ASYMPTOTICS OF PRINCIPAL EVALUATIONS OF SCHUBERT POLYNOMIALS FOR LAYERED PERMUTATIONS

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Abstract. Denote by $u(n)$ the largest principal specialization of the Schubert polynomial:

$$u(n) := \max_{w\in S_n} \mathfrak{S}_w(1,\ldots,1)$$

Stanley conjectured in [Sta] that there is a limit

$$\lim_{n\to\infty} \frac{1}{n^2} \log u(n),$$

and asked for a limiting description of permutations achieving the maximum $u(n)$. Merzon and Smirnov conjectured in [MeS] that this maximum is achieved on layered permutations. We resolve both Stanley's problems restricted to layered permutations.

1. Introduction

Understanding the large-scale behavior of combinatorial objects is so fundamental to modern combinatorics, that it has become routine and no longer requires justification. However, in *algebraic combinatorics*, there are fewer results in this direction, as the objects tend to be have more structure and thus less approachable. This paper studies the asymptotic behavior of the principal evaluation of Schubert polynomials, partially resolving an open problem by Stanley [Sta]. As the reader shall see, the results are surprisingly precise.

Main results. *Schubert polynomials* $\mathfrak{S}_w(x_1,\ldots,x_n)\in\mathbb{N}[x_1,\ldots,x_n]$, $w\in S_n$, were introduced by Lascoux and Schützenberger [LS] to study Schubert varieties. They have been intensely studied in the last two decades and remain a central object in algebraic combinatorics. The principle evaluation of the Schubert polynomials can be defined via *Macdonald's identity* [Mac, Eq. 6.11]:

$$\Upsilon_w := \mathfrak{S}_w(1,\ldots,1) = \frac{1}{\ell!}\sum_{(a_1,\ldots,a_\ell)\in \mathrm{R}(w)} a_1\cdots a_\ell\,. \tag{1.1}$$

Here $\ell=\ell(w)$ is the *length* of w (the number of inversions, and $\mathrm{R}(w)$ denotes the set of *reduced words* of $w\in S_n$: tuples $(a_1,\ldots,a_\ell)$ such that $s_{a_1}\cdots s_{a_\ell}$ is a reduced decomposition of w into simple transpositions $s_i=(i,i+1)$.

Note that Υ_w has a more direct (but less symmetric) combinatorial interpretation as the number of certain *rc-graphs* (also called *pipe dreams*), see e.g. [As]. In particular, we have $\Upsilon_w\in\mathbb{N}$, even though this is not immediately apparent from (1.1) (cf. §4.4).

Denote by $u(n)$ the largest principal specialization of the Schubert polynomial:

$$u(n) := \max_{w\in S_n} \Upsilon_w\,.$$

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Conjecture 1.1 (Stanley [Sta]). *There is a limit*

$$\lim_{n\to\infty} \frac{1}{n^2} \log u(n).$$

In addition, Stanley asked whether the permutations w in S_n achieving the maximum $\Upsilon_w = u(n)$ had a limiting description. There was some evidence in favor of this (see below), but before we turn to positive results let us put this conjecture into context.

One can think of Υ_w as a statistical sum of *weighted random sorting networks* of the permutation w. From a combinatorial point of view, this is a more natural notion, since e.g. $\Upsilon_{w_0} = 1$, where $w_0 = (n, n-1, \ldots, 1)$ is the permutation with maximal length $\ell(w_0) = \binom{n}{2}$. It is thus natural to expect $u(n)$ to have nice asymptotic behavior. In fact, Stanley gave the first order of asymptotics for $u(n)$:

Theorem 1.2 (Stanley [Sta]).

$$\frac{1}{4} \le \liminf_{n\to\infty} \frac{\log_2 u(n)}{n^2} \le \limsup_{n\to\infty} \frac{\log_2 u(n)}{n^2} \le \frac{1}{2}. \tag{1.2}$$

Stanley's proof is nonconstructive and based on the *Cauchy identity* for Schubert polynomials, see [Man, Prop. 2.4.7]. The first constructive lower bound was given by the authors in [MPP1, §6], where the asymptotics of Υ_w was computed for several families of permutations. Notably, for a permutation

$$w(b, n-b) := \big(b, b-1, \ldots, 1, n, n-1, \ldots, b+1\big) \quad \text{where} \quad b = \frac{n}{3},$$

we showed that

$$\frac{1}{n^2} \log_2 \Upsilon_{w(b,n-b)} \longrightarrow C \approx 0.25162 \quad \text{as} \quad n \to \infty.$$

In fact, it is easy to see that the limit C is the largest limit value over all ratios $0 < b/n < 1$. This also gives a small improvement on the lower bound in Stanley's theorem.

