}/(n+1)!)_{n\ge 0}$ is *not* a Stieltjes moment sequence — a fact that we already knew from (3.8) and the H-determinacy.

Let us now consider the even subsequence $(E_{2n}/(2n)!)_{n\ge 0}$. Is it a Hamburger moment sequence? The answer is no, in a very strong sense:

Proposition 3.1. *Define* $\widetilde{E}_n = E_n/n!$. *Then* $(\widetilde{E}_{2n})_{n\ge 0}$ *is not a Hamburger moment sequence. In fact, no arithmetic-progression subsequence* $(\widetilde{E}_{n_0+jN})_{N\ge 0}$ *with* n_0 even *and* $j \ge 1$ *is a Hamburger moment sequence.*

We give two proofs:

First Proof. For any *even* $n_0 \geq 2$ and any $j \geq 1$, the equation (3.8) represents $(\widetilde{E}_{n_0+jN})_{N\geq 0}$ as the moments of a *nonpositive* signed measure supported on $[-2/3\pi, 2/\pi]$. Corollary 2.10 then implies that $(\widetilde{E}_{n_0+jN})_{N\geq 0}$ is not a Hamburger moment sequence. The assertion for $n_0 = 0$ then follows from the assertion for $n_0 = 2j$. □

The second proof is based on the following fact, which is of some interest in its own right:

Proposition 3.2. *Define $\widetilde{E}_n = E_n/n!$. Then $(\widetilde{E}_{2n})_{n\geq 0}$ is a Pólya frequency sequence, i.e. every minor of the infinite Toeplitz matrix*

$$(\widetilde{E}_{2j-2i})_{i,j\geq 0} \;=\; \begin{bmatrix} \widetilde{E}_0 & \widetilde{E}_2 & \widetilde{E}_4 & \widetilde{E}_6 & \cdots \\ 0 & \widetilde{E}_0 & \widetilde{E}_2 & \widetilde{E}_4 & \cdots \\ 0 & 0 & \widetilde{E}_0 & \widetilde{E}_2 & \cdots \\ 0 & 0 & 0 & \widetilde{E}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3.12}$$

is nonnegative. Moreover, a minor using rows $i_1 < i_2 < \ldots < i_r$ and columns $j_1 < j_2 < \ldots < j_r$ is strictly positive if $i_k \leq j_k$ for $1 \leq k \leq r$. In particular, the sequence $(\widetilde{E}_{2n})_{n\geq 0}$ is strictly log-concave.

Proof. It follows from the well-known infinite product representation for $\cos t$ that

$$\sum_{n=0}^{\infty} \widetilde{E}_{2n}\, t^n \;=\; \sec\sqrt{t} \;=\; \prod_{k=0}^{\infty} \left(1 \,-\, \frac{t}{(k+\frac{1}{2})^2\pi^2}\right)^{-1} . \tag{3.13}$$

This implies [51, p. 395] that $(\widetilde{E}_{2n})_{n\geq 0}$ is a Pólya frequency sequence; and it also implies [51, p. 427–430] the statement about strictly positive minors. The strict log-concavity is simply the strict positivity of the 2×2 minors above the diagonal. □

Second Proof of Proposition 3.1. No arithmetic-progression subsequence $(\widetilde{E}_{n_0+jN})_{N\geq 0}$ with n_0 even and $j \geq 1$ can be a Hamburger moment sequence, since its even subsequence $(\widetilde{E}_{n_0+2jN})_{N\geq 0}$ is strictly log-concave and hence cannot be log-convex. □

The even and odd subsequences thus have radically different behavior: the even subsequence $(\widetilde{E}_{2n})_{n\geq 0}$ is strictly log-concave, while the odd subsequence $(\widetilde{E}_{2n+1})_{n\geq 0}$ is strictly log-convex (since it is a Stieltjes moment sequence with a representing measure of infinite support). These two facts are special cases of the following more general inequality that appears to be true:

Conjecture 3.3. *Define $\widetilde{E}_n = E_n/n!$. Then for all $n \geq 0$ and $j, k \geq 1$, we have*

$$(-1)^{n-1}[\widetilde{E}_n\widetilde{E}_{n+j+k} - \widetilde{E}_{n+j}\widetilde{E}_{n+k}] \;>\; 0 . \tag{3.14}$$

I do not know how to prove (3.14), but I have verified it for $n, j, k \le 900$.

