

The γ -Vectors of Pascal-like Triangles Defined by Riordan Arrays

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Abstract

We define and characterize the γ -matrix associated to Pascal-like matrices that are defined by ordinary and exponential Riordan arrays. We also define and characterize the γ -matrix of the reversions of these triangles, in the case of ordinary Riordan arrays. We are led to the γ -matrices of a one-parameter family of generalized Narayana triangles. Thus these matrices generalize the matrix of γ -vectors of the associahedron. The principal tools used are the bivariate generating functions of the triangles and Jacobi continued fractions.

1 Introduction

A polynomial $P_n(x) = \sum_{k=0}^n a_{n,k}x^k$ of degree n is said to be *reciprocal* if

$$P_n(x) = x^n P_n(1/x).$$

Thus we have

$$[x^k]P_n(x) = a_{n,k} = [x^k]x^n P_n(1/x).$$

Now

$$\begin{aligned} [x^k]x^n P_n(1/x) &= [x^{k-n}] \sum_{i=0}^n a_{n,i} \frac{1}{x^i} \\ &= [x^{k-n}] \sum_{i=0}^n a_{n,i} x^{-i} \\ &= a_{n,n-k}. \end{aligned}$$

Thus $P_n(x) = \sum_{k=0}^n a_{n,k}x^k$ defines a family of reciprocal polynomials if and only if $a_{n,k} = a_{n,n-k}$. We shall call a lower-triangular matrix $(a_{n,k})$ *Pascal-like* if

1. $a_{n,k} = a_{n,n-k}$
2. $a_{n,0} = a_{n,n} = 1$.

Such a matrix will then be the coefficient array of a family of monic reciprocal polynomials.

We have the following well-known result [7]

Proposition 1. Let $P_n(x)$ be a reciprocal polynomial of degree n . Then there exists a unique polynomial γ_n of degree $\lfloor \frac{n}{2} \rfloor$ with the property

$$P_n(x) = (1+x)^n \gamma_n \left(\frac{x}{(1+x)^2} \right).$$

If $P_n(x)$ has integer coefficients then so does $\gamma_n(x)$.

By this means, we can associate to every Pascal-like matrix $(a_{n,k})$ a matrix $(\gamma_{n,k})$ so that for all n , we have

$$P_n(x) = \sum_{k=0}^n a_{n,k} x^k = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,k} x^k (1+x)^{n-2k}.$$

We shall call this matrix the γ -matrix associated to the coefficient array $(a_{n,k})$ of the family of polynomials $P_n(x)$.

We can characterize the matrix $(a_{n,k})$ in terms of the γ -matrix $(\gamma_{n,k})$ as follows. Before we do this, we shall change our notation somewhat. In algebraic topology, it is customary to use the notation $h(x)$ for palindromic (reciprocal) polynomials [8, 14]. Thus we shall set $h_n(x) = \sum_{k=0}^n h_{n,k} x^k$, where $(h_{n,k})$ now denotes a Pascal-like matrix. We shall denote by $h(x, y)$ the bivariate generating function of this matrix.

Proposition 2. For a Pascal-like matrix $(h_{n,k})$ we have

$$h_{n,k} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2i}{k-i} \gamma_{n,i}.$$

Proof. We have

$$\begin{aligned} h_{n,k} &= [x^k] \sum_{i=0}^n h_{n,i} x^i \\ &= [x^k] \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,i} x^i (1+x)^{n-2i} \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,i} [x^k] x^i (1+x)^{n-2i} \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,i} [x^{k-i}] \sum_{j=0}^{n-2i} \binom{n-2i}{j} x^j \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,i} \binom{n-2i}{k-i}. \end{aligned}$$

□

Example 3. The identity

$$\binom{n}{k} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2i}{k-i} \delta_{i,0}$$

shows that the matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is the γ -matrix for the binomial matrix $\mathbf{B} = \left(\binom{n}{k} \right)$ [A007318](#). Here, we have used the *Annnnnnn* number of the On-Line Encyclopedia of Integer Sequences [12, 13] for the binomial matrix (Pascal's triangle).

When $(\gamma_{n,k})$ is the γ -matrix for $(h_{n,k})$, we shall say the $(\gamma_{n,k})$ *generates*, or *is the generator of*, the matrix $(h_{n,k})$.