Layered permutations $w(b_k, \ldots, b_1)$ are defined as

$$w(b_k, b_{k-1}, \ldots, b_1) := \big(b_k, b_k-1, \ldots, 1, b_k+b_{k-1}, b_k+b_{k-1}-1, \ldots, b_k+1, \ldots, n, \ldots, n-b_1+1\big),$$

for integers $b_1 + \ldots + b_{k-1} + b_k = n$. They are also called *Richardson* and *pop-stack sortable permutations* in a different contexts, see e.g. [Kit, §2.1.4] and [MeS]. Denote by $\mathcal{L}_n$ the set of layered permutations $w \in S_n$.

Theorem 1.3. *Let*

$$v(n) := \max_{w\in\mathcal{L}_n} \Upsilon_w.$$

Then there is a limit

$$\lim_{n\to\infty} \frac{1}{n^2} \log_2 v(n) = \frac{\gamma}{\log 2} \approx 0.2932362762,$$

where $\gamma \approx 0.2032558981$ *is a universal constant. Moreover, the maximum value* $v(n)$ *is achieved at a layered permutation*

$$w(\ldots, b_2, b_1), \quad \textit{where} \quad b_i \sim \alpha^{i-1}(1-\alpha)n \quad \textit{as} \quad n \to \infty,$$

for every fixed i*, and where* $\alpha \approx 0.4331818312$ *is a universal constant.*

In other words, the *runs* b_i form a geometric distribution in the limit. See Figure 1 for examples of the permutation matrix of such w. A posteriori this is unsurprising, since the weights of reduced words are heavily skewed in favor of having many transpositions at the end.

The story behind the theorem is also quite interesting. Calculations for $n \le 10$ reported in [MeS] and [Sta], prompted Merzon and Smirnov to make the following conjecture:

Conjecture 1.4 ([MeS, Conj. 5.7]). *For every* n*, all permutations* w *attaining the maximum* $u(n)$ *are layered permutations. In particular,* $u(n) = v(n)$*.*

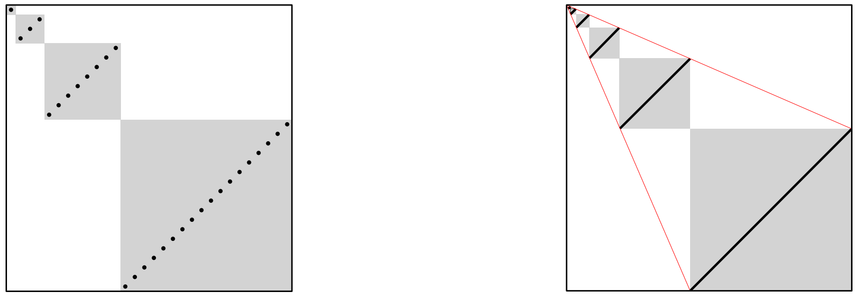

FIGURE 1. Shapes of optimal layered permutations $w(1,3,8,18)$ and $w(2,4,9,20,46,106,246,567)$, of size 30 and 1000, respectively.

In other words, if the Merzon–Smirnov conjecture holds, our Theorem proves Stanley's conjecture with the same limit value and limiting description, as suggested by Stanley (see §4.2 however). Unconditionally, Theorem 1.3 improves a the lower bound for the lim inf in Theorem 1.2 to about 0.2932.

Remark 1.5. We learned about the Merzon–Smirnov conjecture from Hugh Thomas, who used it to compute $v(n)$ and permutations attaining it up to $n = 300$ (see the Appendix). This data allowed us to make a conjecture on the limit shape, which we prove in the theorem.

Exact constants. The constants α and γ in Theorem 1.3 are defined as follows. Consider the function

$$f(x) \,:=\, x^2 \log x \,-\, \frac{1}{2}\,(1-x)^2 \log(1-x) \,-\, \frac{1}{2}\,(1+x)^2 \log(1+x) \,+\, 2x \log 2\,. \tag{1.3}$$

This function is obtained from a double integral that approximates the logarithm of the product formula of Proctor [Pro] for the number of certain plane partitions (Proposition 3.1). Then α is defined as the solution other than $x = 1$ of the equation

$$2x\, f(x) \,+\, (1-x^2)\, f'(x) \,=\, 0,$$

see Figure 3 for plots of $f(x)$ and the equation above. The constant γ is defined as

$$\gamma \,:=\, \frac{f(\alpha)}{1-\alpha^2}\,.$$

One can show that α is transcendental by using Baker's theorem, see [Ba, §2.1], but this goes beyond the scope of this paper. It would be interesting to see if existing technology allows to show that γ is also transcendental.

