ONE-DEPENDENT HARD-CORE PROCESSES AND COLORINGS OF THE STAR GRAPH

THOMAS M. LIGGETT AND WENPIN TANG

Abstract. In this paper we compute the critical point of the one-dependent hard-core process on the d-ray star graph for each $d \geq 2$. This allows us to improve the upper bound for the critical point of infinite connected graphs of maximum degree d for $d \geq 3$. We also discuss the problem of one-dependent colorings of the d-ray star graph.

Key words : Hard-core processes, random colorings, one-dependent processes.
AMS 2010 Mathematics Subject Classification: 60C05, 05C15, 82B27.

Contents

1. Introduction and main results

Let $G = (V, E)$ be a simple graph. A *hard-core process* on G is a process $(J_v; v \in V) \in \{0,1\}^V$ such that almost surely we do not have $J_u = J_v = 1$ whenever u and v are neighbors. It is *k-dependent* if its restriction to two subset of V are independent whenever these subsets are at graph distance larger than k from each other. Define the *critical point*

$$p_h(G) := \sup\{p;\ \exists \text{ 1-dependent hard-core process } J \text{ with } \mathbb{P}(J_v = 1) = p\ \forall v\}. \quad (1.1)$$

Motivated by the study of the repulsive lattice gas, Scott and Sokal [9, 10] proved that for any infinite connected graph G of maximum degree d,

$$\frac{(d-1)^{d-1}}{d^d} \leq p_h(G) \leq \frac{1}{4} \quad \text{for all } d \geq 2. \quad (1.2)$$

It was proved in [8, Lemma 23] that for each $p \leq p_h(G)$ there is a unique 1-dependent hard-core process with marginals $\mathbb{P}(J_v = 1) = p$ for all v. The lower bound in (1.2) is known to be achieved by the infinite d-regular tree [11], which gives a version of the *Lovász local lemma*. An interesting question is whether the upper bound in (1.2) for $d \geq 3$ can be achieved by some subgraph of the d-regular tree with at least one vertex of degree d.

Date: April 20, 2018.

In this paper, we consider for any $d \geq 3$ the *d-ray star graph* $\mathscr{S}^d$ with a distinguished vertex of degree d and all the others of degree 2. The graph consists of d rays $\mathbf{v}^1 = (v_{11}, v_{12}, \ldots)$, $\mathbf{v}^2 = (v_{21}, v_{22}, \ldots), \ldots$, and $\mathbf{v}^d = (v_{d1}, v_{d2}, \ldots)$, emanating from the distinguished vertex v_0. See Figure 1 below for a 3-ray star graph.

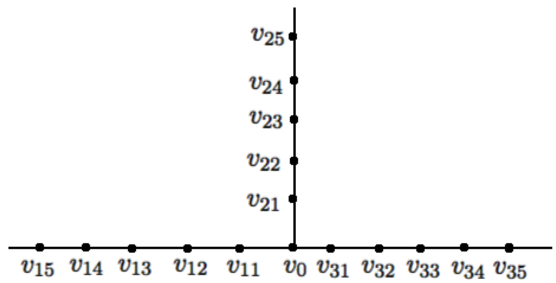

FIGURE 1. The 3-ray star graph

We also identify the 2-ray star graph $\mathscr{S}^2$ with the integers. The main result below gives an explicit formula for the critical point of $\mathscr{S}^d$ for each $d \geq 2$.

Theorem 1.1. *For $d \geq 2$, the critical point $p_h(\mathscr{S}^d)$ is the unique solution on $(0, 1/4]$ to the equation:*

$$\left(\frac{1+\sqrt{1-4p}}{2}\right)^d = p. \tag{1.3}$$

In particular, $p_h(\mathscr{S}^2) = 1/4$, $p_h(\mathscr{S}^3) = \sqrt{5} - 2 \approx 0.236$ and

$$p_h(\mathscr{S}^4) = \frac{1}{3}\left(2 - 11\left(\frac{2}{9\sqrt{93}-47}\right)^{\frac{1}{3}} + \left(\frac{9\sqrt{93}-47}{2}\right)^{\frac{1}{3}}\right) \approx 0.217.$$

We prove Theorem 1.1 in Section 2. It is easy to see that $p_h(\mathscr{S}^d) \sim \ln d/d$ as $d \to \infty$. Table 1 below displays the numerical values of $p_h(\mathscr{S}^d)$ as d ranges from 2 to 12.

d	2	3	4	5	6	7	8	9	10	11	12
$p_h(\mathscr{S}^d)$	0.250	0.236	0.217	0.199	0.185	0.173	0.162	0.153	0.149	0.138	0.131

TABLE 1. Numerical values of $p_h(\mathscr{S}^d)$, $2 \leq d \leq 12$.

