

A combinatorial matrix approach for the generation of vacuum Feynman graphs multiplicities in ϕ^4 theory

Erick Castro^{1,*} and I. Roditi^{1,†}

¹*Centro Brasileiro de Pesquisas Físicas/MCTI, 22290-180, Rio de Janeiro, RJ, Brazil*

From the standard procedure for constructing Feynman vacuum graphs in ϕ^4 theory from the generating functional Z , we find a relation with sets of certain combinatorial matrices, which allows us to generate the set of all Feynman graphs and the respective multiplicities in an equivalent combinatoric way. These combinatorial matrices are explicitly related with the permutation group, which facilitates the construction of the vacuum Feynman graphs. Various insights in this combinatoric problem are proposed, which in principle provide an efficient way to compute the Feynman vacuum graphs and its multiplicities.

PACS numbers: 02.10.Yn; 02.10.Ox; 05.70.Fh

Keywords: Feynman vacuum diagrams; ϕ^4 theory, Counting graphs, Combinatorial Matrix Theory

I. INTRODUCTION

The problem of generating Feynman graphs and their respective multiplicities is recurrent in the quantum field literature. The starting point in such constructions is the generating functional of the correlation functions Z and the associate Legendre transformations, which generates specific types of Feynman graphs (connected and one-particle irreducible, see ref.[1] for a concise introduction). The paradigmatic case in order to introduce Feynman diagrammatic machinery is generally the ϕ^4 theory which is in the heart of second-order phase-transition phenomena. Diagrammatic counting techniques are no exception, one instance is the use of zero-dimensional ϕ^N theory to generate the number of total Feynman graphs in each perturbation order ref.[2] (in particular, connected vacuum diagrams and connected graphs with external legs), extending then to QED and non-abelian field theories.

In the construction of the Feynman graphs by functional derivatives (or equivalently, by contractions in the field operator approach, see for example ref.[3]) some diagrams generated by different derivatives (or different contractions) are equivalent, this appears as a multiplicity of equivalent diagrams. Such multiplicities in a given diagram \mathcal{G} , are related to the symmetry factor of \mathcal{G} , which is in connection with the number of automorphisms $|\text{Aut}(\mathcal{G})|$ of the graph, see ref.[4] and ref.[5] for example. The determination of $|\text{Aut}(\mathcal{G})|$ for a given graph \mathcal{G} is in general a difficult problem [6], therefore, the same can be said with respect to the multiplicities of \mathcal{G} . Nonetheless, this does not prevent the existence of practical computational algorithms for the generation of Feynman graphs [7].

In a global sense, the complete information of the multiplicities is contained in the generating functional of the Feynman graphs. This is evident in the zero dimensional

approach. In this case, for a given order, all the diagrams are equivalent and the number of Feynman graphs appear explicitly multiplying the coupling constant. In the general case, the construction of Feynman diagrams starting from the generating functional is an arduous task for growing perturbation orders, since the number of functional derivatives increase rapidly. Nevertheless, some achievements have been reached in this respect: ref.[8] introduce differential functional calculus to get recursive relations, which, iteratively determine the multiplicities for connected vacuum diagrams. In this reference it is shown that the m -order vacuum connected diagrams determine the other connected m -order diagrams (Feynman graphs with external legs), and therefore the corresponding multiplicities in a simple way.

On the other hand, relations between combinatorics and perturbative quantum field theory have been established gradually, for example, in the nature of renormalization group structure [9] [10] or with problems in combinatorial theory [11]. With respect to the problem of counting of objects in quantum field theories, interesting relationships with the permutation group and string theories have been established (see for example ref.[12]). In this respect, interesting correspondences between N -rooted maps and Feynman diagrams in QED field theory have been put in evidence, see Ref.[13].

In this work, we explore a different route. From the generating functional Z in ϕ^4 theory, we establish subset classes of m -order vacuum Feynman graphs (connected and disconnected) for each perturbative order m . Every diagram in each subset have associated a natural number. When adding all the numbers associated to the same diagram in all the subsets we obtain the corresponding multiplicity. This subsets of Feynman graphs are indexed by the equivalence classes in the total set of certain types of $m \times m$ matrices with equal row and column sums (RC-magic squares). The equivalence classes are induced naturally by the permutation group. In particular, every RC-magic square in the equivalence class determine the associated subset of vacuum diagrams in a simple way. Thus, the problem of constructing vacuum

*erickc@cbpf.br

†roditi@cbpf.br

Feynman graphs is reduced to finding one representative square matrix in each equivalence class of the total RC-magic square set. The relation between Feynman diagrams and RC-magic squares is found directly from the perturbative expansion of the generating functional Z . RC-magic squares are interesting combinatorial objects, the matrices studied here are simpler than those known by Chinese, Indian and Arab civilizations which had the same row, column and diagonal components sum. Here, our RC-magic squares only have identical row and column sum. For matrix combinatorial theory bibliography see for example ref.[14].

This paper is organized as follows. In sec.II we unambiguously establish the relation between certain subset of m -order vacuum Feynman graphs and the total set of $m \times m$ RC-magic squares with sum equal to 4. The subsets are generated directly from the expansion of the functional generator Z and the relation with the RC-magic squares is explicitly demonstrated. The relation is formulated within the adjacency matrix notation for graph theory, which facilitates the computation of the Feynman graphs associated with the respective RC-magic square. In Sec.III we prove that the action of the permutation group, as an interchange between rows, and an interchange between columns of a specific RC-magic square does not change the associated subset of the Feynman vacuum graphs. This fact, induces in the total $m \times m$ RC-magic square set an equivalence relation, whose equivalence classes are formed by equivalent matrices that generates the same subset of m -order Feynman graphs. As an example, we construct all the third-order Feynman graphs using only one representative in each equivalence class in the total set of the 3×3 RC-magic squares. Finally, in sec.IV we discuss the efficiency of an algorithm, which code is implemented in the appendix A, for the calculation of the RC-magic square representatives and the size (number of elements) of each equivalence class. We put in evidence certain properties of the permutation matrices, which together with the Birkhoff-von Neumann theorem could allow some simplification on the construction of the RC-magic squares representatives. Sec.V contains discussion and conclusions.

II. RELATION BETWEEN THE EXPANSION OF THE GENERATING FUNCTIONAL AND RC-MAGIC SQUARES

To avoid the appearance of the imaginary unit, we start our study with the generating functional Z in euclidean ϕ^4 theory, coupled with an external field $J(x)$, $x \in \mathbb{R}^d$. According to the standard procedure [1], we have $Z(J)$ in the formal perturbative form:

$$Z(J) = \exp \left[-\mathcal{V} \left(\frac{\delta}{\delta J} \right) \right] \exp \left(\frac{1}{2} J \Delta J \right), \quad (1)$$

with

$$\frac{1}{2} J \Delta J \rightarrow \frac{1}{2} \int d^d y d^d z J(y) \Delta(y, z) J(z),$$

and

$$\mathcal{V} \left(\frac{\delta}{\delta J} \right) \rightarrow \frac{g}{4!} \int d^d x \frac{\delta^4}{\delta J^4(x)},$$

where $\Delta(y, z)$ is the free propagator and g the coupling constant.