Outline of the paper. In Section 2 we give the necessary background on asymptotics and on the principal evaluation of Schubert polynomials of layered permutations. In Section 3 we prove Theorem 1.3. We conclude with final remarks and open problems in Section 4.

2. BACKGROUND

2.1. **Permutations.** We write permutations of $\{1, 2, \ldots, n\}$ as $w = w_1 w_2 \ldots w_n \in S_n$, where w_i is the image of i. Given two permutations u in S_m and v in S_n we denote by $u \times v$ the following permutation of S_{m+n}:

$$u \times v \,:=\, u_1 u_2 \ldots u_m \,(m+v_1)\,(m+v_2)\, \ldots\, (m+v_n).$$

Similarly, denote by $1^m \times w$ the permutation

$$1^m \times w \,:=\, 12 \ldots m \,(m+w_1)\,(m+w_2)\, \ldots\, (m+w_n)\,.$$

Finally, let $|b| = b_1 + \cdots + b_k$.

2.2. **Product formulas for Υ_w for layered permutations.** In this section we give a product formula for Υ_w when w is a layered permutation $w(b_k,\ldots,b_1)$.

Let w_0 be the longest permutation $(p, p-1,\ldots,1)$ and let

$$F(m,p) \, := \, \Upsilon_{1^m \times w_0}.$$

Fomin–Kirillov [FK] showed that $F(m,p)$ counts the number of plane partitions of shape $(p-1, p-2,\ldots,1)$ with entries at most m. This number of plane partitions has a product formula given by Proctor [Pro].

Theorem 2.1 ([FK, Pro]). *In the notation above, we have:*

$$F(m,p) \, = \prod_{1\le i<j\le p} \frac{2m+i+j-1}{i+j-1}\,.$$

In notation of [MPP2], we have:

$$F(m,p) \, = \, \frac{\Lambda(2m+2p)\,\Lambda(2m+1)\,\Phi(p)}{\Phi(2m+p)\,\Lambda(2p)}\,,$$

where $\Phi(n) := 1!\cdot 2!\cdots(n-1)!$ and $\Lambda(n) := (n-2)!(n-4)!\cdots$

Proposition 2.2. *For nonnegative integers $b_1, b_2,\ldots,b_k$, let $w(b_k,\ldots,b_1)$ be the associated layered permutation then*

$$\Upsilon_{w(b_k,\ldots,b_1)} \, = \, \Upsilon_{w(b_k,\ldots,b_2)} \cdot F\big(|b|-b_1, b_1\big), \tag{2.1}$$

where $|b| = b_1+b_2+\cdots+b_k$.

Proof. The permutation $w(b_k,\ldots,b_1)$ can be written as the product $w(b_k,\ldots,b_2)\times w_0(b_k)$. By properties of Schubert polynomials (e.g. see [Mac, (4.6)] or [Man, Cor. 2.4.6]) we have that

$$\mathfrak{S}_{w(b_k,\ldots,b_1)} = \mathfrak{S}_{w(b_k,\ldots,b_2)} \cdot \mathfrak{S}_{1^{|b|-b_1}\times w_0(b_k)},$$

and the result follows by doing a principal evaluation. □

Remark 2.3. Equation (2.1) can be turned into a dynamic program to find layered permutations $w(b_k,\ldots,b_1)$ that achieves $v(n)$, see the appendix.

3. Asymptotics of the largest $v(n)$

3.1. **The outline.** We will use (2.1) inductively to prove the main result. Let $p := b_1$ and $m := n-p$, so that $m = b_2+\ldots+b_k$. By definition of $v(n)$, we have that

$$v(n) \, = \, \max_{b\,:\,|b|=n} \Upsilon_{w(b)}\,.$$

Next, using (2.1), $v(n)$ becomes

$$v(n) \, = \, \max_{1\le p\le n} \big\{ v(n-p)\,F(n-p,p)\big\}. \tag{3.1}$$

We will need very precise estimates on $\log F(m, n-m)$. Note that the exact asymptotic expansion for the *Barnes G-function*, which can be used to obtain the asymptotics of $\Phi(\cdot)$ and $\Lambda(\cdot)$, see e.g. [AR]. However, these bounds are insufficient as we also need sharp bounds for the error terms which hold for all m and n. We obtain these in the next subsection. These estimates are then combined with Proposition 2.2 to prove Theorem 1.3.

3.2. **Technical estimates.** Let $f(x)$ be the function defined in (1.3). The next lemma gives bounds on $\log F(m, n-m)$ in terms of the function $f(x)$.

