Boundary algebra of a GL_m-dimer

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Abstract

We consider GL_m-dimers of triangulations of regular convex n-gons, which give rise to a dimer model with boundary Q and a dimer algebra Λ_Q. Let e_b be the sum of the idempotents of all the boundary vertices, and $\mathcal{B}_Q := e_b \Lambda_Q e_b$ the associated boundary algebra. In this article we show that given two different triangulations T_1 and T_2 of the n-gon, the boundary algebras are isomorphic, i.e. $e_b \Lambda_{Q_{T_1}} e_b \cong e_b \Lambda_{Q_{T_2}} e_b$.

Contents

1 Introduction

Dimer models with boundary were introduced by Baur, King and Marsh in [1]. In case without boundary the definition is similar to dimer models defined by Bocklandt [2]. Dimer models with boundary are quivers with faces satisfying certain axioms. To any dimer model Q one can associate its dimer algebra A_Q as the path algebra of Q modulo the relations arising from an associated potential.

A source for dimer models are Postnikov diagrams of type (k, n) in the disk, introduced by Postnikov in [7] and used in [1] as a combinatorial approach to Grassmannian cluster categories. In

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general, dimer algebras arising from different (k, n) diagrams are not isomorphic. However, if you consider their boundary algebra, which is the idempotent subalgebra $B_Q := eA_Qe$, where $e := e_1 + \ldots + e_t$ is the sum of all idempotents corresponding to the boundary vertices, then one of the main results of [1] is that for any two (k, n)-diagrams the associated boundary algebras are isomorphic.

In this article, we study another source for dimer models, the so-called GL_m-dimers. They arise from Goncharov's A^*_{m-1}-webs on disks defined in [3]. In the case when S is a disk with n special points on the boundary, A^*_m-webs describe a cluster coordinate systems on the moduli space $\mathrm{Conf}_n(\mathcal{A}^*_{SL_{m+1}})$ as shown in [3]. This moduli space is defined as n-tuples of the moduli space GL_{m+1}/U of all decorated flags in an $m+1$-dimensional vector space V_{m+1}, where U is the upper triangular unipotent subgroup in GL_{m+1}, modulo the diagonal action of the group $SL_m = \mathrm{Aut}(V_m, \Omega_m)$, where Ω_m is a volume form in V_m.

The purpose of this article is to show that on the disk the boundary algebra of any two different GL_m-dimers are all isomorphic.
The paper is structured as follows. After giving the necessary background in Section 2, we will first prove this result for the boundary algebra of a GL_2-dimer in Section 3 and address the general case in Section 4.
For GL_2-dimers, the main result is the following: the quiver of the boundary algebra of any GL_2-dimer on an n-gon is given by $\Gamma(n)$, the following quiver shown in Figure 1.

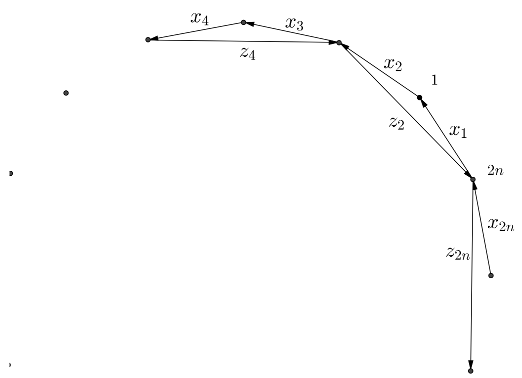

Figure 1: Part of the quiver $\Gamma(n)$.

Any two 3-cycles incident with a common vertex are equivalent. Furthermore, any compo-

sition $z_{2k}z_{2k-2}$ is equivalent to the corresponding composition of $2n-4$ arrows $x_{2k+1}\cdots x_{2k-2}$, reducing modulo $2n$ and considering the composition of paths from left to right.
The strategy to prove this result is to first prove it for boundary algebras arising from fan triangulations and then to show flip invariance.
Throughout this paper, when we consider indices modulo k, we always assume them to be between 1 and k. In particular, 0 is never used as an index.

2 Settings and GL_m-dimer

2.1 Background

Definition 2.1 (quiver with faces)**.** A quiver with faces is a quiver $Q = (Q_0, Q_1)$ together with a set Q_2 of faces and a map

$$\partial : Q_2 \to Q_{cyc},$$

which assigns to each $F \in Q_2$ its boundary $\partial F \in Q_{cyc}$, where Q_{cyc} is the set of oriented cycles in Q (up to cyclic equivalence).

We will always denote a quiver with faces by Q, regarded now as the tuple (Q_0, Q_1, Q_2, s, t). A quiver with faces is called finite if Q_0,Q_1 and Q_2 are finite sets. The (unoriented) *incidence graph* of Q, at a vertex $i \in Q_0$, has vertices given by the arrows incident with i. The edges between two arrows α,β correspond to the paths of the form

$$\xrightarrow{\ \alpha\ } \overset{i}{\bullet} \xrightarrow{\ \beta\ }$$

occurring in a cycle bounding a face.

Definition 2.2 (dimer model with boundary [1])**.** A (finite, oriented) dimer model with boundary is given by a finite quiver with faces $Q = (Q_0, Q_1, Q_2)$ where Q_2 is written as disjoint union $Q_2 = Q_2^+ \cup Q_2^-$, satisfying the following properties:

(a) the quiver Q has no loops, i.e. no 1-cycles, but 2-cycles are allowed,

(b) all arrows in Q_1 have face multiplicity 1 (boundary arrows) or 2 (internal arrows),

(c) each internal arrow lies in a cycle bounding a face in Q_2^+ and in a cycle bounding a face in Q_2^-,

(d) the incidence graph of Q at each vertex is connected.

Note that, by (b), each incidence graph in (d) must be either a line (at a boundary vertex) or an unoriented cycle (at an internal vertex).

2.2 Dimer algebra and boundary algebra

Definition 2.3 (natural potential W)**.** Let $Q = (Q_0, Q_1, Q_2)$ be a dimer model with boundary. Then the following formula

$$W := W_Q := \sum_{\gamma \in Q_2^+} \partial\gamma - \sum_{\gamma \in Q_2^-} \partial\gamma$$

defines the natural potential associated to Q.

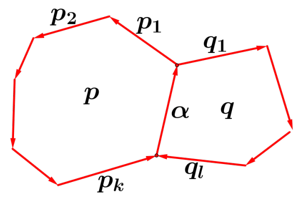

Figure 2: α is part of a positive cycle p and a negative cycle q.

Remark (differentiation of W)**.** Let ∂W be the cyclic derivative with respect to all internal arrows α in Q. That means, if α is both part of the negative (clockwise) cycle $q = \alpha q_1 \ldots q_l$ and the positive (counterclockwise) cycle $p = \alpha p_1 \ldots p_k$ with $k, l \geq 1$ as shown in Figure 2, then the equation

$$\frac{\partial W}{\partial \alpha} : p_1 p_2 \ldots p_k = q_1 q_2 \ldots q_l$$

holds. In this article we use the notation $p_1 p_2 \ldots p_k \overset{\alpha}{\cong} q_1 q_2 \ldots q_l$ for relations obtained by the natural potential W.

Definition 2.4 (dimer algebra)**.** Let $Q = (Q_0, Q_1, Q_2)$ be a dimer model with boundary and let W and ∂W be defined as above. Then the dimer algebra Λ_Q is defined as

$$\Lambda_Q := \mathbb{C}Q/_{\langle \partial W \rangle}.$$

As usual, we write e to denote an idempotent of an algebra and in the path algebra $\mathbb{C}Q$, let e_i be the trivial path of length zero at vertex i. It is an idempotent of $\mathbb{C}Q$. Define

$$e_b := e_1 + \ldots + e_t$$

where $1, \ldots, t$ are the boundary vertices of the quiver; i.e. the vertices that are incident with boundary arrows. Furthermore, we call the remaining vertices of the quiver *internal* (or *inner*) *vertices* and all arrows, that are incident with at least one internal vertex are called *internal* (or *inner*) *arrows.*

Definition 2.5 (boundary algebra)**.** The boundary algebra of a dimer model Q with boundary is the spherical subalgebra consisting of linear combinations of paths which have starting and terminating points on the boundary of the quiver (i.e. one of the idempotent elements $e_1, \ldots e_t$):

$$\mathcal{B}_Q := e_b \Lambda_Q e_b.$$

2.3 GL_m-dimer

Definition 2.6 (triangulation)**.** A triangulation of a regular convex polygon is a subdivision of the n-gon by diagonals into triangles, where each pair of diagonals intersects at most in one of the vertices of the polygon.

Remark. A triangulation of an n-gon consists of its the n edges and $n-3$ diagonals of the polygon.