Example 4. The matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

with $\gamma_{n,0} = 1$, $\gamma_{n,\lfloor \frac{n}{2} \rfloor} = 1$, and 0 otherwise, generates the matrix $(h_{n,k})$ that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 7 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{pmatrix}.$$

2 Pascal-like matrices defined by Riordan arrays

We now wish to characterize the γ -matrices that are generators for the family of Pascal-like matrices that are determined by the one-parameter family of Riordan arrays

$$\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x} \right).$$

We shall also determine the (generalized) γ -matrices associated to the reversion of these triangles. We recall that an ordinary Riordan array $(g(x), f(x))$ is defined [1, 9, 10] by two power series

$$\begin{aligned} g(x) &= 1 + g_1x + g_2x^2 + \dots, \\ f(x) &= x + f_2x^2 + f_3x^3 + \dots, \end{aligned}$$

where the (n, k) -th element of the resulting lower-triangular matrix is given by

$$a_{n,k} = [x^n]g(x)f(x)^k.$$

Such matrices are invertible. When they have integer entries, the inverse again is an integer matrix (note that we have $a_{n,n} = 1$ in our case because $g_0 = 1$ and $f_1 = 1$). The bivariate generating function of the Riordan array (g, f) is given by

$$\frac{g(x)}{1 - yf(x)}.$$

Matrices defined in a similar manner but with $f(x)$ replaced by $\phi(x) = x^2 + \phi_3x^3 + \dots$ are called “stretched” Riordan arrays [5]. They are not invertible but they do possess left inverses.

Example 5. The stretched Riordan array $(\frac{1}{1-x}, x^2)$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is the γ -matrix for the Pascal-like triangle that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 4 & 1 & 0 & 0 & 0 \\ 1 & 5 & 9 & 5 & 1 & 0 & 0 \\ 1 & 6 & 14 & 14 & 6 & 1 & 0 \\ 1 & 7 & 20 & 29 & 20 & 7 & 1 \end{pmatrix}.$$

Example 6. The matrix $\binom{n-k}{k}$ is the stretched Riordan array $(\frac{1}{1-x}, \frac{x^2}{1-x})$ that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 & 0 & 0 \\ 1 & 5 & 6 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

It generates the Pascal-like matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 5 & 1 & 0 & 0 & 0 \\ 1 & 7 & 13 & 7 & 1 & 0 & 0 \\ 1 & 9 & 25 & 25 & 9 & 1 & 0 \\ 1 & 11 & 41 & 63 & 41 & 11 & 1 \end{pmatrix}.$$

We shall see that this is the Riordan array $\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right)$, which is [A008288](#), the triangle of Delannoy numbers.

The bivariate generating function of the stretched Riordan array $(g(x), \phi(x))$ is given by

$$\frac{g(x)}{1 - y\phi(x)}.$$

We have the following proposition [4].

Proposition 7. *The Riordan array $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$ is Pascal-like (for any $r \in \mathbb{Z}$).*

This is clear since in this case we have

$$h_{n,k} = \sum_{j=0}^k \binom{k}{j} \binom{n-j}{n-k-j} r^j = \sum_{j=0}^k \binom{k}{j} \binom{n-k}{n-k-j} (r+1)^j.$$

We can now characterize the γ -matrices that generate these Pascal-like matrices.

Proposition 8. *The γ -matrices that generate the Pascal-like matrices $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$ defined by ordinary Riordan arrays are given by the stretched Riordan arrays*

$$\left(\frac{1}{1-x}, \frac{rx^2}{1-x}\right),$$

with (n, k) -th term

$$\gamma_{n,k} = \binom{n-k}{k} r^k.$$

Proof. The generating function of the Pascal-like matrix $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$ is given by

$$h(x, y) = \frac{1}{1-x} \frac{1}{1 - y \frac{x(1+rx)}{1-x}} = \frac{1}{1 - (1+y)x - rx^2y}.$$

Similarly, the generating function of the matrix $\left(\frac{1}{1-x}, \frac{rx^2}{1-x}\right)$ is given by

$$\gamma(x, y) = \frac{1}{1-x} \frac{1}{1 - y \frac{rx^2}{1-x}} = \frac{1}{1 - x - rx^2y}.$$

We now have

$$h(x, y) = \gamma \left((1 + y)x, \frac{y}{(1 + y)^2} \right).$$

□

We recall that for a generating function $f(x)$, its $\text{INVERT}(\alpha)$ transform is the generating function

$$\frac{f(x)}{1 + \alpha x f(x)}.$$

Note that

$$\frac{\frac{v}{1 + \alpha x v}}{1 - \alpha x \frac{v}{1 + \alpha x v}} = v,$$

and thus the inverse of the $\text{INVERT}(\alpha)$ transform is the $\text{INVERT}(-\alpha)$ transform.