The above expressions make sense only perturbatively. Thus, expanding the two exponentials and taking the formal limit $J \rightarrow 0$, $Z(J \rightarrow 0)$ generates the vacuum Feynman diagrams. Particularly, the m -order diagrams are in the expression

$$\frac{1}{(2m)! 2^{2m}} \frac{g^m}{(4!)^m m!} \int d^d x_1 \cdots d^d x_m \frac{\delta^4}{\delta J^4(x_1)} \cdots \times \frac{\delta^4}{\delta J^4(x_m)} \left[(J(y) \Delta(y, z) J(z))^{2m} \right]. \quad (2)$$

Note that the number of functional derivatives in the above expression is equal to the number of J 's in the derivative argument, therefore, this is the only term that survives for all m when $J \rightarrow 0$. We define

$$B(J, J, J, J) = (J \Delta J)^2 \quad (3)$$

applying the chain rule in (2) over the B products and adding the terms that generate the same function, we obtain a number $\mathcal{N}(m)$ of summing terms. Over each B in these terms act four functional derivatives. Assuming $\Delta(x, y) = \Delta(y, x)$ and defining

$$\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} \frac{\delta}{\delta J(z)} \frac{\delta}{\delta J(w)} [B] \equiv \aleph(x, y, z, w) \quad (4)$$

we have

$$\begin{aligned} \aleph(x, y, z, w) = & 8\Delta(x, y)\Delta(z, w) \\ & + 8\Delta(x, z)\Delta(y, w) + 8\Delta(x, w)\Delta(y, z). \end{aligned} \quad (5)$$

In particular we have

$$\aleph(x, x, y, y) = 8\Delta(x, x)\Delta(y, y) + 16\Delta^2(x, y) \quad (6)$$

$$\aleph(x, x, y, z) = 8\Delta(x, x)\Delta(y, z) + 16\Delta(x, y)\Delta(x, z) \quad (7)$$

$$\aleph(x, x, x, y) = 24\Delta(x, x)\Delta(x, y) \quad (8)$$

$$\aleph(x, x, x, x) = 24\Delta^2(x, x). \quad (9)$$

If we change the order of the variables in (5), we generate the same function. Now, each one of the $\mathcal{N}(m)$ summing terms have m products of \aleph 's functions. Let us think about the following problem, be a set of $4m$ balls

of m different colors such that, for each color, we have 4 balls. Be a box with m compartments such that every compartment only supports 4 balls. In how many different ways can these $4m$ balls be distributed in the box? It is evident that (2) is also a generating function of this combinatorial problem; there are $4m$ variables introduced by the $4m$ functional derivatives, the m variables $\{x_1, x_2, \dots, x_m\}$ correspond to the m colors, every \aleph is a compartment with four variables and the chain rule of the derivatives, after adding identical contributions, gives all the possibilities of distribution. Therefore, there are $\aleph(m)$ ways to distribute the $4m$ balls in the box.

An arbitrary configuration in the $4m$ distribution problem can be represented by a matrix (see TABLE I). The a_{ij} components are the number of j -color balls presents in the compartment i , evidently $a_{ij} \in \{0, 1, 2, 3, 4\}$ for all i and j . Every compartment has 4 balls, therefore, we have

$$\sum_{j=1}^m a_{ij} = 4,$$

for each i . In the same way, exists 4 balls of each color, then

$$\sum_{i=1}^m a_{ij} = 4$$

for each j . This means that each term of (2) in the \aleph 's multiplicative form can be represented by a matrix with equal row and column sum (RC-magic square).

TABLE I: An arbitrary configuration in the combinatorial $4m$ balls distribution problem.

	color 1	color 2	...	color m
Comp 1	a_{11}	a_{12}	...	a_{1m}
Comp 2	a_{21}	a_{22}	...	a_{2m}
...
Comp m	a_{m1}	a_{m2}	...	a_{mm}

An example for $m = 4$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{aligned} &\aleph(x_1, x_2, x_3, x_4) \times \aleph(x_1, x_1, x_1, x_2) \\ &\times \aleph(x_2, x_2, x_3, x_4) \times \aleph(x_3, x_3, x_4, x_4) \end{aligned} \quad (10)$$

This relation is implicit using the Table I, substituting the compartments by the \aleph functions, and the color i by the variable x_i . In our particular case, TABLE II exemplifies the association between RC-magic square and the \aleph 's product in (10).

TABLE II: The construction of the \aleph product associated with the RC-magic square in 10.

	x_1	x_2	x_3	x_4
First \aleph	1	1	1	1
Second \aleph	3	1	0	0
Third \aleph	0	2	1	1
Fourth \aleph	0	0	2	2

As we mentioned before, the chain rule in (2) contains all the possibles ways for constructing a RC-magic square. Some ways produce identical results, for example in (4) there are $4!$ ways of constructing $\aleph(x, y, z, w)$. In the general case, for an arbitrary RC-magic square $m \times m$, the multiplicity is the following product

$$\prod_{i=1}^m \left[\frac{4!}{a_{i1}! a_{i2}! \dots a_{im}!} \right], \quad (11)$$

for (10), we have the multiplicity $24 \times 4 \times 12 \times 6 = 6912$.

Now, for any RC-magic square the next question arises: what are the associated Feynman graphs? For a $m \times m$ matrix is necessary to multiply the m \aleph 's functions associated, this generates a sum of free propagators products, which correspond to the Feynman graphs. There is an algebraic way to implement this, each \aleph function have at most four x_i different variables, with $1 \leq i \leq m$, which are the vertices of a certain sum of propagators product terms given by one of the equations (5)-(9); every one of this terms is a graph, which can be represented by an $n \times n$ adjacency matrix (with n the different vertices of the graph, thus $1 \leq n \leq 4$). Given $\{x_a, x_b, x_c, x_d\}$ with $1 \leq a < b < c < d \leq m$, the possible adjacency matrices are

$$\aleph(x_a, x_b, x_c, x_d) \rightarrow 8 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + 8 \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + 8 \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (12)$$

$$\aleph(x_a, x_a, x_b, x_b) \rightarrow 8 \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + 16 \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad (13)$$

$$\aleph(x_a, x_a, x_b, x_c) \rightarrow 8 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 16 \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (14)$$

$$\aleph(x_a, x_b, x_b, x_c) \rightarrow 8 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} + 16 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (15)$$

$$\aleph(x_a, x_b, x_c, x_c) \rightarrow 8 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} + 16 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad (16)$$

$$\aleph(x_a, x_a, x_a, x_b) \rightarrow 24 \exp \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad (17)$$

$$\aleph(x_a, x_b, x_b, x_b) \rightarrow 24 \exp \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \quad (18)$$

$$\aleph(x_a, x_a, x_a, x_a) \rightarrow 24 \exp \begin{bmatrix} 4 \end{bmatrix} \quad (19)$$

Note that the usual order in the indices of the variables $\{x_a, x_b, x_c, x_d\}$ was chosen, this only to facilitate the construction of the adjacency matrices. The notation with brackets in the matrices means that we are working with adjacency matrices. Adjacency matrices are always symmetric square matrices (do not confuse with the RC-magic squares) whose coefficients represents the number of lines (edges) joining two vertices. So, $a_{ij} = a_{ji}$ is the number of lines joining the vertices x_i and x_j . If $j = i$ the number of lines of x_i in itself (loops) is usually multiplied by 2, therefore the diagonals coefficients are even, see ref.[15].

