A note on number triangles that are almost their own production matrix

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Abstract

We characterize a family of number triangles whose production matrices are closely related to the original number triangle. We study a number of such triangles that are of combinatorial significance. For a specific subfamily, these triangles relate to sequences that have interesting convolution recurrences and continued fraction generating functions.

1 Preliminaries

In this note, we shall be concerned with infinite lower-triangular matrices with integer entries. Thus we will be working with matrices $(T_{n,k})$ where $T_{n,k} = 0$ if k > n. We shall index these matrices starting with $T_{0,0}$ in the upper left corner. The Riordan group of ordinary Riordan arrays [1, 7, 8] is composed of such matrices, where an ordinary Riordan array is defined by two power series,

$$g(x) = 1 + g_1 x + g_2 x^2 + \cdots, \quad f(x) = x + f_2 x^2 + f_3 x^3 + \cdots,$$

where

$$T_{n,k} = [x^n]g(x)f(x)^k$$

Here, $[x^n]$ is the operator that extracts the coefficient of the term in x^n . Similarly, an exponential Riordan array [g(x), f(x)] [1, 4] is defined by two power series

$$g(x) = 1 + g_1 \frac{x}{1!} + g_2 \frac{x^2}{2!} + \cdots, \quad f(x) = \frac{x}{1!} + f_2 \frac{x^2}{2!} + f_3 \frac{x^3}{3!} + \cdots,$$

where the general (n, k)-th element of [g(x), f(x)] is given by

$$T_{n,k} = \frac{n!}{k!} [x^n] g(x) f(x)^k$$

In dealing with the infinite matrix $(T_{n,k})$, we display a suitable truncation in the text that follows.

Many of the sequences that we shall encounter have ordinary generating functions that can be expressed as Stieltjes or Jacobi continued fractions [2, 11]. For these we use the following notation. The Stieltjes continued fraction

$$\frac{1}{1 - \frac{ax}{1 - \frac{\alpha x}{1 - \frac{bx}{1 - \frac{\beta x}{1 - \cdots}}}}$$

will be denoted by $\mathcal{S}(a, \alpha, b, \beta, \ldots)$ or where appropriate as $\mathcal{S}(a, b, \ldots; \alpha, \beta, \ldots)$.

The Jacobi continued fraction

$$\frac{1}{1-ax-\frac{bx^2}{1-cx-\frac{dx^2}{1-\cdots}}}$$

will be denoted by $\mathcal{J}(a, c, \ldots; b, d, \ldots)$.

Sequences and triangles, where known, will be referenced by their Annnnn number in the On-Line Encyclopedia of Integer Sequences [9, 10].

2 Introduction

For an invertible lower-triangular matrix M, we define its production matrix [3] to be the matrix

$$P_M = M^{-1}\overline{M} = M^{-1} \cdot U \cdot M,$$

where U denotes the infinite shift matrix that begins

$\int 0$	1	$\begin{array}{c} 0 \\ 1 \end{array}$	0	0	0	0 \
0	0	1	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0
0	0			0	0	1
$\int 0$	0	0	0	0	0	0 /

Thus \overline{M} is the matrix M with its top row removed.

By construction, the matrix P_M will have a non-zero superdiagonal, so it will differ from M in this respect (note that P_M is a Hessenberg matrix). For the purposes of this note, we shall say that the matrix M is almost its own production matrix P_M if, apart from the extra super-diagonal in P_M , the two matrices only differ otherwise in their first column.

In order to define a class of number triangles that are almost their own production matrix, we introduce some notation. Given a sequence b_0, b_1, b_2, \ldots with generating function g(x), we define the corresponding *sequence array* to be the lower triangular matrix with (n, k)-term equal to b_{n-k} , for $k \leq n$, and 0 otherwise. In the language of ordinary Riordan arrays, this is the array (g(x), x), a member of the Appell subgroup.

We define V to be the infinite matrix with -n on the sub-diagonal, and 0 otherwise. Thus V begins

$\int 0$	0	0	0	0	0	0 \
0	0	0	0	0	0	0
0	-1	0	0	0	0	0
0	0	-2	0	0	0	0
0	0	0	-3	0	0	0
0	0	0	0	-4	0	0
$\int 0$	0	0	0	0	-5	0 /

We have the following proposition.