Proposition 3.1. *For all integers $n \geq m \geq 0$, we have:*

$$-2n \;\leq\; \log F(m, n-m) \,-\, n^2 f(m/n) \;\leq\; 0.$$

We split the proof into two lemmas, one for the upper bound and the other for the lower bound.

Lemma 3.2. *For all integers $n \geq m \geq 0$, we have:*

$$\log F(m, n-m) - n^2 f(m/n) \leq 0.$$

Proof. We use the product formula for $F(m,p)$ in Theorem 2.1.

$$\begin{aligned} \log F(m,p) &= \sum_{1 \leq i < j \leq p} \Big(\log(2m + i + j - 1) - \log(i + j - 1) \Big) \\ &= \sum_{1 \leq i \leq j' \leq p-1} \Big(\log(2m + i + j') - \log(i + j') \Big), \end{aligned} \tag{3.2}$$

where we changed the index to $j' = j - 1$. Next, we approximate this sum using a double integral. Let

$$g(x,y) := \log(2m + x + y) - \log(x + y).$$

Notice that the function $g(x,y)$ is constant along the lines $x + y = k$ for constant k. Therefore, we can shift the terms of the sum in the RHS of (3.2) by $(i,j) \mapsto (i - 1/\sqrt{2}, j + 1/\sqrt{2})$ without changing the sum (see center of Figure 2)

$$\log F(m,p) \;=\; \sum_{(i,j) \in S} \Big(\log(2m + i + j') - \log(i + j') \Big), \tag{3.3}$$

where $S = \big\{\mathbb{Z}^2 + (-1/\sqrt{2}, 1/\sqrt{2})\big\} \cap \big\{(x,y) \;:\; 0 \leq x \leq p, x < y \leq p\big\}$.

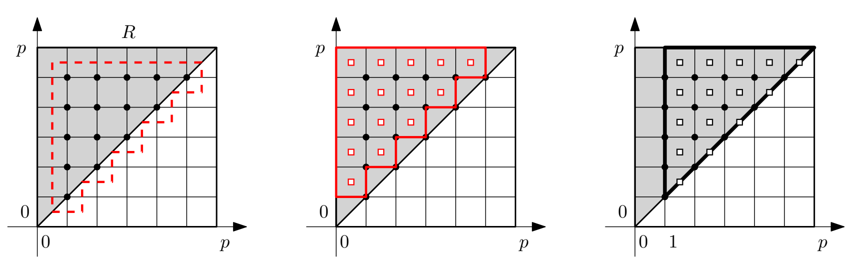

FIGURE 2. Illustration of the proof of the upper and lower bounds of Proposition 3.1 for $\log F(m,p)$ for $p = 5$. The lattice points $\bullet$ on the left are the support of the sum $\sum_{i \leq j} g(i,j)$. This sum remains the same if the support is shifted by $\big(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\big)$, giving points $\square$ in the middle. The original sum is bounded below by the sum over the support shifted by $\big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\big)$, giving points $\square$ in the right.

Next, compute the Hessian H of $g(x,y)$. We have:

$$\mathrm{H} \;=\; C \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{where} \quad C \;=\; \frac{1}{(x+y)^2} \;-\; \frac{1}{(2m + x + y)^2}\,.$$

Matrix H has eigenvalues 0 and $2C$ that are nonnegative in $[0,p] \times [0,p]$. Thus $g(x,y)$ is convex in this region. The modified sum in (3.3) is the sum of values of $g(x,y)$ over centers of the unit squares which fit entirely in R. By convexity, each such value of $g(x,y)$ is less than the average value of $g(x,y)$ over its square. Hence the sum in (3.3) is bounded above by the double integral,

$$\log F(m,p) \leq \int_0^p \int_y^p \left(\log(2m+x+y) - \log(x+y)\right)\, dx\, dy.$$

Next, we compute this double integral and obtain

$$\int_0^p \int_y^p \left(\log(2m+x+y) - \log(x+y)\right)\, dx\, dy \,=\, (m+p)^2 f(m/(m+p)), \tag{3.4}$$

for $f(x)$ defined in (1.3). This proves the upper bound. □

Lemma 3.3. *For all integers $n \geq m \geq 0$, we have:*

$$\log F(m, n-m) - n^2 f(m/n) \,\geq\, -2n.$$

Proof. Since the function $g(x,y)$ is decreasing along the x direction and y direction then each value $g(i,j)$ in the sum is bigger than the average value of $g(x,y)$ over the unit square with center $(i+1/\sqrt{2}, j+1/\sqrt{2})$ (see right of Figure 2). Hence the original sum in (3.2) is bounded below by the double integral

$$\log F(m,p) \,=\, \sum_{1\leq i<j\leq p} g(i,j) \geq \int_1^p \int_x^p g(x,y)\, dy\, dx. \tag{3.5}$$

