NEW PROPERTIES OF THE EDELMAN-GREENE BIJECTION

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ABSTRACT. Edelman and Greene constructed a correspondence between reduced words of the reverse permutation and standard Young tableaux. We prove that for any reduced word the shape of the region of the insertion tableau containing the smallest possible entries evolves exactly as the upper-left component of the permutation's (Rothe) diagram. Properties of the Edelman-Greene bijection restricted to 132-avoiding and 2143-avoiding permutations are presented. We also consider the Edelman-Greene bijection applied to non-reduced words.

1. INTRODUCTION

In 1982, Richard Stanley conjectured, and later proved algebraically in [23] that the number of different reduced words for the reverse permutation in the symmetric group $\mathfrak{S}_n$ is equal to the number of staircase shape standard Young tableaux. Motivated to find a bijective proof, Edelman and Greene [9] constructed a correspondence based on the celebrated Robinson-Schensted-Knuth (RSK) algorithm and Schützenberger's jeu de taquin. See also the work of Haiman on dual equivalence [12]. Later, Little [19] found another bijection based on the Lascoux-Schützenberger tree, [16], which was proved to be equivalent to the Edelman-Greene (EG) correspondence by Hamaker and Young in [14]. Recently, reduced words of the reverse permutation have been studied under the name of *sorting networks*. Uniformly random sorting networks are the topic of, for example, [1] by Angel, Holroyd, Romik, and Virág, and the subsequent papers, in particular the recent work by Dauvergne and Virág [7] and Dauvergne [6] announcing proofs of the conjectures in [1]. See an example of a sorting network illustrated by its *wiring diagram* in Figure 1.

Our main result, Theorem 3.3, is that the shape of the empty area (Rothe diagram) in the upper left corner of the permutation matrix is exactly the same as a region in the tableaux generated by the EG-correspondence which we call the *frozen region*. See Figure 2. One consequence of this is Conjecture 1, a reformulation of a part of [1, Conjecture 2] directly in terms of the EG-bijection.
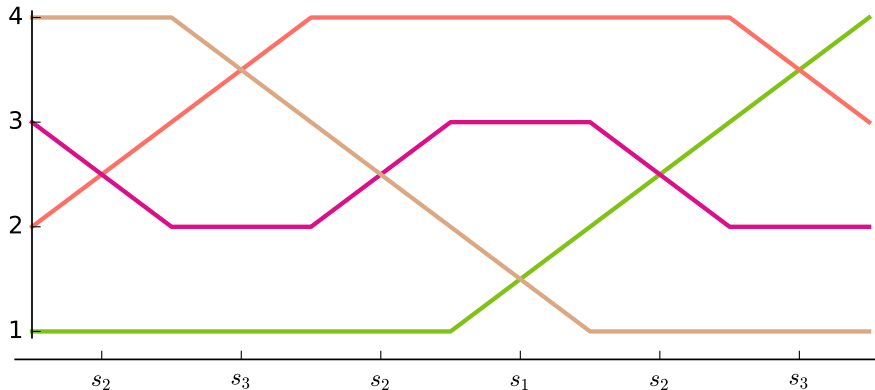
FIGURE 1. The wiring diagram of 232123.

As a side-product of Theorem 3.3 we obtain some new observations and simple reproofs of previous results on the reduced words of 132-avoiding permutations in Corollary 3.8, Corollary 3.9 and Proposition 3.10. We also consider sorting networks whose intermediate permutations are required to be 132-avoiding. These can be viewed as chains of maximum length in the Tamari lattice [2], and have recently been studied by Fishel and Nelson [10], and Schilling, Thiéry, White and Williams [21]. The results in this paper are used to study limit phenomena of random 132-avoiding sorting networks in [18].

In Section 4 we consider the Edelman-Greene bijection applied to non-reduced words. In particular, we study the sets of words yielding the same pairs of Young tableaux under the Edelman-Greene correspondence and study a natural partial order on this set which turns out to have some nice and surprising properties. Note that there is a different generalization of the Edelman-Greene bijection for non-reduced words called Hecke insertion [3].

2. PRELIMINARIES

This section briefly reviews the basic definitions and background of this paper.

2.1. **Reduced words and the weak Bruhat order on $\mathfrak{S}_n$.** The *symmetric group* $\mathfrak{S}_n$ contains all permutations $\sigma = \sigma(1) \ldots \sigma(n)$ on $[n] = \{1, \ldots, n\}$. The set of *inversions* of a permutation $\sigma \in \mathfrak{S}_n$ is defined as $\mathrm{Inv}(\sigma) = \{(i,j) : 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}$. The *weak Bruhat order* is then defined by $\sigma \preceq_w \tau$ for $\sigma, \tau \in \mathfrak{S}_n$ if $\mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\tau)$. The *reverse permutation* $n(n-1) \ldots 21$ is the unique maximal element of $(\mathfrak{S}_n, \preceq_w)$ and the identity permutation $\mathrm{id} = (1, \ldots, n)$ the unique minimal element.

Each $\sigma \in \mathfrak{S}_n$ can be written as a composition of at least $\mathrm{inv}(\sigma) = |\mathrm{Inv}(\sigma)|$ adjacent tranpositions, $s_i = (i \ i+1)$. Hence $\sigma \in \mathfrak{S}_n$ can

be written as a word $w = w_1 \ldots w_m$, $m \geq \mathrm{inv}(\sigma)$, with letters $1 \leq w_i \leq n - 1$ corresponding to transpositions s_{w_i}. The notation $w \in \mathbb{N}^*$ means that w is a finite word with positive integer letters. We define $\mathrm{len}(w) = m$, the length of w. When $\mathrm{len}(w) = \mathrm{inv}(\sigma)$, we say that w is a *reduced word* of σ. Note that each reduced word $w = w_1 w_2 \ldots w_m$, $m = \mathrm{inv}(\sigma)$, of $\sigma \in \mathfrak{S}_n$ can be identified with a chain $\mathrm{id} \preceq_w s_{w_1} \preceq_w s_{w_1} s_{w_2} \preceq_w \cdots \preceq_w s_{w_1} s_{w_2} \cdots s_{w_m} = \sigma$ in the weak Bruhat order on $\mathfrak{S}_n$. We denote the set of reduced words of $\sigma \in \mathfrak{S}_n$ by $\mathcal{R}(\sigma)$, and, for convenience, in the case of $\sigma = n(n-1) \ldots 21$ use the abbreviation $\mathcal{R}(n)$.

We will adopt the convention that the permutation matrix corresponding to $\sigma \in \mathfrak{S}_n$ has 1s in entries $(\sigma(i), i)$, $i = 1 \ldots n$, see the example below. It is important to note that we consider the transpositions acting on positions and perform the compositions of s_{w_i} corresponding to a word $w = w_1 \ldots w_m$ from the left in our arguments. (Equivalently one could compose from the right and consider them acting on values.) As an example, consider $\mathfrak{S}_4$ and the reduced word 1213. Composing $s_1 s_2 s_1 s_3$ from the left yields the permutation 3241. In terms of permutation matrices, we would have, for example,

$$
s_1 = \begin{matrix} \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{4} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}
\quad
s_1 s_2 = \begin{matrix} \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{4} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}
\quad
s_1 s_2 s_1 s_3 = \begin{matrix} \mathbf{3} & \mathbf{2} & \mathbf{4} & \mathbf{1} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix}
$$

where we can see that s_i corresponds to swapping the columns i and $i + 1$.

2.2. **Standard Young tableaux.** Recall that a *partition* λ of $m \in \mathbb{N}$ is a tuple $(\lambda_1, \ldots, \lambda_\ell)$ of positive integers $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ such that $\lambda_1 + \cdots + \lambda_\ell = m$. The *length* of λ is the number of parts in it: $\mathrm{len}(\lambda) = \ell$. A partition can be represented by its *Young diagram* (also called *Ferrers diagram*) which is the set $\{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\}$ and is often (in the so-called English notation) drawn as a collection of square boxes corresponding to the *cells* (i, j) with i increasing downwards and j to the right.

A *Young tableau* T of shape $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a filling of the Young diagram of λ, typically with positive integer *entries*, denoted $T_{i,j}$. Such a tableau T is called *standard* if the entries $1, \ldots, \lambda_1 + \cdots + \lambda_\ell$ appear exactly once each, and the rows and columns of T are strictly increasing. We let $\mathrm{SYT}(\lambda)$ be the set of standard Young tableaux of the shape λ.

2.3. **The Edelman-Greene bijection.** The Edelman-Greene correspondence is a bijection between $\mathcal{R}(n)$, that is, maximal chains in the

weak Bruhat order on $\mathfrak{S}_n$, and standard Young tableaux of the *staircase shape* $sc_n = (n-1, n-2, \ldots, 1)$.