Let $\mathbb{N} := \{1, 2, \ldots\}$ be the set of positive integers. For $\mathbf{n} := (n_1, \ldots, n_d) \in \mathbb{N}^d$, let $\mathscr{S}^d_{\mathbf{n}}$ be the finite subgraph of the d-ray star graph, with d rays of lengths $n_1, \ldots, n_d$ respectively. It is easy to deduce from the proof of Theorem 1.1 that

$$p_h(\mathscr{S}^d_{\mathbf{n}}) = \sup\left\{p;\, p \leq \prod_{i=1}^{d} \inf_{k \leq n_i} a_k(p)\right\}, \tag{1.4}$$

where $(a_k(p);\, k \geq 1)$ is defined recursively by

$$a_1(p) = 1 - p, \quad \text{and} \quad a_k(p) + \frac{p}{a_{k-1}(p)} = 1 \text{ for } k \geq 2. \tag{1.5}$$

Note that any infinite connected graph of maximum degree d contains a copy of $\mathscr{S}^d_{(1,\dots,1,n)}$ for each n. This leads to the following corollary.

Corollary 1.2. *For $d \geq 2$, let G be an infinite connected graph of maximum degree d. Then*

$$\frac{(d-1)^{d-1}}{d^d} \leq p_h(G) \leq p_\star(d), \tag{1.6}$$

where $p_\star(2) = p_\star(3) = 1/4$ and for $d \geq 4$, $p_\star(d)$ is the unique solution on $(0, 1/4]$ to the equation:

$$\frac{1}{2}(1-p)^{d-1}(1+\sqrt{1-4p}) = p. \tag{1.7}$$

Note that the bounds in (1.6) are tight, and the upper bound is improved for all $d \geq 4$. It is also easy to see that

$$\frac{(d-1)^{d-1}}{d^d} \sim \frac{1}{ed} \quad \text{and} \quad p_\star(d) \sim \frac{\ln d}{d} \quad \text{as } d \to \infty. \tag{1.8}$$

Table 2 below displays the numerical values of $p_\star(d)$ as d ranges from 2 to 12.

d	2	3	4	5	6	7	8	9	10	11	12
$p_\star(d)$	0.250	0.250	0.245	0.229	0.212	0.197	0.183	0.172	0.162	0.153	0.146

TABLE 2. Numerical values of $p_\star(d)$, $2 \leq d \leq 12$.

As indicated in [8], there is a close connection between hard-core processes and proper colorings. A stochastic process $X = (X_v; v \in V)$ indexed by vertices is called a *proper q-coloring* if each X_v takes values in $[q] := \{1, \dots, q\}$ and almost surely $X_u \neq X_v$ for adjacent vertices u and v. So if X is a q-coloring of G, then $J_v := 1_{\{X_v = x\}}$ defines a hard-core process for any given color $x \in [q]$; if X is 1-dependent, then so is J. Holroyd and Liggett [8] found a stationary 1-dependent 4-coloring of the integers which is invariant under permutations of the colors. Further in [7], they provided an algebraic construction for each $q \geq 4$ of a stationary 1-dependent q-coloring which is invariant under permutations of the colors and under reflection. See also [1, 2] for literature on one-dependent processes, and [3, 4, 5, 6] for recent progress of finitely dependent colorings.

An interesting question is whether a symmetric 1-dependent q-coloring exists for general graphs. It is an open problem to find a fully automorphism invariant 1-dependent coloring of the d-regular tree for each $d \geq 3$. The lower bound in (1.2) or (1.6) implies that any symmetric 1-dependent coloring of the infinite d-regular tree requires at least $d^d/(d-1)^{d-1}$ colors, e.g. 7 colors for the 3-regular tree, and 10 colors for the 4-regular tree. The following corollary gives a lower bound for the number of colors needed for a symmetric 1-dependent coloring on the d-ray star graph.