Proposition 1. Let a_n be a sequence with generating function $f(x) = a_0 + a_1x + a_2x^2 + \cdots$. Then the lower-triangular matrix

$$M = ((1 - xf(x), x) + V)^{-1}$$

is almost its own production matrix. In fact, we have that

$$P_M - M$$

takes the form of the matrix that begins

$(a_0 - 1)$	1	0	0	0	0 \
a_1	0	1	0	0	0
a_2	0	0	1	0	0
a_3	0	0	0	1	0
a_4	0	0	0	0	1
a_5	0	0	0	0	0 /

Example 2. We let a_n be the sequence of indecomposable permutations <u>A003319</u> that begins

 $1, 1, 3, 13, 71, 461, 3447, 29093, 273343, 2829325, 31998903, \ldots$

This sequence is the INVERT(1) transform of the shifted factorials (n + 1)!. The (ordinary) generating function f(x) of this sequence is given by the Stieltjes continued fraction

$$\frac{1}{1 - \frac{x}{1 - \frac{2x}{1 - \frac{2x}{1 - \frac{3x}{1 - \frac{3x}{1 - \dots}}}}}}$$

with coefficients $1, 2, 2, 3, 3, 4, 4, \dots$ The matrix $M = ((1 - xf(x), x) + V)^{-1}$ then begins

1	/ 1	0	0	0	0	0	0	-1	(1	0	0	0	0	0	0	
	-1	1	0	0	0	0	0		1	1	0	0	0	0	0	
	-1	-2	1	0	0	0	0		3	2	1	0	0	0	0	
	-3	-1	-3	1	0	0	0	=	13	7	3	1	0	0	0	.
	-13	-3	-1	-4	1	0	0		71	33	13	4	1	0	0	
	-71	-13	-3	-1	-5	1	0		461	191	71	21	5	1	0	
	-461	-71	-13	-3	-1	-6	1 /		3447	1297	461	133	31	6	1 /	

This is <u>A104980</u> [Paul D. Hanna, [9]]. The production matrix P_M of M then begins

	/ 1	1	0	0	0	0	0 \	
	2	1	1	0	0	0	0	
	6	2	1	1	0	0	0	
	26	7	3	1	1	0	0	
	142	33	13	4	1	1	0	
	922	191	71	21	5	1	1	
,	6894	2 7 33 191 1297	461	133	31	6	1 /	

We have that $P_M - M$ begins

	$ \begin{array}{r} 1 \\ 2 \\ 6 \\ 26 \\ 142 \\ 922 \\ 6894 \end{array} $	$ \begin{array}{c} 1 \\ 2 \\ 7 \\ 33 \\ 191 \\ 1297 \end{array} $	$egin{array}{c} 0 \\ 1 \\ 3 \\ 13 \\ 71 \\ 461 \end{array}$	$egin{array}{c} 0 \ 0 \ 1 \ 1 \ 4 \ 21 \ 133 \end{array}$	$egin{array}{c} 0 \\ 0 \\ 1 \\ 1 \\ 5 \\ 31 \end{array}$	$egin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 6 \end{array}$	$\left(\begin{array}{c} 0\\ 0\\ 0\\ 0\\ 1\\ 1\end{array}\right)$			1 1 2 1 7 46 34	8 3 1 51	$egin{array}{c} 0 \\ 1 \\ 2 \\ 7 \\ 33 \\ 191 \\ 1297 \end{array}$	$egin{array}{c} 0 \\ 0 \\ 1 \\ 3 \\ 13 \\ 71 \\ 461 \end{array}$	$egin{array}{c} 0 \\ 0 \\ 1 \\ 4 \\ 21 \\ 133 \end{array}$	$egin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 5 \\ 31 \end{array}$	$egin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 6 \end{array}$	$\left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right)$
				=		$0\\1\\13\\71\\461\\3447$	$ \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0$	$egin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}$	0 0 1 0 0 0 0	$ \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} $	0 0 0 1 0 0	$\left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{array} \right) .$					