Definition 2.1 (The Edelman-Greene insertion)**.** *Suppose that P is a Young tableau with strictly increasing rows $P_1, \ldots, P_\ell$ and $x_0 \in \mathbb{N}$ is to be inserted in P. The insertion procedure is as follows for each $0 \le i \le \ell$:*

- *If $x_i > z$ for all $z \in P_{i+1}$, place x_i at the end of P_{i+1} and stop.*
- *If $x_i = z'$ for some $z' \in P_{i+1}$, insert $x_{i+1} = z' + 1$ in P_{i+2}.*
- *Otherwise, $x_i < z$ for some $z \in P_{i+1}$, and we let z' be the least such z, replace it by x_i and insert $x_{i+1} = z'$ in P_{i+2}. In both this and the case above we say that x_i **bumps** z'.*

Repeat the insertion until for some i the x_i is inserted at the end of P_{i+1} and the algorithm stops. This could be a previously empty row $P_{\ell+1}$.
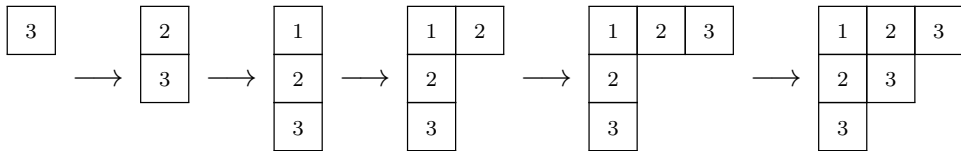
We should mention that our definition of the insertion differs from that of [9], where it is called the *Coxeter-Knuth insertion*. However, using for example the proof of [9, Lemma 6.23], one can show that the two definitions coincide for reduced words. In our formulation the tableaux are increasing in rows and columns also for non-reduced words. Note also that except for a difference in handling equal elements bumping, the Edelman-Greene insertion and the RSK insertion are the same.

Definition 2.2 (The Edelman-Greene correspondence)**.** *Suppose that $w = w_1 \ldots w_i \ldots w_m \in \mathbb{N}^*$. Initialize $P^{(0)} = \emptyset$.*

- *For each $1 \le i \le m$, insert w_i in $P^{(i-1)}$ and denote the result by $P^{(i)}$.*

Let $P^{(m)} = P(w)$ and let $Q(w)$ be the Young tableau obtained by setting $Q(w)_{i,j} = k$ for the unique cell $(i,j) \in P^{(k)} \setminus P^{(k-1)}$. Set $\mathrm{EG}(w) = Q(w)$.

As an example, consider the reduced word $w = 321232$. Then the $P^{(k)}, 1 \le k \le 6$, form the following sequence

$$
\begin{array}{c}
\boxed{3}
\end{array}
\longrightarrow
\begin{array}{c}
\boxed{2} \\ \boxed{3}
\end{array}
\longrightarrow
\begin{array}{c}
\boxed{1} \\ \boxed{2} \\ \boxed{3}
\end{array}
\longrightarrow
\begin{array}{c}
\boxed{1}\ \boxed{2} \\ \boxed{2} \\ \boxed{3}
\end{array}
\longrightarrow
\begin{array}{c}
\boxed{1}\ \boxed{2}\ \boxed{3} \\ \boxed{2} \\ \boxed{3}
\end{array}
\longrightarrow
\begin{array}{c}
\boxed{1}\ \boxed{2}\ \boxed{3} \\ \boxed{2}\ \boxed{3} \\ \boxed{3}
\end{array}
$$

so that

$$
P(321232) = \begin{array}{c}
\boxed{1}\ \boxed{2}\ \boxed{3} \\ \boxed{2}\ \boxed{3} \\ \boxed{3}
\end{array}
\quad\text{and}\quad
\mathrm{EG}(321232) = Q(321232) = \begin{array}{c}
\boxed{1}\ \boxed{4}\ \boxed{5} \\ \boxed{2}\ \boxed{6} \\ \boxed{3}
\end{array}.
$$

The tableau $P(w)$ is called the *insertion tableau* and the tableau $Q(w)$ the *recording tableau*. Note that $P(w)$ and $Q(w)$ are always of

the same shape for a fixed w. To state one of the main results of Edelman and Greene, let the *reading word* $r(P)$ of an insertion tableau P be the word obtained by collecting the entries of P row by row from left to right starting from the bottom row.

Theorem 2.3 ([9, Theorem 6.25]). *The correspondence*

$$w \mapsto (P(w), Q(w))$$

is a bijection between $\cup_{\sigma \in \mathfrak{S}_n} \mathcal{R}(\sigma)$ and the set of pairs of tableaux (P, Q) such that P is row and column strict, $r(P)$ is reduced, P and Q have the same shape, and Q is standard.

Each of the $P^{(k)}$, $1 \le k \le m$, is going to contain some amount of entries such that $P_{i,j}^{(k)} = i + j - 1$. We call the region of $P^{(k)}$ formed by such entries the *frozen region* and say that an insertion tableau is *frozen* if the tableau is entirely a frozen region. The reason for using this terminology is that the frozen region does not change during the Edelman-Greene insertion. See P in Figure 2. The frozen region is white in the example. It turns out that $P(w)$ is always frozen when $w \in \mathcal{R}(n)$, and in fact, as we will see later in Corollary 3.7, more generally if and only if $w \in \mathcal{R}(\sigma)$ with σ 132-avoiding. Frozen tableaux have previously appeared in the literature on the combinatorics of K-theory under the name *minimal increasing tableaux*, see, for example, [4] and [13].

Theorem 2.4 ([9, Theorem 6.26]). *Suppose $w \in \mathcal{R}(n)$. Then $P(w)$ is frozen and $Q(w) \in \mathrm{SYT}(\mathrm{sc}_n)$. The map $\mathrm{EG}(w) : w \mapsto Q(w)$ is a bijection from $\mathcal{R}(n)$ to $\mathrm{SYT}(\mathrm{sc}_n)$.*

Continuing in the setting of Theorem 2.4, if $w \in \mathcal{R}(n)$, the inverse to the Edelman-Greene bijection takes a very special form. To define it, we have to introduce Schützenberger's *jeu de taquin*. For a good introduction, we refer to [24] or [20], although the terminology is slightly different.

Let T be a partially filled Young diagram with increasing rows and columns, and each entry $1 \le k \le \max_{(i,j) \in T} T_{i,j}$ occurring exactly once. The *evacuation path* of T is a sequence of cells $\pi_1, \ldots, \pi_s$ such that

- $\pi_1 = (i_{\max}, j_{\max})$, the location of the largest entry of T,
- if $\pi_k = (i, j)$, then $\pi_{k+1} = (i', j') \in T$ such that $T_{i',j'} = \max\{T_{i,j-1}, T_{i-1,j}\} > -\infty$ with the convention $T_{i,j} = -\infty$ for $(i, j) \notin T$ and for unlabeled $(i, j) \in T$.

Next, define the tableau T^∂ by

- removing the label of T_{π_1},
- and sliding the labels along the evacuation path: $T_{\pi_1} \leftarrow T_{\pi_2} \leftarrow \cdots \leftarrow T_{\pi_s}$.

A single application of ∂ is called an *elementary promotion*. Whenever a label $1 \le \ell \le T_{\pi_1}$ slides from some cell (i, j) to $(i, j+1)$ (respectively

$(i+1, j))$ in applying ∂ until all labels have been removed is referred to as a *right slide* (respectively *downslide*). For $w \in \mathcal{R}(n)$, the inverse to the Edelman-Greene bijection can then be defined as follows.

Theorem 2.5 ([9, Theorem 7.18]). *Suppose $Q \in \mathrm{SYT}(\mathrm{sc}_n)$. Apply ∂ until all labels have been cleared and say that $\pi_1^{(k)} = (i_k, j_k)$ is the first cell of the evacuation path $\pi^{(k)}$ for the k:th iteration. Then $\mathrm{EG}^{-1}(Q) = j_{\binom{n}{2}} \cdots j_k \cdots j_1$.*

Consider again the example following Definition 2.2. Applying ∂ yields the sequence

$$Q = \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \quad \xrightarrow{\partial} \quad \begin{array}{|c|c|c|} \hline & 1 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \quad \xrightarrow{\partial} \quad \begin{array}{|c|c|c|} \hline & & 1 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \quad \xrightarrow{\partial} \quad \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 2 \\ \cline{2-2} 3 \\ \cline{1-1} \end{array}$$

$$\xrightarrow{\partial} \quad \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 2 \\ \cline{2-2} \\ \hline \end{array} \quad \xrightarrow{\partial} \quad \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & \\ \hline \\ \hline \end{array} \quad \xrightarrow{\partial} \quad \begin{array}{|c|c|c|} \hline & & \\ \hline & \\ \hline \\ \hline \end{array}.$$

The largest entries are in the cells $\pi_1^{(1)} = (2,2), \pi_1^{(2)} = (1,3), \pi_1^{(3)} = (2,2), \pi_1^{(4)} = (3,1), \pi_1^{(5)} = (2,2)$ and $\pi_1^{(6)} = (1,3)$. Hence, $\mathrm{EG}^{-1}(Q) = 321232$ as expected.