This integral can be written in terms of the original integral, computed in (3.4), as follows

$$\begin{aligned} \int_1^p \int_x^p g(x,y)\, dy\, dx \,&=\, \int_0^p \int_x^p g(x,y)\, dy\, dx \,-\, \int_0^1 \int_x^p g(x,y)\, dy\, dx \\ &=\, (m+p)^2 f(m/(m+p)) \,-\, \int_0^1 \int_x^p g(x,y)\, dy\, dx. \end{aligned} \tag{3.6}$$

Since the function $g(x,y)$ is decreasing in the x direction then the double integral in the RHS above is bounded by the following single integral

$$-\int_0^1 \int_x^p g(x,y)\, dy\, dx \,\geq\, -\int_0^p g(0,y)\, dy. \tag{3.7}$$

We evaluate this single integral and use Jensen's inequality to obtain

$$\begin{aligned} -\int_0^p g(0,y) dy \,&=\, 2m\log(2m) \,+\, p\log(p) \,-\, (2m+p)\log(2m+p) \\ &\geq\, (2m+p)\big(\log(2m+p) \,-\, \log 2\big) \,-\, (2m+p)\log(2m+p). \end{aligned} \tag{3.8}$$

Combining (3.5),(3.6), (3.2), and (3.8) we have

$$\log F(m,p) \,\geq\, (m+p)^2 f(m/(m+p)) + (2m+p)\left(\log(2m+p) \,-\, \log(2)\right) - (2m+p)\log(2m+p).$$

The RHS is greater than or equal to $(m+p)^2 f(m/(m+p)) - 2(m+p)$, as desired. □

3.3. **Optimizing constants.** Our goal is to show that $\lim_{n\to\infty} \log_2 v(n)/n^2$ is a constant. In the previous lemma we gave bounds on the error of approximating $\log F(m, n-m)$ by $n^2 f(x)$ where $x = m/n$ in $[0,1]$. We now find a unique constant γ such that $f(x) + \gamma x^2$ has a unique maximum over $x \in [0,1)$.

Lemma 3.4. *There exist a unique $\gamma > 0$ and $\alpha \in (0,1)$, such that:*

(1) $2\gamma\alpha + f'(\alpha) = 0$,
(2) $\gamma\alpha^2 + f(\alpha) = \gamma$ *with* $2\gamma + r''(\alpha) \leq 0$.

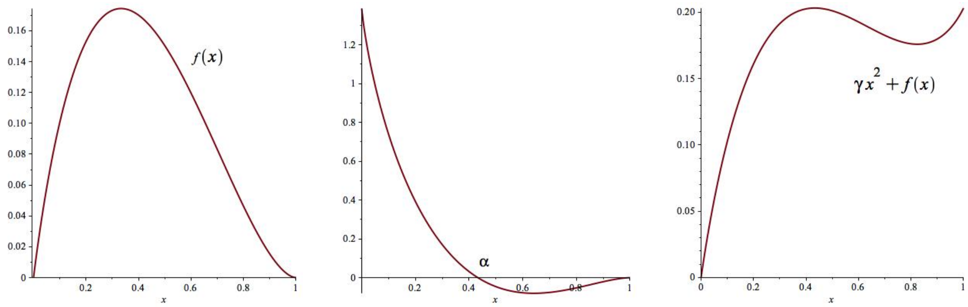

FIGURE 3. Graphs of the functions $f(x)$, $q(x)$ and $\gamma x^2 + f(x)$, on $(0,1)$.

And for this γ, the maximum of $f(x) + \gamma x^2$ over $x \in [0,1)$ is achieved at the given α, and the value is precisely γ. That is,

$$\max_{x\in[0,1)} \left(f(x) + \gamma x^2\right) \,=\, f(\alpha) + \gamma\alpha^2 \,=\, \gamma.$$

Proof. First, it is straightforward to show that $\lim_{x\to 0} f(x) = \lim_{x\to 1} f(x) = 0$ and that $f(x) > 0$ for $x \in (0,1)$ (see plot of $f(x)$ on the left of Figure 3).

Let α be a solution to the equation $q(x) = 0$ where

$$\begin{aligned} q(x) :=\ & f(x)2x + f'(x)(1-x^2) \\ =\ & (1-x)^2\log(1-x) - (1+x)^2\log(1+x) + 2x\log(x) + 2(1+x^2)\log(2). \end{aligned}$$

This function on the RHS above has one root $\alpha = 0.4331818312..$ and the other is $x = 1$, as easily seen from the plot, but also can be shown analytically. Then we set

$$\gamma := \frac{f(\alpha)}{1-\alpha^2} = -\frac{f'(\alpha)}{2\alpha},$$

so γ and α now satisfy conditions (1) and (2).