We can think of (12)-(19) as the 8 basic blocks to build all the Feynman graphs, namely, when we multiply the \aleph 's functions of a RC-magic square, the distributive mul-

tiplication property offers all the allowed ways to “assemble” the blocks. How we can interpreted this process in the adjacency matrix notation? Every one of this blocks have one $m \times m$ matrix representation, for example, to $\aleph(x_2, x_2, x_4, x_4)$ the 5×5 representation of (13) for $x_a = x_2$ and $x_b = x_4$ is

$$\aleph(x_2, x_2, x_4, x_4) \rightarrow 8 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + 16 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (20)$$

or, the 3×3 representation of (19) for $x_a = x_2$ is

$$\aleph(x_2, x_2, x_2, x_2) \rightarrow 24 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (21)$$

when we multiply two \aleph 's functions, the corresponding products of propagators are multiplied in every possible way (according to the distributive property) and this is equivalent to add the associated adjacency matrices in the $m \times m$ representation. This can be computationally implemented as follows: instead of replacing the propagator products by the associated adjacency matrices, we replace them by the exponential of a undefined functions

associated univocally with each $m \times m$ adjacency matrix. The computer program must interpret these functions as just numbers. The multiplicative property of the exponential guarantees that functions belonging to different \aleph 's be added. At the end, replacing each undefined function by the respective adjacency matrix, we get a sum of exponential matrices whose arguments are the adjacency matrices of the searched m -order Feynman graph. With this procedure we essentially work as if the adjacency matrices were all commuting which give us the right result. As an example, let us look at the Fourth order Feynman graphs associated with (10), in the adjacency matrix 4×4 representation. Abusing the notation:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \rightarrow \left\{ 8 \exp \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + 8 \exp \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + 8 \exp \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\} \times \left\{ 24 \exp \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} \\ \times \left\{ 8 \exp \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + 16 \exp \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right\} \times \left\{ 8 \exp \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} + 16 \exp \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \right\} \quad (22)$$

where it is clear that in (22) we are not dealing directly with matrices, but with their "representatives". After applying the distributive property we have 12 exponentials



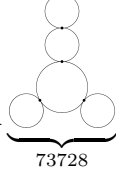
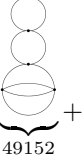
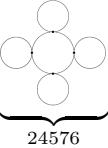

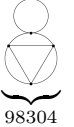
of an adjacency matrix. Some of this adjacency matrices generates the same Feynman graphs. For instance one of the terms is:

$$\rightarrow \left\{ 8 \exp \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right\} \times \left\{ 24 \exp \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} \times \left\{ 8 \exp \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right\} \times \left\{ 8 \exp \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \right\} \quad (23)$$

$$\rightarrow 8 \times 24 \times 8 \times 8 \exp \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \right\} \quad (24)$$

$$\rightarrow 12288 \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \rightarrow \underbrace{\text{○○○○}}_{12288} \quad (25)$$

In particular we have 7 different Feynman graphs with the respective multiplicities:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \rightarrow \underbrace{\text{diagram 1}}_{12288} + \underbrace{\text{diagram 2}}_{24576} + \underbrace{\text{diagram 3}}_{73728} + \underbrace{\text{diagram 4}}_{49152} + \underbrace{\text{diagram 5}}_{24576} + \underbrace{\text{diagram 6}}_{49152} + \underbrace{\text{diagram 7}}_{98304} \quad (26)$$








Note that the first two diagrams are disconnected. This process must be repeated for all the others possible RC-magic squares. At the end, after multiplying each Feynman graph subset by the respective factor (11), we add the multiplicities of all the equivalent diagrams generated. If we divide by $(2m)!2^{2m}$ all the multiplicities we obtain the Feynman graphs multiplicities used in ref.[8]. On the other hand, if we divide by $(4!)^m m!(2m)!2^{2m}$ we obtain for each diagram $1/s$, with s the corresponding Feynman graph symmetry factor (see this multiplicative factors directly in (2)).

III. THE PERMUTATION GROUP AND THE SET OF RC-MAGIC SQUARES

We see that the total set of RC-magic squares with row and column sum equal to 4 provide all the vacuum Feynman graphs and the respective multiplicities in ϕ^4 theory. As we will see later, the number $\mathcal{N}(m)$ of total $m \times m$ RC-magic squares grows rapidly with m . In particular for $m = 1, 2, 3, 4, 5 \dots$ we have for $\mathcal{N}(m)$

$$1, 5, 120, 10147, 2224955 \dots \quad (27)$$

these are many matrices. For example, in 4-order we should apply the procedure used in (26) over the remaining 10146 RC-magic squares.

However, there are possible shortcuts. Let us show that for growing m and for every $m \times m$ RC-magic square, there is a big number of equivalent RC-magic squares, which generates the same subset of vacuum Feynman graphs. This decreases considerably the number of RC-magic squares required to build all the vacuum Feynman graphs in m -order.

Suppose that one has performed a permutation of two different rows of a RC-magic square, evidently this is also a RC-magic square. In the \aleph representation this is equivalent to interchange the two \aleph 's function. Only that this generates the same product of \aleph 's functions since the product is the usual (commutative multiplication). Now, suppose that two different columns are permuted, as we see the columns are associated with one variable x_i , so in principle the permutations change the \aleph 's functions that contain the x_i and x_j variables. At the end, when applying the distributive property in the new RC-magic square, the free propagators products terms are

the same except that the variables x_i and x_j are interchanged. Thus, the column permutation correspond with one interchange of the vertices x_i and x_j in all the diagrams corresponding with the initial RC-magic square. Therefore columns and row permutation in a RC-magic square do not change the associated subset of vacuum Feynman graphs. It is evident that the multiplicity factor (11) is identical for RC-magic squares related by a row or column permutation.

Given an arbitrary matrix \mathbf{A} of dimension $m \times m$, the possible $m!$ row permutations can be expressed by means of the left action $\mathbf{P}_i \cdot \mathbf{A}$, with $\mathbf{P}_i \in \mathcal{P}_{m \times m}$ and $\mathcal{P}_{m \times m}$ the set of $m \times m$ permutation matrices. The right action $\mathbf{A} \cdot \mathbf{P}_i$ induces the $m!$ possible column permutations. Thus, given an matrix \mathbf{A} the set of different RC-magic squares obtained by row or column permutation $\mathbf{P}_i \cdot \mathbf{A} \cdot \mathbf{P}_j$ (including the matrix \mathbf{A}) contains at most $(m!)^2$ elements, (if there are matrices \mathbf{P}_a and \mathbf{P}_b such that $\mathbf{P}_a \cdot \mathbf{A} \cdot \mathbf{P}_b = \mathbf{A}$, this number is smaller). As we see, this different RC-magic squares generates identical sets of Feynman graphs.

The row and column permutation group induces an equivalence relation in the set of RC-magic squares with equivalence classes of at most $(m!)^2$ elements. At the end, we choose an arbitrary element (RC-magic square) \mathbf{A}_i in every equivalence class. Carrying out the procedure (22) and (26) in each matrix \mathbf{A}_i , we obtain a set of Feynman diagrams $\mathcal{F}[\mathbf{A}_i]$ (with the respective generated multiplicities). The corresponding total multiplicities and all the possible diagrams in m -order are generated by

$$\sum_i \mathfrak{N}_{\mathbf{A}_i} \times \prod_{j=1}^m \left[\frac{4!}{a_{j1}! a_{j2}! \dots a_{jm}!} \right]_i \times \mathcal{F}[\mathbf{A}_i], \quad (28)$$

where i indexes each equivalence class, $\mathfrak{N}_{\mathbf{A}_i}$ is the number of elements present in the equivalence class i , and a_{jk} are the components of the chosen matrix \mathbf{A}_i . As discussed below, the number of equivalence classes, and therefore, the number of RC-magic squares required to generate the Feynman graphs for $m = 1, 2, 3, 4, 5, \dots$ is respectively

$$1, 3, 9, 43, 264 \dots \quad (29)$$

at least until 5-order, the number of equivalence classes match with the OEIS sequence A232216 generated in [16]. Given the total set of RC-magic squares for every

order and the permutation matrices, the Burnside lemma in group theory [17] allows to find this sequence.

As an example, we generate all the third-order Feyn-

man graphs in ϕ^4 theory. There are 9 equivalence classes of 3×3 RC-magic squares whose row and column sum is 4. We choose the following nine representatives

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad (30)$$

with the number of elements $\mathfrak{N}_{3 \times 3}$ $\{6, 18, 9, 18, 12, 36, 6, 9, 6\}$ in each equivalence class respectively, let us determine $\mathcal{F}[\mathbf{A}_i]$ for this nine matri-

ces, in accordance with the procedure in (22) and (26) for $m = 3$.