Corollary 9. *The generating function $h(x, y)$ of the Pascal-like matrix $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x} \right)$ is the $\text{INVERT}(y)$ transform of the generating function $\gamma(x, y)$ of the corresponding γ -matrix.*

Proof. A direct calculation shows that for $\gamma(x, y) = \frac{1}{1-x-rx^2y}$ we have

$$\frac{\gamma(x, y)}{1 - yx\gamma(x, y)} = \frac{1}{1 - (y + 1)x - rx^2y} = h(x, y).$$

□

Equivalently, we can say that the generating function of the γ -matrix is the $\text{INVERT}(-y)$ transform of the generating function of the corresponding Pascal-like matrix.

We make the following observation, which will be relevant when we discuss a family of generalized Narayana triangles. The γ -matrix corresponding to the signed Pascal-like matrix

$$\left(\frac{1}{1+x}, \frac{-x(1+rx)}{1+x} \right)$$

has generating function

$$\frac{1}{1 + x + rx^2y}.$$

This is the matrix with general term $(-1)^{n-k} r^k \binom{n-k}{k}$. By a signed Pascal-like matrix in this case we mean that $a_{n,k} = a_{n,n-k}$ but we now have $a_{n,0} = a_{n,n} = (-1)^n$.

We close this section by recalling the formula

$$\gamma_n = (1 + x)^n \gamma_n \left(\frac{x}{(1 + x)^2} \right).$$

We now note that the inverse of the Riordan array

$$\left(1, \frac{x}{(1 + x)^2} \right)$$

is given by

$$(1, xc(x)^2),$$

where

$$c(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

is the generating function of the Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$ [A000108](#). In fact, we have the following result [\[8\]](#).

Proposition 10. (*Zeilberger's Lemma*).

$$\gamma_{n,k} = [x^k] \frac{h_n(xc(x)^2)}{c(x)^n}.$$

We can use this result to find an explicit formula for $\gamma_{n,k}$ in terms of $h_{n,k}$. We let $\alpha_{n,k}$ be the general (n, k) -th element of the Riordan array $(1, xc(x)^2)$. We have

$$\alpha_{n,k} = \binom{2n-1}{n-k} \frac{2k + 0^{n+k}}{n+k + 0^{n+k}}.$$

We let $\beta_{n,k}$ be the general (n, k) -th term of the Riordan array $(1, \frac{x}{c(x)})$. We have $\beta_{n,n} = 1$, and

$$\beta_{n,k} = \sum_{j=0}^{n-k} \frac{(-1)^j j}{n-k} \binom{k+j-1}{j} \binom{2(n-k)}{n-k-j},$$

otherwise. This is essentially [A271875](#). Then we have the following result.

Corollary 11. *We have*

$$\gamma_{n,k} = \sum_{i=0}^k \left(\sum_{j=0}^n h_{n,j} \alpha_{i,j} \right) \beta_{n+k-i,n}.$$

Proof. We have

$$\begin{aligned} [x^k][x^k] \frac{h_n(xc(x)^2)}{c(x)^n} &= \sum_{i=0}^n [x^i] \sum_{j=0}^n h_{n,j} (xc(x)^2)^j [x^{k-i}] \frac{1}{c(x)^n} \\ &= \sum_{i=0}^k \left(\sum_{j=0}^n h_{n,j} [x^i] (xc(x)^2)^j \right) [x^{k-1+n}] \frac{x^n}{c(x)^n} \\ &= \sum_{i=0}^k \left(\sum_{j=0}^n h_{n,j} \alpha_{i,j} \right) \beta_{n+k-i,n}. \end{aligned}$$

□

This gives us the following formula.