Remark. A special case of triangulation is the so called *fan triangulation*, where each diagonal of the triangulation contains a given fixed vertex of the polygon.

We recall Goncharov's definition of bipartite graphs $\Gamma_{A^*_{m-1}}(T)$ of an m-triangulation of a decorated surface S as in [3]. We use these graphs[1] to define a family of dimer models with boundary.

Definition and construction 2.7 (GL_m-dimer)**.** Take an arbitrary triangulation of the polygon.Every triangle is subdivided with $(m-1)$-lines in equidistance parallel to each of its sides, as in Figure 3 for $m=4$. Each triangle of the triangulation is now subdivided into small triangles of two kinds, namely *upwards* and *downwards* triangles w. r. t. an arbitrary edge. The subdivided triangle in the case of $m=4$ consists of 10 upwards and 6 downwards triangles for example.

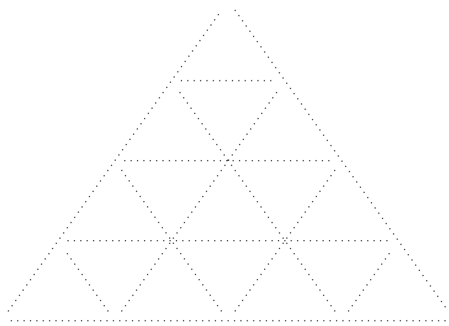

Figure 3: Subdivision of a triangle for $m=4$

From such a subdivision we create a bipartite graph: We apply the following procedure to each triangle of the triangulation (see Figure 4).

- Put black points on the midpoints of the short segments of the sides of the original triangle (e.g. either diagonals of the triangulation or edges of the polygon) and put black points into every downwards triangle.
- Put a white point inside every upwards triangle.

Finally two points are connected if they differ in color and the points belong to the same small triangle or their small triangles have a side in common.
We call the resulting graph a *GL_m-dimer.*

According to the first point of this list, there are exactly m black points on each of the diagonals of the triangulation and on the edges of the initial polygon. Figure 5 shows a GL_2-dimer of a triangulated pentagon.

The resulting graph is bipartite and its complement splits the original surface into several connected components.

[1] up to a changed condition at the boundary

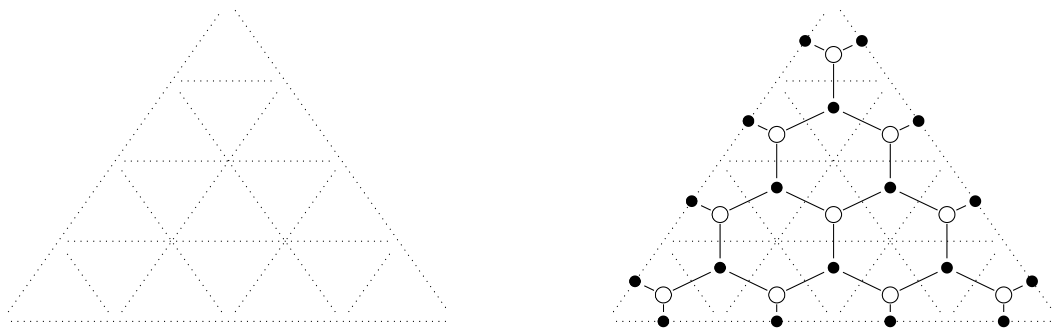

Figure 4: Constructing the GL_m-dimer on a triangle. Here $m = 4$.

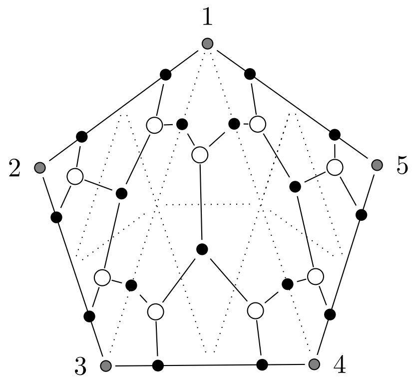

Figure 5: GL_2-dimer of a pentagon.

2.4 Boundary algebra of a GL_m-dimer

We can associate a dimer model with boundary to a GL_m-dimer:

Put a vertex in each connected component of the complement of the GL_m-dimer.

Then connect adjacent components by arrows such that the white point of the dimer is on the left hand side of the arrow, shown in Figure 6. Since the GL_m-dimer is bipartite, the quiver arising from it is a dimer model with boundary; each white vertex sits in a counterclockwise face and each black vertex in a clockwise face. Note that Q_2^+ and Q_2^- are the set of all faces whose boundaries are oriented counterclockwise and clockwise respectively.

The quiver Q of the GL_m-dimer of a triangle fulfills all aspects of Definition 2.2 and hence it is a dimer model with boundary[2]. An example of the quiver of the GL_2-dimer of a triangle is shown in Figure 7. The boundary vertices of the quiver are denoted by $1, \ldots, 6$.

Definition 2.8 (chordless cycle)**.** A chordless cycle of a quiver Q is a cycle such that the full subquiver on its vertices is also a cycle.

[2]We will often use the notation quiver instead of dimer model with boundary for readability of the article.

Figure 6: The white point of the dimer is on the left hand side of the arrow

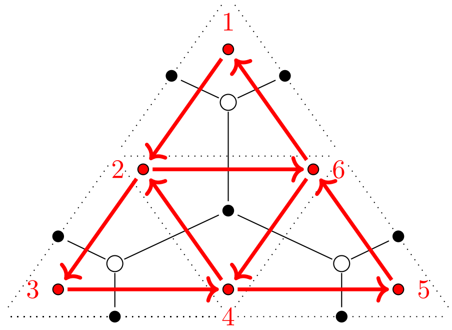

Figure 7: Quiver of the GL_2-dimer of a triangle.

Remark 2.9. The following fact is due to the definition of a dimer model with boundary: Let Q be the dimer model of a GL_m-dimer and Λ_Q the corresponding dimer algebra. Let k be an arbitrary vertex of Λ_Q with at least two incoming and two outgoing arrows. Then, up to ∂W, $c_1 = c_2$ for any two chordless cycles c_1, c_2 starting at k.

Examples of quivers of GL_m-dimers are shown in Figure 7 for $m = 2$ and in Figure 16 for $m = 5$. In our particular setting, the number of faces (cycles) incident with a vertex i are always 1 or 3 for boundary vertices and 4 or 6 for internal vertices.

By Remark 2.9, all chordless cycles at a given vertex are equal and hence it makes sense to refer to any one of them as the cycle at this vertex.

Definition 2.10 (short cycle u)**.** Let i be a vertex of Q, then we write u_i for a chordless cycle at i.

3 Boundary algebras of dimer models of GL_2-dimers of arbitrary triangulations of the n-gon are isomorphic

Recall that $\mathcal{B}_Q = e_b \Lambda_Q e_b$ is the boundary algebra obtained from the quiver Q of a GL_2-dimer of a triangulation of an n-gon, where $e_b = e_1 + \ldots + e_{2n}$ denotes the sum of all boundary idempotents. We also recall the quiver $\Gamma(n)$ from the introduction. For $n = 5$, it has the following form $\Gamma(5)$:

Theorem 3.1 (Main Theorem)**.** *The quiver of $\mathcal{B}_Q$ with relations ∂W is isomorphic to $\Gamma(n)$ subject to the following relations (writing compositions of paths from left to right), for $i = 1, \ldots, n$,*

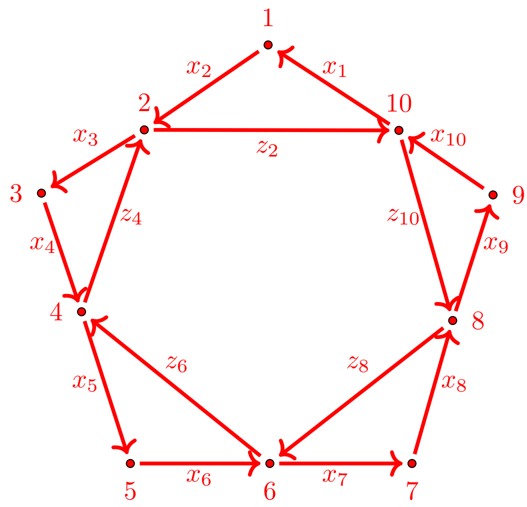

Figure 8: $\Gamma(5)$.

where indices are considered modulo $2n$*:*

$$\begin{aligned} x_{2i+1}x_{2i+2}z_{2i+2} &= z_{2i}x_{2i-1}x_{2i} \\ z_{2i}z_{2i-2} &= x_{2i+1}x_{2i+2}\cdots x_{2i+2\cdot(n-2)}. \end{aligned}$$

Furthermore the element

$$t := \sum_{i=1}^{n} x_{2i-1}x_{2i}z_{2i} + \sum_{i=1}^{n} x_{2i}z_{2i}x_{2i-1}$$

is central in $\mathcal{B}_Q$*.*

The proof of Theorem 3.1 is split into two main steps. First, we consider fan triangulations and show by induction, that in this case the boundary algebra $\mathcal{B}$ has the desired structure for any n. The second step is to show that the flip of a diagonal does not change the boundary algebra i.e. $\mathcal{B}$ is flip-invariant. Using the fact that every triangulation of a polygon can be reached from any starting triangulation under application of finitely many flips (Theorem (a) in [5]), we get the claimed result.