Though $(E_{2n}/(2n)!)_{n\ge 0}$ is not a Hamburger moment sequence, one could try multiplying it by a Hamburger (or Stieltjes) moment sequence $(b_n)_{n\ge 0}$; the result $(b_n E_{2n}/(2n)!)_{n\ge 0}$ *might* be a Hamburger (or even a Stieltjes) moment sequence. For instance, the central binomial coefficients $\binom{2n}{n} = (2n)!/(n!)^2$ are a Stieltjes moment sequence, with representation

$$\binom{2n}{n} \;=\; \frac{1}{\pi} \int\limits_0^4 x^n \, x^{-1/2} \, (4-x)^{-1/2} \, dx \;. \qquad (3.15)$$

Might $(E_{2n}/(n!)^2)_{n\ge 0}$ be a Hamburger moment sequence? The answer is no, because the 7×7 Hankel determinant $\det(a_{i+j})_{0\le i,j\le 6}$ for $a_n = E_{2n}/(n!)^2$ is negative. Unfortunately I do not know any simpler proof.

But if we multiply by another factor of $n!$, then the result $(E_{2n}/n!)_{n\ge 0}$ *is* a Hamburger — and indeed a Stieltjes — moment sequence. To see this, start by rewriting (3.3) as

$$\frac{E_{2n}}{(2n)!} \;=\; 2 \sum_{k=0}^{\infty} (-1)^k \left[(k+\tfrac{1}{2})\pi \right]^{-(2n+1)} \;. \qquad (3.16)$$

Now multiply this by the Stieltjes integral representation

$$\frac{(2n)!}{n!} \;=\; 2^n \, (2n-1)!! \;=\; \frac{2^n}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} x^{2n} \, e^{-\frac{1}{2}x^2} \, dx \qquad (3.17)$$

to get

$$\frac{E_{2n}}{n!} \;=\; \frac{2^{n+1}}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dx \, e^{-\frac{1}{2}x^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\frac{1}{2})\pi} \left(\frac{x}{(k+\frac{1}{2})\pi} \right)^{2n} \;. \qquad (3.18)$$

Change variable to $y = x/[(k+\frac{1}{2})\pi]$ and interchange integration and summation; this leads to

$$\frac{E_{2n}}{n!} \;=\; \frac{2^{n+1}}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dy \, y^{2n} \sum_{k=0}^{\infty} (-1)^k \, \exp\!\left[-\tfrac{1}{2}(k+\tfrac{1}{2})^2 \pi^2 y^2 \right] \;. \qquad (3.19)$$

The density here is positive because each term with even k dominates the term $k+1$:

$$\exp\!\left[-\tfrac{1}{2}(k+\tfrac{1}{2})^2 \pi^2 y^2 \right] \;\ge\; \exp\!\left[-\tfrac{1}{2}(k+1+\tfrac{1}{2})^2 \pi^2 y^2 \right] \;. \qquad (3.20)$$

It follows that $(E_{2n}/n!)_{n\ge 0}$ is a Stieltjes moment sequence. Its ordinary generating function is therefore given by a Stieltjes-type continued fraction with coefficients $\alpha_i > 0$; but no explicit formula for these coefficients seems to be known.

4 Euler numbers, part 2

Thus far we have considered the sequence of Euler numbers E_n divided by factorials. Now we consider the sequence of Euler numbers E_n *tout court*, along with its even and odd subsequences.

We have already seen that $(E_{n+1}/(n+1)!)_{n\ge 0}$ is a Hamburger moment sequence. Since $((n+1)!)_{n\ge 0}$ is also a Hamburger (in fact a Stieltjes) moment sequence, it follows that their product $(E_{n+1})_{n\ge 0}$ is again a Hamburger moment sequence. Similarly, we have seen that $(E_{2n+1}/(2n+1)!)_{n\ge 0}$ is a Stieltjes moment sequence; and since $((2n+1)!)_{n\ge 0}$ is a Stieltjes moment sequence, it follows that their product $(E_{2n+1})_{n\ge 0}$ is a Stieltjes moment sequence. And finally, we have seen that $(E_{2n}/n!)_{n\ge 0}$ is a Stieltjes moment sequence; and since $(n!)_{n\ge 0}$ is a Stieltjes moment sequence, it follows that their product $(E_{2n})_{n\ge 0}$ is a Stieltjes moment sequence. In this section we will obtain explicit expressions for these sequences' representing measures and for the continued-fraction expansions of their ordinary generating functions.