$$\gamma_{n,k} = \sum_{i=0}^k \sum_{j=0}^n h_{n,j} \binom{2i-1}{i-j} \frac{2j + 0^{i+j}}{i+j + 0^{i+j}} \text{If} \left(k = i, 1, \sum_{m=0}^{k-i} \frac{m(-1)^m}{k-i} \binom{n-1+m}{m} \binom{2(k-i)}{k-i-m} \right).$$

Example 12. If we take $(h_{n,k})$ to be the triangle of Eulerian numbers [A008292](#) that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 11 & 11 & 1 & 0 & 0 & 0 \\ 1 & 26 & 66 & 26 & 1 & 0 & 0 \\ 1 & 57 & 302 & 302 & 57 & 1 & 0 \\ 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \end{pmatrix}$$

we find that the γ -matrix $(\gamma_{n,k})$ is the triangle [A101280](#) that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 8 & 0 & 0 & 0 & 0 & 0 \\ 1 & 22 & 16 & 0 & 0 & 0 & 0 \\ 1 & 52 & 136 & 0 & 0 & 0 & 0 \\ 1 & 114 & 720 & 272 & 0 & 0 & 0 \end{pmatrix}.$$

This is the triangle of γ -vectors for the permutahedra (of type A). It also gives the number of permutations of n objects with k descents such that every descent is a peak [11].

Example 13. We consider the Pascal-like matrix $(h_{n,k}) = \left(\frac{1}{1-x}, x\right)$ that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We note that the row elements are constant. We have that

$$\gamma_{n,k} = \sum_{i=0}^k \sum_{j=0}^n \binom{2i-1}{i-j} \frac{2j+0^{i+j}}{i+j+0^{i+j}} \text{If} \left(k = i, 1, \sum_{m=0}^{k-i} \frac{m(-1)^m}{k-i} \binom{n-1+m}{m} \binom{2(k-i)}{k-i-m} \right).$$

We find that the γ -matrix in this case begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 3 & 0 & 0 & 0 & 0 \\ 1 & -5 & 6 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

This is the matrix $\binom{n-k}{k}(-1)^k$. Thus

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2i}{k-i} \binom{n-i}{i} (-1)^i = \text{If}[k \leq n, 1, 0].$$

3 Stretched Riordan arrays as γ -matrices

Every stretched Riordan array of the form

$$\left(\frac{1}{1-x}, x^2 g(x) \right),$$

where

$$g(x) = 1 + g_1 x + g_2 x^2 + \dots$$

can be used to generate a Pascal-like matrix. Thus to each power series $g(x)$ above we can associate a Pascal-like matrix whose γ -matrix is given by this stretched Riordan array.

In this section, we shall concentrate on the case when $g(x) = \frac{1+rx}{1-x}$.

Example 14. For $r = 1$, we obtain the γ -matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 5 & 0 & 0 & 0 & 0 \\ 1 & 9 & 13 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding Pascal-like matrix then begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 6 & 1 & 0 & 0 & 0 \\ 1 & 9 & 17 & 9 & 1 & 0 & 0 \\ 1 & 12 & 36 & 36 & 12 & 1 & 0 \\ 1 & 15 & 64 & 101 & 64 & 15 & 1 \end{pmatrix}.$$

The row sums of this matrix, which begin

$$1, 2, 5, 14, 37, 98, 261, \dots$$

give [A077938](#), with generating function

$$\frac{1}{1 - 2x - x^2 - 2x^3}$$

The diagonal sums, which begin

$$1, 1, 2, 4, 8, 16, 31, \dots$$

are the Pentanacci numbers [A001591](#) with generating function

$$\frac{1}{1 - x - x^2 - x^3 - x^4 - x^5}.$$

We have the following proposition.

Proposition 15. *The Pascal-like triangle that begins*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & r+5 & r+5 & 1 & 0 & 0 & 0 \\ 1 & 2r+7 & 4r+13 & 2r+7 & 1 & 0 & 0 \\ 1 & 3r+9 & 11r+25 & 11r+25 & 3r+9 & 1 & 0 \\ 1 & 4r+11 & r^2+22r+41 & 2r^2+36r+63 & r^2+22r+41 & 4r+11 & 1 \end{pmatrix}$$

with γ -matrix given by the stretched Riordan array $\left(\frac{1}{1-x}, \frac{x^2(1+rx)}{1-x}\right)$, has row sums with generating function

$$\frac{1}{1 - 2x - x^2 - 2rx^3},$$

and diagonal sums given by the generalized Pentanacci numbers with generating function

