FURTHER DEVELOPMENT OF “NON-PYTHAGOREAN” MUSICAL SCALES BASED ON LOGARITHMS

THOMAS MORRILL

Abstract. We offer some refinements to the logarithmic series and a resulting “non-Pythagorean” musical scale, as given by Robert Schneider, the former of which we treat analogously to the harmonic series. In order to yield pleasing resonances within chords in this logarithmic mode, we produce two subseries of the logarithmic series and some logarithmic musical scales, all of which contain many of their beat frequencies. We also demonstrate that a beat scale which contains all its beat frequencies necessarily consists solely of integer ratios.

1. Background

We seek to produce *beat scales*, sets of pitches $S = \{f_1, f_2, \ldots, f_k\}$ for which many of the *beat frequencies* [1] [6] occur in S, up to octave equivalence. That is, we want $2^k|f_j - f_i| \in S$ for some k. Schneider [7] invites us to compose with the pitches

$$F\log(4), F\log(5), F\log(6), \ldots,$$

in order to take advantage of the property $\log(m) - \log(n) = \log(m/n)$. Here F is an arbitrary reference pitch. As this scale is infinite, Schneider also gives a logarithmic scale which divides the octave into 12 pitches suited for piano keyboards. If we normalize so that the scale begins on a ratio of 1:1, and omit the reference pitch F, Schneider's scale is given by:

Ratio	Cents
1	0
$\log_4(5)$	258.38
$\log_4(6)$	444.17
$\log_4(7)$	587.05
$\log_4(8)$	701.95
$\log_4(9)$	797.33
$\log_4(10)$	878.42
$\log_4(11)$	948.64
$\log_4(12)$	1010.35
$\log_4(13)$	1065.24
$\log_4(14)$	1114.55
$\log_4(15)$	1159.22
2	1200.00

This normalization also allows us to include $\log(2)$ and $\log(3)$ while avoiding negative cents between scale pitches. We call

$$\log(2), \log(3), \log(4), \ldots \tag{1}$$

the *logarithmic series*, to treat it analogously to the harmonic series[1], $1, 2, 3, \ldots$.

Many tuning systems arise from the harmonic series, such as the major Pythagorean scale,

$$1, \frac{9}{8}, \frac{81}{64}, \frac{4}{3}, \frac{3}{2}, \frac{27}{16}, \frac{243}{128}, 2,$$

which consists of repeated ratios of 2 and 3 under octave equivalence. Although logarithms do not behave as well under repeated ratios, we present some other techniques for creating novel scales from the logarithmic series.

2. Mapping Integers to Chords

We begin with a number-theoretic technique for composing with the logarithmic series. Let A be a positive integer with the unique prime factorization

$$A = \prod_{i=1}^{k} p_i^{a_i}.$$

Because of the property

$$\log(A) = \sum_{i=1}^{k} a_i \log(p_i),$$

each positive integer corresponds to a unique pitch set in the logarithmic series. We call

$$C_A = \{a_1 \log(p_1), a_2 \log(p_2), \ldots, a_k \log(p_k)\}$$

the *factored chord* corresponding to A. Two factored chords C_{A_1} and C_{A_2} relate to each other harmonically depending on the divisibility properties of A_1 and A_2. For example, moving between C_{2016} and C_{4752} creates pure intervals:

$$\begin{aligned} 5\log(2) &\mapsto 4\log(2) \\ 2\log(3) &\mapsto 3\log(3) \\ \log(7) &\mapsto \log(11), \end{aligned}$$

a major third in the $\log(2)$ voice, and a major fifth in the $\log(3)$ voice. This reflects the fact that $\gcd(2016, 4752) = 2^4 3^2$. This author has composed a number of short pieces which transform integer sequences into melodies and chord progressions [3].

Note that a single factored chord cannot contain any pure intervals, as the primes are distinct. This also means that a factored chord does not produce beat frequencies which fall in the logarithmic series.

3. Beat Series

When two pitches f_1 and f_2 are heard at the same time, an interference frequency $f_1 - f_2$ can sometimes be heard [1]. This interference is perceived as a beating sound when $f_1 - f_2$ is below the audible threshold, or as a distinct pitch when $f_1 - f_2$ is larger. Thus, the *beat frequency* of two pitches is $f_1 - f_2$. For logarithmic pitches, $f_1 = \log(m)$ and $f_2 = \log(n)$,

[1] In fact, the logarithmic series contains copies of the harmonic series via the property $\log(n^k) = k\log(n)$.

this simplifies to $\log(m/n)$. If m/n is a proper fraction, then it is apparent that $\log(m/n)$ does not occur in the logarithmic series, nor do any of its octave equivalents

$$2^k \log(m/n) = \log\left(\frac{m^{2^k}}{n^{2^k}}\right).$$

3.1. **Integer Beat Series.** We seek a sequence of integers in which any fraction f_i/f_j reduces to an integer. There are many integer valued functions we may consider. One option is to take logarithms of the factorial function [4], which normalizes to:

Ratio	Cents
1	0
$\log_2(6)$	1644.17
$\log_2(24)$	2636.29
$\log_2(120)$	3345.64
$\log_2(720)$	3896.028
$\log_2(5040)$	4344.59
$\log_2(40320)$	4722.46
$\vdots$	$\vdots$

The first differences of this series are $\log((n+1)!/n!) = \log(n)$, which gives the logarithmic series. Another is to take the values of the Chebyshev theta function,

$$\vartheta(x) := \sum_{p_i \leq x} \log(p_i) = \log\left(p_1 p_2 \cdots p_{\lfloor x \rfloor}\right),$$

where p_i is the ith prime number. The corresponding series of integers is known as the *primorial numbers* [5]. The Chebyshev series normalizes to:

Ratio	Cents
1	0
$\log_2(6)$	1644.17
$\log_2(30)$	2753.77
$\log_2(210)$	3537.03
$\log_2(2310)$	4178.44
$\log_2(30030)$	4673.68
$\log_2(570570)$	5108.60
$\vdots$	$\vdots$

The differences of this series are of the form $\log(p_i p_{i+1} \cdots p_j)$, where the argument can be any product of consecutive primes. However, the beat frequencies of both these series fall in the full logarithmic series, not in the subseries themselves.

3.2. **Restricted Beat Series.** We could instead choose a sequence of integers $d_1, d_2, \ldots, d_k$ to be a repeating sequence of beat frequencies for building a beat series. Then the series

$$\log(d_1), \log(d_1 d_2) \ldots, \log(d_1 d_2 \cdots d_k), \log(d_1^2 d_2 \cdots d_k), \log(d_1^2 d_2^2 \cdots d_k), \ldots$$

contains all of its beat frequencies, with the exception of $\log(d_2)$, $\log(d_3)$, ..., $\log(d_k)$, which may be appended[2]. For example, with $k = 2$ and $d_1 = 3$, $d_2 = 5$ normalizes to:

Ratio	Cents
1	0
$\log_3(5)$	661.049
$\log_3(15)$	1561.89
$\log_3(45)$	2151.41
$\log_3(225)$	2761.89
$\log_3(675)$	3081.62
$\vdots$	$\vdots$

4. Logarithmic Beat Scales

Here we seek a set of integers $n_1 < n_2 < \ldots < n_k$ to produce a scale which divides the octave, that is,

$$1, \log_{n_1}(n_2), \log_{n_1}(n_3), \ldots, 2.$$

However, there is the potential to over-correct, which loses the transcendental properties of the logarithm.

Proposition 1. *Let S be a finite set of real numbers such that for all $x > y \in S$, there exists a $z \in S$ and an integer t such that*

$$x - y = 2^t z. \tag{2}$$

If S is a finite set, then there exists $F > 0$ so that $S \subset F\mathbb{Q}$. That is, if a finite beat scale contains all of its beat frequencies under octave equivalence, then the scale consists of integer ratios of some frequency F.

Proof. We induct on $|S| = n$. The case $n = 1$ is trivial. Suppose the proposition holds for all scales of size n, and consider an S for which $|S| = n + 1$. Let $x, y, z \in S$ as in (2). Then $S \subseteq \mathrm{span}_{\mathbb{Q}}(S) \subseteq \mathrm{span}_{\mathbb{Q}}(S - \{z\}) = F\mathbb{Q}$. □

We offer some methods for producing logarithmic scales which contain many, but not all, of their beat frequencies.

4.1. Root Approximation Scales.

Choose integers n and m so that $n^{2^k} \approx m$ for some positive integer k. This scale consists of pitches

$$\{\log(m), \log(nm), \log(n^2 m), \ldots, \log(n^{2^{2k-1}} m), \log(m^2)\} \cup \{\log(n^{2^{2k}})\}.$$

In this scale, many of the beat frequencies are multiples of $\log(n)$, which is k octaves below $\log(n^{2^k})$. For example, if we let $n = 2$, $m = 17$, and $k = 4$, then the corresponding scale

[2] Appending these introduces new beat frequencies not present in the revised series.

normalizes to:

Ratio	Cents
1	0
$\log_{17}(34)$	378.89
$\log_{17}(68)$	689.56
$\log_{17}(136)$	952.88
$\log_{17}(256)$	1162.55
2	1200

This method can also be modified to approximate integer ratios while maintaining the transcendental sound of the logarithmic mode: If $n^{a-b} \approx m^a$, then $\log_m(nm) \approx a/b$.