Start by rewriting (3.7b) as

$$\frac{E_n}{n!} \;=\; 2\left[\sum_{k=0}^{\infty}\left(\frac{2}{(4k+1)\pi}\right)^{n+1} \;-\; (-1)^n \sum_{k=0}^{\infty}\left(\frac{2}{(4k+3)\pi}\right)^{n+1}\right] . \tag{4.1}$$

Now multiply by the Stieltjes integral representation $n! = \int\limits_0^\infty x^n\, e^{-x}\, dx$ to get

$$E_n \;=\; 2\left[\int\limits_0^\infty dx\, e^{-x} \sum_{k=0}^{\infty} \frac{2}{(4k+1)\pi}\left(\frac{2x}{(4k+1)\pi}\right)^n \right. \\ \left. -\; (-1)^n \int\limits_0^\infty dx\, e^{-x} \sum_{k=0}^{\infty} \frac{2}{(4k+3)\pi}\left(\frac{2x}{(4k+3)\pi}\right)^n\right] . \tag{4.2}$$

Change variable to $y = 2x/[(4k+1)\pi]$ in the first term, and $y = 2x/[(4k+3)\pi]$ in the second, and interchange integration and summation; this leads to

$$E_n \;=\; 2\left[\int\limits_0^\infty \frac{e^{-(\pi/2)y}}{1-e^{-2\pi y}}\, y^n\, dy \;-\; \int\limits_0^\infty \frac{e^{-(3\pi/2)y}}{1-e^{-2\pi y}}\, (-y)^n\, dy\right] \tag{4.3a}$$

$$=\; \int\limits_{-\infty}^{\infty} y^n\, \frac{e^{(\pi/2)y}}{\sinh \pi y}\, dy\, . \tag{4.3b}$$

The integral (4.3b) is absolutely convergent for $n \ge 1$; for $n = 0$ it is valid as a principal-value integral at $y = 0$. In particular we have

$$E_{n+1} \;=\; \int\limits_{-\infty}^{\infty} y^n\, \frac{y\, e^{(\pi/2)y}}{\sinh \pi y}\, dy\, , \tag{4.4}$$

which represents E_{n+1} as the nth moment of a positive measure on $\mathbb{R}$. Hence $(E_{n+1})_{n\ge 0}$ is a Hamburger moment sequence. It is H-determinate by virtue of (1.6) and Corollary 2.9.

Note also that multiplying (4.3b) by $t^n/n!$ and summing $\sum_{n=0}^{\infty}$, we recover the two-sided Laplace transform

$$\sec t + \tan t \;=\; \int\limits_{-\infty}^{\infty} \frac{e^{(t+\pi/2)y}}{\sinh \pi y}\, dy\, , \tag{4.5}$$

which is valid for $-3\pi/2 < \operatorname{Re} t < \pi/2$ as a principal-value integral [29, 6.2(8)], or equivalently (by symmetrizing)

$$\sec t + \tan t \;=\; \int\limits_{-\infty}^{\infty} \frac{\sinh(t+\pi/2)y}{\sinh \pi y}\, dy \;. \tag{4.6}$$

For n even we can combine the $y \ge 0$ and $y \le 0$ contributions in (4.3b) to obtain [42, 3.523.4] [56, 24.7.6]

$$E_{2n} \;=\; \int\limits_{0}^{\infty} y^{2n} \operatorname{sech}\Bigl(\frac{\pi}{2} y\Bigr)\, dy \;, \tag{4.7}$$

while for n odd a similar reformulation gives [42, 3.523.2]

$$E_{2n+1} \;=\; \int\limits_{0}^{\infty} y^{2n}\, y \operatorname{csch}\Bigl(\frac{\pi}{2} y\Bigr)\, dy \;. \tag{4.8}$$

We have thus explicitly expressed $(E_{2n})_{n\ge 0}$ and $(E_{2n+1})_{n\ge 0}$ as Stieltjes moment sequences. By (1.6) and Theorem 2.3(b) they are S-determinate. And since the measures in (4.7)/(4.8) are continuous, Proposition 2.4 implies that these sequences are also H-determinate.

The moment representations (4.7)/(4.8) can also be expressed nicely in terms of the *Lerch transcendent* (or *Lerch zeta function*) [42, §9.55] [56, §25.14], which we take to be defined by the integral representation

$$\Phi(z,s,\alpha) \;=\; \frac{1}{\Gamma(s)} \int\limits_{0}^{\infty} \frac{t^{s-1}\, e^{-\alpha t}}{1 \,-\, z e^{-t}}\, dt \tag{4.9}$$

for $\operatorname{Re} s > 0$, $\operatorname{Re} \alpha > 0$, and $z \in \mathbb{C} \setminus [1,\infty)$. For $|z| < 1$ we can expand the integrand in a Taylor series in z and then interchange integration with summation: this yields

$$\Phi(z,s,\alpha) \;=\; \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s} \;, \tag{4.10}$$

valid for $\operatorname{Re} s > 0$, $\operatorname{Re} \alpha > 0$, and $|z| < 1$. Moreover, an application of Lebesgue's dominated convergence theorem to the same series expansion shows that (4.10) holds also for $|z| = 1$ with the exception of $z = 1$.[6] And under the stronger hypothesis $\operatorname{Re} s > 1$ we can take $z \uparrow 1$ and conclude that (4.10) holds also for $z = 1$.