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \rightarrow \underbrace{\text{○○○○○○}}_{13824} \quad (31)$$

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} \rightarrow \underbrace{\text{○○○○○○}}_{13824} \quad (32)$$

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \underbrace{\text{○○○○○○}}_{1536} + \underbrace{\text{○○○○○○}}_{6144} + \underbrace{\text{○○○○○○}}_{6144} \quad (33)$$

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix} \rightarrow \underbrace{\text{○○○○○○}}_{4608} + \underbrace{\text{○○○○○○}}_{9216} \quad (34)$$

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 3 \end{pmatrix} \rightarrow \underbrace{\text{○○○○○○}}_{13824} \quad (35)$$

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix} \rightarrow \underbrace{\text{○○○○○○}}_{1536} + \underbrace{\text{○○○○○○}}_{3072} + \underbrace{\text{○○○○○○}}_{3072} + \underbrace{\text{○○○○○○}}_{6144} \quad (36)$$

$$\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{pmatrix} \rightarrow \underbrace{\text{○○○○○○}}_{512} + \underbrace{\text{○○○○○○}}_{3072} + \underbrace{\text{○○○○○○}}_{6144} + \underbrace{\text{○○○○○○}}_{4096} \quad (37)$$

$$\begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} \rightarrow \underbrace{\text{[5 circles in a row]}}_{512} + \underbrace{\text{[2 circles with a line between them]}}_{1024} + \underbrace{\text{[4 circles in a row]}}_{2048} + \underbrace{\text{[triangle with a circle inside]}}_{4096} + \underbrace{\text{[3 circles in a triangle]}}_{2048} + \underbrace{\text{[2 circles with a line between them]}}_{4096} \quad (38)$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \rightarrow \underbrace{\text{[3 circles in a triangle]}}_{512} + \underbrace{\text{[4 circles in a row]}}_{3072} + \underbrace{\text{[2 circles with a line between them]}}_{6144} + \underbrace{\text{[triangle with a circle inside]}}_{4096} \quad (39)$$

the size $\mathfrak{N}_{\mathbf{A}_i} \leq 36$ of each equivalence class is respectively 6, 18, 9, 18, 12, 36, 6, 9, 6 and the product (11) associated with each matrix (which is identical for all the matrices in the equivalence class) is respectively 1, 16, 36, 192, 64, 288, 216, 864, 1728. Thus, applying the formula (28) and adding the topological equivalent graphs we obtain the total multiplicities \mathcal{M}_T of each third order vacuum graph. If we divide \mathcal{M}_T by $6! \times 2^6$ we obtain the multiplicities \mathcal{M}_K obtained by Kleinert in ref.[8], and dividing by $(4!)^3 \times 3! \times 6! \times 2^6$ we obtain $1/s$ with s the corresponding symmetry factor of each graph. In particular

$$\underbrace{\text{[5 circles in a row]}}_{\mathcal{M}_T=1244160, \mathcal{M}_K=27, s=3072} \quad (40)$$

$$\underbrace{\text{[4 circles in a row]}}_{\mathcal{M}_T=29859840, \mathcal{M}_K=648, s=128} \quad (41)$$

$$\underbrace{\text{[2 circles with a line between them]}}_{\mathcal{M}_T=9953280, \mathcal{M}_K=216, s=384} \quad (42)$$

$$\underbrace{\text{[3 circles in a triangle]}}_{\mathcal{M}_T=79626240, \mathcal{M}_K=1728, s=48} \quad (43)$$

$$\underbrace{\text{[4 circles in a row]}}_{\mathcal{M}_T=119439360, \mathcal{M}_K=2592, s=32} \quad (44)$$

$$\underbrace{\text{[2 circles with a line between them]}}_{\mathcal{M}_T=159252480, \mathcal{M}_K=3456, s=24} \quad (45)$$

$$\underbrace{\text{[triangle with a circle inside]}}_{\mathcal{M}_T=79626240, \mathcal{M}_K=1728, s=48} \quad (46)$$

IV. THE SET OF RC-MAGIC SQUARES

In our construction, the calculus of the $m \times m$ RC-magic square representatives is important to determine the set and multiplicities of all the m -order Feynman graphs. This calculation can be implemented by the following algorithm, which consists of two parts: the first one build a set \mathcal{A} of matrices which contains at least one member of each equivalence class. The second one discard the equivalent elements and only preserve one element matrix of each equivalence class. At first sight, and in order to move forward we are forced to use a “maximal” construction of the set \mathcal{A} which build the matrices, testing all the possible constructions. In the $m \times m$ case such construction can be arduous, since when testing all the possibilities some must be discarded for not satisfying the equal row and column sum; what would be equivalent to analyzing a vast number of possibilities.

Fortunately, the equivalence relation induced by the permutation group can be induced in any $n \times m$ set of matrices. In this case, the left action operation $\mathbf{P}_i \cdot \mathbf{A}$ is induced by the set $\mathcal{P}_{n \times n}$ of permutation matrices. Given a set of $n \times m$ matrix representatives of the total $n \times m$ matrix set with row sum equal to 4 and column sum less or equal to 4 and $n < m$, we can use the algorithm to calculate the matrices representatives in the same total set of $(n+1) \times m$ matrices. In this case, the set $\mathcal{A} = \mathcal{A}_{n+1}$ is constructed adding a new $n+1$ row to all the $n \times m$ matrices representatives and testing all the possibles $n+1$ rows such that the $(n+1) \times m$ matrix constructed have row sum equal to 4 and column sum less or equal to 4. The set \mathcal{A}_{n+1} is formed by all the $(n+1) \times m$ constructed matrices that satisfy such condition. This set contains at least one element of each equivalence class of the $(n+1) \times m$ matrix total set with the mentioned sum condition.

Since we take all the possible new row additions in each $n \times m$ matrix representative, this must be the case.

The second part of the algorithm work in the following way, listing each element of \mathcal{A}_{n+1} in a list, we take all the possible permutations for each matrix \mathbf{A}_i in \mathcal{A}_{n+1} this produce the set \mathfrak{P}_i . The set $\mathcal{G}_i = \mathfrak{P}_i \cap \mathcal{A}_{n+1}$ is formed by all the $(n+1) \times m$ matrices equivalent to \mathbf{A}_i (including \mathbf{A}_i) contained in \mathcal{A}_{n+1} . Be \mathfrak{H}_i the position of each element of \mathcal{G}_i in the list \mathcal{A}_{n+1} , \mathfrak{H}_i is a set of different positive integers. For two equivalent matrices \mathbf{A}_i and \mathbf{A}_j we have $\mathfrak{H}_i = \mathfrak{H}_j$, instead if \mathbf{A}_i and \mathbf{A}_j are not equivalent we have $\mathfrak{H}_i \cap \mathfrak{H}_j = \emptyset$. Taking all the different \mathfrak{H}_i sets, we choose one element (for example the first) in every \mathfrak{H}_i .

Taking the corresponding elements in the list \mathcal{A}_{n+1} , we obtain a matrix representative set of the $(n+1) \times m$ total set of matrices with row sum equal to 4 a column sum less or equal to 4.