Corollary 1.3. *Suppose that there exists a 1-dependent q-coloring X of the d-ray star graph $\mathscr{S}^d$ with $(X_v; v \in V)$ identically distributed. Then*

$$q \geq \frac{1}{p_h(\mathscr{S}^d)}, \tag{1.9}$$

where $p_h(\mathscr{S}^d)$ is given in Theorem 1.1. Consequently, there is no 1-dependent 4 coloring of the 3-ray star graph $\mathscr{S}^3$ which satisfies:

(i). the coloring is invariant in distribution under permutations of the colors,

(ii). the restriction to any copy of the integers (i.e. $\mathbf{v}^i v_0 \mathbf{v}^j$ for each $i \neq j \in \{1, 2, 3\}$) is a 1-dependent 4-coloring which is invariant in law under permutations of the colors, under translation and reflection.

In Section 3 we give a direct proof that there is no 1-dependent 4-coloring of the 3-ray star graph, which satisfies the conditions (i)-(ii) above. It still remains unknown whether one can find a 1-dependent q-coloring satisfying (i)-(ii) for any $q \geq 5$ of the 3-ray star graph, and a fully automorphism invariant 1-dependent q-coloring for any $q \geq 7$ of the 3-regular tree. We leave these questions as open.

2. The critical point of the d-ray star graph

In this section we aim to prove Theorem 1.1. For $\mathbf{x} = (x_v; v \in \mathscr{S}^d_{\mathbf{n}}) \in \{0,1\}^{\mathscr{S}^d_{\mathbf{n}}}$ with d rays $\mathbf{x}_1 := (x_{11}, \ldots, x_{1n_1}), \ldots, \mathbf{x}_d := (x_{d1}, \ldots, x_{dn_d})$ emanating from x_0, write

$$P(\mathbf{x}) := \mathbb{P}(J_v = x_v \text{ for all } v \in \mathscr{S}^d_{\mathbf{n}}).$$

If there exists a 1-dependent hard-core process J on $\mathscr{S}^d$ with marginals $\mathbb{P}(J_v = 1) = p$ for all v, then the collection $(P(\mathbf{x}); \mathbf{x} \in \{0,1\}^{\mathscr{S}^d_{\mathbf{n}}}, \mathbf{n} \in \mathbb{N}^d)$ is a non-negative solution to the 1-dependence equations at all internal vertices, and the consistency equations at all d leaves. The following proposition shows that these cylinder probabilities are uniquely determined by the 1-dependence and consistency conditions.

Proposition 2.1. *If there exists a 1-dependent hard-core process J on $\mathscr{S}^d$ with marginals $\mathbb{P}(J_v) = p$ for all v, then for each $\mathbf{x} \in \mathscr{S}^d_{\mathbf{n}}$,*

$$P(\mathbf{x}) = \begin{cases} \prod_{i=1}^d P(\mathbf{x}_i) & \text{if } \quad x_{i1} = 1 \text{ for some } i \\ p \prod_{i=1}^d P(\widehat{\mathbf{x}}_{i1}) & \text{if } \quad x_0 = 1 \text{ and } x_{i1} = 0 \text{ for all } i \\ \prod_{i=1}^d P(\mathbf{x}_i) - p \prod_{i=1}^d P(\widehat{\mathbf{x}}_{i1}) & \text{if } \quad x_0 = 0 \text{ and } x_{i1} = 0 \text{ for all } i, \end{cases} \tag{2.1}$$

where $\widehat{\mathbf{x}}_{i1} := (x_{i2}, \ldots, x_{in_i})$ is obtained by deleting the first entry of $\mathbf{x}_i$, and we adopt the convention $P(\emptyset) := 1$. Conversely if

$$\prod_{i=1}^d P(\mathbf{x}_i) - p \prod_{i=1}^d P(\widehat{\mathbf{x}}_{i1}) \geq 0 \quad \text{for all } \mathbf{x}_i \in \{0\} \times \{0,1\}^{n_i} \text{ with } n_i \geq 0, \tag{2.2}$$

then the collection $(P(\mathbf{x}); \mathbf{x} \in \{0,1\}^{\mathscr{S}^d_{\mathbf{n}}}, \mathbf{n} \in \mathbb{N}^d)$ given by (2.1) defines the 1-dependent hard-core process J on $\mathscr{S}^d$ with marginals $\mathbb{P}(J_v) = p$ for all v.