Another important operator will be the so-called *evacuation S*, which is in some sense dual to promotion. If T is a standard Young tableau, T^S is defined by setting $T_{i,j}^S = k$ if and only if (i,j) is not labeled in T^{∂^k} but is labeled in $T^{\partial^{k-1}}$. Thus T^S records when cells become empty in iterating the elementary promotion ∂ for T. Returning to the previous example, we would have

$$Q^S = \begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}.$$

In his original work [22], Schützenberger proved a remarkable property of the operator S: it is an involution.

3. Frozen regions and diagrams

This section aims to prove our main result. Before proceeding with the proof, we need to recall some additional properties of the Edelman-Greene bijection. The results below are due to Edelman and Greene.

Lemma 3.1 ([9, Lemma 6.22]). *If P is row and column strict, then $P(r(P)) = P$.*

Lemma 3.2 ([9, a part of Lemma 6.23]). *If $w \in \mathcal{R}(\sigma)$, then $P(w)$ is row and column strict, and $r(P(w)) \in \mathcal{R}(\sigma)$.*

Our goal is to show that the shape of the frozen region of $P^{(k)}$ corresponds to the shape of one part of the so-called diagram of $\sigma = s_{w_1} s_{w_2} \ldots s_{w_k}$. The (Rothe) *diagram* $D(\sigma)$ of a permutation σ is the set of cells left unshaded when we shade all the cells weakly to the east and south of 1-entries in the permutation matrix $M(\sigma)$. In particular, we consider the (possibly empty) connected component of $D(\sigma)$ containing $(1,1)$ which we call the *top-left component* of the diagram and denote by $D_{(1,1)}(\sigma)$. The top-left component induces a partition which is denoted by $\lambda(\sigma)$. Similarly, the frozen region of the insertion tableau of a reduced word induces a partition $\lambda_f(w)$ since by Theorem 2.3 the tableau is row and column strict. See Figure 2 for an example.

$$D(\sigma) = \begin{array}{|c|c|c|c|c|c|}
\hline
0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\qquad
P = \begin{array}{|c|c|c|}
\hline
1 & 2 & 4 \\
\hline
2 & 3 & 5 \\
\cline{1-2}
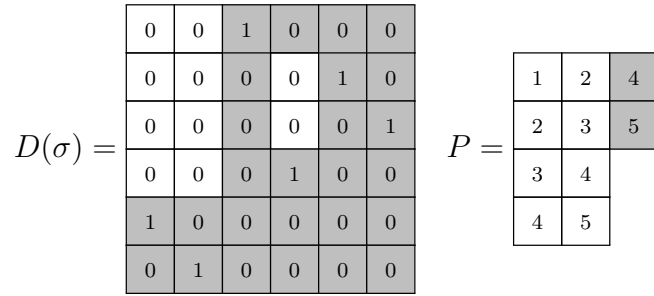3 & 4 \\
\cline{1-2}
4 & 5 \\
\hline
\end{array}$$

FIGURE 2. The diagram $D(\sigma)$ and $P = P(w)$ for any $w \in \mathcal{R}(\sigma)$ for $\sigma = 561423$. The top-left component $D_{(1,1)}(\sigma)$ induces the partition $\lambda(\sigma) = (2,2,2,2)$ and the frozen region of P the partition $\lambda_f(w) = (2,2,2,2)$.

The following is one of our main results.

Theorem 3.3. *If $w = w_1 \cdots w_\ell$ is reduced, then $\lambda(s_{w_1} \ldots s_{w_\ell}) = \lambda_f(w)$. That is, the top-left component of the diagram of $s_{w_1} \ldots s_{w_\ell}$ has the same shape as the frozen region of $P(w)$.*

Proof. By Lemma 3.1 and Lemma 3.2, it is enough to consider the case when $w = r(P(w)) = p_m \ldots p_2 p_1$ where p_i is the word formed by the letters in $P_i(w)$, the i:th row of $P(w)$. The remark below will be useful throughout.

Remark. *Let $\sigma(w) = s_{w_1} \cdots s_{w_k}$ for a word $w = w_1 \ldots w_k$. Since $r_i, 1 \le i \le m$, is a strictly increasing word, a number in $\sigma(w)$ can move at most one step to the left in $\sigma(w)\sigma(r_i)$. The number in position j moves k steps to the right if $j(j+1)\cdots(j+k-1)$ is a subword of r_i.*

Let $\sigma = s_{w_1} \cdots s_{w_\ell}$. We will start by showing that $\lambda_f(w)_1 = \lambda(\sigma)_1$ and $\text{len}(\lambda_f(w)) = \text{len}(\lambda(\sigma))$. For the first statement, note that the topmost 1 of $M(\sigma)$ is in the cell $(1, \lambda(\sigma)+1)$. On the other hand, none of the rows of $P(w)$ below the first can contain a 1 as the columns of $P(w)$ are strictly increasing. Hence, by the remark, the sequence $s_1 \cdots s_{\lambda_f(w)_1}$ of transpositions coming from p_1 moves the number 1 to

position $\lambda_f(w)_1 + 1$ in σ. Note that this means the same thing as moving the 1 in the first row to column $\lambda_f(w)_1 + 1$ in $M(\sigma)$. Thus, $\lambda_f(w)_1 = \lambda(\sigma)_1$.

Similarly, we can show that there is a 1 in the cell $(\text{len}(\lambda_f(w))+1, 1)$ in $M(\sigma)$, where $\text{len}(\lambda_f(w))$ is the index of the last row containing a frozen part, as follows. By column and row-strictness, no transpositions in $p_m \ldots p_i, i = \text{len}(\lambda_f(w))+1$, affect the number $\text{len}(\lambda_f(w))+1$ in the permutation. Hence, the first numbers $\text{len}(\lambda_f(w)), \ldots, 1$ of the rows $\text{len}(\lambda_f(w)), \ldots, 1$ form a sequence of transpositions $s_{\text{len}(\lambda_f(w))}, \ldots, s_1$ moving the number $\text{len}(\lambda_f(w))+1$ to position 1 in σ. Thus $\text{len}(\lambda_f(w)) = \text{len}(\lambda(\sigma))$.

It remains to show that $\lambda_f(w)_i = \lambda(\sigma)_i$ for $1 < i \leq \text{len}(\lambda_f(w))$. Consider the row i with frozen part of length $\lambda_f(w)_i$, and the permutation σ^i corresponding to the reduced word $w^i = p_m \cdots p_i$. By row and column-strictness, the letter i does not appear in w^{i+1}. By the remark, the effect of the transpositions at indices $(p_i)_1 = i, \ldots, (p_i)_{\lambda_f(w)_i} = (i + \lambda_f(w)_i - 1)$ is to move the 1 in row i to column $i + \lambda_f(w)_i$ in $M(\sigma^i)$. Suppose $\lambda_f(w)_j = \lambda_f(w)_i$ for $i' \leq j \leq i$, and $\lambda_f(w)_j > \lambda_f(w)_i$ for all $j < i'$. We will now prove that $\lambda(\sigma)_j = \lambda_f(w)_i$ for $i' \leq j \leq i$. This situation is illustrated in Figure 3.

	1		$\lambda_f(w)_i$	$\lambda_f(w)_i + 1$	
i'	0	...	0	1	...
	⋮	⋱	⋮	0	...
i	0	...	0	0	...
	⋮	1	...	0	...

FIGURE 3. The top-left component of the diagram of σ.

First, note that by the remark, in fact, for all $i' \leq j \leq i$, the 1 in row j is moved by p_j to column $j + \lambda_f(w)_i$ in $M(\sigma^j)$.

Next, consider i'. We have $P_{(j, \lambda_f(w)_{i'}+1)} = j + \lambda_f(w)_i$ for $j \leq i$ since it is in the frozen part. These entries for $j = i'-1, \ldots, 1$ will move the number i' back to $\lambda_f(w)_i + 1$. Hence $\sigma(i') = \lambda_f(w)_i + 1$.

Finally, by the remark, the number j, $i' < j \leq i$, can be moved at most $j - 1$ steps to the left by $p_{j-1}, \ldots, p_1$. Hence $\sigma(j) > \lambda_f(w)_i + 1$ for $i' < j \leq i$, and the claim follows. This implies that $\lambda_f(w)_i = \lambda(\sigma)_i$ for any $1 < i \leq \text{len}(\lambda_f(w))$. $\square$

Given $w = w_1 \ldots w_k \in \mathcal{R}(\sigma)$, let $w^{\text{rev}} = w_k \ldots w_1 \in \mathcal{R}(\sigma^{-1})$. This corresponds to reflecting the wiring diagram in the vertical axis through the midpoint. Edelman and Greene proved the following lemmas.