Thus, the $m \times m$ representatives can be obtained applying the algorithm $m-1$ times starting to the $1 \times m$ matrix representatives. Obtained the set of $(m-1) \times m$ representatives exist a unique way to build each element of \mathcal{A}_m which is a set of $m \times m$ RC-magic squares. Applying the second part of the algorithm to \mathcal{A}_m , we obtain the $m \times m$ RC-magic squares representatives. For $m \geq 4$ we have five $1 \times m$ representatives which correspond with the sum partitions of 4

$$(4, 0, \dots, 0), \quad (3, 1, 0, \dots, 0), \quad (2, 2, 0, \dots, 0), \quad (2, 1, 1, 0, \dots, 0), \quad (1, 1, 1, 1, 0, \dots, 0) \quad (47)$$

if $m = 2$ the 1×2 representatives are $(4, 0), (3, 1), (2, 2)$ and if $m = 3$ we have $(4, 0, 0), (3, 1, 0), (2, 2, 0), (2, 1, 1)$.

This algorithm, which code is implemented in APPENDIX.A, allows the calculation of the matrices representatives and the number of elements in each equivalence class. Anyhow, the algorithm realize left and right multiplications of matrices with the total set of permutations in the matricial representation (which, implements the column and row permutations). For each matrix, we have $(m!)^2$ products of matrices, thus for growing m the computation represents a huge challenge, but a one that may have some ways of dealing with.

The group of permutations not only induces equivalence classes in the total set of RC-magic squares, it has a deeper relationship with this set due to the Birkhoff-von Neumann theorem[14]. This theorem, applied to our case, states that any $m \times m$ RC-magic square \mathbf{A} is a convex combination of permutations matrices

$$\mathbf{A} = \lambda_1 \mathbf{P}_1 + \dots + \lambda_k \mathbf{P}_k, \quad \sum_{j=1}^k \lambda_j = 4 \quad (48)$$

with $\mathbf{P}_j \in \mathcal{P}_{m \times m}$ and $\lambda_j \in \{1, 2, 3, 4\}$. In particular the only possible λ 's sets in one convex decomposition are $\{4\}$, $\{3, 1\}$, $\{2, 2\}$, $\{2, 1, 1\}$ and $\{1, 1, 1, 1\}$. For some RC-magic squares, this decomposition is not unique. Apart from this possible repetition of the matrices, combining (48) in all the possible ways, we generate the total set of $m \times m$ RC-magic squares.

We devise, below, a way to characterize the set of representatives in a more efficient way with the help of the Birkhoff-von Neumann theorem. As an example at third order there are the following six permutations matrices

$$\begin{aligned} \mathbf{P}_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \mathbf{P}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathbf{P}_4 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \mathbf{P}_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathbf{P}_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned} \quad (49)$$

The convex decompositions of the 9 representative matrices (30) are

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4\mathbf{P}_1 \quad (50)$$

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} = 3\mathbf{P}_1 + \mathbf{P}_2 \quad (51)$$

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} = 2\mathbf{P}_1 + 2\mathbf{P}_2 \quad (52)$$

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix} = \mathbf{P}_1 + \mathbf{P}_4 + 2\mathbf{P}_2 \quad (53)$$

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 3 \end{pmatrix} = 3\mathbf{P}_1 + \mathbf{P}_4 \quad (54)$$

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix} = 2\mathbf{P}_1 + \mathbf{P}_4 + \mathbf{P}_2 \quad (55)$$

$$\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{pmatrix} = 2\mathbf{P}_1 + 2\mathbf{P}_4 \quad (56)$$

$$\begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{P}_4 \quad (57)$$

the last matrix have two possible descompositions:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = 2\mathbf{P}_1 + \mathbf{P}_4 + \mathbf{P}_5 = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{P}_6 \quad (58)$$

If we take all the subsets of $\mathcal{P}_{3 \times 3}$ with 1, 2, 3 and 4 elements respectively, we see that the elements in the subsets have a property that characterize in an unambiguous way each representative:

- The one-element subsets characterize completely the equivalence class represented by (50), in this equation any matrix \mathbf{P}_j establish one valid representative.
- The two-elements subsets can be divided in two parts with two properties: the two elements have one identical component or do not have any common component. The two permutation matrices in (51) and (52) have one component in common, instead the two permutation matrices in 54 and 56 do not have any common component. If we change this two permutation matrices by another satisfying the same property, we obtain an equivalent RC-magic square (that is to say another RC-magic square in the same equivalence class). The interchange of the λ 's only produces one permutation in the RC-magic square.
- The three-elements subsets can be divided in two parts: The three permutations matrices have common components (in particular, one permutation matrix have two different components in common, one component with each one of the other matrices, and the other two matrices do not have any common component between them) or do not have any common component. The three permutation

matrices in (53) have the first property, with the fact that $\lambda = 2$ is associated with the matrix that have two component in common with the other two. The three permutation matrices in (55) also have the first property with the difference that $\lambda = 2$ is associated with anyone of the two matrices that do not have common components between them. In (58) the three permutations matrices do not have components in common. Here also the substitution of the permutation matrices by other with the same property produce an equivalent RC-magic square.

- At last, the four element subsets can be divided in two parts corresponding with two different properties in the number of common components (here the properties are in the number of common elements). The permutations matrices in (57) and (58) have this properties respectively and the substitution by other matrices with the same properties produce equivalent RC-magic squares.

This construction can be generalized to larger orders. Particularly, it allows to build a technique to know if two arbitrary magic squares are not equivalent. Suppose that the arbitrary $m \times m$ RC-magic square \mathbf{A} have n different convex decompositions, each one with a particular common component property between the corresponding permutation matrices. Any RC-magic square \mathbf{A}' equivalent to \mathbf{A} satisfy $\mathbf{A}' = \mathbf{P}_a \cdot \mathbf{A} \cdot \mathbf{P}_b$, with \mathbf{P}_a and $\mathbf{P}_b \in \mathcal{P}_{m \times m}$. Thereby, for a particular convex decomposition we have

$$\mathbf{A}' = \mathbf{P}_a \cdot \left(\sum_{j=1}^k \lambda_j \mathbf{P}_j \right) \cdot \mathbf{P}_b = \sum_{j=1}^k \lambda_j \mathbf{P}_a \cdot \mathbf{P}_j \cdot \mathbf{P}_b = \sum_{j=1}^k \lambda_j \mathbf{P}'_j \quad (59)$$

Since the permutation operation is identical for all the permutation matrices \mathbf{P}_j , (59) will be one convex decomposition for \mathbf{A}' with the same common component property that the convex decomposition $\sum_{j=1}^k \lambda_j \mathbf{P}_j$ for \mathbf{A} . Since we take one arbitrary convex decomposition, the same is valid for all the other $n - 1$ decompositions.

Suppose now that \mathbf{A}' have other different convex decomposition in addition to the n convex decompositions founded. Since $\mathbf{A} = \mathbf{P}_a^{-1} \cdot \mathbf{A}' \cdot \mathbf{P}_b^{-1}$, this implies that \mathbf{A} have $n + 1$ convex decompositions. This is a contradiction since by hypothesis \mathbf{A} have only n different decompositions. Thus, defining $\mathcal{D}_\mathbf{A}$ and $\mathcal{D}_\mathbf{B}$ as the sets that contains all the convex decomposition of \mathbf{A} and \mathbf{B} respectively, we have the following result

- If the RC-magic square \mathbf{A} and \mathbf{B} are equivalent, then the two convex decompositions sets $\mathcal{D}_\mathbf{A}$ and $\mathcal{D}_\mathbf{B}$ have the same number of elements and an one to one correspondence between $\mathcal{D}_\mathbf{A}$ and $\mathcal{D}_\mathbf{B}$ such that the decompositions related with the correspondence have the same common component property.

The contrapositive equivalent formulation of this result

is more useful:

- If the sets $\mathcal{D}_\mathbf{A}$ and $\mathcal{D}_\mathbf{B}$ have different number the elements or if the elements of this sets can not be matched in an one to one correspondence with the common component property, then \mathbf{A} and \mathbf{B} are not equivalent.