Next, we see that $\gamma = f(\alpha)/(1-\alpha^2) \approx 0.2032558981$. To prove that this is indeed a maximum for $f(x)+\gamma x^2$, we check that the second derivative, $d^2(\gamma x^2 + f(x))/dx^2 = 2\gamma + r''(x) < 0$ for $x = \alpha$. We have that $r''(x) = \log(x^2/(1-x^2))$. Since $\alpha \le 0.45$, we have that $x^2/(1-x^2) < 0.26$ and so $r''(\alpha) < -1.3 < -2\gamma$ and so the value is a local maximum and by condition (2) it is equal to γ. □

3.4. **Proof of Theorem 1.3.** The theorem follows immediately from the following lemma.

Lemma 3.5. *For all $n \ge 2$ we have:*

$$\left|\log v(n) - \gamma n^2\right| \,\le\, 4n\,.$$

Conversely, suppose for a layered permutation $w(b) \in S_n$ we have

$$\left|\log \Upsilon_w - \gamma n^2\right| \,\le\, 4n\,.$$

Then $b = (....,b_2,b_1)$, s.t. $b_i \sim (1-\alpha)\alpha^{i-1}n$ for all fixed $i \ge 1$.

Proof. We proceed by induction to show that $|\log v(n) - \gamma n^2| \le 4n$ holds for all $n \ge 2$. The base cases $n = 2$ can be checked directly (see exact values in the appendix).

We start with (3.1) and use the induction hypothesis and the upper bound of Proposition 3.1 to obtain

$$\begin{aligned}\log v(n) &= \max_{m<n}\big(\log v(m) + \log F(m, n-m)\big)\\ &\le \max_{m<n}\big(\gamma m^2 + \log F(m, n-m) + 2m\big),\\ &\le n^2 \max_{x\in[0,1)}\big(f(x)+\gamma x^2\big) + 2n.\end{aligned}$$

By Lemma 3.4, the maximum value of $f(x)+\gamma x^2$ is equal to γ. Thus, the above inequality becomes

$$\log v(n) \le \gamma n^2 + 2n.$$

This maximum is achieved when $x=\alpha$, i.e. when $m = n\alpha$ and $p = b_1 = (1-\alpha)n$. By the definition of $v(n)$, for this value of m we have that

$$\log v(n) \ge \log v(n\alpha) + \log F(n\alpha, n-n\alpha).$$

By the induction hypothesis and the lower bound of Proposition 3.1, the above inequality becomes

$$\begin{aligned}\log v(n) &\ge \big(\gamma n^2\alpha^2 - 4n\alpha\big) + \big(n^2 f(\alpha) - 2n\big)\\ &= \gamma n^2 - 2(1+2\alpha)n \ge \gamma n^2 - 4n.\end{aligned}$$

Here we again used the fact that $f(\alpha)+\gamma\alpha^2 = \gamma$ and that $\alpha \le 1/2$. In summary,

$$\big|\log v(n) - \gamma n^2\big| \le 4n,$$

and this bound is attained when $b_1 \sim (1-\alpha)n$. Recursively, we obtain $b_i \sim (1-\alpha)\alpha^{i-1}n$ for every fixed $i = 2, 3, \ldots$. □

Remark 3.6. Note that the appendix shows rather slow rate of convergence for $h(n) := \frac{1}{n^2}\log_2 v(n)$, giving only $h(300) \approx 0.2904$. This suggests that $h(n) = \gamma/(\log 2) - 1/n - o(1/n)$, so that the bound in Lemma 3.5 is quite sharp.

4. Final remarks

4.1. Stanley's Conjecture 1.1 remains open but is very likely to hold. Denote by

$$a(n) = \sum_{w\in S_n} \Upsilon_w$$

the total number of rc-graphs (pipe dreams) of size n. Since

$$u(n) \le a(n) \le n!\,u(n),$$

we conclude that it suffices to prove the asymptotics result for a_n. This suggests connections to counting general tilings (see e.g. [AS]), as pipe dreams can be viewed as tilings of a staircase shape with two types of tiles, but with one global condition (strains can intersect at most once). The problem is especially similar to counting *Knutson-Tao puzzles* enumerating the *Littlewood–Richardson coefficients*, whose maximal asymptotics was recently studied in [PPY].