Lemma 3.4 ([9, Corollary 7.22]). *Suppose $w = w_1 \ldots w_k$ is a reduced word. Then*

- $P(w^{\text{rev}}) = P(w)^t$, where t is the transpose, and
- $Q(w^{\text{rev}}) = Q(w)^S$.

A similar statement holds for taking complements. This time the wiring diagram picture would be to reflect the diagram in the horizontal axis through the middle.

Lemma 3.5 ([9, Corollary 7.21]). *Suppose $w = w_1 \ldots w_k \in \mathcal{R}(n)$ and let $\bar{w} = \bar{w}_1 \ldots \bar{w}_k$, where $\bar{w}_i = n - w_i$ for $1 \leq i \leq k$. Then $\bar{w} \in \mathcal{R}(n)$, and $Q(\bar{w}) = Q(w)^t$.*

Note that if $s_{w_1} \cdots s_{w_k} = \sigma$, then $s_{\bar{w}_1} \cdots s_{\bar{w}_k} = (\sigma^r)^c = (\sigma^c)^r$, where $\sigma^r = (\sigma(n), \ldots, \sigma(1))$, which corresponds to flipping the permutation matrix about its horizontal axis, and $\sigma^c = (n+1-\sigma(1), \ldots, n+1-\sigma(n))$, which corresponds to doing the same about the vertical axis.

The symmetries above yield the reformulation of a part of [1, Conjecture 2] below. We state it informally. The reader is referred to [1] for the details on their conjecture.

Conjecture 1 (Reformulation of a consequence of [1, Conjecture 2]). *Let w be a random sorting network. For all $t \in (0,1)$, the limit shape of the scaled frozen region*

$$F_t = \{(\frac{2j}{n} - 1, 1 - \frac{2i}{n}) \in \mathbb{R}^2 : (i,j) \in \lambda_f(w_1 \ldots w_{\lfloor t\binom{n}{2} \rfloor})\}$$

is $\{(x,y) \in \mathbb{R}^2 : x \leq -\cos(\pi t), y \geq \cos(\pi t), \sin^2(\pi t) - 2xy\cos(\pi t) - x^2 - y^2 = 0\}$.

A proof of the corresponding part of [1, Conjecture 2] has been announced recently in [7]. See also a stronger version in [6, Theorem 2]. Conjecture 1 and [1, Conjecture 2] are illustrated in Figure 4.

3.1. **Pattern avoidance.** Theorem 3.3 also connects our work with the study of pattern-avoiding permutations. The permutation $\sigma \in \mathfrak{S}_n$ *contains* the pattern $p = p_1 \ldots p_k \in \mathbb{N}^*$ if there exist indices $1 \leq i_1 < i_2 \cdots < i_k \leq n$ such that $\sigma(i) < \sigma(j)$ if and only if $p_i < p_j$ for all $i < j$, $i, j \in \{i_1, \ldots, i_k\}$. If σ does not contain p, it is called *p-avoiding*. The set of 132-avoiding permutations of $[n]$, $\mathfrak{S}_n(132)$, is of particular interest here. The reason is an observation of Fulton.

Lemma 3.6 ([11, Proposition 9.19]). *Let $\sigma \in \mathfrak{S}_n$. Then σ is 132-avoiding if and only if $D(\sigma) = D_{(1,1)}(\sigma)$.*

Since the length of a reduced word of $\sigma \in \mathfrak{S}_n$ is exactly the number of inversions in σ, that is $\text{inv}(\sigma)$, Lemma 3.6 suggests we also need the following well-known fact: If $\sigma \in \mathfrak{S}_n$, then $|D(\sigma)| = \text{inv}(\sigma)$. Note that by Lemma 3.6, this can also be stated as $\lambda(\sigma) \vdash \text{inv}(\sigma)$ for $\sigma \in \mathfrak{S}_n(132)$, meaning that $\lambda(\sigma)$ is a partition of $\text{inv}(\sigma)$. We then obtain the characterization below.
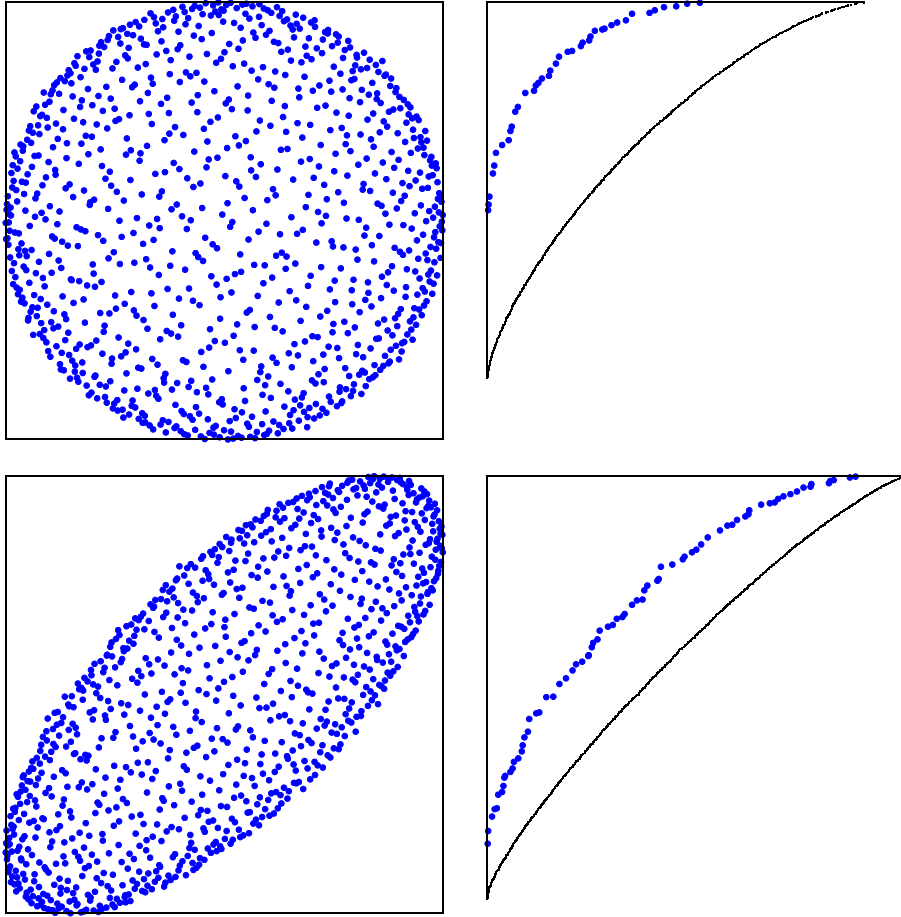
FIGURE 4. A comparison at times $t = \frac{1}{2}$ and $\frac{3}{4}$ illustrating how the same shapes occur in both permutation matrices $M(\sigma_t)$ and frozen regions $\lambda_f(w_1 \ldots w_{\lfloor t\binom{n}{2}\rfloor})$, where σ_t is the permutation defined by $w_1 \ldots w_{\lfloor t\binom{n}{2}\rfloor}$ for a random sorting network w.

Corollary 3.7. *Let $w \in \mathcal{R}(\sigma)$. The insertion tableau $P(w)$ is frozen if and only if σ is 132-avoiding.*

Somewhat related, Tenner showed in [25, Theorem 5.15] that the set of 132-avoiding permutations of any length with k inversions is in bijection with partitions of k. The proof is by constructing a bijection by filling the Young diagram of $\lambda \vdash k$ in such a way that the result is a frozen tableau (it is called antidiagonal filling in the paper). Then the reading words of these tableaux are shown to be reduced, as also follows by Lemma 3.1 and Lemma 3.2, and moreover to yield the 132-avoiding permutations. This would then imply the "only if"-direction of Corollary 3.7 by Lemma 3.2.

The corollary below is mostly a reproof of consequences of results by Stanley [23, Theorem 4.1], and Edelman and Greene [9, Theorem 8.1]. We have added the observation that each shape $\lambda \subset \mathrm{sc}_n$ appears for exactly one $\sigma \in \mathfrak{S}_n(132)$ (and the consequent second bijection), which also follows from their works by properties of 132-avoiding permutations but is not discussed.