Thus, we have one useful and simple criterion to check if two arbitrary RC-magic squares are not equivalent. The known Hall theorem in combinatorial theory [15] offers a way to find the complete set of convex decomposition of an arbitrary RC-magic square. The converse of this result is not true, that is to say, if the sets $\mathcal{D}_\mathbf{A}$ and $\mathcal{D}_\mathbf{B}$ have the same number of elements and an one to one correspondence that maps each element of $\mathcal{D}_\mathbf{A}$ in an respective element of $\mathcal{D}_\mathbf{B}$ with the same common component property, then the matrices \mathbf{A} and \mathbf{B} are not necessarily equivalent. As an example, consider the following two 4×4 RC-magic squares with an unique convex decomposition

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 2 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (60)$$

and

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (61)$$

The two convex decompositions in each equation (60) and (61) have the same common component property, and each RC-magic square an unique convex decomposition. But this matrices are not equivalent.

V. DISCUSSION AND CONCLUSION

The construction of Feynman graphs in quantum field theory is essentially a combinatorial problem. The question of how many ways exist to construct Feynman graphs given an number of vertices, edges and explicit rules (incidence numbers in graph theoretical language) for connecting vertices and edges, refers to traditional

combinatorics problems which can be thought independently of any physical application. In combinatorics it is usual to find equivalences and relations between apparently different problems, in this work we find one combinatorial relation between counting vacuum Feynman graphs in ϕ^4 theory, and the counting of a specific type of combinatorial matrices whose columns and rows coefficients sum 4, which in turn defines one combinatorial distribution problem (the $4m$ distribution ball in sec.II). From a mathematical perspective, this relation can be understood as an application of combinatorial matrix theory in graph theory.

From a physical perspective, two questions arise: does the combinatorial approach of this work simplify the calculation of the multiplicities? (or the symmetry factor) and, does the specific combinatorial character shown here have some direct and physical implication manifested in the integrals represented by the Feynman graphs?

The first question is related to the importance in knowing the multiplicities for non-perturbative calculations, which appears explicitly in sequences of Feynman graphs what could have some physical content. In this respect, the weight in Feynman diagrams are important. Only algorithmic and recursive calculations are known, this work transfers the algorithmic calculus in Feynman diagrams to the set of RC-magic squares (which are more tractable objects). There are many RC-magic squares, and we show that the relation with the permutation group decreases the number of RC-magic squares required for constructing the Feynman graphs. In appendix A we implement a code for the algorithm described at the beginning of sec.IV for the calculation of the $m \times m$ RC-magic squares representatives and the size \mathfrak{N}_i of each equivalence class, which works with all the elements of the permutation group $\mathcal{P}_{m \times m}$. We consider that an efficient computation of the RC-magic squares representatives and of the number \mathfrak{N}_i would simplify considerably the construction

of the Feynman graphs and the respective multiplicities. The Birkhoff-von Neumann theorem, could be the beginning of a more efficient algorithm. We believe that the present work offers an initial insight for such algorithmic construction.

The second question merits further study. We hope to investigate this in the near future. It is obvious that our construction can be easily generalised for any ϕ^N theory for $N \geq 3$ integer. The philosophy of this work can also be useful for any other perturbative quantum field theory, since the generating functional and the derivative functional construction process also are present.

ACKNOWLEDGMENTS

The authors thank the Brazilian agencies CNPq and Capes for partial financial support.

Appendix A: Computation of the RC-magic squares representatives and the size of the equivalence classes

Until now we saw that finding one representative matrix \mathbf{A}_i in each equivalence class, and finding the size $\mathfrak{N}_{\mathbf{A}_i}$ of each equivalence class determine the m -order vacuum Feynman graphs (see this in (28)). In this appendix we implement the straightforward algorithm mentioned at the beginning of sec.IV for the determination of \mathbf{A}_i and $\mathfrak{N}_{\mathbf{A}_i}$ using the program MATHEMATICA [18], which works, in principle, for all orders. The code builds up the RC-magic squares row by row, beginning with five $1 \times m$ matrices for $m \geq 4$ (for $m < 4$ we have a lower number of $1 \times m$ matrices). For 4-order these matrices are $(4, 0, 0, 0), (3, 1, 0, 0), (2, 2, 0, 0), (2, 1, 1, 0), (1, 1, 1, 1)$. To obtain the matrices 2×4 we define

```

A[b1_, b2_, b3_, b4_] := DeleteCases[Flatten[Table[MatrixForm[{ {
  {b1, b2, b3, b4 },
  {a1, a2, a3, a4}
} ]*KroneckerDelta[a1 + a2 + a3 + a4, 4]*
HeavisideTheta[-(a1 + b1) + 4.1]*
HeavisideTheta[-(a2 + b2) + 4.1]*
HeavisideTheta[-(a3 + b3) + 4.1]*
HeavisideTheta[-(a4 + b4) + 4.1]], {a1, 0, 4}, {a2, 0,
4 - a1}, {a3, 0, 4 - a2 - a1}, {a4, 0, 4 - a3 - a2 - a1}]], MatrixForm[{ {
{0, 0, 0, 0},
{0, 0, 0, 0}
} ]]]

```

Note that the function A has four variables, which correspond with the components of the five 1×4 matrices.

Thus, we define the set

A2=Union[A[4,0,0,0],A[3,1,0,0],A[2,2,0,0],A[2,1,1,0],A[1,1,1,1]]

It is important locate all the matrices of A2 in the TraditionalForm format, what is achieved by applying the command

MatrixForm [...]

in each matrix of A2. After, we define B2 = A2 in

StandardForm. A2 is the form appropriate in which MATHEMATICA interpret correctly the matrix elements of the set. While B2 is the appropriate form for multiply matrices. Defining the permutation matrices sets $\mathcal{P}_{2 \times 2}$, $\mathcal{P}_{3 \times 3}$, $\mathcal{P}_{4 \times 4}$ in StandardForm as b, c and d respectively

b={{1, 0}, {0, 1}}, {{0, 1}, {1, 0}}

c= {{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}, {{1, 0, 0}, {0, 0, 1}, {0, 1, 0}},
 {{0, 1, 0}, {1, 0, 0}, {0, 0, 1}}, {{0, 1, 0}, {0, 0, 1}, {1, 0, 0}},
 {{0, 0, 1}, {1, 0, 0}, {0, 1, 0}}, {{0, 0, 1}, {0, 1, 0}, {1, 0, 0}}}

d= {{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}},
 {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}},
 {{1, 0, 0, 0}, {0, 0, 1, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}},
 {{1, 0, 0, 0}, {0, 0, 0, 1}, {0, 1, 0, 0}, {0, 0, 1, 0}},
 {{1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}},
 {{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}},
 {{0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}},
 {{0, 1, 0, 0}, {0, 0, 1, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}},
 {{0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}},
 {{0, 1, 0, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 0, 1, 0}},
 {{0, 1, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}, {1, 0, 0, 0}},
 {{0, 0, 1, 0}, {1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}},
 {{0, 0, 1, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}, {0, 1, 0, 0}},
 {{0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 1}},
 {{0, 0, 1, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}},
 {{0, 0, 1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, 0, 1, 0}, {0, 0, 0, 1}, {0, 1, 0, 0}, {1, 0, 0, 0}},
 {{0, 0, 0, 1}, {1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}},
 {{0, 0, 0, 1}, {1, 0, 0, 0}, {0, 0, 1, 0}, {0, 1, 0, 0}},
 {{0, 0, 0, 1}, {0, 1, 0, 0}, {1, 0, 0, 0}, {0, 0, 1, 0}},
 {{0, 0, 0, 1}, {0, 1, 0, 0}, {0, 0, 1, 0}, {1, 0, 0, 0}},
 {{0, 0, 0, 1}, {0, 0, 1, 0}, {1, 0, 0, 0}, {0, 1, 0, 0}},
 {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}}}

which have 2, 6, 24 permutation matrices.