By analogy with the tilings, one can ask if $u(n)$ satisfies some sort of super-multiplicativity property. Formally, let $w\otimes 1^c$ denote the *Kronecker product permutation* of size cn, whose permutation matrix equals the Kronecker product of the permutation matrix P_w and the identity I_c (see [MPP1]).

Conjecture 4.1. *For $w \in S_n$, we have $\Upsilon_{w\otimes 1^2} \ge \Upsilon_w^4$.*

We verified the conjecture for all $w \in S_n$ where $n \le 5$, but perhaps more computational evidence would be helpful.

4.2. Similarly, the Merzon–Smirnov Conjecture 1.4 remains open. In our opinion, the numerical evidence in favor of the conjecture is insufficient, and it would be interesting to verify it for larger n. To speedup the computation, perhaps, there are large classes of permutations $u \in S_n$ which can be proved to be non-maximal, i.e. there exists $w \in S_n$, s.t. $\Upsilon_u \le \Upsilon_w$. Such permutations can then be ignored in the exhaustive search.

In fact, Prop. 6.5 in [MPP1] gives explicit constructions of large families of permutations $w \in S_n$, for which $\log \Upsilon_w = \Theta(n)$. These permutations are very far from being layered (in the transposition distance), suggesting that if true, proving Conjecture 1.4 might not be easy.

4.3. In [Sta], Stanley also considered the case when Υ_w is small. It is well known that $\Upsilon_w = 1$ if and only if w is *dominant* [Man], i.e. 132-avoiding. Stanley conjectured that $\Upsilon_w = 2$ if and only if w has exactly one instance of the pattern 132. This was recently proved by Weigandt [Wei], who also showed that $\Upsilon_w - 1$ is greater than or equal the number of instances of the pattern 132 in w.

This suggests the problem of finding permutations where the number of patterns 132 is maximal. In the field of pattern avoidance, this problem can be rephrased as asking for permutations $w \in S_n$ with maximal *packing density* of the pattern 132, see [Kit, §8.3.1]. The solution due to Stromquist is extremely well understood, and has been both refined and generalized, see [A+, BSV, HSV], [Pri, §5.1] and [OEIS, A061061]. The maximal packing density is attained at a layered permutation $w(b_1, b_2, \ldots)$, where the runs b_i have a geometric distribution:

$$b_i \,\sim\, \rho(1-\rho)^{i-1} n, \;\; i = 1, 2, \ldots \quad \text{where} \quad \rho \,=\, \frac{\sqrt{3}-1}{2} \,\approx\, 0.366025$$

While, of course, $v(n)$ are attained at somewhat different layered permutations, the similarities to this work are rather striking and go beyond coincidences. They are rooted in the recursive nature of optimal permutations in both cases, which are solutions of similar (but different!) maximization problems.

4.4. The bounds for $u(n)$ from Theorem 1.2 are obtained from the Cauchy identity of Schubert polynomials which gives

$$\sum_{w_0 = v^{-1}u} \Upsilon_u \Upsilon_v = 2^{\binom{n}{2}}. \tag{4.1}$$

One could then ask for large values of $\Upsilon_w \Upsilon_{ww_0^{-1}}$. Let $u'(n) := \max_{w \in S_n}\{\Upsilon_w \cdot \Upsilon_{ww_0^{-1}}\}$. The table below has the values of $u'(n)$ for $n = 2, \ldots, 9$ and the permutations w (up to multiplying by w_0^{-1}) that achieve that value $u'(n)$.

n	$u'(n)$	w
3	2	132
4	6	1423
5	33	15243
6	286	162534
7	4620	1736254
8	162360	18527364
9	9057090	195283746

Note that for a layered permutation $w(b)$, the permutation $w(b)w_0^{-1}$ is dominant and so $\Upsilon_{w(b)w_0^{-1}} = 1$.

There is a combinatorial proof of (4.1) by Bergeron and Billey [BB] involving taking a double *rc-graph* of w_0 ($2^{\binom{n}{2}}$ many) and reading from each half of it permutations u and v satisfying $w_0 = v^{-1}u$. All such double rc-graphs of w_0 can be obtained from an initial double rc-graph via certain local transformations (see [BB, Sec. 4]). One can use these local transformations in a Markov chain to obtain a random double rc-graph of w_0 and from it read off a permutation u;

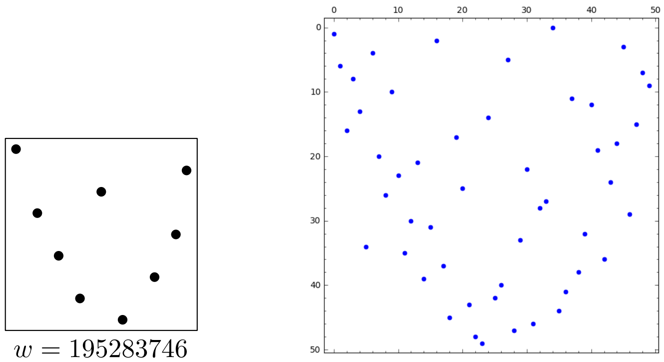

FIGURE 4. Permutation matrices of 195283746 and of a permutation $u \in S_{50}$ from the random double rc-graph.

see Figure 4. We conjecture that the permutation matrix of random permutations u has a parabolic frozen region.