Rearranging, we obtain that the production matrix P_M can then be expressed as beginning

1	1	1	0	0	0	0	0 \	
	2	1	1	0	0	0	0	
	6	2	1	1	0	0	0	
	26	7	3		1	0	0	=
	142	33	13	4	1	1	0	
	922	191	71	21	5	1	1	
ĺ	6894	1297	461	133	31	6	1 /	

1	´ 1	0	0	0	0	0	0 \		(0	1	0	0	0	0	0 \	
	1	1	0	0	0	0	0		1	0	1	0	0	0	0	
I	3	2	1	0	0	0	0		3	0	0	1	0	0	0	
	13	7	3	1	0	0	0	+	13	0	0	0	1	0	0	
	71	33	13	4	1	0	0		71	0	0	0	0	1	0	
	461	191	71	21	5	1	0		461	0	0	0	0	0	1	
	3447	1297	461	133	31	6	1 /		3447	0	0	0	0	0	0 /	

3 Proof of the proposition

The proof of the proposition hinges on two lemmas.

Lemma 3. The production matrix of an element $(g(x), x)^{-1}$ of the Appell subgroup of the Riordan group, where $g(x) = 1 + b_1 x + b_2 x + \dots$, begins

1	$-b_1$	1	0	0	0	0	0 \
	$-b_2$	0	1	0	0	0	0
	$-b_3$	0	0	1	0	0	0
	$-b_4$	0	0	0	1	0	0
	$-b_5$	0	0	0	0	1	0
	$-b_6$	0	0	0	0	0	1
	$-b_7$	0	0	0	0	0	0 /

Corollary 4. The production matrix of the element $(1 - xf(x), x)^{-1}$ of the Appell subgroup, where $f(x) = 1 + a_1x + a_2x + \cdots$, begins

(a_0	1	0	0	0	0	0 \
	a_1	0	1	0	0	0	0
	a_2	0	0	1	0	0	0
	a_3	0	0	0	1	0	0
	a_4	0	0	0	0	1	0
	a_5	0	0	0	0	0	1
ſ	a_6	0	0	0	0	0	0 /

For the next lemma, we let W be the infinite matrix with 1 in its top left position, and 0 elsewhere. The identity matrix will be denoted by I.

Lemma 5. We have the following identity of matrices.

$$V \cdot U - U \cdot V = I - W.$$

We can now prove the proposition of the previous section. We are required to prove that

$$P_M - M = P_{A^{-1}} - W,$$

where A = (1 - xf(x), x). We shall let A_0 denote the matrix whose first column is that of A, and all of whose other entries are 0. Thus we have

$$P_{A^{-1}} = A_0 + U.$$

The desired equality

$$P_M - M = P_{A^{-1}} - W$$

can now be written

$$(A+V)U(A+V)^{-1} - (A+V)^{-1} = A_0 + U - W.$$

Multiply both sides by A+V on the right, to obtain an equivalent statement that must be proved. We get

$$(A + V)U - I = A_0(A + V) + U(A + V) - W(A + V),$$

which is equivalent to

$$(A + V)U - I = A_0 + U(A + V) - W$$

Equivalently, we require that

$$AU + VU - I = A_0 + UA + UV - W.$$

But this is so since VU - UV = I - W. The result is thus proven. Example 6. We let $f(x) = \frac{1}{1-x}$, so that the sequence a_n begins

 $1, 1, 1, 1, 1, \dots$

We find that

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 11 & 7 & 3 & 1 & 0 & 0 & 0 \\ 261 & 165 & 69 & 21 & 5 & 1 & 0 \\ 1631 & 1031 & 431 & 131 & 31 & 6 & 1 \end{pmatrix},$$

$$P_M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 12 & 7 & 3 & 1 & 1 & 0 & 0 \\ 12 & 7 & 3 & 1 & 1 & 0 & 0 \\ 12 & 7 & 3 & 1 & 1 & 0 & 0 \\ 50 & 31 & 13 & 4 & 1 & 1 & 0 \\ 262 & 165 & 69 & 21 & 5 & 1 & 1 \\ 1632 & 1031 & 431 & 131 & 31 & 6 & 1 \end{pmatrix}.$$

and that

$$P_M - M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The self-building nature of these triangles is illustrated below: For any element of a noninitial column, the elements above it in the same column, prepended by 1, are used as multipliers on the elements in the row above it, starting in the column to its left.