3.1 Boundary algebra of a fan triangulation

The goal of this chapter is to show that the boundary algebra of a fan triangulation is isomorphic to the algebra $\mathcal{B}$. Before describing the boundary algebra, the structure of the quiver of a fan triangulation of an n-gon will be determined. We will write $Q_F(n)$ for the dimer model of the GL_2-dimer of a fan triangulation of the n-gon.

Proposition 3.2. *Let* $Q_F(n)$ *be the dimer model with boundary of a fan triangulation of the* n*-gon,* $n \geq 3$*. Then* $Q_F(n)$ *has the following form:*
It consists of $2n$ *vertices on the boundary, labelled anticlockwise by* $1, \ldots, 2n$*, and* $n-3$ *internal*

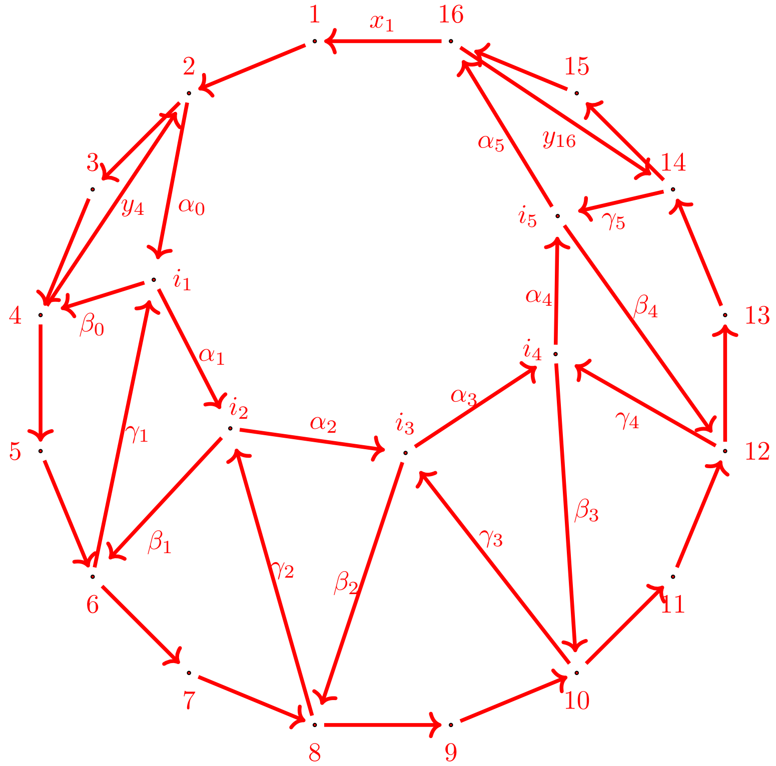

Figure 9: $Q_F(8)$: Quiver of a fan triangulation of the octagon.

vertices labelled $i_1, \ldots, i_{n-3}$.
Furthermore it has $2n+2$ *arrows between the boundary vertices, with* $k \in [1, 2n]$ *and indices taken modulo* $2n$,

$$\begin{aligned} x_k &: k-1 \to k \\ y_4 &: 4 \to 2 \\ y_{2n} &: 2n \to 2n-2, \end{aligned}$$

and arrows

$$\begin{aligned} \alpha_0 &: 2 \to i_1 && \\ \alpha_k &: i_k \to i_{k+1} && 1 \leq k < n-3 \\ \alpha_{n-3} &: i_{n-3} \to 2n && \\ \beta_{k-1} &: i_k \to 2k+2 && 1 \leq k \leq n-3 \\ \gamma_k &: 2k+4 \to i_k && 1 \leq k \leq n-3. \end{aligned}$$

Remark. The quiver $Q_F(n)$ has $2n-2$ faces.

The quiver $Q_F(8)$ is illustrated in Figure 9.

Proof. The claim follows inductively by removing 2-cycles of the dimer model using the relations obtained by the natural potential W. Note that on the dimer, this reduction corresponds to replacing every subgraph of the form shown on left hand side in Figure 10 to the form shown on the right hand side. The case $n = 3$ is the induction basis. The quiver of $Q_F(3)$ has the claimed

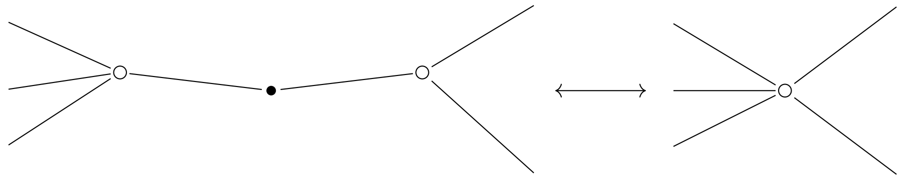

Figure 10: Reduction procedure.

form, see Figure 7.
Assume that $Q_F(n)$ has the described form for $n \geq 3$ and consider the GL_2-dimer of the fan triangulation of the $n+1$-gon. It can be obtained from the fan triangulation of the n-gon by adding a triangle to it at vertex 1 to the end of the fan. The dimer is obtained analogously. Observe that the same reduction steps can be done for the GL_2-dimer of the $n+1$-gon as for the one of the n-gon, because the only difference between the two triangulations is the additional triangle between 1, n and $n+1$, which does not change the GL_2-dimer of the former n-gon. The reduction steps are the one described in Figure 10. Figure 11 shows the relevant part of the reduced dimer, i.e. the new part obtained by increasing the number of vertices of the polygon. Note that it is possible to reduce the new dimer as the regions I and II indicate. This leads to the reduced dimer shown in Figure 12, containing the reduced quiver $Q_F(n+1)$. The new quiver has 2 additional faces, the chordless cycles $2n$,$2n+1$,$2n+2$ and $2n$,i_{n-2},$2n+2$. So it has the claimed structure.

□

Remark. Whenever we have a 2-cycle of internal arrows in the quiver of a GL_m-dimer, we can remove it using the relations from the potential. From now on we will always tacitly remove such 2-cycles from our dimer model and will use the phrases GL_m-dimer and quiver instead of "reduced GL_m-dimer" and "reduced quiver".

Knowing the structure of the quiver of a fan triangulation in detail, it is now possible to describe the boundary algebra of the n-gon.

Definition 3.3. We define paths $z_2, \ldots, z_{2n}$ as follows:

$$z_{2k} := \gamma_{k-2}\beta_{k-3} \text{ for } k = 3, \ldots, n-1 \tag{1}$$
$$z_4 := y_4 \tag{2}$$
$$z_2 := \alpha_0\alpha_1 \ldots \alpha_{n-3} \tag{3}$$
$$z_{2n} := y_{2n}. \tag{4}$$

Lemma 3.4. *The x_i, $i = 1, \ldots, 2n$ together with the z_{2k}, $k = 1, \ldots, n$ as defined in Definition 3.3 generate the boundary algebra of $Q_F(n)$.*

Remark. The set $\{\{x_i\}_{1\leq i\leq 2n}, \{z_{2j}\}_{1\leq j\leq n}\}$ is minimal in the sense that any proper subset does not suffice.