---

[6] Proof. $\left| \sum\limits_{n=0}^{N} u^n \right| \;=\; \left| \dfrac{1-u^{N+1}}{1-u} \right| \;\le\; \dfrac{2}{|1-u|}$ whenever $|u| \le 1$. Applying this with $u = ze^{-t}$ shows that the dominated convergence theorem applies to the Taylor expansion in z whenever $|z| \le 1$ and $z \ne 1$.

Let us now use (4.10) for $z = \pm 1$: then (3.16) and (3.4) can be written as

$$\frac{E_{2n}}{(2n)!} \;=\; \frac{2}{\pi^{2n+1}}\, \Phi(-1, 2n+1, \tfrac{1}{2}) \tag{4.11}$$

$$\frac{E_{2n+1}}{(2n+1)!} \;=\; \frac{2}{\pi^{2n+2}}\, \Phi(1, 2n+2, \tfrac{1}{2}) \tag{4.12}$$

Using (4.9) to express $\Gamma(s)\, \Phi(z, s, \alpha)$ as an integral, we recover (4.7) and (4.8).

We can also obtain continued fractions for the ordinary generating functions of these three sequences. For $(E_{n+1})_{n \ge 0}$ we have the Jacobi-type continued fraction

$$\sum_{n=0}^{\infty} E_{n+1}\, t^n \;=\; \cfrac{1}{1 - t - \cfrac{t^2}{1 - 2t - \cfrac{3t^2}{1 - 3t - \cfrac{6t^2}{1 - 4t - \cfrac{10t^2}{1 - \cdots}}}}} \tag{4.13}$$

with coefficients $\gamma_n = n+1$ and $\beta_n = n(n+1)/2$. This continued fraction ought to be classical, but the first mention of which I am aware is a 2006 contribution to the OEIS by an amateur mathematician, Paul D. Hanna, who found it empirically [44]; it was proven a few years later by Josuat-Vergès [49] by a combinatorial method (which also yields a q-generalization).

Remark. The J-fraction (4.13) does not arise by contraction from any S-fraction. Indeed, if we use the contraction formula (2.9) and solve for $\boldsymbol{\alpha}$, we find $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (1, 1, 1, 3, 0)$, but then $\alpha_5 \alpha_6 = \beta_3 = 6$ has no solution. ■

For the even and odd subsequences, we have Stieltjes-type continued fractions:

$$\sum_{n=0}^{\infty} E_{2n}\, t^n \;=\; \cfrac{1}{1 - \cfrac{1^2 t}{1 - \cfrac{2^2 t}{1 - \cfrac{3^2 t}{1 - \cdots}}}} \tag{4.14}$$

with coefficients $\alpha_n = n^2$, and

$$\sum_{n=0}^{\infty} E_{2n+1}\, t^n \;=\; \cfrac{1}{1 - \cfrac{1 \cdot 2t}{1 - \cfrac{2 \cdot 3t}{1 - \cfrac{3 \cdot 4t}{1 - \cdots}}}} \tag{4.15}$$

with coefficients $\alpha_n = n(n+1)$. These formulae were found by Stieltjes [69, p. H9] in 1889 and by Rogers [61, p. 77] in 1907. They were given beautiful combinatorial proofs by Flajolet [32] in 1980.