$$\frac{1}{1 - x - x^2 - x^3 - rx^4 - rx^5}.$$

4 Reverting triangles

Let $h(x, y)$ be the generating function of the lower-triangular matrix $h_{n,k}$, with $h_{0,0} = 1$. By the reversion of this triangle, we shall mean the triangle whose generating function $h^*(x, y)$ is given by

$$h^*(x, y) = \frac{1}{x} \text{Rev}_x(xh(x, y)).$$

Procedurally, this means that we solve the equation

$$uh(u, y) = x$$

and then we divide the solution $u(x, y)$ that satisfies $u(0, y) = 0$ by x .

Proposition 16. *The generating function of the reversion of the Pascal-like matrix defined by the Riordan array $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$ is given by*

$$h^*(x, y) = \frac{1}{1 + x(y+1)} c\left(\frac{-rx^2y}{(1 + x(y+1))^2}\right),$$

where

$$c(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

is the generating function of the Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$. ([A000108](#)).

Proof. Solving the equation

$$\frac{u}{1 - u(y+1) - ru^2y} = x$$

gives us

$$h^*(x, y) = \frac{-1 - x(y+1) + \sqrt{1 + 2x(y+1) + x^2(1 + 2y(2r+1) + y^2)}}{2rx^2y}.$$

Thus

$$h^*(x, y) = \frac{1}{1 + x(y+1)} c\left(\frac{-rx^2y}{(1 + x(y+1))^2}\right).$$

□

We note that we can now calculate an expression for the terms of the reverted triangle, since, using the language of Riordan arrays, we have

$$h^*(x, y) = \left(\frac{1}{1 + y(x+1)}, \frac{-rx^2y}{(1 + x(y+1))^2} \right) \cdot c(x).$$

Proposition 17. *We have*

$$[x^n][y^i]h^*(x, y) = h_{n,i}^* = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^n (-r)^k \binom{n}{2k} C_k \binom{n-2k}{i-k}.$$

The γ -matrix of the reverted triangle $(h_{n,k}^*)$ is given by

$$\gamma_{n,k}^* = (-1)^n (-r)^k \binom{n}{2k} C_k.$$

The γ -matrix $(\gamma_{n,k}^*)$ of the reverted triangle $(h_{n,k}^*)$ is the reversion of the triangle $\gamma_{n,k}$.

Proof. The expression for $h_{n,k}^*$ results from a direct calculation. Reverting the expression $\gamma(x, y) = \frac{1}{1 - x - rx^2y}$ in the sense above gives us

$$\gamma^*(x, y) = \frac{1}{1+x} c\left(\frac{-rx^2y}{(1+x)^2}\right),$$

from which we deduce the other statements. □

Example 18. For $r = -1, 0, 1$, the triangles $(h_{n,k})$ begin, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 5 & 1 & 0 & 0 & 0 \\ 1 & 7 & 13 & 7 & 1 & 0 & 0 \\ 1 & 9 & 25 & 25 & 9 & 1 & 0 \\ 1 & 11 & 41 & 63 & 41 & 11 & 1 \end{pmatrix}.$$

The corresponding reverted triangles $(h_{n,k}^*)$, are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ -1 & -6 & -6 & -1 & 0 & 0 & 0 \\ 1 & 10 & 20 & 10 & 1 & 0 & 0 \\ -1 & -15 & -50 & -50 & -15 & -1 & 0 \\ 1 & 21 & 105 & 175 & 105 & 21 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -3 & -3 & -1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ -1 & -5 & -10 & -10 & -5 & -1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & -2 & -4 & -2 & 1 & 0 & 0 \\ -1 & 5 & 10 & 10 & 5 & -1 & 0 \\ 1 & -9 & -15 & -15 & -9 & 1 & 1 \end{pmatrix}.$$

Note that for $r = -1$, the reverted triangle is $(-1)^n$ times the Narayana triangle [A001263](#).