Proof. For the convenience of the reader, Figure 9 shows $Q_F(8)$ in order to make it easier to follow the argumentation by using relations obtained by the natural potential W for different

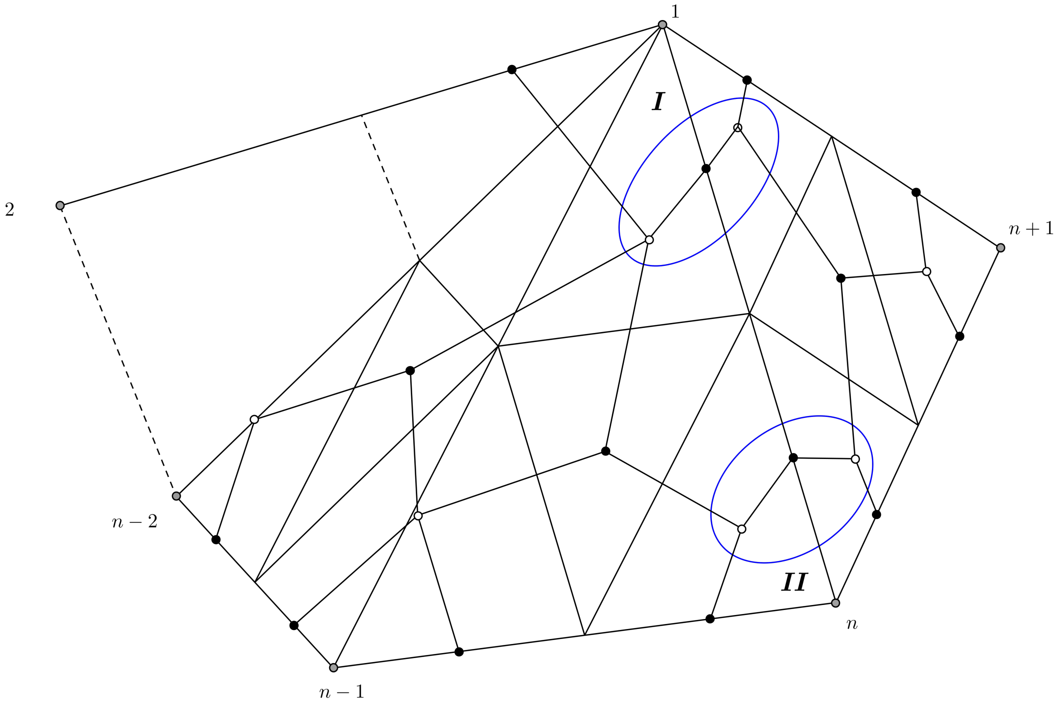

Figure 11: Fan triangulation of the $n+1$-gon and new part of the dimer. Labelling corresponds to the vertices of the $n+1$-gon.

internal arrows.
We show that each path from 2 to $2n$ factors through z_2. Assume to the contrary that a path δ from 2 to n does not factor through z_2. We can assume that δ does not contain cycles. Then the following two cases can occur: Either $\delta = x_3x_4 \cdots x_{2n}$ or there exists a k, $1 \leq k \leq n-3$ such that α_k is not an arrow of δ, w.l.o.g let k be minimal, i.e. $\delta = \alpha_0 \dots \alpha_{k-1}\widetilde{\delta}$. In the first case we get the equivalence of the following paths:

$$x_3x_4 \cdots x_{2n} \overset{y_4}{\cong} \alpha_0\beta_0x_5x_6 \cdots x_{2n} \overset{\gamma_1}{\cong} \dots \overset{\gamma_{n-3}}{\cong} \alpha_0 \cdots \alpha_{n-3}y_{2n}x_{2n-1}x_{2n} = z_2u_{2n}. \tag{5}$$

The last equality holds as the path $y_{2n}x_{2n-1}x_{2n}$ is a chordless cycle at $2n$. Hence δ factors through z_2 in the first case. In the second case, as δ does not contain cycles, $\widetilde{\delta}$ must be of the form $\beta_{k-1}x_{2k+1}x_{2k+2}\widetilde{\delta}'$ for some path $\widetilde{\delta}'$. Then

$$\widetilde{\delta} = \beta_{k-1}x_{2k+1}x_{2k+2}\widetilde{\delta}' \overset{\gamma_k}{\cong} \alpha_k\beta_k\widetilde{\delta}'$$

which is a contradiction to the minimality of k. Hence every path from 2 to $2n$ factors through z_2. The rest of the statement follows with similar arguments. □

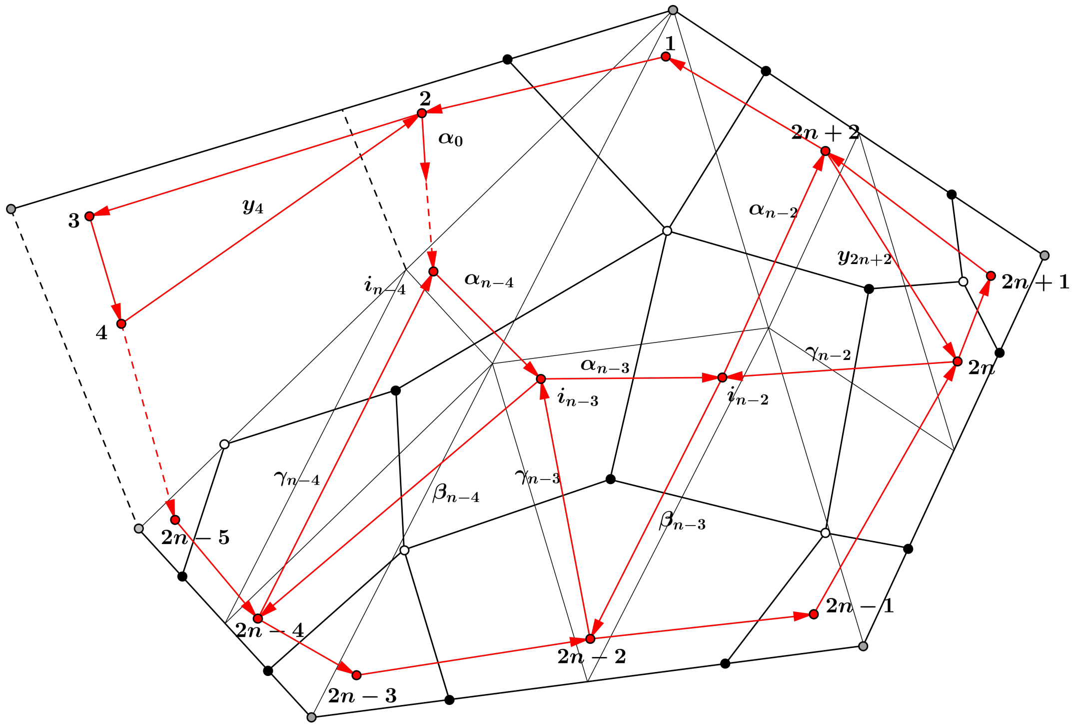

Figure 12: Part of reduced GL_2-dimer of the $n+1$-gon and the quiver $Q(n+1)$.

Now we want prove that the relations between the arrows, stated in the previous section are fulfilled for the boundary algebra of the fan triangulation of a polygon. Let $\Lambda_{Q_F(n)}$ be the dimer algebra of $Q_F(n)$ and let $e_b = \sum_{k=1}^{2n} e_k$. Note that we will always reduce indices modulo $2n$.

Proposition 3.5. *The boundary algebra $e_b \Lambda_{Q_F(n)} e_b$ satisfies the following relations for $k \in [1, n]$:*

(I.) $z_{2k} z_{2k-2} = x_{2k+1} x_{2k+2} \dots x_{2k+2\cdot(n-2)}$

(II.) $x_{2k+1} x_{2k+2} z_{2k+2} = z_{2k} x_{2k-1} x_{2k}$.

Proof. By the same calculations as in (5) of Lemma 3.4 we immediately get the equality of the paths

$$x_3 x_4 \cdots x_{2n-3} x_{2n-2} = z_2 y_{2n} = z_2 z_{2n}.$$

All other relations of type (I.) follow from Lemma 3.4 in the same way. The second kind of relation has already been stated in Remark 2.9, as both sides of the equation are short cycles u_{2k} at boundary vertex $2k$. $\square$

Thus we described the boundary algebra of the GL_2-dimer of a fan triangulation for arbitrary large n in detail and it remains to show flip invariance in order to get the main result for $m = 2$.

3.2 Flips in boundary algebras

This section starts with the definition of a diagonal flip of a triangulation in order to show that a flip does not change the structure of the boundary algebra itself. As already shown the boundary algebra of the fan triangulation has the structure given in Theorem 3.1. Together with the main result of this section (Theorem 3.9), this proves that all boundary algebras arising from GL_2-dimers of arbitrary triangulations of an n-gon are isomorphic.

Definition 3.6 (Diagonal flip of triangulation.)**.** For a triangulation a diagonal flip is defined as follows. Let (l, j) be a diagonal of the triangulation of the n-gon. Then two triangles l,j,k and l,j,i belong to the triangulation. A flip, as shown in Figure 13,

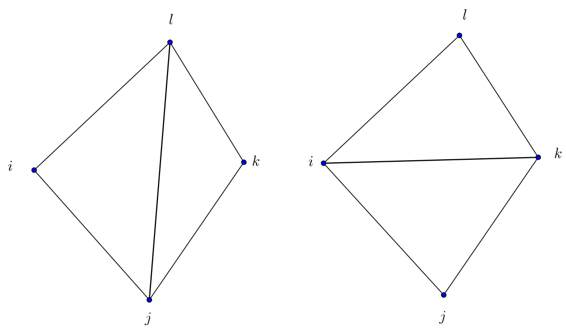

Figure 13: Diagonal flip of a triangulation.

is the removal of the diagonal (l, j) replacing it by the diagonal (i, k).