Since $(E_{n+1})_{n\ge 0}$ is a Hamburger moment sequence, it is natural to ask about the full sequence $(E_n)_{n\ge 0}$. Is it a Hamburger moment sequence? The answer is no, because the 3×3 Hankel matrix $\begin{bmatrix} E_0 & E_1 & E_2 \\ E_1 & E_2 & E_3 \\ E_2 & E_3 & E_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix}$ has determinant -1. But a much stronger result is true:

Proposition 4.1. *No arithmetic-progression subsequence* $(E_{n_0+jN})_{N\ge 0}$ *with* n_0 even *and* j odd *is a Hamburger moment sequence.*

PROOF. For $n_0 \ge 1$, (4.4) yields

$$E_{n_0+jN} \;=\; \int\limits_{-\infty}^{\infty} y^{jN}\, \frac{y^{n_0}\, e^{(\pi/2)y}}{\sinh \pi y}\, dy\;. \tag{4.16}$$

When n_0 is even (≥ 2) and j is odd, this represents $(E_{n_0+jN})_{N\ge 0}$ as the moments of a *nonpositive* signed measure on $\mathbb{R}$ with exponential decay. Corollary 2.10 then implies that $(E_{n_0+jN})_{N\ge 0}$ is not a Hamburger moment sequence. The assertion for $n_0 = 0$ then follows from the assertion for $n_0 = 2j$. $\Box$

Since $(E_{n+1})_{n\ge 0}$ is a Hamburger moment sequence with a representing measure of infinite support, it follows that all the Hankel determinants $\Delta_n^{(m)} = \det(E_{i+j+m})_{0\le i,j\le n-1}$ for m *odd* are strictly positive. On the other hand, the $j = 1$ case of Proposition 4.1 implies that for every *even* m there must exist at least one n such that $\Delta_n^{(m)} < 0$. But which one(s)? The question of the sign of $\Delta_n^{(m)}$ for m even seems to be quite delicate, and I am unable to offer any plausible conjecture.

Remarks. 1. Although the sequence $(E_n)_{n\ge 0}$ of Euler numbers is not a Stieltjes or even a Hamburger moment sequence, it is log-convex. This can be proven inductively from the recurrence (1.4) [55, Example 2.2]. Alternatively, it can be proven by observing that the tridiagonal matrix (2.7) associated to the continued fraction (4.13) is totally positive of order 2, i.e. $\beta_n \ge 0$, $\gamma_n \ge 0$ and $\gamma_n\gamma_{n+1} - \beta_{n+1} \ge 0$ for all n. This implies [66, 78] that $(E_{n+1})_{n\ge 0}$ is log-convex. And since $E_0E_2 - E_1^2 = 0$, it follows that also $(E_n)_{n\ge 0}$ is log-convex.

2. Dumont [24, Proposition 5] found a nice Jacobi-type continued fraction also for the sequence of Euler numbers with some sign changes:

$$\sum_{n=0}^{\infty} (-1)^{n(n-1)/2}\, E_{n+1}\, t^n \;=\; \cfrac{1}{1 - t + \cfrac{3t^2}{1 + \cfrac{5t^2}{1 - t + \cfrac{14t^2}{1 + \cfrac{18t^2}{1 - t - \cdots}}}}} \tag{4.17}$$

with coefficients $\gamma_{2k} = 1$, $\gamma_{2k+1} = 0$, $\beta_{2k-1} = -k(4k-1)$ and $\beta_{2k} = -k(4k+1)$. ■

5 Springer numbers

We now turn to the sequence of Springer numbers. Since $\cos t - \sin t = \sqrt{2}\cos(t+\pi/4)$, the partial-fraction expansion (3.1) for secant yields

$$\frac{1}{\cos t - \sin t} \;=\; \frac{1}{\sqrt{2}} \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{(-1)^k}{(k+\frac{1}{4})\pi - t} \;. \tag{5.1}$$

Inserting this into the exponential generating function (1.7) of the Springer numbers and extracting coefficients of powers of t on both sides, we obtain

$$\frac{S_n}{n!} \;=\; \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} (-1)^k \, \big[(k+\tfrac{1}{4})\pi\big]^{-(n+1)} \tag{5.2}$$

(with the interpretation $\lim\limits_{N\to\infty} \sum\limits_{k=-N}^{N}$ when $n=0$). Since (5.2) represents every arithmetic-progression subsequence $(\widetilde{S}_{n_0+jN})_{N\ge 0}$ [where $\widetilde{S}_n = S_n/n!$] as the moments of a *non-positive* signed measure supported on $[-4/3\pi, (4/\pi)^j]$, it follows by Corollary 2.10 that no such sequence is a Hamburger moment sequence.