Proof. Suppose that there is a 1-dependent hard-core process J on $\mathscr{S}^d$ with marginals $\mathbb{P}(J_v) = p$ for all v. For $\mathbf{x} \in \mathscr{S}^d_{\mathbf{n}}$, there are three cases.

Case 1: $x_0 = 0$ and one of the neighbors to x_0 is 1. By the 1-dependence condition at x_0, we get the desired result.

Case 2: $x_0 = 1$ and all the neighbors to x_0 are 0. By the 1-dependence condition at x_{11}, we have

$$\begin{aligned} P(\mathbf{x}) &= P(\widehat{\mathbf{x}}_{11})P(\widehat{\mathbf{x}}^1) \\ &= P(\widehat{\mathbf{x}}_{11}) \cdot p \prod_{i=2}^{d} P(\widehat{\mathbf{x}}_{i1}), \end{aligned}$$

where $\widehat{\mathbf{x}}^1$ is obtained by deleting the ray $\mathbf{x}_1$ from $\mathbf{x}$, and the second equality follows by applying the 1-dependence condition successively at $x_{21}, \ldots, x_{d1}$.

Case 3: $x_0 = 0$ and all the neighbors to x_0 are 0. By the 1-dependence condition at x_0 and the result in Case 2, we get the desired result.

For the converse part, it is easy to check that the collection $(P(\mathbf{x});\, \mathbf{x} \in \{0,1\}^{\mathscr{S}^d_{\mathbf{n}}}, \mathbf{n} \in \mathbb{N}^d)$ given by (2.1) satisfies the 1-dependence and consistency conditions. We leave the full detail to readers. The non-negativity condition is guaranteed by (2.2). We conclude by applying Kolmogorov's extension theorem. □

As a consequence of Proposition 2.1, we have

$$\begin{aligned} p_h(\mathscr{S}^d) &= \sup\left\{p;\, p \le \frac{\prod_{i=1}^d P(\mathbf{x}_i)}{\prod_{i=1}^d P(\widehat{\mathbf{x}}_{i1})} \text{ for all } \mathbf{x}_i \in \{0\} \times \{0,1\}^{n_i} \text{ with } n_i \ge 0\right\} \\ &= \sup\left\{p;\, p \le \left(\frac{P(\mathbf{x})}{P(\widehat{\mathbf{x}}_1)}\right)^d \text{ for all } \mathbf{x} \in \{0\} \times \{0,1\}^{n} \text{ with } n \ge 0\right\}. \end{aligned} \tag{2.3}$$

For $\mathbf{x} \in \{0\} \times \{0,1\}^n \setminus \{0\}^{n+1}$, let

$$k(\mathbf{x}) := \inf\{k;\, x_k = 1\} - 1$$

be the number of 0's before the first 1 in $\mathbf{x}$. Write $\mathbf{x} = 0_{k(\mathbf{x})}\mathbf{R}$ with $\mathbf{R} \in \{1\} \times \{0,1\}^{n-k(\mathbf{x})}$. By the 1-dependence condition at the $k(x)^{th}$ position, we have

$$P(\mathbf{x}) = P(0_{k(\mathbf{x})-1})P(\mathbf{R}),$$

with the convention $P(0_0) := 1$. So for $\mathbf{x} \in \{0\} \times \{0,1\}^n \setminus \{0\}^{n+1}$,

$$\frac{P(\mathbf{x})}{P(\widehat{\mathbf{x}}_1)} = \begin{cases} 1 & \text{if} \quad k(\mathbf{x}) = 1 \\ P(0_{k(\mathbf{x})-1})/P(0_{k(\mathbf{x})-2}) & \text{if} \quad k(\mathbf{x}) \ge 2. \end{cases} \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$p_h(\mathscr{S}^d) = \sup\left\{p;\, p \le \inf_{k \ge 1}\left(\frac{P(0_k)}{P(0_{k-1})}\right)^d\right\}. \tag{2.5}$$

Thus the value of $p_h(\mathscr{S}^d)$ is entirely determined by the sequence $(P(0_k);\, k \ge 0)$. The following lemma gives an explicit recursion of the sequence $(P(0_k);\, k \ge 0)$.