The corresponding γ -matrices $(\gamma_{n,k})$ are given by, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 3 & 0 & 0 & 0 & 0 \\ 1 & -5 & 6 & -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 & 0 & 0 \\ 1 & 5 & 6 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding reverted γ -matrices $(\gamma_{n,k}^*)$ are then, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 2 & 0 & 0 & 0 & 0 \\ -1 & -10 & -10 & 0 & 0 & 0 & 0 \\ 1 & 15 & 30 & 5 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & -6 & 2 & 0 & 0 & 0 & 0 \\ -1 & 10 & -10 & 0 & 0 & 0 & 0 \\ 1 & -15 & 30 & -5 & 0 & 0 & 0 \end{pmatrix}.$$

It is interesting to represent the generating functions of the $(\gamma_{n,k}^*)$ and the $(h_{n,k}^*)$ triangles as Jacobi continued fractions. We have

Proposition 19. *The generating function $h^*(x, y)$ can be expressed as the Jacobi continued fraction*

$$\mathcal{J}(-(y+1), -(y+1), -(y+1), \dots; -ry, -ry, -ry, \dots).$$

The generating function $\gamma^(x, y)$ can be expressed as the Jacobi continued fraction*

$$\mathcal{J}(-1, -1, -1, \dots; -ry, -ry, -ry, \dots).$$

Proof. We solve the continued fraction equation

$$u = \frac{1}{1 + (y+1)x + rx^2u}$$

to retrieve the generating function $h^*(x, y)$. Similarly, we solve the continued fraction equation

$$u = \frac{1}{1 + x + rx^2u}$$

to retrieve the generating function $\gamma^*(x, y)$. □

Note that we have used the notation $\mathcal{J}(a, b, c, \dots; r, s, t, \dots)$ to denote the Jacobi continued fraction [2, 15]

$$\frac{1}{1 - ax - \frac{rx^2}{1 - bx - \frac{sx^2}{1 - cx - \frac{tx^2}{1 - \dots}}}}$$

We can now express the relationship between the generating functions $h^*(x, y)$ and $\gamma^*(x, y)$ in terms of repeated binomial transforms.

Corollary 20. *The generating function $h^*(x, y)$ is the $(-y)$ -th binomial transform of the γ generating function $\gamma^*(x, y)$:*

$$h^*(x, y) = \frac{1}{1 + xy} \gamma^* \left(\frac{x}{1 + xy}, y \right).$$

Equivalently, the γ generating function $\gamma^(x, y)$ is the y -th binomial transform of the generating function $h^*(x, y)$:*

$$\gamma^*(x, y) = \frac{1}{1 - xy} h^* \left(\frac{x}{1 - xy}, y \right).$$

This reflects the general assertion that the reversion of an INVERT transform is a binomial transform.

5 The γ -vectors of generalized Narayana numbers

The Riordan array $\left(\frac{1}{1+x}, \frac{-x(1+rx)}{1+x} \right)$, with bivariate generating function

$$\frac{1}{1 + x(y + 1) + rx^2y},$$

has a γ -matrix with generating function

$$\frac{1}{1 + x + rx^2y}.$$

We shall call elements of the reversions of the Riordan array $\left(\frac{1}{1+x}, \frac{-x(1+rx)}{1+x} \right)$ r -Narayana numbers. The Narayana numbers $N_{n,k} = \frac{1}{k+1} \binom{n+1}{k} \binom{n}{k}$ are then the 1-Narayana numbers. The bivariate generating function for the r -Narayana numbers is given by

$$\frac{1}{1 - x(y + 1)} \left(\frac{rx^2y}{(1 - x(y + 1))^2} \right).$$

The bivariate generating function for the γ -matrix of the r -Narayana numbers is then obtained by reverting the generating function $\frac{1}{1+x+rx^2y}$. We thus obtain the following result.

Proposition 21. *The γ -matrix for the r -Narayana numbers has generating function*

$$\frac{1}{1-x}c\left(\frac{rx^2y}{(1-x)^2}\right).$$

This is the matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & r & 0 & 0 & 0 & 0 & 0 \\ 1 & 3r & 0 & 0 & 0 & 0 & 0 \\ 1 & 6r & 2r^2 & 0 & 0 & 0 & 0 \\ 1 & 10r & 10r^2 & 0 & 0 & 0 & 0 \\ 1 & 15r & 30r^2 & 5r^3 & 0 & 0 & 0 \end{pmatrix},$$