We now consider the sequence of Springer numbers *tout court*. Start by rewriting (5.2) as

$$\frac{S_n}{n!} \;=\; \frac{1}{\sqrt{2}} \left[\sum_{k=0}^{\infty} (-1)^k \, \big[(k+\tfrac{1}{4})\pi\big]^{-(n+1)} \;+\; (-1)^n \sum_{k=0}^{\infty} (-1)^k \, \big[(k+\tfrac{3}{4})\pi\big]^{-(n+1)} \right] . \tag{5.3}$$

Now multiply by the Stieltjes integral representation $n! = \int\limits_0^\infty x^n\, e^{-x}\, dx$ to get

$$S_n \;=\; \frac{1}{\sqrt{2}} \left[\int\limits_0^\infty dx\, e^{-x} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\frac{1}{4})\pi} \left(\frac{x}{(k+\frac{1}{4})\pi} \right)^{\! n} \right.$$
$$\left. +\; (-1)^n \int\limits_0^\infty dx\, e^{-x} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\frac{3}{4})\pi} \left(\frac{x}{(k+\frac{3}{4})\pi} \right)^{\! n} \right] . \tag{5.4}$$

Change variable to $y = x/[(k+\frac{1}{4})\pi]$ in the first term, and $y = x/[(k+\frac{3}{4})\pi]$ in the second, and interchange integration and summation; this leads to

$$S_n \;=\; \frac{1}{\sqrt{2}} \left[\int\limits_0^\infty \frac{e^{-(\pi/4)y}}{1+e^{-\pi y}} \, y^n \, dy \;+\; \int\limits_0^\infty \frac{e^{-(3\pi/4)y}}{1+e^{-\pi y}} \, (-y)^n \, dy \right] \tag{5.5a}$$

$$=\; \frac{1}{2\sqrt{2}} \int\limits_{-\infty}^{\infty} y^n \, \frac{e^{(\pi/4)y}}{\cosh(\pi y/2)} \, dy \;, \tag{5.5b}$$

which is absolutely convergent for all $n \ge 0$. It follows that $(S_n)_{n\ge 0}$ is a Hamburger moment sequence. It is H-determinate by virtue of (1.9) and Corollary 2.9. Since the unique representing measure has support equal to all of $\mathbb{R}$, it follows that $(S_n)_{n\geq 0}$ is not a Stieltjes moment sequence. This can alternatively be seen from the fact that the 3×3 once-shifted Hankel matrix $\begin{bmatrix} S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \\ S_3 & S_4 & S_5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 11 \\ 3 & 11 & 57 \\ 11 & 57 & 361 \end{bmatrix}$ has determinant -96.

Note also that multiplying (5.5b) by $t^n/n!$ and summing $\sum_{n=0}^{\infty}$, we recover the two-sided Laplace transform

$$\frac{1}{\cos t - \sin t} \;=\; \frac{1}{2\sqrt{2}} \int\limits_{-\infty}^{\infty} \frac{e^{(t+\pi/4)y}}{\cosh(\pi y/2)} \, dy \;, \tag{5.6}$$

which is valid for $-3\pi/4 < \operatorname{Re} t < \pi/4$ [29, 6.2(11)].

For n even we can combine the $y \geq 0$ and $y \leq 0$ contributions in (5.5b) to obtain

$$S_{2n} \;=\; \frac{1}{\sqrt{2}} \int\limits_{0}^{\infty} y^{2n} \, \frac{\cosh(\pi y/4)}{\cosh(\pi y/2)} \, dy \;, \tag{5.7}$$

while for n odd a similar reformulation gives

$$S_{2n+1} \;=\; \frac{1}{\sqrt{2}} \int\limits_{0}^{\infty} y^{2n} \, \frac{y \, \sinh(\pi y/4)}{\cosh(\pi y/2)} \, dy \;. \tag{5.8}$$

We have thus explicitly expressed $(S_{2n})_{n\geq 0}$ and $(S_{2n+1})_{n\geq 0}$ as Stieltjes moment sequences. By (1.9) and Theorem 2.3(b) they are S-determinate. And since the measures in (5.7)/(5.8) are continuous, Proposition 2.4 implies that these sequences are also H-determinate.

We can also obtain continued fractions for the ordinary generating functions of these three sequences. For $(S_n)_{n\geq 0}$ we have the Jacobi-type continued fraction

$$\sum_{n=0}^{\infty} S_n \, t^n \;=\; \cfrac{1}{1 - t - \cfrac{2\cdot 1^2\, t^2}{1 - 3t - \cfrac{2\cdot 2^2\, t^2}{1 - 5t - \cfrac{2\cdot 3^2\, t^2}{1 - 7t - \cfrac{2\cdot 4^2\, t^2}{1 - \cdots}}}}} \tag{5.9}$$

with coefficients $\gamma_n = 2n+1$ and $\beta_n = 2n^2$. This formula was proven a few years ago by Josuat-Vergès [49], by a combinatorial method that also yields a q-generalization; it was independently found (empirically) by an amateur mathematician, Sergei N. Gladkovskii [38]. The fact that $\beta_n > 0$ for all n tells us again that $(S_n)_{n\geq 0}$ is a Hamburger moment sequence.