Note that a diagonal flip is always a local operation that only changes the structure around vertices corresponding to the edges of the quadrilateral, and hence a local operation on the GL_2-dimer and the corresponding dimer algebra. Furthermore, every triangulation of a polygon can be reached from any starting triangulation by application of finitely many flips, see Theorem (a) in [5]. Recall that $Q_{F(n)}$ denotes the quiver of the GL_2-dimer of a fan triangulation of an n-gon with dimer algebra $\Lambda_{Q_{F(n)}}$ and e_b the sum of the boundary idempotents of $\Lambda_{Q_{F(n)}}$.

Lemma 3.7. *Let $j \in [3, n-1]$ and μ be the flip of the diagonal $(1, j)$ of $Q_F(n)$. Let e_b' be the sum of the boundary idempotents of $\Lambda_{\mu Q_F(n)}$. Then there is an isomorphism*

$$e_{b'}\Lambda_{\mu Q_F(n)}e_{b'} \cong e_b\Lambda_{Q_F(n)}e_b.$$

Proof. By Lemma 3.4 the x_i, $i = 1, \ldots, 2n$ together with the z_{2k}, $k = 1, \ldots, n$ generate the boundary algebra $\mathcal{B}(Q_F(n))$ and these elements fulfil relations (I.) and (II.) by Proposition 3.5. We consider the flip μ of the diagonal $(1, j)$ of the n-gon and the new dimer algebra $\Lambda_{\mu Q(F)}$, the relevant part is shown in Figure 14.

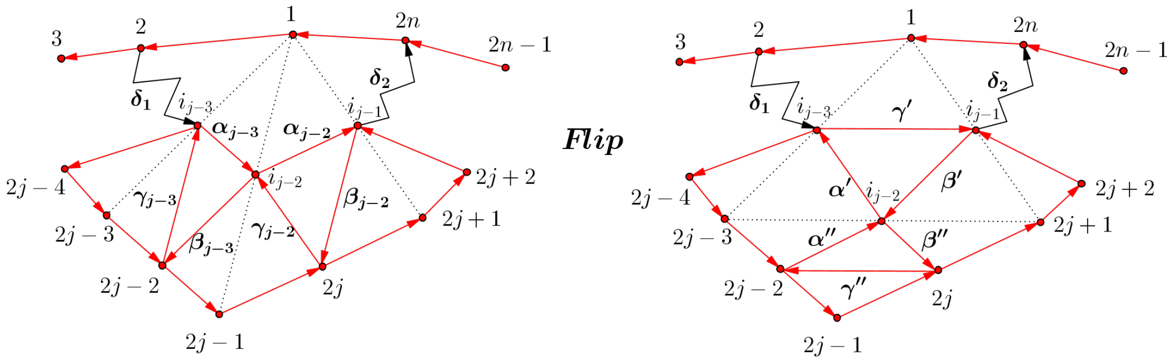

Figure 14: Flip of diagonal $(1,j)$ changes the dimer algebra to $\Lambda_{\mu Q_F(n)}$.

Here the paths δ_1 and δ_2 are

$$\delta_1 = \begin{cases} \alpha_0 \cdots \alpha_{j-4} & 4 \le j \le n-1 \\ e_2 & j = 3 \end{cases}$$

$$\delta_2 = \begin{cases} \alpha_{j-1} \cdots \alpha_{n-3} & 3 \le j \le n-2 \\ e_{2n} & j = n-1 \end{cases}$$

and hence might be empty. Furthermore, if $j = 3$, then $\gamma_{j-3} = y_4$, $\beta_{j-4} = e_2$, $i_{j-3} = 2$ and if $j = n-1$, then $\gamma_{j-1} = e_{2n}$, $\beta_{j-2} = y_{2n}$ and $i_{j-1} = 2n$.
We know that every path of $\mathcal{B}(Q_F(n))$ from 2 to $2n$ factors through $z_2 = \delta_1\alpha_{j-3}\alpha_{j-2}\delta_2$. The boundary algebra $e_{b'}\Lambda_{Q_F(n)}e_{b'} =: \mathcal{B}'$ has generators in terms of Lemma 3.4, which we now want to describe in detail. First notice that $x'_k := x_k$ for $k \in [1, 2n]$ are also generators in $\mathcal{B}'$, because a diagonal flip does not change the boundary.
Claim: Every path from 2 to $2n$ in $\mathcal{B}'$ factors through z'_2, with

$$z'_2 := \delta_1\gamma'\delta_2. \tag{6}$$

Proof of the Claim. We can assume w.l.o.g. that z'_2 does not contain a cycle (otherwise, by removing the cycle, every path from 2 to $2n$ would still factor through it). Furthermore, because the flip does not change the rest of the arrows (apart from the internal arrows α_{j-3}, α_{j-2}, β_{j-3}, β_{j-2}, γ_{j-3} and γ_{j-2}), every path from 2 to $2n$ still factors through δ_1 and δ_2 in $\mathcal{B}'$.
The generator z'_2 for paths from 2 to $2n$ must contain at least one arrow of the new quadrilateral, because otherwise this generator would already have existed in the original dimer algebra, a contradiction to the fact that all paths from 2 to $2n$ factor through $z_2 = \delta_1\alpha_1\beta_1\delta_2$ in $\mathcal{B}(Q_F(n))$.
The arrows α' or β' immediately lead to a cycle at i_{j-3} or i_{j-1} respectively, so they can't be part of the generator z'_2. As β' is not part of z'_2, if γ' is part of the generator, it has to be the only arrow of the new quadrilateral.

Assume now that γ' is not part of z'_2. Then at least one of the arrows α'', β'' or γ'' has to be part of the generator z'_2.
If α'' was part of it, then either α' or β'' were also part of the generator. The former is a contradiction as mentioned above, the latter to z'_2 using an arrow of the new quadrilateral, as

$$\alpha''\beta'' \overset{\gamma''}{\cong} x'_{2j-1}x'_{2j}.$$

The other cases lead to contradictions similarly.
Analogously, we can define

$$z'_{2j+2} := \gamma_{j-1}\beta'\beta''$$
$$z'_{2j} := \gamma''$$
$$z'_{2j-2} := \alpha''\alpha'\beta_{j-4}.$$

Furthermore, as every path, which does not contain arrows of the new quadrilateral remains unchanged, we can define $z'_{2k} := z_{2k}$ for all $k \in [1,n] \setminus \{1, j-1, j, j+1\}$. Then the set $\left\{\{x'_i\}_{1\leq i\leq 2n}, \{z'_{2j}\}_{1\leq j\leq n}\right\}$ generates $\mathcal{B}'$ in a minimal way as in Remark after Lemma 3.4.
It remains to show, that the relations are fulfilled in $\mathcal{B}'$. Of course, the relations $x'_{2k+1}x'_{2k+2}z'_{2k+2} = z'_{2k}x'_{2k-1}x'_{2k}$ hold in $\mathcal{B}'$ by Remark 2.9.
We have to check the relations

$$z'_{2i}z'_{2i-2} = x'_{2i+1}x'_{2i+2}\dots x'_{2i+2\cdot(n-2)}.$$

for all paths involving arrows of the new quadrilateral. Let $i = j$, then

$$z'_{2j}z'_{2j-2} \overset{\beta''}{\cong} x'_{2j+1}x'_{2j+2}\gamma_{j-1}\beta'\alpha'\beta_{j-4} \overset{\gamma'}{\cong} x'_{2j+1}x'_{2j+2}\delta_2 x'_1 x'_2\delta_1\beta_{j-4} = x_{2j+1}x_{2j+2}\delta_2 x_1 x_2\delta_1\beta_{j-4}.$$

This last path does not contain an arrow of the new quadrilateral, hence we can apply Proposition 3.5 and get the desired result:

$$x_{2j+1}x_{2j+2}\delta_2 x_1 x_2\delta_1\beta_{j-4} \overset{3.5}{\cong} x_{2j+1}x_{2j+2}\dots x_{2j-4} = x'_{2j+1}x'_{2j+2}\dots x'_{2j-4}.$$

All further relations involving arrows of the quadrilateral follow analogously. □

A direct consequence of the proof of Lemma 3.7 is:

Corollary 3.8. *The isomorphism of Lemma 3.7 is induced by*

$$x_k \mapsto x'_k \text{ for } k \in [1,2n] \tag{7}$$
$$z_{2k} \mapsto z'_{2k} \text{ for } k \in [1,n]. \tag{8}$$

Theorem 3.9. *Let Q be the quiver of the GL_2-dimer of an arbitrary triangulation of the n-gon, with dimer algebra Λ_Q and $e_{b'}$ the sum of the boundary idempotents of Λ_Q. Then there is an isomorphism*

$$e_{b'}\Lambda_Q e_{b'} \cong e_b\Lambda_{Q_F(n)}e_b.$$

Proof. We prove the claim by induction over the number of flips.
The induction basis is Lemma 3.7.
For the induction step, we use that we can reach any triangulation of an n-gon by a finite number of diagonal flips. So let Q be the quiver of an arbitrary triangulation and

$$Q = \mu_t\mu_{t-1}\dots\mu_1 Q_F(n),$$

where $\mu_1, \dots, \mu_t$ are t flips of diagonals (writing flips from right to left).