Lemma 2.2. *For each $k \ge 2$,*

$$P(0_k) = P(0_{k-1}) - pP(0_{k-2}). \tag{2.6}$$

In particular,

$$P(0_1) = 1 - p, \quad P(0_2) = 1 - 2p, \quad P(0_3) = 1 - 3p + p^2, \quad P(0_4) = 1 - 4p + 3p^2 \ldots$$

Proof. By the consistency condition of 0_k, we have

$$P(0_k) = P(0_{k-1}) - P(0_{k-1}1).$$

Further by the 1-dependence condition of $0_{k-1}1$ at the $(k-1)^{th}$ position, we get

$$P(0_{k-1}1) = P(0_{k-2})P(1),$$

which yields (2.6). □

For $k \geq 1$, let $a_k(p) := P(0_k)/P(0_{k-1})$. The recursion (2.6) implies (1.5). It follows by standard analysis that for $p \leq 1/4$, the sequence $a_k(p)$ decreases to the limit $(1 + \sqrt{1-4p})/2$. Combining this with (2.5) yields Theorem 1.1.

3. No symmetric 1-dependent 4-coloring of the 3-ray star graph

In this section we give a direct proof that there is no symmetric 1-dependent 4-coloring of the the 3-ray star graph as stated in Corollary 1.3.

Suppose by contradiction that such a coloring exists. Let $\mathbf{T}$ and $\mathbf{T}'$ be the 3-ray star graphs colored as follows:

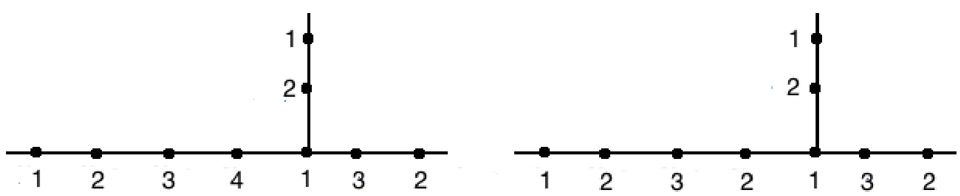

Figure 2. Two colorings $\mathbf{T}$ (left) and $\mathbf{T}'$ (right)

The 1-dependence condition of $\mathbf{T}$ at the distinguished vertex of degree 3 implies that

$$P(\mathbf{T}) = P(1234)P(12)P(32) = P(1234)(P(12))^2,$$

while the 1-dependence condition of $\mathbf{T}$ at the left neighbor to the distinguished vertex implies that

$$P(\mathbf{T}) + P(\mathbf{T}') = P(123)P(12132).$$

Consequently,

$$P(123)P(12132) \geq P(1234)(P(12))^2. \tag{3.1}$$

Recall from [7] that for a 1-dependent 4-coloring of the integers which is symmetric in the colors, translation invariant and invariant under reflection, the cylinder probabilities of length $k \leq 4$ are given by

$$\begin{aligned}
&P(1) = \frac{1}{4}, \\
&P(12) = \frac{1}{12}, \\
&P(121) = \frac{1}{48}, \quad P(123) = \frac{1}{32}, \\
&P(1212) = \frac{\alpha}{48}, \quad P(1213) = \frac{1-\alpha}{96}, \quad P(1231) = \frac{1}{96}, \quad P(1234) = \frac{1+\alpha}{96},
\end{aligned} \tag{3.2}$$

i.e. the cylinder probabilities of length $k \leq 3$ are uniquely determined, while those of length $k = 4$ are given by a one parameter family indexed by α.

Now we consider the cylinder probabilities of length $k = 5$. There are 10 equivalence classes under permutations of colors and under reflection: 12134, 12314, 12324, 12341, 12131, 12123, 12132, 12312, 12321 and 12121. Solving the 1-dependence and consistency conditions for these cylinder probabilities gives the following result.

Lemma 3.1. *For a* 1*-dependent* 4*-coloring of* $\mathbb{Z}$ *which is invariant in law under permutations of the colors, under translation and reflection, with the cylinder probabilities of length* $k \leq 4$ *given by* (3.2)*, the cylinder probabilities of length* $k = 5$ *are uniquely determined by*

$$\begin{aligned}
&P(12134) = \frac{1}{288}, \quad P(12314) = \frac{5}{1152}, \quad P(12324) = \frac{1}{288}, \quad P(12341) = \frac{1+4\alpha}{384}, \\
&P(12131) = \frac{5-12\alpha}{1152} \quad P(12123) = \frac{1}{576}, \quad P(12132) = \frac{1}{384}, \quad P(12312) = \frac{1}{288}, \\
&P(12321) = \frac{1-2\alpha}{192}, \quad P(12121) = \frac{6\alpha-1}{288}.
\end{aligned} \tag{3.3}$$

So the nonnegative condition for cylinder probabilities requires $1/6 \leq \alpha \leq 5/12$, which contradicts $\alpha \leq 1/8$ implied by (3.1).