Corollary 3.8. *If σ is 132-avoiding, then $P(w)$ is frozen and has the same shape $\lambda(\sigma)$ for all $w \in \mathcal{R}(\sigma)$. Furthermore, each shape $\lambda \subset \mathrm{sc}_n$ appears for exactly one $\sigma \in \mathfrak{S}_n(132)$. Hence, $\mathrm{EG}(w) : w \mapsto Q(w)$ defines a bijection*

$$\mathcal{R}(\sigma) \to \mathrm{SYT}(\lambda(\sigma)),$$

and a bijection

$$\bigcup_{\sigma \in \mathfrak{S}_n(132)} \mathcal{R}(\sigma) \to \bigcup_{\lambda \subset \mathrm{sc}_n} \mathrm{SYT}(\lambda).$$

Corollary 3.9. *Let $f^\lambda = |\mathrm{SYT}(\lambda)|$. Then*

$$\left| \bigcup_{\sigma \in \mathfrak{S}_n(p)} \mathcal{R}(\sigma) \right| = \sum_{\lambda \subset \mathrm{sc}_n} f^\lambda,$$

where $p \in \{132, 213\}$.

This is implied by Corollary 3.8 and symmetries proved by Edelman and Greene. However, we have not been able to simplify the sum on the right-hand side.

3.2. **132-avoiding sorting networks.** Having in mind that the insertion tableau $P(w)$ becomes frozen for any reduced word w of the reverse permutation, it could be interesting to restrict to *132-avoiding sorting networks*, that is, those reduced words $w = w_1 \dots w_{\binom{n}{2}} \in \mathcal{R}(n)$ such that for any $1 \leq i \leq \binom{n}{2}$ the permutation $s_{w_1} \cdots s_{w_i}$ is 132-avoiding, or, equivalently, $P(w_1 \dots w_i)$ is frozen. This corresponds to considering the maximum length chains in the weak Bruhat order on $\mathfrak{S}_n$ restricted to 132-avoiding permutations. Björner and Wachs showed in [2] that the restriction yields a sublattice isomorphic to the *Tamari lattice* $\mathcal{T}_n$.

Using results from the next section, we can characterize 132-avoiding sorting networks in terms of *shifted standard Young tableaux*, which was first proved by Fishel and Nelson [10, Theorem 4.6]. These are standard Young tableaux for which each row i can be shifted $(i - 1)$ steps to the right without breaking the rule that the columns are increasing downwards. For example,

Proposition 3.10. [10, Theorem 4.6] *Let $w = w_1 \ldots w_{\binom{n}{2}}$ be a sorting network.*
It is 132-avoiding if and only if $Q_{i,j} > Q_{i-1,j+1}$ for all $(i,j), (i-1, j+1) \in Q$, or in other words, Q is a shifted standard Young tableau of the shape sc_n, where $Q = \mathrm{EG}(w)$.
It is 213-avoiding if and only if $Q_{i,j} < Q_{i-1,j+1}$ for all $(i,j), (i-1, j+1) \in Q$ where $Q = \mathrm{EG}(w)$.

Proof. Suppose $Q_{i,j} > Q_{i-1,j+1}$ for all $(i,j), (i-1, j+1) \in Q$. We shall see in Proposition 4.3 that then $w = c_1 c_2 \ldots c_{\binom{n}{2}}$ where c_i is the number of the column of $1 \le i \le \binom{n}{2}$ in Q. This implies that $P^{(k)}$ is frozen for all $1 \le k \le \binom{n}{2}$, since each letter c_i is inserted in column number c_i on the first row. For example, consider $w = 121321$. Its insertion forms the sequence
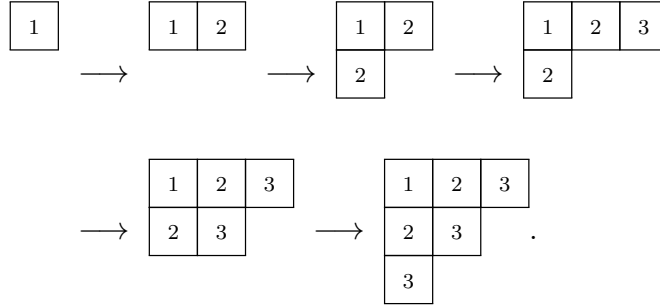
For the other direction, assume $P^{(k)}$ is frozen for all $1 \le k \le \binom{n}{2}$ and suppose w is not of the form $c_1 c_2 \ldots c_{\binom{n}{2}}$. Then some letter w_i is inserted in column $c_j > c_i$ on the first row. The letter w_i bumps c_j. Otherwise the insertion tableau was not frozen. This means $c_j + 1$ is inserted in the second row. Either it is the largest on the row or bumps a $c_j + 1$ since the insertion tableau has to be frozen. Using this argument inductively, we see that at no point in the insertion can a letter be inserted into a column other than c_j. This is a contradiction. Hence $w = c_1 c_2 \ldots c_{\binom{n}{2}}$, but then by Proposition 4.3, $Q_{i,j} > Q_{i-1,j+1}$ for all $(i,j), (i-1, j+1) \in Q$.

The second statement follows from the first by symmetries. $\qquad\square$

This subclass of sorting networks has also been studied by Schilling, Thiéry, White and Williams in [21]. Note in particular the observation that 132-avoiding sorting networks form a commutation class, that is, each 132-avoiding sorting network is reachable from another by a sequence of commutations: $s_i s_j \mapsto s_j s_i$ if $|i - j| > 1$. They also observed that by [21, Lemma 2.2] n-element 132-avoiding sorting networks are in bijection with reduced words of the signed permutation $-(n-1) -(n-2) \ldots -1$ by $s_i \mapsto s_{i-1}$.

Another characterization of 132-avoiding sorting networks is in terms of lattice words (also called lattice permutations or Yamanouchi words).

A *lattice word* of type $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a word $w = w_1 \ldots w_m$ in which for each $2 \leq i+1 \leq m$ there is at least one i before it, and i occurs λ_i times in w.

Proposition 3.11. *Let $w = w_1 \ldots w_{\binom{n}{2}}$ be a sorting network and let $\bar{w} = \bar{w}_1 \ldots \bar{w}_k$, where $\bar{w}_i = n - w_i$ for $1 \leq i \leq k$. Then w is 132-avoiding if and only if w (or equivalently, w^{rev}) is a lattice word of type sc_n. It is 213-avoiding if and only if $\bar{w}$ (or equivalently, $\bar{w}^{rev}$) is a lattice word of type sc_n.*

Proof. The proof borrows from the proof of Proposition 3.10. Suppose w is a 132-avoiding sorting network. Then, by Proposition 3.10 and Proposition 4.3, $w = c_1 c_2 \ldots c_{\binom{n}{2}}$ where c_i is the column of $1 \leq i \leq \binom{n}{2}$ in $Q(w)$. This implies that w is a lattice word of type sc_n. For the other direction, note that if w is a lattice word of type sc_n, then the $P^{(k)}$ obtained in computing $\mathrm{EG}(w)$ are frozen for all $1 \leq k \leq \binom{n}{2}$. By Corollary 3.7, w is 132-avoiding.

The second statement follows from the first. $\square$

Fishel and Nelson proved the "$\Rightarrow$"-direction of Proposition 3.11 in [10, Corollary 4.5]. Note that if $w = w_1 \ldots w_k$ is a 132-avoiding sorting network, $w^{\mathrm{rev}} = w_k \ldots w_1$ is a 132-avoiding sorting network as well, since $Q(w^{\mathrm{rev}})$ can be obtained by shifting $Q(w)$, reflecting the result anti-diagonally, complementing the entries: $m \mapsto \binom{n}{2} - m + 1$, and (un)shifting back.

We should emphasize that 132-avoiding and 312-avoiding sorting networks coincide.

Proposition 3.12. *A sorting network is 132-avoiding if and only if it is 312-avoiding. Similarly, a sorting network is 213-avoiding if and only if it is 231-avoiding.*

Proof. Suppose that a 132-avoiding sorting network is not 312-avoiding. This means that an intermediate permutation contains the pattern 312. It must have been created by swapping the 1 and the 3. Hence, a previous intermediate permutation contains the pattern 132, a contradiction. If a 312-avoiding sorting network is not 132-avoiding, an intermediate permutation contains the pattern 132. The 1 and the 3 are swapped in a later intermediate permutation, which leads to a contradiction. A similar argument applies to 213-avoiding and 231-avoiding sorting networks. $\square$

The following enumerative result was, stated in another form, first obtained by Fishel and Nelson [10, Corollary 3.4] who enumerated the maximum length chains in $\mathcal{T}_n$ using a different set of methods. However, it is also a reformulation of Corollary 4.4 by Proposition 3.10.

Corollary 3.13 ([10, Corollary 3.4]). *The number of 132-avoiding sorting networks of length $\binom{n}{2}$ is*

$$\binom{n}{2}!\frac{1!2!\dots(n-2)!}{1!3!\dots(2n-3)!}.$$

The same holds for 213-avoiding sorting networks.