By induction hypothesis, we know that

$$\mathcal{B}(\mu_{t-1}\dots\mu_1 Q_F(n)) \cong \mathcal{B}(Q_F(n)).$$

The induction step follows in a similar way as in the proof of Lemma 3.7. Note that Figure 15 illustrates the effect of an arbitrary flip, the paths $\delta_1, \dots, \delta_8$ from and to the quadrilateral involved may differ from those in Lemma 3.7 in general. However, the arguments for checking do not change. Hence we get the desired result,

$$\mathcal{B}(\mu_t \mu_{t-1} \dots \mu_1 Q_F(n)) \cong \mathcal{B}(\mu_{t-1} \dots \mu_1 Q_F(n)).$$

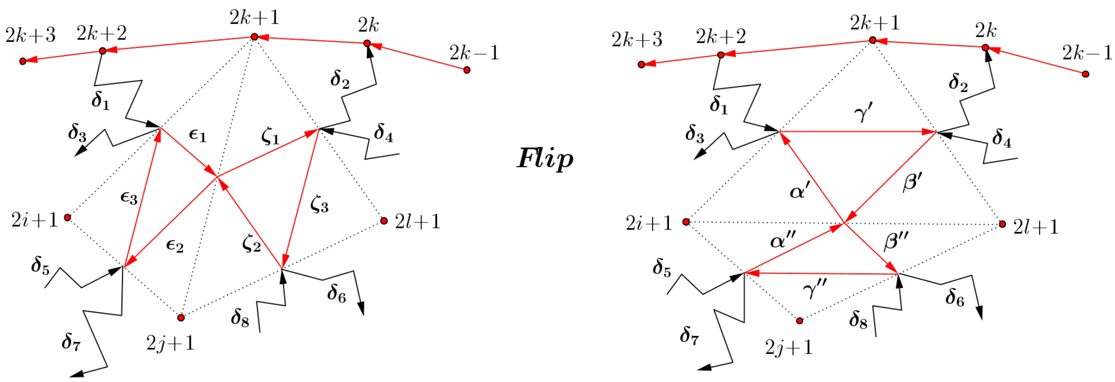

Figure 15: Effect of an arbitrary flip on arrows of type z_{2k}.

□

Corollary 3.10. *Consider the boundary algebra $\mathcal{B}_Q$ of a dimer model Q of a GL_2-dimer of an arbitrary triangulation of the n-gon. Then the element t,*

$$t := \sum_{i=1}^{n} x_{2i-1} x_{2i} z_{2i} + \sum_{i=1}^{n} x_{2i} z_{2i} x_{2i-1},$$

is a central element of this algebra.

Proof. The element t is the sum of exactly one chordless cycle for every boundary vertex and hence commutes with every element of $\mathcal{B}_Q$. □

4 The general case

In this section, we describe the boundary algebras for arbitrary m.

Starting with the GL_m-dimer as defined in Section 2, we can reduce the dimer and achieve the quiver of an n-gon equivalently as for $m = 2$ for arbitrary m. From now on, we will assume that the dimer (and hence the quiver) is always reduced.

Figure 16 shows the quiver of the GL_5-dimer of a triangulation of the quadrilateral. This example already shows, that there is a new type of generators arising for describing the boundary algebra: Let's have an informal look at boundary vertex 10. In contrast to the case where $m = 2$, there are not only paths to 11 and 9 but also an additional path to 2 which cannot be reduced by any of the relations. Hence we need a third type of generators. In order to introduce a formal notation for the generators, we have a closer look at the internal vertices. We can give a formula for the number of internal vertices, depending on m and n.

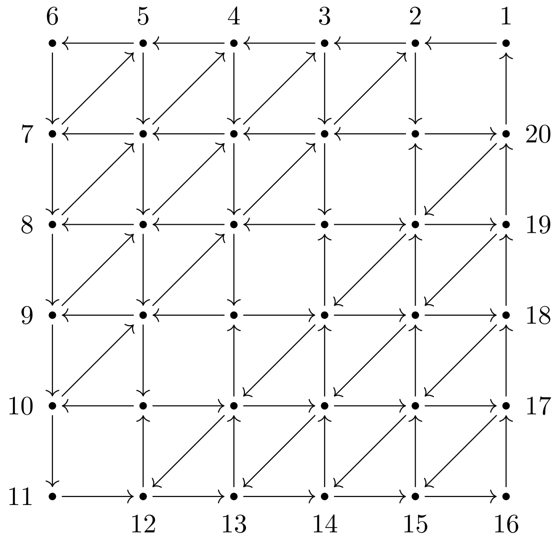

Figure 16: Reduced quiver of the GL_5-dimer of a quadrilateral with diagonal incident with 1.

Definition 4.1 (polygonal numbers of second order)**.** For a polygon with s vertices, the k^{th} s-gonal number of second order is

$$P_2(s,k) = \frac{k^2 \cdot (s-2) + k \cdot (s-4)}{2}.$$

Remark. Using the setting as in definition above, the polygonal number (of first order) is defined as $P(s,k) = \frac{k^2 \cdot (s-2) - k \cdot (s-4)}{2}$, see A057145 in the OEIS [6].

Proposition 4.2. *The number of internal vertices of the dimer model of the* GL_m*-dimer of the fan triangulation of the* n*-gon is* $P_2(n, m-1)$ *for* $n > 3$*.*

Proof. The proof is done by induction on n and m. First let m be fixed. For induction basis let $n = 4$. The number of internal vertices equals $(m-1)^2$ by construction, which coincides with $P_2(4, m-1)$. Let the number of internal vertices $V_{n,m}$ of the GL_m-dimer of the fan triangulation of the n-gon be $P_2(n, m-1)$. Consider the $n+1$-gon, where we add a triangle to the fan. By construction of the GL_m-dimer, we get

$$1 + 2 + \ldots + (m-2) + (m-1).$$

additional vertices by adding a triangle. By using the induction hypothesis

$$\begin{aligned} V_{n+1,m} &= P_2(n, m-1) + 1 + 2 + \ldots + (m-2) + m - 1 = \\ &= \frac{(m-1)^2 \cdot (n-2) + (m-1) \cdot (n-4)}{2} + \frac{(m-1) \cdot m}{2} = \\ &= \frac{(m-1)^2 \cdot (n-2) + (m-1) \cdot (n-4)}{2} + \frac{(m-1)^2}{2} + \frac{m-1}{2} = \\ &= \frac{(m-1)^2 (n-1) + (m-1) \cdot (n-3)}{2} = P_2(n+1, m-1), \end{aligned}$$

we achieve the desired result.
Now let n be fixed. For $m = 2$, the number of inner vertices of the GL_2-dimer of an n-gon is $(n-3)$ and coincides with $P_2(n,2)$. Again, let the number $V_{n,m}$ of internal vertices of the GL_m dimer of the n-gon be $P_2(n, m-1)$. If we increase m by one, the number of internal vertices increases by $m \cdot n - 2m - 1$, by construction of the GL_m-dimer. So the number $V_{n,m+1}$ of internal vertices of quiver of the GL_{m+1}-dimer is by using induction hypothesis

$$V_{n,m+1} = P_2(n, m-1) + m \cdot n - 2m - 1 = \frac{m^2 \cdot (n-2) + m \cdot (n-4)}{2} = P_2(n,m)$$

and the proof is done. □

Remark. The number of internal vertices is the same for any triangulation of the n-gon.