For the even Springer numbers we have the Stieltjes-type continued fraction [24, Corollary 3.3]

$$\sum_{n=0}^{\infty} S_{2n} t^n \;=\; \cfrac{1}{1 - \cfrac{1 \cdot 3t}{1 - \cfrac{4 \cdot 4t}{1 - \cfrac{5 \cdot 7t}{1 - \cfrac{8 \cdot 8t}{1 - \cdots}}}}} \tag{5.10}$$

with coefficients $\alpha_{2k-1} = (4k-3)(4k-1)$ and $\alpha_{2k} = (4k)^2$. For the odd Springer numbers we have the Jacobi-type continued fraction [36]

$$\sum_{n=0}^{\infty} S_{2n+1} t^n \;=\; \cfrac{1}{1 - 11t - \cfrac{16 \cdot 1 \cdot 3 \cdot 5t^2}{1 - 75t - \cfrac{16 \cdot 4 \cdot 7 \cdot 9t^2}{1 - 203t - \cfrac{16 \cdot 9 \cdot 11 \cdot 13t}{1 - \cdots}}}} \tag{5.11}$$

with coefficients $\gamma_n = 32n^2+32n+11$ and $\beta_n = (4n-1)(4n)^2(4n+1)$. This formula can be obtained as a specialization of a result of Stieltjes [70] (see [8]); it can alternatively be obtained from [24, Propositions 7 and 8] by the transformation formula for Jacobi-type continued fractions under the binomial transform [9, Proposition 4] [66]. Since the odd Springer numbers are a Stieltjes moment sequence, their ordinary generating function is also given by a Stieltjes-type continued fraction with coefficients $\alpha_i > 0$; these coefficients can in principle be obtained from (5.11) by solving (2.9), but no explicit formula for them seems to be known (and maybe no simple formula exists).

Remarks. 1. Although the sequence $(S_n)_{n\ge 0}$ of Springer numbers is not a Stieltjes moment sequence, it is log-convex. This follows [66, 78] from the fact that the tridiagonal matrix (2.7) associated to the continued fraction (5.9) is totally positive of order 2.

2. Dumont [24, Corollary 3.2] also found a nice Jacobi-type continued fraction for the sequence of Springer numbers with some sign changes:

$$\sum_{n=0}^{\infty} (-1)^{n(n-1)/2}\, S_n\, t^n \;=\; \cfrac{1}{1 - t + \cfrac{4t^2}{1 - t + \cfrac{16t^2}{1 - t + \cfrac{36t^2}{1 - \cdots}}}} \tag{5.12}$$

with coefficients $\gamma_n = 1$ and $\beta_n = -4n^2$. This formula follows from (4.14) with t replaced by $4t^2$, combined with the identity

$$(-1)^{n(n-1)/2}\, S_n \;=\; \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (-4)^k\, E_{2k} \tag{5.13}$$

(which follows from the exponential generating functions [24, p. 275]) and a general result about how Jacobi-type continued fractions behave under the binomial transform [9, Proposition 4] [66]. ■

6 What next?

Enumerative combinatorialists are not content with merely counting sets; we want to refine the counting by measuring one or more statistics. To take a trivial example, an n-element set has 2^n subsets, but we can classify these subsets according to their cardinality, and say that there are $\binom{n}{k}$ subsets of cardinality k. We then collect these refined counts in a *generating polynomial*

$$P_n(x) \;\stackrel{\text{def}}{=}\; \sum_{A \subseteq [n]} x^{|A|} \;=\; \sum_{k=0}^{n} \binom{n}{k} x^k \;, \tag{6.1}$$

which in this case of course equals $(1+x)^n$. To take a less trivial example, the number of ways of partitioning an n-element set into nonempty blocks is given by the Bell number B_n; but we can refine this classification by saying that the number of ways of partitioning an n-element set into k nonempty blocks is given by the Stirling number $\left\{ {n \atop k} \right\}$, and then form the Bell polynomial

$$B_n(x) \;=\; \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k \;. \tag{6.2}$$

We can then study generating functions, continued-fraction expansions, moment representations and so forth for $B_n(x)$, generalizing the corresponding results for $B_n = B_n(1)$.