We define the two functions

G[n_] := Intersection[
 Flatten[Table[
 MatrixForm[b[[i]].B2[[n]].d[[j]]], {i, 1, 2}, {j, 1, 24}]], A2]

H[n_] := Flatten[
 Table[Position[A2, G[n][[j]]], {j, 1,
 First[First[Position[G[n], Last[G[n]]]]]]]

Finally we get the desired set of 2×4 matrices by

Table[A2[[n]], {n,
 DeleteDuplicates[Table[First[H[n]], {n, 1, Length[A2]}]]]

Producing a total of 32 matrices 2×4 . To get the matrices 3×4 we repeat the procedure, the function A will have now 8 variables

```

A[b1_, b2_, b3_, b4_, c1_, c2_, c3_, c4_] :=
DeleteCases[Flatten[Table[MatrixForm[( {
{b1, b2, b3, b4},
{c1, c2, c3, c4},
{a1, a2, a3, a4}
} )*KroneckerDelta[a1 + a2 + a3 + a4, 4]*
HeavisideTheta[-(a1 + b1 + c1) + 4.1]*
HeavisideTheta[-(a2 + b2 + c2) + 4.1]*
HeavisideTheta[-(a3 + b3 + c3) + 4.1]*
HeavisideTheta[-(a4 + b4 + c4) + 4.1]], {a1, 0, 4}, {a2, 0,
4 - a1}, {a3, 0, 4 - a2 - a1}, {a4, 0, 4 - a3 - a2 - a1}]],
MatrixForm[( {
{0, 0, 0, 0},
{0, 0, 0, 0},
{0, 0, 0, 0}
} )]]

```

in the A3 set we use the 32 matrices 2×4

```

A3= Union[A[1, 1, 1, 1, 0, 0, 1, 3], A[1, 1, 1, 1, 0, 0, 2, 2],
A[1, 1, 1, 1, 0, 1, 1, 2], A[1, 1, 1, 1, 1, 1, 1, 1],
A[2, 1, 1, 0, 0, 0, 0, 4], A[2, 1, 1, 0, 0, 0, 1, 3],
A[2, 1, 1, 0, 0, 0, 2, 2], A[2, 1, 1, 0, 0, 0, 3, 1],
A[2, 1, 1, 0, 0, 1, 1, 2], A[2, 1, 1, 0, 0, 1, 2, 1],
A[2, 1, 1, 0, 0, 1, 3, 0], A[2, 1, 1, 0, 0, 2, 2, 0],
A[2, 1, 1, 0, 1, 0, 0, 3], A[2, 1, 1, 0, 1, 0, 2, 1],
A[2, 1, 1, 0, 1, 0, 3, 0], A[2, 1, 1, 0, 1, 1, 2, 0],
A[2, 1, 1, 0, 2, 0, 0, 2], A[2, 1, 1, 0, 2, 0, 1, 1],
A[2, 1, 1, 0, 2, 0, 2, 0], A[2, 1, 1, 0, 2, 1, 1, 0],
A[2, 2, 0, 0, 0, 0, 0, 4], A[2, 2, 0, 0, 0, 0, 1, 3],
A[2, 2, 0, 0, 0, 0, 2, 2], A[2, 2, 0, 0, 0, 1, 0, 3],
A[2, 2, 0, 0, 0, 2, 0, 2], A[2, 2, 0, 0, 2, 2, 0, 0],
A[3, 1, 0, 0, 0, 0, 0, 4], A[3, 1, 0, 0, 0, 0, 1, 3],
A[3, 1, 0, 0, 0, 1, 0, 3], A[3, 1, 0, 0, 0, 3, 0, 1],
A[3, 1, 0, 0, 1, 3, 0, 0], A[4, 0, 0, 0, 0, 0, 0, 4]]

```

with A3 in the traditional form, B3 = A3 in standard form. Applying consecutively the last three commands

```

G[n_] := Intersection[
Flatten[Table[
MatrixForm[c[[i]].B3[[n]].d[[j]]], {i, 1, 6}, {j, 1, 24}]], A3]

H[n_] := Flatten[
Table[Position[A3, G[n][[j]]], {j, 1,
First[First[Position[G[n], Last[G[n]]]]]}]]

Table[A3[[n]], {n,
DeleteDuplicates[Table[First[H[n]], {n, 1, Length[A3]}]]}]

```

we obtain 103 matrices 3×4 . repeating the procedure again we obtain the desired 43 RC-magic squares representatives of dimension 4×4 . The process is generalisable

for others orders.

To determine the equivalence class size $\mathfrak{N}_{\mathbf{R}}$ of a representative $m \times m$ matrix \mathbf{R} we use

```

Length[DeleteDuplicates[
Flatten[Table[
MatrixForm[d[[i]].R.d[[j]], {i, 1, m!}, {j, 1, m!}]]]]]

```

with \mathbf{R} written in Standard Form. For example for

$$\mathbf{R} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix} \quad (\text{A1})$$

```

Length[DeleteDuplicates[
Flatten[Table[
MatrixForm[d[[i]].
{{2, 1, 1, 0}, {0, 0, 1, 3}, {0, 2, 1, 1}, {2, 1, 1, 0}}.d[[j]],
{i, 1, 24}, {j, 1, 24}]]]]]

```

obtaining $\mathfrak{N}_{\mathbf{R}} = 288$. The sequence (27) also can be obtained summing all the $\mathfrak{N}_{\mathbf{A}_i}$ for each order. For example, in four order

$$\sum_{i=1}^{43} \mathfrak{N}_{\mathbf{A}_i} = 10147 \quad (\text{A2})$$

Appendix B: All the fourth and fifth-order Feynman vacuum graphs multiplicities

We will now to write the multiplicities of all the connected Feynman vacuum graphs for orders four and five. We use the program MATHEMATICA to perform explicitly the multiplicative process (22). As we see, for each RC-magic square representative, we get one sum of exponentials, whose multiplicative coefficient are the multiplicities and the argument the adjacency matrix of a graph; at the end we use (28) and we get a single set of adjacency matrices with the respective total multiplicities. Listing the multiplicities and the adjacency matrices in two list in such a way that each adjacency matrix and the corresponding multiplicity be indexed by the same positive integer. We adding the multiplicities of all the topologically equivalent graphs. In particular, if two graphs \mathcal{G}_1 and \mathcal{G}_2 of order m are equivalent, then the corresponding $m \times m$ adjacency matrices $\mathbf{A}_{\mathcal{G}_1}$ and $\mathbf{A}_{\mathcal{G}_2}$ are connected by an row-column permutation of the kind

$$\mathbf{A}_{\mathcal{G}_2} = \mathbf{P}_a \cdot \mathbf{A}_{\mathcal{G}_1} \cdot \mathbf{P}_b \quad (\text{B1})$$

with \mathbf{P}_a and \mathbf{P}_b two permutation matrices in $\mathcal{P}_{m \times m}$ such that $\mathbf{P}_a \cdot \mathbf{P}_b = \mathbf{I}$ with \mathbf{I} the $m \times m$ identity. Given $\mathbf{A}_{\mathcal{G}}$,

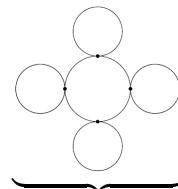
how do calculate the equivalent diagrams? We list all the different products $\mathbf{P}_i \cdot \mathbf{A}_{\mathcal{G}} \cdot \mathbf{P}_j$, and intersects with the previous adjacency matrix list. This gives all the adjacency matrices of the equivalent graphs, included $\mathbf{A}_{\mathcal{G}}$. The MATHEMATICA function Position[...] determine the position of all this matrices in the adjacency matrix list. Using this information in the multiplicity list, (since each adjacency matrix and the associated multiplicity are indexed by the same number in the two lists) we add all the multiplicities corresponding to \mathcal{G} giving the total multiplicity.