(1	0	0	0	0	0	0	
	1	1	0	0	0	0	0	
	13	$^{2}2$	⁴ 1	0	0	0	0	
	11	7	3	1	0	0	0	
	¹ 49	$^{1}31$	$^{2}13$	$^{7}4$	$^{31}1$	0	0	
	261	165	69	¹ 21	¹ 5	⁵ 1	0	
	1631	1031	431	131	31	6	1	

For the initial column, we use the elements in the first column of P_M in a similar fashion, only this time starting in the row immediately above. (Note that for consistency, we could have prepended a 1 to this sequence, and used it on an all-zero column to the left of the initial column of the matrix).

The construction is thus as follows. We start with a sequence

$$a_0, a_1, a_2, \ldots$$

We set $T_{0,0} = a_0$, and $T_{n,k} = 0$ if k > n. Then we have

$$T_{n,0} = (a_0 + m_0 - 1)T_{n-1,0} + \sum_{j=1}^{n-1} (a_j + m_j)T_{n-1,j},$$

and for k > 0,

$$T_{n,k} = T_{n-1,k-1} + \sum_{j=0}^{n-k-1} T_{k+j,k} T_{n-1,k+j}$$

Here, the sequence m_0, m_1, \ldots is the first column of M, and the sequence $a_n + m_n - 0^n$ is the first column of the production matrix P_M .

Note that the sequence $1, 3, 11, 49, \ldots$ <u>A001339</u> is the binomial transform of (n + 1)!, with exponential generating function $\frac{e^x}{(1-x)^2}$. Its ordinary generating function is given by the Jacobi continued fraction $\mathcal{J}(3, 5, 7, 9, \ldots; 2, 6, 12, 20, \ldots)$.

Example 7. We set $f(x) = \frac{1}{1-2x}$, so that the sequence a_0, a_1, a_2, \ldots is the sequence $a_n = 2^n$. We find that

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 & 0 & 0 \\ 18 & 8 & 3 & 1 & 0 & 0 & 0 \\ 92 & 40 & 14 & 4 & 1 & 0 & 0 \\ 536 & 232 & 80 & 22 & 5 & 1 & 0 \\ 3552 & 1536 & 528 & 144 & 32 & 6 & 1 \end{pmatrix},$$

and

$$P_M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 8 & 2 & 1 & 1 & 0 & 0 & 0 \\ 26 & 8 & 3 & 1 & 1 & 0 & 0 \\ 108 & 40 & 14 & 4 & 1 & 1 & 0 \\ 568 & 232 & 80 & 22 & 5 & 1 & 1 \\ 3616 & 1536 & 528 & 144 & 32 & 6 & 1 \end{pmatrix}.$$

Thus we have

$$P_M - P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 1 & 0 & 0 \\ 16 & 0 & 0 & 0 & 0 & 1 & 0 \\ 32 & 0 & 0 & 0 & 0 & 0 & 1 \\ 64 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The sequence 1, 4, 18, 92, 536, 3552, ... <u>A081923</u> has exponential generating function $\frac{e^{2x}}{(1-x)^2}$, and is the binomial generating function of the sequence in the previous example. Its ordinary generating function is thus given by $\mathcal{J}(4, 6, 8, 10, \ldots; 2, 6, 12, 20, \ldots)$.