As indicated in [7], the construction of a 1-dependent q-coloring of $\mathbb{Z}$ requires finding a nonnegative solution to an infinite set of nonlinear equations. The unknowns are the cylinder probabilities, which are assumed to be symmetric in the colors, translation invariant and invariant under reflection. For each $q \geq 4$ and $k \geq 1$, let

$$L_{q,k} := \#\text{ unknowns corresponding to the cylinder probabilities of length } k.$$

So $L_{q,k}$ is the number of the equivalence classes of q-colorings of $[k]$ under permutations of the colors and under reflection. The number of equations for these $L_{q,k}$ unknowns is then $(k-1)L_{q,k}$, among which $(k-2)L_{q,k}$ are 1-dependence conditions, and $L_{q,k}$ are consistency conditions. For $q = 4$, the sequence $(L_{4,k};\, k \geq 1)$ is given by the OEIS A001998 [12]. In particular,

$$L_{4,k} = \frac{1 + 3^{k-1} + 3^{(k-2)/2}(2+\sqrt{3} - (-1)^{k-1}(2-\sqrt{3}))}{4} \quad \text{for } k \geq 1. \tag{3.4}$$

To illustrate, $L_{4,1} = 1$, $L_{4,2} = 1$, $L_{4,3} = 2$, $L_{4,4} = 4$, $L_{4,5} = 10$, $L_{4,6} = 25$, $L_{4,7} = 70$, $L_{4,8} = 196$, $L_{4,9} = 574$, $L_{4,10} = 1681\ldots$ By further solving the equations for cylinder probabilities of length $k = 6$, we get a two parameter family of (α, β) for $L_{4,6} = 25$ unknowns. But it is not obvious there is a unique nonnegative solution to these equations unless the non-negativity conditions force $\alpha = 1/5$ as $k \to \infty$.

Acknowledgement: We thank Alexander Holroyd for helpful discussions.

References

[1] J. Aaronson, D. Gilat, M. Keane, and V. de Valk. An algebraic construction of a class of one-dependent processes. *Ann. Probab.*, 17 (1):128–143, 1989.

[2] A. Borodin, P. Diaconis, and J. Fulman. On adding a list of numbers (and other one dependent determinantal processes). *Bull. Amer. Math. Soc.*, 47 (4):639–670, 2010.

[3] Alexander Holroyd, Tom Hutchcroft, and Avi Levy. Finitely dependent cycle coloring. *arXiv:1707.09374*, 2017.

[4] Alexander Holroyd, Tom Hutchcroft, and Avi Levy. Mallows permutations and finite dependence. *arXiv:1706.09526*, 2017.

[5] Alexander Holroyd, Oded Schramm, and David Wilson. Finitary coloring. *arXiv:1412.2725*, 2014.

[6] Alexander E. Holroyd. One-dependent coloring by finitary factors. *Ann. Inst. Henri Poincar Probab. Stat.*, 53 (2):753–765, 2017.

[7] Alexander E. Holroyd and Thomas M. Liggett. Symmetric 1-dependent colorings of the integers. *Electron. Commun. Probab.*, 20:no. 31, 8, 2015.

[8] Alexander E. Holroyd and Thomas M. Liggett. Finitely dependent coloring. *Forum Math. Pi*, 4:e9, 43, 2016.

[9] Alexander Scott and Alan Sokal. On dependency graphs and the lattice gas. *J. Stat. Phys*, 118 (5-6):1151–1261, 2005.

[10] Alexander Scott and Alan Sokal. On dependency graphs and the lattice gas. *Combin. Probab. Comput.*, 15 (1-2):253 – 279, 2010.

[11] James B. Shearer. On a problem of Spencer. *Combinatorica*, 5(3):241–245, 1985.

[12] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences, A001998. Available at https://www.oeis.org/A001998.

Department of Mathematics, UCLA. Email:
E-mail address: `tml@math.ucla.edu`

Department of Mathematics, UCLA. Email:
E-mail address: `wenpintang@math.ucla.edu`