The study of 132-avoiding sorting networks is continued in [18].

3.3. **Vexillary permutations.** The proof method of [9, Theorem 8.1, part 2] would also lead to a proof of Corollary 3.7. Moreover, it in fact allows us to prove something stronger. A permutation is said to be *vexillary* if it is 2143-avoiding. For $(i,j) \in D(\sigma)$, let $r(i,j)$ be the *rank* of (i,j), the number of 1s north-west of (i,j) in $M(\sigma)$. We have the following result.

Theorem 3.14. *Let $w \in \mathcal{R}(\sigma)$. If σ is vexillary, then the cell (i,j) of $P(w)$ contains the entry $(i+j-1)+k$, $k \geq 0$, if and only if $(i+k, j+k)$ is in $D(\sigma)$, where $k = r(i+k, j+k)$. Furthermore, if the set consisting of the cells $(i+k, j+k)$ for entries $(i+j-1)+k$, $k \geq 0$, in cells (i,j) in $P(w)$ is the diagram of a vexillary permutation, then σ is vexillary.*

Proof. We prove this by using a modification of the construction in the proof of [9, Theorem 8.1, part 2]. The idea is as follows. For a permutation σ, create a row (with possible empty spaces) of cells, the columns x containing the positions x such that $\sigma(y) < \sigma(x)$ for some $y > x$. Next, for each x in the row, add $x+1, \dots, x+r_x(\sigma)-1$, where $r_x = |\{y : y > x, \sigma(y) < \sigma(x)\}|$, below x in the same column. Note that r_x is just the number of inversions whose smaller component is x. Denote this configuration of cells by $T_0(\sigma)$. Finally, left-justify the rows and call the resulting increasing tableau $T(\sigma)$. It follows from [9, Theorem 8.1, part 1] and the proof of [9, Theorem 8.1, part 2] that for σ vexillary, $T(\sigma) = P(w)$ for all $w \in \mathcal{R}(\sigma)$. As an example, $\sigma = 813975246$ is considered in Figure 5.

Consider the connected component $D_{(i+k,j+k)}$ in the diagram of a vexillary permutation σ having its north-west corner in $(i+k, j+k)$, where k is the number of 1s north-west of $(i+k, j+k)$. Note that k is well-defined. Assume that $D_{(i+k,j+k)}$ has column lengths $c_0, \dots, c_{l-1}$.

We first show that for $0 \leq m \leq l$, column $j+k+m$ of $T_0(\sigma)$ has at least c_m entries weakly south of row i. These entries are then by construction $(i+j-1)+k+m, \dots, (i+j-1)+k+m+c_m-1$ as required in $P(w)$. It is clear that there are at least c_m 1s east of column $j+k+m$, north of row $(i+k)+c_m-1$ and weakly south of $i+k$, whereas the 1-entry of column $j+k+m$ must lie weakly south of $(i+k)+c_m-1$. See Figure 7. Furthermore, there are exactly k 1s north-west of $(i+k, j+k)$ in the permutation matrix. Hence $i-1$ 1s are strictly north-east of the component with north-west corner in
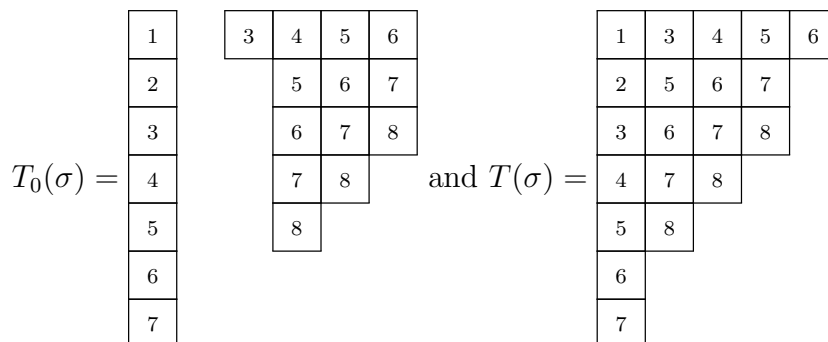
$$T_0(\sigma) = \begin{array}{|c|}\hline 1 \\\hline 2 \\\hline 3 \\\hline 4 \\\hline 5 \\\hline 6 \\\hline 7 \\\hline\end{array} \quad \begin{array}{|c|c|c|c|}\hline 3 & 4 & 5 & 6 \\\hline & 5 & 6 & 7 \\\hline & 6 & 7 & 8 \\\hline & 7 & 8 \\\hline & 8 \\\hline\end{array} \quad \text{and } T(\sigma) = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 5 & 6 \\\hline 2 & 5 & 6 & 7 \\\hline 3 & 6 & 7 & 8 \\\hline 4 & 7 & 8 \\\hline 5 & 8 \\\hline 6 \\\hline 7 \\\hline\end{array}$$

FIGURE 5. The construction used in the proof of Theorem 3.14 for $\sigma = 813975246$. Compare with Figure 6.

$$D(\sigma) = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\\hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\\hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\\hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\\hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\\hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\\hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\\hline\end{array}$$
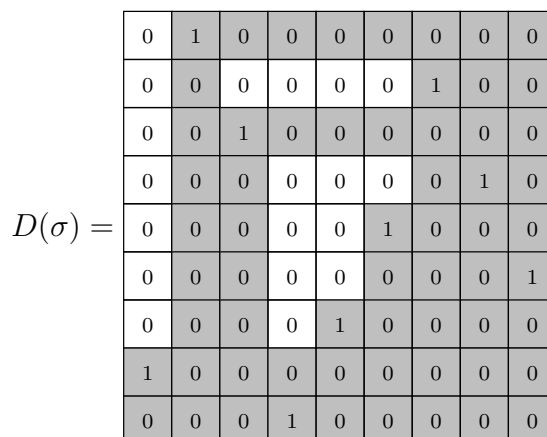
FIGURE 6. The diagram of $\sigma = 813975246$. Compare with Figure 5.

$(i + k, j + k)$. Hence column $j + k + m$ of $T_0(\sigma)$ contains at least the entries $j + k + m, \ldots (j + k) + (i - 2) + m, (i + j - 1) + k + m, \ldots, (i + j - 1) + k + m + c_m - 1$. This proves the claim.

Next we show that the part of $T_0(\sigma)$ corresponding to $D_{(i+k,j+k)}$ is shifted to the left by k steps in $T(\sigma)$. No 1s appear west of the component. Hence all columns of $T_0(\sigma)$ left of $j + k$ have either length shorter than i or longer than $i + c_0$. Since $r(i + k, j + k) = k$, exactly k columns are shorter. This proves that $(x, y) \in T(\sigma)$ contains $(x + y - 1) + k$ for all $(x + k, y + k) \in D_{(i+k,j+k)}$.

For the other direction, note that the map above is surjective from the set of all diagrams of vexillary permutations to insertion tableaux of reduced words of vexillary permutations. Furthermore, it is injective since $D(w) : P(w) \mapsto D \subset \mathbb{N}^2$ defined by sending (i, j) with entry $(i + j - 1) + k$ to $(i + k, j + k)$ is the inverse. This proves the " $\Rightarrow$ "-direction.
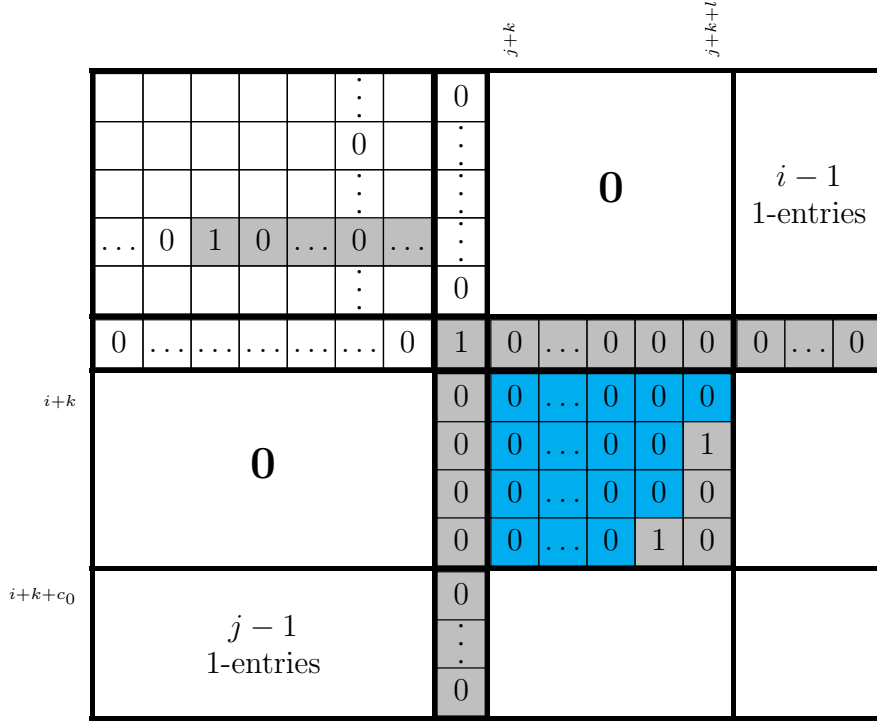
FIGURE 7. An example illustrating the proof of Theorem 3.14. The component $D_{(i+k,j+k)}(\sigma)$ is in cyan and the entries shaded are not in $D(\sigma)$.