with general term

$$\binom{n}{2k}r^kC_k.$$

For $r = -1, 0, 1$, the matrices $\left(\frac{1}{1+x}, \frac{-x(1+rx)}{1+x}\right)$ begin, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ -1 & -5 & -5 & -1 & 0 & 0 & 0 \\ 1 & 7 & 13 & 7 & 1 & 0 & 0 \\ -1 & -9 & -25 & -25 & -9 & -1 & 0 \\ 1 & 11 & 41 & 63 & 41 & 11 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -3 & -3 & -1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ -1 & -5 & -10 & -10 & -5 & -1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The corresponding matrices of r -Narayana numbers are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -2 & -4 & -2 & 1 & 0 & 0 \\ 1 & -5 & -10 & -10 & -5 & 1 & 0 \\ 1 & -9 & -15 & -15 & -9 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 6 & 1 & 0 & 0 & 0 \\ 1 & 10 & 20 & 10 & 1 & 0 & 0 \\ 1 & 15 & 50 & 50 & 15 & 1 & 0 \\ 1 & 21 & 105 & 175 & 105 & 21 & 1 \end{pmatrix}.$$

This last matrix, as expected, is the Narayana triangle [A001263](#). The corresponding γ -matrices for these r -Narayana triangles are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 \\ 1 & -6 & 2 & 0 & 0 & 0 & 0 \\ 1 & -10 & 10 & 0 & 0 & 0 & 0 \\ 1 & -15 & 30 & -5 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 2 & 0 & 0 & 0 & 0 \\ 1 & 10 & 10 & 0 & 0 & 0 & 0 \\ 1 & 15 & 30 & 5 & 0 & 0 & 0 \end{pmatrix}.$$

This last matrix is [A055151](#). The rows of this triangle are the γ -vectors of the n -dimensional (type A) associahedra [8]. We have seen that its elements are given by

$$\gamma_{n,k} = \sum_{i=0}^k \sum_{j=0}^n N_{n,j} \binom{2i-1}{i-j} \frac{2j+0^{i+j}}{i+j+0^{i+j}} \text{If} \left(k=i, 1, \sum_{m=0}^{k-i} \frac{m(-1)^m}{k-i} \binom{n-1+m}{m} \binom{2(k-i)}{k-i-m} \right),$$

where $N_{n,k}$ denotes the (n, k) -th Narayana number [A001263](#).

The relationship between the γ -matrix and the r -Narayana numbers can be further clarified as follows.

Proposition 22. *The generating function of the r -Narayana numbers can be expressed as the Jacobi continued fraction*

$$\mathcal{J}((y+1), (y+1), (y+1), \dots; ry, ry, ry, \dots).$$

The generating function of the corresponding γ -matrix can be expressed as the Jacobi continued fraction

$$\mathcal{J}(1, 1, 1, \dots; ry, ry, ry, \dots).$$

Corollary 23. *The generating function of the r -Narayana numbers is the y -th binomial transform of the generating function of the corresponding γ -matrix.*

$$h^*(x, y) = \frac{1}{1 - xy} \gamma^* \left(\frac{x}{1 - xy}, y \right).$$

Equivalently, the γ generating function $\gamma^(x, y)$ is the $(-y)$ -th binomial transform of the generating function $h^*(x, y)$:*

$$\gamma^*(x, y) = \frac{1}{1 + xy} h^* \left(\frac{x}{1 + xy}, y \right).$$

6 Pascal-like triangles defined by exponential Riordan arrays

We recall that an exponential Riordan array $[g(x), f(x)]$ [1, 6] is defined by two exponential generating functions

$$g(x) = 1 + g_1 \frac{x}{1!} + g_2 \frac{x^2}{2!} + \cdots,$$

and

$$f(x) = \frac{x}{1!} + f_2 \frac{x^2}{2!} + \cdots,$$

with its (n, k) -th term $a_{n,k}$ given by

$$a_{n,k} = \frac{n!}{k!} [x^n] g(x) f(x)^k.$$

In the context of Pascal-like matrices, we have that the exponential Riordan array

$$[e^x, x(1 + rx/2)],$$

with general term

$$h_{n,k} = \frac{n!}{k!} \sum_{j=0}^k \frac{r^j}{(n - k - j)! 2^j},$$

is a Pascal-like matrix [3]. This matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & r+2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3r+3 & 3r+3 & 1 & 0 & 0 & 0 \\ 1 & 6r+4 & 3r^2+12r+6 & 6r+4 & 1 & 0 & 0 \\ 1 & 10r+5 & 15r^2+30r+10 & 15r^2+30r+10 & 10r+5 & 1 & 0 \\ 1 & 15r+6 & 45r^2+60r+15 & 15r^3+90r^2+90r+20 & 45r^2+60r+15 & 15r+6 & 1 \end{pmatrix}.$$

We have the following result.