Before we can state the structure of the quiver $Q_F(m,n)$ of the GL_m-dimer of the fan triangulation of the n-gon, we have to introduce some notation.
The boundary vertices are labelled by $1, \dots, nm$ anticlockwise.
The quiver $Q_F(m,n)$ contains $m-1$ disjoint nested oriented paths from $2+i$ to $nm-i$ for $i = 0, \dots, m-2$ formed by successive arrows. We will denote them by α_P, where P is the sink of the arrow. The internal vertices are labelled by a 3-tuple (a,b,c) depending on their position along these oriented paths as follows:

$a \in [1, n-2]$ denotes the triangle, to which the internal vertex can be assigned to.

$b \in [1, m-1]$ is one less than the starting vertex of the nested path.

$c \in [1, b]$ counts the number of internal vertices up to b in every triangle.

Figure 17 shows the labelling of all internal vertices of $Q_F(4,6)$. The arrows are all indexed by their sinks. Along the boundary they are $x_k : k-1 \to k$. All the other arrows are as follows:

Notation 4.3.

$$\begin{aligned}
\alpha_{(1,i,1)} &:= i+1 \to (1,i,1) \quad i \in [1, m-1] \\
\alpha_{(a,b,1)} &:= (a-1,b,b) \to (a,b,1) \quad a \in [2, n-2], b \in [1, m-2] \\
\alpha_{(a,b,c)} &:= (a, b.c-1) \to (a,b,c) \quad a \in [1, n-2], b \in [2, m-2], c \in [2, b] \\
\alpha_{m \cdot n - i} &:= (n-2, i+1, i) \to m \cdot n - i \quad i \in [1, m-2] \\
\alpha_{m \cdot n} &:= (n-3, 1, 1) \to m \cdot n \\
\beta_m &:= m+2 \to m \\
\beta_i &:= (1,i,1) \to i \quad i \in [2, m-1] \\
\beta_{(k-1,m-1,i-2)} &:= k \cdot m + i \to (k-1, m-1, i-2) \quad i \in [3, m], k \in [2, n-1] \\
\beta_{(k-2,m-1,m-1)} &:= k \cdot m + 1 \to (k-2, m-1, m-1) \\
\beta_{(a,b,c)} &:= \begin{cases} (a, b+1, c+1) \to (a,b,c) & a \in [1, n-2] \;\; b \in [2, m-1] \;\; c \in [1, m-3] \\ (a+1, b+1, 1) \to (a,b,c) & a \in [1, n-2] \;\; b \in [2, m-1] \;\; c = b \end{cases} \\
\gamma_{m \cdot (n-1)} &:= m \cdot (n-1) + 2 \to m \cdot (n-1) \\
\gamma_{(n-2,i+1,i)} &:= m \cdot n - i + 1 \to (n-2, i+1, i) \quad i \in [1, m-2] \\
\gamma_{m \cdot k+1+i} &:= (k, m-1, i) \to m \cdot k + 1 + i \quad k \in [1, n-3] i \in [1, m-1] \\
\gamma_{m \cdot (n-2)+1+i} &:= (n-2, m-1, i) \to m \cdot k + 1 + i \quad i \in [1, m-2] \\
\gamma_{(a,b,c)} &:= (a, b-1, c) \to (a,b,c) \quad a \in [1, n-2], \;\; b \in [2, m-1], \;\; c = b-1
\end{aligned}$$

In Figure 17 the definition of the different types of arrows is shown in the case of the GL_4-dimer of a hexagon: Beside the black boundary arrows x_k $k \in [1, m \cdot n]$, the remaining arrows can be identified as follows: The bold black arrows are named α_i, the blue arrows β_i and the green arrows γ_i, where the index i always coincides with the labelling of the sink of the arrow. This means i is either a triple (a, b, c) or some natural number in $[1, m \cdot n]$ with $i \not\equiv 1 \mod m$. Note that additionally to the arrows and vertices of the quiver the diagonals of the original triangulation are drawn as dotted lines to emphasize the idea of labelling the internal vertices (a, b, c).

Proposition 4.4. *Let $Q_F(m, n)$ be the quiver of the fan triangulation of the n-gon, $n \geq 3$. Then $Q_F(m, n)$ has the following form:*
It consists of $m \cdot n$ vertices on the boundary, labelled anticlockwise by $1, \ldots, m \cdot n$, and $P_2(n, m-1)$ internal vertices labelled (a, b, c), with a,b and c as described above.
Furthermore it has $m \cdot n + 2$ arrows between the boundary vertices

$$\begin{aligned} x_k &: k-1 \to k \\ y_m &:= \beta_m \\ y_{m\cdot(n-1)} &:= \gamma_{m\cdot(n-1)} \end{aligned}$$

and the internal arrows as described in 4.3.

Proof. The proof is similar to the proof of 3.2. □

Equivalently to former section, we define some elements of the boundary algebra and show, that these elements are generating the algebra.

Definition 4.5. We define paths $z_{m\cdot j+k}$ and $y_{m\cdot j+k}$ as follows:

$$y_k := \left(\prod_{i=k}^{m-1} \beta_{(1,m-1,m-k)}\right) \cdot \beta_k, \quad k \in [2, m] \tag{9}$$

$$y_{m\cdot j+k} := \left(\prod_{i=k}^{m-1} \beta_{(j+1,m+1-k,m-k)}\right) \cdot \beta_{(j,k-1,k-1)} \cdot \left(\prod_{i=k}^{m-1} \gamma_{(j,i,k-1)}\right) \cdot \gamma_{(m\cdot j+k)}, \quad j \in [1, n-3] \quad k \in [2, m] \tag{10}$$

$$y_{m\cdot(n-2)+k} := \left(\prod_{i=k}^{m-1} \gamma_{(n-2,i,k-1)}\right) \cdot \gamma_{(m\cdot(n-2)+k)}, \quad k \in [2, m] \tag{11}$$

$$y_{m\cdot n-k} := \left(\prod_{i=1}^{n-2} \prod_{j=0}^{k} \alpha_{((i,k+1,j))}\right) \cdot \alpha_{m\cdot n-k}, \quad k \in [0, m-2] \tag{12}$$

$$z_k := \gamma_{(1,k,1)} \beta_k, \quad k \in [2, m-1] \tag{13}$$

$$z_{m\cdot j+k} := \beta_{(j,m-1,k-1)} \gamma_{m\cdot j+k}, \quad k \in [2, m-1] \tag{14}$$

$$z_{m\cdot n-k} := \gamma_{n-2,k+1,k} \beta_{m\cdot n-k}, \quad k \in [2, m-1]. \tag{15}$$

Remark. The products above might be empty, which happens in equations $(9) - (11)$, when $k = m$. In these cases, only a single arrow remains. These are exactly the arrows

$$\begin{aligned} y_m &:= \beta_m \\ y_{m\cdot j+m} &:= \beta_{(j,k-1,k-1)} \cdot \gamma_{m\cdot j+m} \\ y_{m\cdot(n-2)+m} &:= \gamma_{m\cdot(n-2)+m}. \end{aligned}$$