For the final statement, note that in the vexillary case no two of the k 1s north-west of $(i+k, j+k)$ can form a decreasing subsequence. Hence (i, j) is the first cell of $\mathbb{N}^2$ on the diagonal of $(i+k, j+k)$, not in the set D obtained after the components of $D(\sigma)$ with their north-west corners (i', j') on the same diagonal have been shifted diagonally north-west to the first available cells in order of increasing (i', j'). This gives an alternative description of the map $P(\sigma) : D(\sigma) \mapsto P(w)$: send the cells $(i+k, j+k) \in D(\sigma)$ in increasing order along the same diagonal to the first available cell, (i, j), and put the label $(i+j-1)+k$ into (i, j). Then, P can be defined for any permutation σ, and for $w \in \mathcal{R}(\sigma)$, the map $D(w) : P(w) \mapsto D \subset \mathbb{N}^2$ defined by sending (i, j) with entry $(i+j-1)+k$ to $(i+k, j+k)$ is invertible with P as its inverse. This proves the last part. $\qquad\square$

Note that the entries with $k = 0$ are in the frozen region of $P(w)$.

4. Non-reduced words

The Edelman-Greene bijection takes as its argument a reduced word. In order to understand the insertion better, we study its interaction

with non-reduced words as well. Simultaneously, we obtain Proposition 4.3 which can be used to prove Proposition 3.10 and Proposition 3.11.

Fix a standard Young tableau Q and let $\mathcal{W}_Q = \{w \in \mathbb{N}^* : \mathrm{EG}(w) = Q, P(w) \text{ frozen}\}$. Recall that by Corollary 3.7 the reduced words in the sets $\mathcal{W}_Q$ are reduced words of 132-avoiding permutations. Note that since the tableau $Q(w) = \mathrm{EG}(w)$ has $\mathrm{len}(w)$ entries, all words in $\mathcal{W}_Q$ have the same length. Also, since the Edelman-Greene correspondence is a bijection between $\mathcal{R}(\sigma)$ and $\mathrm{SYT}(\lambda(\sigma))$ for $\sigma \in \mathfrak{S}_n(132)$, $\mathcal{W}_Q$ contains exactly one reduced word.

We define the poset $\mathcal{P}_Q = (\mathcal{W}_Q, \preceq)$ by setting $v \preceq w$ for $v, w \in \mathcal{W}_Q$ if $v_i \leq w_i$ for all $1 \leq i \leq \mathrm{len}(v) = \mathrm{len}(w)$. Figure 8 contains some examples.
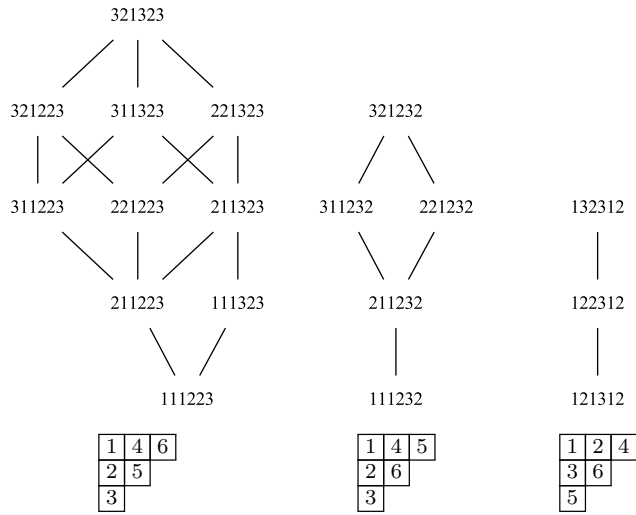

FIGURE 8. Some examples of the 16 posets $\mathcal{P}_Q$ for $Q \in \mathrm{SYT}(\mathrm{sc}_4)$. The bottom elements are the column words of the respective tableaux below.

4.1. **Properties of $\mathcal{P}_Q$.** First, we extend a result of Edelman and Greene. The *descents* of a standard Young tableau T are entries k such that if $T_{i,j} = k$, then $T_{i',j'} = k + 1$ for $i' > i$, in other words $k + 1$ is strictly south of k. Let $D(T) = \{k : k \text{ is a descent of } T\}$ be the set of descents of T. Correspondingly, for $w \in \mathbb{N}^*$, let $D(w) = \{1 \leq i \leq \mathrm{len}(w) - 1 : w_i \geq w_{i+1}\}$. The elements of $D(w)$ are called the *weak descents* of w.

At times, in particular in the following proof, we refer to bumping paths. Consider constructing $\mathrm{EG}(w)$ for an arbitrary word $w = w_1 \cdots w_m$. When w_k is inserted, some entries $P_{i,j}^{(k-1)}$ of $P^{(k-1)}$ may be bumped. We let the *bumping path* $p_{w_k}^w$ of w_k be the set of the corresponding cells $(i,j) \in P^{(k-1)}$.

Proposition 4.1. *For all $w \in \mathcal{P}_Q$, $D(w) = D(Q)$.*

Proof. The proof is based on an extension of Lemma 6.28 in [9] (which is analogous to a property of the RSK correspondence). Let $w = w_1 \ldots w_i \ldots w_m$. First suppose $i \in D(w)$. In running the Edelman-Greene insertion, when x_i and y_i, $x_i \geq y_i$, are inserted consecutively on row i, x_i either becomes the last entry of that row or bumps some $x' \geq x_i$, and y_i bumps some y', $x_i \geq y' \geq y_i$. Hence, $x_{i+1} \geq x_i + 1$ and $y_{i+1} \leq x_i + 1$. Using this argument inductively shows that $p^w_{w_{i+1}}$ is weakly to the left of $p^w_{w_i}$. Thus $i + 1$ ends up on a lower row than i in $Q = \mathrm{EG}(w)$, so a weak descent of the word becomes a descent in Q.

For the converse, suppose $1 \leq i \leq m - 1$ is not a weak descent in w, which means $w_i < w_{i+1}$. Again, when x_i and y_i, $x_i < y_i$, are inserted consecutively on row i, x_i either becomes the last entry of that row or bumps some $x' \geq x_i$. Next, since $y_i > x_i$, it is either inserted at the end of the row or bumps some $y' \geq y_i > x_i$. Furthermore, since the insertion tableaux always have increasing rows, $y' > x'$. Hence, $x_{i+1} < y_{i+1}$, except possibly in the case x_i bumped an x_i, and $y' = x_i + 1$. But then necessarily $y_i = x_i + 1$, meaning that $x_{i+1} = x_i + 1$ and $y_{i+1} = x_i + 2$, so $x_{i+1} < y_{i+1}$. Repeating this argument inductively, we get that $p^w_{w_{i+1}}$ is strictly to the right of $p^w_{w_i}$. Hence, $i + 1$ cannot end up in a lower row than i, and i is not a descent in Q. $\square$

Suppose Q is a standard Young tableau with m entries. Define $c(Q) = c_1 \ldots c_i \ldots c_m$, where c_i is the column of i in Q for $1 \leq i \leq m$. Then we say that $c(Q)$ is the *column word* of Q. See Figure 8 for examples. Note that this term is used differently by other authors. Column words of standard Young tableaux are, by their definition, lattice words.

Proposition 4.2. *For $Q \in \mathrm{SYT}(\mathrm{sc}_n)$, $\hat{0} = c(Q)$ is the unique minimal element in $\mathcal{P}_Q$.*

Proof. Since Q is a standard Young tableau, if c_i are the columns of the entries i of Q, $|\{i \leq j : c_i = x\}| \geq |\{i \leq j : c_i = x + 1\}|$ for $1 \leq j \leq \binom{n}{2}, 1 \leq x \leq n - 2$. Otherwise there is a row of Q which is not increasing. Since $c(Q)$ has this form, each letter x will end up in the x:th column in the Edelman-Greene insertion, $P(w)$ will be of the frozen form, and the Q-tableau has the entry i in column c_i. Hence, $c(Q) \in \mathcal{P}_Q$. By the same argument, if any of the c_i's is replaced by a smaller number, the shape of P (and Q) changes. Thus $c(Q)$ is a minimal element in $\mathcal{P}_Q$. Since the columns of the insertion tableaux are always strictly increasing, the bumping paths in the Edelman-Greene insertion go down and to the left. If there is another minimal element w in $\mathcal{P}_Q$, then it has to have a letter $w_i < c_i$. But then w_i is inserted to a cell strictly before the c_i:th cell on the first row in the insertion tableau and i cannot end up in the column c_i as it does in Q. Hence $c(Q)$ is the unique minimal element in $\mathcal{P}_Q$. $\square$

We conjecture that $\mathrm{EG}^{-1}(Q)$ is maximal in $\mathcal{P}_Q$. However, in general it is not the unique maximal element. As an example, take a reduced word of the reverse permutation in $\mathfrak{S}_6$ starting $4521343\ldots$ and a non-reduced word $2431343\ldots$ in the same poset $\mathcal{P}_Q$, both ending with the same subword. They are incomparable in $\mathcal{P}_Q$.