Following from these two examples, we have the general result that if $f(x) = \frac{1}{1-rx}$, and hence $a_n = r^n$, then the first column of M is given by

$$m_n = \sum_{k=0}^n \binom{n-1}{k} (n-k)! r^k = \sum_{k=0}^n \binom{n-1}{n-k} r^{n-k} k!,$$

with coefficient array that begins

(1	0	0	0	0	0	0 \	
	1	0	0	0	0	0	0	
	2	1	0	0	0	0	0	
	6	4	1	0	0	0	0	
	24	18	6	1	0	0	0	
	120	96	36	8	1	0	0	
ĺ	720	600	240	60	10	1	0 /	

This is essentially <u>A132159</u>. The sequence m_{n+1} then has exponential generating sequence $\frac{e^{rx}}{(1-x)^2}$ and ordinary generating function $\mathcal{J}(r+2, r+4, r+6, 10, \ldots; 2, 6, 12, 20, \ldots)$. It is thus the *r*-th binomial transform of (n+1)!, with $m_{n+1} = \sum_{k=0}^{n} {n \choose k} r^{n-k} (k+1)!$

4 A family of Hanna triangles

Paul D. Hanna has defined a one-parameter family of triangles that are almost their own production matrix, one of which we have already seen. For a parameter $r \in \mathbb{N}$, these triangles

H(r) can be defined as follows: if n < 0 or k < 0, then $H_{n,k} = 0$; if n = k then $H_{n,k} = 1$; if n = k + 1 then $H_{n,k} = n$; otherwise we have

$$H_{n,k} = kH_{n,k+1} + \sum_{j=0}^{n-k-1} H_{j+r,r}H_{n,j+k+1}$$

For r = 1, 2, 3, 4, we get the following triangles.

	/ 1	0	0	0	0	0	0		(1	0	0	0	0	0	0 \	
	1	1	0	0	0	0	0			1	1	0	0	0	0	0	
	3	2	1	0	0	0	0			4	2	1	0	0	0	0	
	13	7	3	1	0	0	0	,		22	8	3	1	0	0	0	,
	71	33	13	4	1	0	0			148	44	14	4	1	0	0	
	461	191	71	21	5	1	0			1156	296	84	22	5	1	0	
	3447	1297	461	133	31	6	1 /		ĺ	10192	2312	600	148	32	6	1/	
,							、		,								
(1	0	0	0	0	0	0 \		(1	0	0	0	0	0	0)	
(1 1	$\begin{array}{c} 0 \\ 1 \end{array}$	0 0	$\begin{array}{c} 0 \\ 0 \end{array}$	0 0	0 0	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$		(1 1	$\begin{array}{c} 0 \\ 1 \end{array}$	$\begin{array}{c} 0 \\ 0 \end{array}$	$\begin{array}{c} 0 \\ 0 \end{array}$	$\begin{array}{c} 0 \\ 0 \end{array}$	$\begin{array}{c} 0 \\ 0 \end{array}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	
$\left(\right)$	$egin{array}{c} 1 \\ 1 \\ 5 \end{array}$	$\begin{array}{c} 0 \\ 1 \\ 2 \end{array}$	-	-			Ŭ.		$\left(\right)$	1 1 6			-	•	•	Ŭ,	
	$ \begin{array}{c} 1 \\ 1 \\ 5 \\ 33 \end{array} $	1	-	0	0	0	0	,	$\left(\right)$	$\begin{array}{c}1\\1\\6\\46\end{array}$	1		0	0	0	0	
		1 2	0 1	0 0	0 0	0 0	0 0	,			$\frac{1}{2}$	0 1	0 0	0 0	0 0	0 0	•
	33	1 2 9	0 1 3	0 0 1	0 0	0 0 0	0 0 0	,		46	1 2 10	$\begin{array}{c} 0 \\ 1 \\ 3 \end{array}$	0 0 1	0 0	0 0 0	0 0 0	-

These are <u>A104980</u>, <u>A111536</u>, <u>111544</u>, <u>A111553</u> respectively. The corresponding first column elements are <u>A003319</u>, <u>A111529</u>, <u>A111530</u> and <u>A111531</u>, respectively.