In a similar way, the Euler and Springer numbers can be refined into polynomials that count alternating permutations or snakes of type B_n according to one or more statistics. For instance, consider the polynomials $E_{2n}(x)$ defined by

$$(\sec t)^x \;=\; \sum_{n=0}^{\infty} E_{2n}(x) \, \frac{t^{2n}}{(2n)!} \tag{6.3}$$

where x is an indeterminate. They satisfy the recurrence [48, p. 123]

$$E_{2n+2}(x) \;=\; x \sum_{k=0}^{n} \binom{2n+1}{2k} \, E_{2n-2k-1} \, E_{2k}(x) \tag{6.4}$$

with initial condition $E_0(x) = 1$. It follows that $E_{2n}(x)$ is a polynomial of degree n with nonnegative integer coefficients, which we call the *secant power polynomial*. The first few secant power polynomials are [57, A088874/A085734/A098906]

$$E_0(x) \;=\; 1 \tag{6.5a}$$

$$E_2(x) \;=\; x \tag{6.5b}$$

$$E_4(x) \;=\; 2x + 3x^2 \tag{6.5c}$$

$$E_6(x) \;=\; 16x + 30x^2 + 15x^3 \tag{6.5d}$$

$$E_8(x) \;=\; 272x + 588x^2 + 420x^3 + 105x^4 \tag{6.5e}$$

Since $E_{2n}(1) = E_{2n}$, these are a polynomial refinement of the secant numbers. Moreover, since $\tan' = \sec^2$ and $(\log \sec)' = \tan$, we have $E_{2n}(2) = E_{2n+1}$ and $E'_{2n}(0) =$

E_{2n-1}, so these are also a polynomial refinement of the tangent numbers. Carlitz and Scoville [19] proved that $E_{2n}(x)$ enumerates the alternating (down-up) permutations of $[2n]$ or $[2n+1]$ according to the number of records:

$$\sum_{\sigma \in \mathrm{Alt}_{2n}} x^{\mathrm{rec}(\sigma)} \;=\; E_{2n}(x) \tag{6.6}$$

and

$$\sum_{\sigma \in \mathrm{Alt}_{2n+1}} x^{\mathrm{rec}(\sigma)} \;=\; x\, E_{2n}(1+x)\;. \tag{6.7}$$

Here a *record* (or *left-to-right maximum*) of a permutation $\sigma \in \mathfrak{S}_n$ is an index i such that $\sigma_j < \sigma_i$ for all $j < i$. (In particular, when $n \ge 1$, the index 1 is always a record. This explains why the $E_{2n}(x)$ for $n > 0$ start at order x.)

It turns out that the ordinary generating function of the secant power polynomials is given by a beautiful Stieltjes-type continued fraction, which was found more than a century ago by Stieltjes [69, p. H9] and Rogers [61, p. 82] (see also [10, 49]):

$$\sum_{n=0}^{\infty} E_{2n}(x)\, t^n \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{2(x+1)t}{1 - \cfrac{3(x+2)t}{1 - \cdots}}}} \tag{6.8}$$

with coefficients $\alpha_n = n(x+n-1)$. When $x = 1$ this reduces to the expansion (4.14) for the secant numbers; when $x = 2$ it becomes the expansion (4.15) for the tangent numbers.

The nonnegativity of the coefficients α_n in (6.8) for $x \ge 0$ implies, by Theorem 2.2, that for every $x \ge 0$, the sequence $(E_{2n}(x))_{n \ge 0}$ is a Stieltjes moment sequence. In fact, $(E_{2n}(x))_{n \ge 0}$ has the explicit Stieltjes moment representation [22, pp. 179–181][7]

$$E_{2n}(x) \;=\; \frac{2^{x-1}}{\pi\, \Gamma(x)} \int\limits_0^\infty s^{2n} \left| \Gamma\Big(\frac{x+is}{2}\Big) \right|^2 ds\;, \tag{6.9}$$

which reduces to (4.7) when $x = 1$, and to (4.8) when $x = 2$.

The continued fraction (6.8) also implies, by Theorem 2.2, that for every $x \ge 0$, every minor of the Hankel matrix $(E_{2i+2j}(x))_{i,j \ge 0}$ is a nonnegative real number. But a vastly stronger result turns out to be true [66]: namely, every minor of the Hankel matrix $(E_{2i+2j}(x))_{i,j \ge 0}$ is a polynomial in x with nonnegative integer coefficients! This *coefficientwise* Hankel-total positivity arises in a wide variety of sequences of combinatorial polynomials (sometimes in many variables) — in some cases provably, in other cases conjecturally. But that is a story for another day.

[7] See [56, eq. 5.13.2] for the normalization.

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