The *height* $h(P)$ of a poset P is the length of its longest chain. Let $[\cdot,\cdot]$ denote an interval in $\mathcal{P}_Q$ and $\ell_Q = h([c(Q), \mathrm{EG}^{-1}(Q)])$. In other words, ℓ_Q is the length of a maximum length chain from $c(Q)$ to $\mathrm{EG}^{-1}(Q)$. Then $\ell_Q \leq \sum_{i=1}^{\mathrm{len}(c(Q))}(\mathrm{EG}^{-1}(Q)_i - c(Q)_i)$. However, computations suggest that we have equality for $Q \in \mathrm{SYT}(\mathrm{sc}_n)$.

Conjecture 2. *For $Q \in \mathrm{SYT}(\mathrm{sc}_n)$, we conjecture that $\mathrm{EG}^{-1}(Q)$ is a maximal element in $\mathcal{P}_Q$ and $\ell_Q = \sum_{i=1}^{\mathrm{len}(c(Q))}(\mathrm{EG}^{-1}(Q)_i - c(Q)_i)$.*

Note that $\sum_{i=1}^{\mathrm{len}(c(Q))}(\mathrm{EG}^{-1}(Q)_i - c(Q)_i)$ is the amount of right slides when performing EG^{-1} on Q. Hence $\ell_Q \leq \binom{n}{3}$ for the shape sc_n. Let $\eta_{n,i}$ denote the number of $Q \in \mathrm{SYT}(\mathrm{sc}_n)$ such that $\ell_Q = i, 0 \leq i \leq \binom{n}{3}$. Table 1 lists some of these values.

i	0	1	2	3	4	5	6	7	8	9	10
$\eta_{3,i}$	1	1									
$\eta_{4,i}$	2	2	8	2	2						
$\eta_{5,i}$	12	14	38	108	142	140	142	108	38	14	12

TABLE 1. The values of $\eta_{n,i}$ for $n = 3, 4, 5$.

The tableaux Q contributing to $\eta_{n,0}$ are simple to characterize. Then $\mathcal{P}_Q$ only contains the column word $c(Q)$.

Proposition 4.3. *If $Q \in \mathrm{SYT}(\mathrm{sc}_n)$, then $\ell_Q = 0$ if and only if $Q_{i,j} > Q_{i-1,j+1}$ for all $(i,j),(i-1,j+1) \in Q$.*

Proof. For the if-direction, assume $Q_{i,j} > Q_{i-1,j+1}$ for all $(i,j),(i-1,j+1) \in Q$. This means that all the anti-diagonals (sets of cells with sums of the coordinates fixed) have entries increasing downwards. Suppose that there is at least one right slide, and consider the first such occurrence. Then either $x < y$ have moved to the same anti-diagonal, into some cells (i,j) and $(i-1,j+1)$, respectively, or the right slide occurs at the top of column $j + 1$. Since this is the first occurrence of a right slide, no evacuation paths starting from columns $c > j + 1$ can have crossed to column $j + 1$. Hence both cases would imply that the evacuation paths have started from column $j+1$ more often than from column j, a contradiction since the anti-diagonals of Q are increasing to the left. Hence, all evacuation paths are vertical and the labels only slide down, so $\mathrm{EG}^{-1}(Q) = c(Q)$ and $\ell_Q = 0$.

For the other direction, suppose there are some $x = Q_{i,j} < Q_{i-1,j+1} = y$. If there are no right slides, then x, y, and the labels above them have

to stay in the same columns. If at some point x and y are on the same row, then there must have been a right slide involving the some element in the column of x. Hence x and y have to end up on the bottom anti-diagonal, but then an evacuation path has to start from y before x and some entry of the column of x slides right, a contradiction. □

Staircase standard Young tableaux satisfying the transpose of the condition in Proposition 4.3 have been enumerated in [26] and can also be reinterpreted in terms of several other combinatorial objects, for example Gelfand-Tsetlin patterns (see the entry A003121 in the OEIS [15]).

Corollary 4.4. *We have*
$$\eta_{n,0} = \binom{n}{2}! \frac{1!2!\dots(n-2)!}{1!3!\dots(2n-3)!}.$$

We end with some consequences of Conjecture 2.

Proposition 4.5. *Assume Conjecture 2 holds and $Q \in \mathrm{SYT}(\mathrm{sc}_n)$. Then*
   a) $\ell_{Q^t} = \binom{n}{3} - \ell_Q$, so the sequence $\eta_{n,i}, 0 \le i \le \binom{n}{3}$, is symmetric,
   b) the Schützenberger involution S satisfies $\ell_Q = \ell_{Q^S}$,
   c) the number $\eta_{n,i}$ is even for all $n \ge 4$, $0 \le i \le \binom{n}{3}$,
   d) and $\ell_Q = \binom{n}{3}$ if and only if $Q_{i,j} < Q_{i-1,j+1}$ for all $(i,j), (i-1,j+1) \in Q$.

Proof. a) By Lemma 3.5, $\ell_{Q^t} = \sum_{i=1}^{\mathrm{len}(w)}(n-1-w_i-c(Q)_i) = \binom{n}{2}(n-1) - \binom{n+1}{3} - \sum_{i=1}^{\mathrm{len}(w)} w_i = \binom{n}{3} - (\sum_{i=1}^{\mathrm{len}(w)} w_i - \binom{n}{3}) = \binom{n}{3} - \ell_Q$.

b) Let $w \in \mathcal{R}(n)$ and $Q = Q(w)$. By Lemma 3.4, $Q(w^{rev}) = Q^S$, and by Conjecture 2, we have $\ell_Q = \sum_{i=1}^{\mathrm{len}(w)}(w_i - c(Q)_i) = \sum_{i=1}^{\mathrm{len}(w)} w_i - \binom{n+1}{3} = \sum_{i=1}^{\mathrm{len}(w^{rev})} w_i^{rev} - \binom{n+1}{3} = \ell_{Q^S}$.

c) By b), the involution S satisfies $\ell_Q = \ell_{Q^S}$. Thus it suffices to prove that it is fixed-point-free for $\mathrm{SYT}(\mathrm{sc}_n)$, $n \ge 4$. This can be seen from Lemma 3.4: $Q(w^{\mathrm{rev}}) = Q(w)^S$ for $w \in \mathcal{R}(n)$, so if S had a fixpoint, there would exist $w \in \mathcal{R}(n)$ such that $w = w^{\mathrm{rev}}$. We show by induction on the length of w that every $w = w^{\mathrm{rev}}$ is a reduced word of the same permutation as (in other words, is Coxeter equivalent to) $i\ (i+1)\ \dots\ (j-1)\ j\ (j-1)\ \dots\ (i+1)\ i$ for some $1 \le i < j \le n$, in which case $w \in \mathcal{R}(n)$ only for $n \le 3$, when $i = 1, j = n-1$. Clearly w has to be of odd length. The base case is then that $w = (j-1)\ j\ (j-1)$ or $(j+1)\ j\ (j+1) \approx j\ (j+1)\ j$ where $\approx$ denotes Coxeter equivalence. Consider adding the letter x: $x\ i\ (i+1)\ \dots\ (j-1)\ j\ (j-1)\ \dots\ (i+1)\ i\ x$. We have $i < x < j$, $x = j, x = j+1$, or $x = i-1$. Using commutations in the first case gives $i\ (i+1)\ \dots\ x\ (x-1)\ x\ \dots\ (j-1)\ j\ (j-1)\ \dots\ x\ (x-1)\ x\ \dots\ (i+1)\ i$,

which is non-reduced. If $x = j$, we get $j\,(j-1)\,j\,(j-1)\,j$ in the middle, which is also non-reduced. Hence either $x = i - 1$ and we are done, or $x = j + 1$, in which case we get $(j + 1)\,j\,(j + 1) \approx j\,(j + 1)\,j$ in the middle, and are also done.

d) This follows from Proposition 4.3 by transposition. $\square$

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