The defining sequences a_0, a_1, a_2, \ldots of these triangles can be characterised as follows. For a given $r \in \mathbb{N}$, the sequence a_0, a_1, a_2, \ldots is the sequence with generating function given by the Stieltjes continued fraction

$$S(r, r+1, r+2, \ldots; 2, 3, 4, \ldots).$$

Thus when r = 0, we have f(x) = 1, and we obtain the triangle <u>A094587</u>

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 6 & 6 & 3 & 1 & 0 & 0 & 0 \\ 120 & 120 & 60 & 20 & 5 & 1 & 0 \\ 720 & 720 & 360 & 120 & 30 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 1 \end{pmatrix}^{-1}$$

with production matrix

1	1	1	0	0	0	0	0 \	
	1	1	1	0	0	0	0	
	2	2	1	1	0	0	0	
	6	6	3	1	1	0	0	
	24	24	12	4	1	1	0	
	120	120	60	20	5	1	1	
ſ	720	720	360	120	30	6	1 /	

This is the exponential Riordan array $\left[\frac{1}{1-x}, x\right]$. In this special case, we have

$$P_M - M = U.$$

Returning to the general case, the sequences m_n (the initial columns of the triangles H(r) in question) have their generating function given by

$$S(1,2,3,\ldots;r+1,r+2,r+2,\ldots).$$

These sequences are solutions to the convolution recurrence

$$m_n = (n-r)m_{n-1} + r\sum_{i=0}^{n-1} m_i m_{n-i-1},$$

with $m_0 = 1, m_1 = 1$. For general r, this sequence begins

$$1, 1, r+2, r^{2}+6r+6, r^{3}+12r^{2}+34r+24, r^{4}+20r^{3}+110r^{2}+210r+120, \dots,$$

with coefficient array that begins

(1	0	0	0	0	0	0	0 \	
	1	0	0	0	0	0	0	0	
	2	1	0	0	0	0	0	0	
	6	6	1	0	0	0	0	0	
	24	34	12	1	0	0	0	0	
	120	210	110	20	1	0	0	0	
	720	1452	974	270	30	1	0	0	
	5040	11256	8946	3248	560	42	1	0 /	

In the Deléham notation, this is a variant of the triangle

$$[0, 2, 1, 3, 2, 4, 3, 5, \ldots] \Delta [1, 0, 1, 0, 1, 0, 1, 0, \ldots]$$

which is <u>A111184</u>. For r = 1, we illustrate below the "almost its own production matrix" property of H(1).

1								
1	1	0	0	0	0	0	0	
	1	1	0	0	0	0	0	
	$^{1}3$	$^{2}2$	⁶ 1	0	0	0	0	
	13	7	3	1	0	0	0	
	¹ 71	¹ 33	$^{2}13$	$^{7}4$	³³ 1	0	0	
	461	191	71	¹ 21	$^{1}5$	⁵ 1	0	
	3447	1297	461	133	31	6	1	

The sequences a_n and m_n of this section actually belong to a single family of sequences that are specified in general by three parameters (two of these parameters are equal to 1 in our case). These have been studied by Martin and Kearney [5, 6]. If we change the indexing of these sequences from 0 to 1, then a_n corresponds to

$$S(1, r-3, 1)$$

using the notation of [5], while m_n corresponds to

$$S(1, -(r+1), 1).$$

When r = 1, we obtain $a_n = m_n$, the sequence of indecomposable permutations.

The sequence $S(\alpha, \beta, \gamma) = (u_n)_{n=1}^{\infty}$ satisfies the recurrence

$$u_n = (\alpha n + \beta)u_{n-1} + \gamma \sum_{j=1}^{n-1} u_j u_{n-j},$$

with $u_1 = 1$.

We posit that these moment sequences have ordinary generating function

$$\mathcal{S}(2\alpha + \beta + \gamma, 3\alpha + \beta + \gamma, 4\alpha + \beta + \gamma, \dots; \alpha + \gamma, 2\alpha + \gamma, 3\alpha + \gamma, \dots),$$

or equivalently,

 $\mathcal{J}(2\alpha+\beta+\gamma,4\alpha+\beta+2\gamma,6\alpha+\beta+2\gamma,\ldots;(\alpha+\gamma)(2\alpha+\beta+\gamma),(2\alpha+\beta)(3\alpha+\beta+\gamma),(3\alpha+\beta)(4\alpha+\beta+\gamma),\ldots).$

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(Concerned with sequences

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