For disconnected graphs we verify for fourth and fifth order the rule shown in [5]. Particularly for a disconnected graph with l connected components of which r are different, we verify that the symmetry factor is

$$s_d = n_1! \cdots n_r! \times s_1^{n_1} \cdots s_r^{n_r} \quad (\text{B2})$$

with s_i the symmetry factor of the component i , n_i the times it repeats and $n_1 + \cdots + n_r = l \geq r$. Thus, we will only write the multiplicities for the connected Feynman graphs.

1. Fourth-order connected diagrams



$$\mathcal{M}_{\mathbf{T}}=642105999360, \mathcal{M}_{\mathbf{K}}=62208, s=128 \quad (\text{B3})$$

$\mathcal{M}_T=2568423997440, \mathcal{M}_K=248832, s=32$

(B4)

$\mathcal{M}_T=642105999360, \mathcal{M}_K=62208, s=128$

(B11)

$\mathcal{M}_T=1712282664960, \mathcal{M}_K=165888, s=48$

(B5)

$\mathcal{M}_T=2568423997440, \mathcal{M}_K=248832, s=32$

(B12)

2. Fifth-order connected diagrams

$\mathcal{M}_T=1712282664960, \mathcal{M}_K=165888, s=48$

(B6)

$\mathcal{M}_T=27738979172352000, \mathcal{M}_K=7464960, s=128$

(B13)

$\mathcal{M}_T=2568423997440, \mathcal{M}_K=248832, s=32$

(B7)

$\mathcal{M}_T=110955916689408000, \mathcal{M}_K=29859840, s=32$

(B14)

$\mathcal{M}_T=5136847994880, \mathcal{M}_K=497664, s=16$

(B8)

$\mathcal{M}_T=221911833378816000, \mathcal{M}_K=59719680, s=16$

(B15)

$\mathcal{M}_T=1284211998720, \mathcal{M}_K=124416, s=64$

(B9)

$\mathcal{M}_T=221911833378816000, \mathcal{M}_K=59719680, s=16$

(B16)

$\mathcal{M}_T=570760888320, \mathcal{M}_K=55296, s=144$


(B10)

$\mathcal{M}_T=11095591668940800, \mathcal{M}_K=2985984, s=320$

(B17)

$\mathcal{M}_T=110955916689408000, \mathcal{M}_K=29859840, s=32$

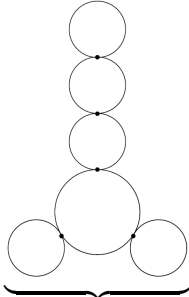
(B18)



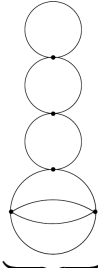
(B19) $\mathcal{M}_T=221911833378816000, \mathcal{M}_K=59719680, s=16$



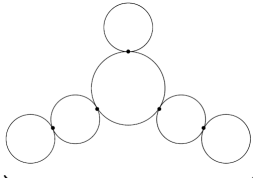
(B20) $\mathcal{M}_T=29588244450508800, \mathcal{M}_K=7962624, s=120$



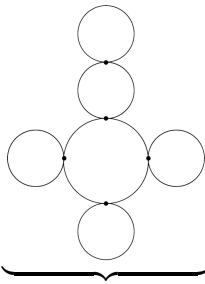
(B21) $\mathcal{M}_T=55477958344704000, \mathcal{M}_K=14929920, s=64$



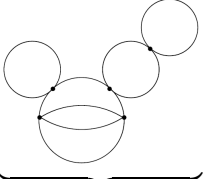
(B22) $\mathcal{M}_T=36985305563136000, \mathcal{M}_K=9953280, s=96$



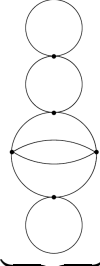
(B23) $\mathcal{M}_T=55477958344704000, \mathcal{M}_K=14929920, s=64$



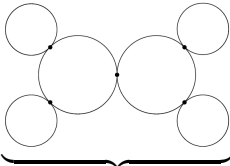
(B24) $\mathcal{M}_T=55477958344704000, \mathcal{M}_K=14929920, s=64$



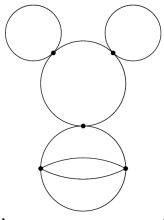
(B25) $\mathcal{M}_T=73970611126272000, \mathcal{M}_K=19906560, s=48$



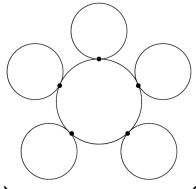
(B26) $\mathcal{M}_T=110955916689408000, \mathcal{M}_K=29859840, s=32$



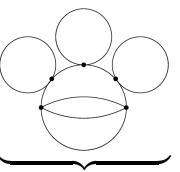
(B27) $\mathcal{M}_T=27738979172352000, \mathcal{M}_K=7464960, s=128$



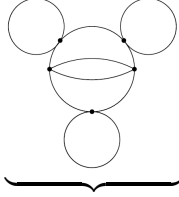
(B28) $\mathcal{M}_T=36985305563136000, \mathcal{M}_K=9953280, s=96$



(B29) $\mathcal{M}_T=11095591668940800, \mathcal{M}_K=2985984, s=320$

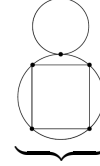


(B30) $\mathcal{M}_T=36985305563136000, \mathcal{M}_K=9953280, s=96$



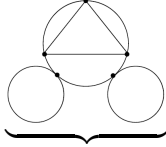
$\mathcal{M}_T=110955916689408000, \mathcal{M}_K=29859840, s=32$

(B31)



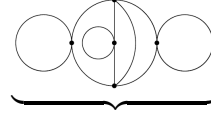
$\mathcal{M}_T=110955916689408000, \mathcal{M}_K=29859840, s=32$

(B37)



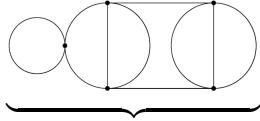
$\mathcal{M}_T=110955916689408000, \mathcal{M}_K=29859840, s=32$

(B32)



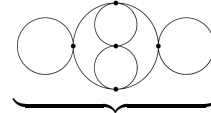
$\mathcal{M}_T=36985305563136000, \mathcal{M}_K=9953280, s=96$

(B38)



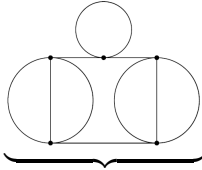
$\mathcal{M}_T=73970611126272000, \mathcal{M}_K=19906560, s=48$

(B33)



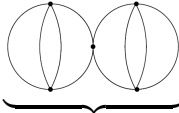
$\mathcal{M}_T=55477958344704000, \mathcal{M}_K=14929920, s=64$

(B39)



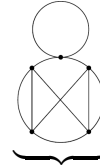
$\mathcal{M}_T=24656870375424000, \mathcal{M}_K=6635520, s=144$

(B34)



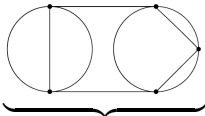
$\mathcal{M}_T=12328435187712000, \mathcal{M}_K=3317760, s=288$

(B35)



$\mathcal{M}_T=221911833378816000, \mathcal{M}_K=59719680, s=16$

(B40)



$\mathcal{M}_T=73970611126272000, \mathcal{M}_K=19906560, s=48$

(B36)

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