4.2. **Composite Scales.** Choose an integer $N = \prod_{i\in I} p_i^{a_i}$ with factors

$$1 = a_1, a_2, \ldots, a_k = N.$$

This scale consists of pitches

$$\{\log(N), \log(a_2 N), \ldots, \log(a_{k-1}N), \log(N^2)\} \cup \{\log(p_i^{2^{b_i}}) \mid i \in I\},$$

where $b_i = \lceil \log_2(\log_{p_i}(N)) \rceil$. For example, letting $N = 108 = 2^2 3^3$ produces the scale:

Ratio	Cents
1	0
$\log_{108}(216)$	239.01
$\log_{108}(256)$	292.88
$\log_{108}(324)$	364.91
$\log_{108}(432)$	448.99
$\log_{108}(648)$	560.96
$\log_{108}(972)$	666.13
$\log_{108}(1296)$	737.05
$\log_{108}(1944)$	832.32
$\log_{108}(2916)$	922.63
$\log_{108}(3888)$	983.96
$\log_{108}(5832)$	1066.86
$\log_{108}(6561)$	1090.22
2	1200

Many of the beat frequencies are integer multiples of some $\log(p_i)$, which is b_i octaves below $\log(p_i^{2^{b_i}})$.

4.3. **Projective Pitch Sets.** We take a cue from Wendy Carlos's α, β, and γ scales [2], and offer a family of logarithmic scales which do not contain redundant intervals, and do not span one octave. Rather than construct beat frequencies which fall in the logarithmic series, we start with rational numbers whose numerators and denominators are only divisible by a fixed set of primes, then exclude these pitches based on what exponent appears with each prime.

Choose bases $b_1, b_2, \ldots, b_k$ and heights $h_{b_1}, h_{b_2}, \ldots, h_{b_k}$. Let S be the set of rational numbers of the form

$$\prod_{i=1}^{k} b_i^{\alpha_i},$$

where:

(1) $\gcd(\alpha_1, \alpha_2, \ldots, \alpha_k) = 1$
(2) $\prod_{i=1}^{k} b_i^{\alpha_i} > 1$.

As with the scales introduced by Carlos [2], this excludes the possibility of the pitch set containing redundant octaves, fourths, fifths, or any integer ratio for that matter.

For example, choosing bases $2, 3$ and heights $h_2 = 2$, $h_3 = 1$ produces the normalized pitch set:

Ratio	Cents
1	0
$\log_{4/3}(3/2)$	594.12
$\log_{4/3}(2)$	1522.42
$\log_{4/3}(3)$	2319.76
$\log_{4/3}(6)$	3166.60
$\log_{4/3}(12)$	3732.77,

which spans over three octaves.

In this example, the beat frequencies take the form $\log_{4/3}(2^i 3^j)$, which falls in the pitch set with the exception of

$$\log_{4/3}(3/2) - \log_{4/3}(4/3) = \log_{4/3}(9/8)$$
$$\log_{4/3}(6) - \log_{4/3}(4/3) = \log_{4/3}(9/2).$$

Note that we use octave equivalence to treat the beat frequencies

$$\log_{4/3}(3) - \log_{4/3}(4/3) = 2\log_{4/3}(3/2)$$
$$\log_{4/3}(12) - \log_{4/3}(4/3) = 2\log_{4/3}(3).$$

5. Conclusion

The use of the logarithm translates beat frequencies from an additive problem to a multiplicative one. This in turn opens up applications of ideal theory, projective geometry, and naïve height theory in the construction of beat scales and in the composition of such scales.

References

[1] Difference Tone Training. http://www.faculty.ucr.edu/~eschwitz/SchwitzPapers/TitchDemo030417.htm, 2005. Accessed: 2018-05-01.
[2] Wendy Carlos. Three asymmetric divisions of the octave. http://www.wendycarlos.com/resources/pitch.html. Accessed: 2018-04-16.
[3] Jocko Homomorphism. Logarithmic Composition. https://soundcloud.com/jocko-homomorphism/sets/logarithmic- 2017. Accessed: 2018-04-22.
[4] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. https://oeis.org/A000142, 2018. Accessed: 2018-04-16.
[5] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. https://oeis.org/A002110, 2018. Accessed: 2018-04-16.

[6] Gareth E. Roberts. *From Music to Mathematics: Exploring the Connections*. 2016.
[7] Robert P. Schneider. A Non-Pythagorean Musical Scale Based on Logarithms. *Proceedings of Bridges: Mathematics, Music, Art, Architecture, Culture Conference*, pages 549–552, 2012.

Department of Physical, Environmental & Mathematical Sciences, University of New South Wales Canberra, PO Box 7916, Canberra BC ACT 2610, Australia
E-mail address: t.morrill@adfa.edu.au