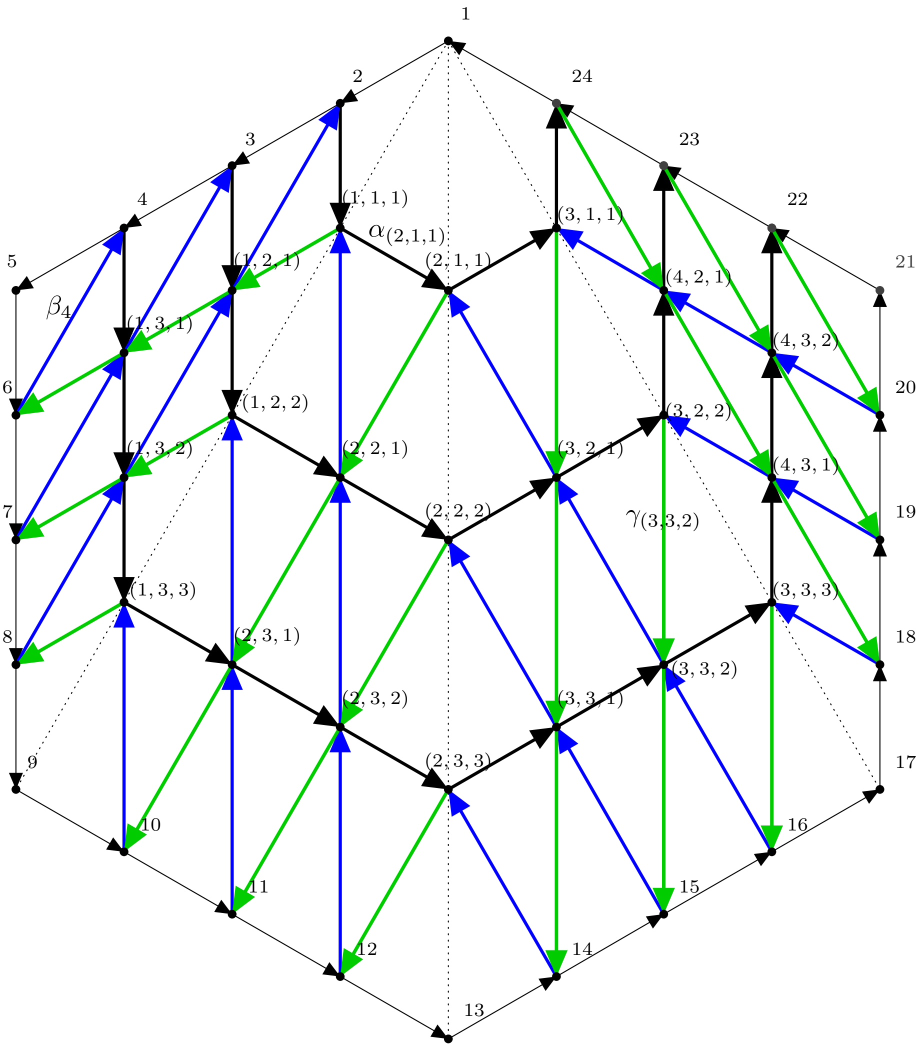

Figure 17: $Q_F(4,6)$: Quiver of the GL_4-dimer of the fan triangulation of a hexagon. For illustration, some arrows are labelled.

Note that the notation of the arrows z_i is different to the previous section, because of the new type of generators.

We will now define a quiver and show, that the boundary algebra of $Q_F(m,n)$ is isomorphic to it up to some relations.

Definition 4.6. The quiver $\Gamma(m,n)$ is defined by $m \cdot n$ vertices labelled anticlockwise and

$3 \cdot n \cdot (m-1)$ arrows

$$\begin{aligned} &x_k : k-1 \to k, && k \in [1, m \cdot n] \\ &y_k : k+2-2i \to k, && k \equiv -i \mod m,\ k \not\equiv 1 \mod m, \text{ and } -i \in [0, m-2] \\ &z_k : k+1 \to k, && k \not\equiv 1, 0 \mod m. \end{aligned}$$

Figure 18 shows the quiver $\Gamma(4,6)$.

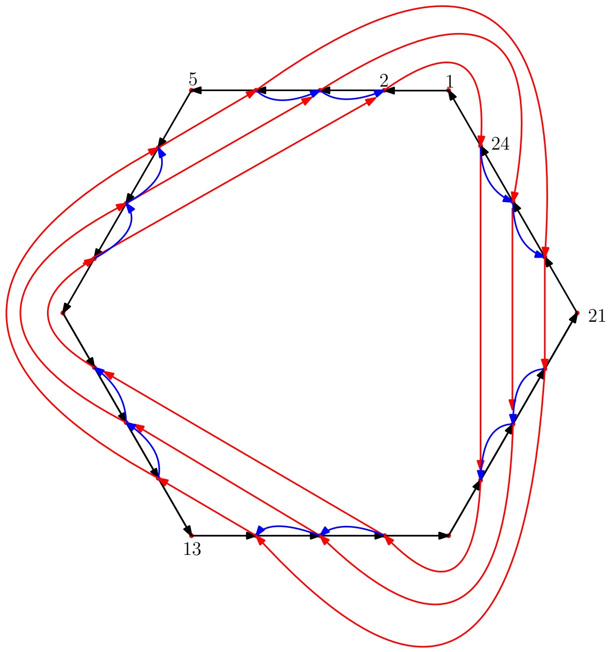

Figure 18: $\Gamma(4,6)$ as in Definition 4.6
Arrows x are black, arrows y are red and arrows z are blue.

Remark. The indices of the tail l and head $k = j \cdot m - k'$ with $k' \in [0, m-2]$ of y_k fulfil the equation

$$\begin{aligned} &l + k = 2 \cdot (j \cdot m + 1) \\ &\Leftrightarrow l = m \cdot j + 2 + k' = k + 2 + 2k', \end{aligned}$$

which is equivalent to the description of y_k in Definition 4.6.

As in the previous section we first describe the boundary algebra of the dimer algebra $\Lambda_{Q_F(m,n)}$, where $Q_F(m,n)$ is the quiver of the GL_m-dimer of the fan triangulation of the n-gon.

Proposition 4.7 (Boundary algebra of the fan triangulation)**.** *The boundary algebra* $\mathcal{B}_{Q_F(m,n)}$

is isomorphic to $\Gamma(m,n)$ *satisfying the following relations obtained by the natural potential* W*:*

$$\begin{aligned}
x_{k+2-2i}y_k &= y_{k+1}z_k,\ k \not\equiv 0,1 \mod m,\ k \equiv -i \mod m,\ \textit{and}\ -i \in [1, m-2] \\
x_{k+1}z_k &= z_{k-1}x_k,\ k \not\equiv 0,1,2 \mod m \\
x_{k+1}z_k &= y_{k-2}x_{k-1}x_k,\ k \equiv 2 \mod m \\
x_{k+1}x_{k+2}y_k &= z_{k-1}x_k,\ k \equiv 0 \mod m \\
y_{k+2-2j}y_k &= x_{k+2m+1}x_{k+2m+2}\cdots x_k, k \equiv -j \mod m,\ k \not\equiv 1 \mod m,\ \textit{and}\ -j \in [0, m-2]
\end{aligned}$$

where $k \in [1, m \cdot n]$ *and indices always considered modulo* $m \cdot n$*.*

Proof. The proof is done by combining Remark 2.9, Lemma 3.4 and Proposition 3.5 applied to the general case. □

The flip-invariance in case of $m > 2$ is a direct consequence of Theorem 3.9.

Theorem 4.8 (Flip equivalence)**.** *Let* $Q_F(m,n)$ *be the quiver of the* GL_m*-dimer of the fan triangulation of the* n*-gon, let* Q' *be the quiver of the* GL_m*-dimer of an arbitrary triangulation of the* n*-gon, with* $\Lambda_{Q_F(m,n)}$ *and* $\Lambda_{Q'}$ *the corresponding dimer algebras and* e_b *respectively* $e_{b'}$ *the sum of the boundary idempotents for* $Q_F(m,n)$ *and for* Q' *respectively. Then there is an isomorphism*

$$e_b \Lambda_{Q_F(m,n)} e_b \cong e_{b'} \Lambda_{Q'} e_{b'}.$$

Proof. Combine the structure of the fan triangulation shown in Proposition 4.7 with the proof of Theorem 3.9 for the general case. □

Corollary 4.9. *Consider the boundary algebra* $\mathcal{B}_Q$ *of a dimer model* Q *of a* GL_m*-dimer of an arbitrary triangulation of the* n*-gon. Then the element* t*,*

$$t := \sum_{i=1}^{m \cdot n} u_i,$$

is a central element of this algebra.

Proof. The element t is the sum of exactly one chordless cycle for every boundary vertex and hence commutes with every element of $\mathcal{B}_Q$. □

Hence, as a consequence of Proposition 4.7 and Theorem 4.8 we can state the main result for the general case:

Theorem 4.10 (Main Theorem)**.** *Let* $\mathcal{B}_Q$ *be the boundary algebra obtained from the dimer model* Q *of a* GL_m*-dimer of an arbitrary triangulation of the* n*-gon. Then the quiver of* $\mathcal{B}_Q$ *with relations* ∂W *is isomorphic to* $\Gamma(m,n)$ *subject to the following relations, for* $k \in [1, m \cdot n]$ *and indices are considered modulo* $m \cdot n$*:*

$$\begin{aligned}
x_{k+2-2i}y_k &= y_{k+1}z_k,\ k \not\equiv 0,1 \mod m,\ k \equiv -i \mod m,\ \textit{and}\ -i \in [1, m-2] \\
x_{k+1}z_k &= z_{k-1}x_k,\ k \not\equiv 0,1,2 \mod m \\
x_{k+1}z_k &= y_{k-2}x_{k-1}x_k,\ k \equiv 2 \mod m \\
x_{k+1}x_{k+2}y_k &= z_{k-1}x_k,\ k \equiv 0 \mod m \\
y_{k+2-2j}y_k &= x_{k+2m+1}x_{k+2m+2}\cdots x_k,\ k \equiv -j \mod m,\ k \not\equiv 1 \mod m,\ \textit{and}\ -j \in [0, m-2].
\end{aligned}$$

Furthermore the element

$$t := \sum_{i=1}^{m\cdot n} u_i$$

is central in $\mathcal{B}_Q$*.*

Proof. Proposition 4.7, Lemma 3.7 and Theorem 4.8, together with Corollary 4.9 give the desired result. □

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