Proposition 24. *The γ -matrix of the the Pascal-like exponential Riordan array $[e^x, x(1 + rx/2)]$ is the matrix with general term*

$$\binom{n}{2k} r^k (2k - 1)!!$$

In fact, the generating function of the exponential Riordan array $[e^x, x(1 + rx/2)]$ is given by

$$\mathcal{J}(y + 1, y + 1, y + 1, \dots; ry, 2ry, 3ry, \dots)$$

while that of its γ -matrix is given by

$$\mathcal{J}(1, 1, 1, \dots; ry, 2ry, 3ry, \dots).$$

Proposition 25. *The generating function of the γ -matrix of the Pascal-like exponential Riordan array $[e^x, x(1 + rx/2)]$ has generating function*

$$e^{x(1+rx/2)}.$$

Proof. By the theory of exponential Riordan arrays, the generating function of the Riordan array $[e^x, x(1 + rx/2)]$ is given by

$$e^x e^{xy(1+rx/2)}.$$

Taking the $(-y)$ -th binomial transform of this, we obtain

$$e^{x(1+rx/2)}.$$

□

Example 26. For $r = 1$, we get the γ -matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 3 & 0 & 0 & 0 & 0 \\ 1 & 10 & 15 & 0 & 0 & 0 & 0 \\ 1 & 15 & 45 & 15 & 0 & 0 & 0 \end{pmatrix}.$$

This is [A100861](#), the triangle of Bessel numbers that count the number of k -matchings of the complete graph $K(n)$. The corresponding Pascal-like matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 6 & 1 & 0 & 0 & 0 \\ 1 & 10 & 21 & 10 & 1 & 0 & 0 \\ 1 & 15 & 55 & 55 & 15 & 1 & 0 \\ 1 & 21 & 120 & 215 & 120 & 21 & 1 \end{pmatrix}.$$

This is [A100862](#), which counts the number of k -matchings of the corona $K'(n)$ of the complete graph $K(n)$ and the complete graph $K(1)$.

Example 27. For $r = 2$, we obtain the γ -matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 12 & 12 & 0 & 0 & 0 & 0 \\ 1 & 20 & 60 & 0 & 0 & 0 & 0 \\ 1 & 30 & 180 & 120 & 0 & 0 & 0 \end{pmatrix}.$$

This is [A059344](#), where row n consists of the nonzero coefficients of the expansion of $2^n x^n$ in terms of Hermite polynomials with decreasing subscripts. The corresponding Pascal-like matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 9 & 9 & 1 & 0 & 0 & 0 \\ 1 & 16 & 42 & 16 & 1 & 0 & 0 \\ 1 & 25 & 130 & 130 & 25 & 1 & 0 \\ 1 & 36 & 315 & 680 & 315 & 36 & 1 \\ de \end{pmatrix}.$$

The row sums of this matrix are given by [A000898](#), the number of symmetric involutions of $[2n]$ (Deutsch).

7 Conclusion

It is the case that the set of Pascal-like matrices defined by Riordan arrays is a restricted one. Nevertheless, we hope that this note indicates that they have interesting properties, including in particular their generating γ -matrices. In the case of Pascal-like matrices defined by ordinary Riordan arrays, we have seen that by reverting them, we find additional (signed) Pascal-like triangles, including triangles of Narayana type. The γ -matrices of these new triangles are again the reversions of the original triangles' γ -matrices.

We have also shown that stretched Riordan arrays play a useful role, and in particular can lead to further (non-Riordan) Pascal-like matrices. We have also found it useful to use Riordan array techniques to find an explicit closed form formula for the elements $\gamma_{n,k}$ of the γ -matrix in terms of $h_{n,k}$.

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(Concerned with sequences [A000108](#), [A000898](#), [A001263](#), [A001591](#), [A007318](#), [A008288](#), [A008292](#), [A055151](#), [A059344](#), [A077938](#), [A100861](#), [A100862](#), [A101280](#), and [A271875](#).)