The second permutation in Figure 4 is obtained by running a Markov chain for $5 \cdot 10^9$ local moves on a double rc-graph of $v^{-1}u = w_0 \in S_{50}$, described in [BB, Sec. 4]. Half of the resulting double rc-graph given in Figure 5 is then converted into a permutation $u \in S_{50}$.

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Table of exact values for $n \leq 300$. Below we present table of tuples b of layered permutations $w(b)$ maximizing $v(n)$. The third column is $f(n) := \frac{1}{n^2}\log_2 v(n)$.

n	$(\ldots,b_2,b_1)$	$f(n)$
1	(1)	0.000000
2	(1,1)	0.000000
3	(1,2)	0.111111
4	(1,3)	0.145121
5	(1,1,3)	0.152294
6	(1,1,4)	0.177564
7	(1,2,4)	0.191149
8	(1,2,5)	0.206317
9	(1,2,6)	0.213824
10	(1,3,6)	0.220771
11	(1,3,7)	0.227005
12	(1,3,8)	0.229879
13	(1,1,3,8)	0.233769
14	(1,1,4,8)	0.237048
15	(1,1,4,9)	0.241677
16	(1,1,4,10)	0.244446
17	(1,2,4,10)	0.246954
18	(1,2,4,11)	0.249509
19	(1,2,5,11)	0.251966
20	(1,2,5,12)	0.254240
21	(1,2,5,13)	0.255575
22	(1,2,6,13)	0.257354
23	(1,2,6,14)	0.258685
24	(1,3,6,14)	0.260063
25	(1,3,6,15)	0.261360
26	(1,3,7,15)	0.262425
27	(1,3,7,16)	0.263673
28	(1,3,7,17)	0.264435
29	(1,3,8,17)	0.265233
30	(1,3,8,18)	0.266034
31	(1,1,3,8,18)	0.266811
32	(1,1,3,8,19)	0.267619
33	(1,1,4,8,19)	0.268165
34	(1,1,4,8,20)	0.268973
35	(1,1,4,9,20)	0.269675
36	(1,1,4,9,21)	0.270460
37	(1,1,4,9,22)	0.270978
38	(1,1,4,10,22)	0.271548
39	(1,1,4,10,23)	0.272081
40	(1,2,4,10,23)	0.272523
41	(1,2,4,10,24)	0.273065
42	(1,2,4,11,24)	0.273453
43	(1,2,4,11,25)	0.273996
44	(1,2,4,11,26)	0.274357
45	(1,2,5,11,26)	0.274862
46	(1,2,5,11,27)	0.275235
47	(1,2,5,12,27)	0.275654
48	(1,2,5,12,28)	0.276036
49	(1,2,5,12,29)	0.276277
50	(1,2,5,13,29)	0.276634

n	$(\ldots,b_2,b_1)$	$f(n)$
51	(1,2,5,13,30)	0.276896
52	(1,2,6,13,30)	0.277275
53	(1,2,6,13,31)	0.277550
54	(1,2,6,14,31)	0.277807
55	(1,2,6,14,32)	0.278094
56	(1,3,6,14,32)	0.278322
57	(1,3,6,14,33)	0.278618
58	(1,3,6,14,34)	0.278815
59	(1,3,6,15,34)	0.279103
60	(1,3,6,15,35)	0.279313
61	(1,3,7,15,35)	0.279525
62	(1,3,7,15,36)	0.279747
63	(1,3,7,16,36)	0.279962
64	(1,3,7,16,37)	0.280192
65	(1,3,7,16,38)	0.280344
66	(1,3,7,17,38)	0.280532
67	(1,3,7,17,39)	0.280698
68	(1,3,8,17,39)	0.280862
69	(1,3,8,17,40)	0.281038
70	(1,3,8,18,40)	0.281178
71	(1,3,8,18,41)	0.281363
72	(1,3,8,18,42)	0.281486
73	(1,1,3,8,18,42)	0.281670
74	(1,1,3,8,18,43)	0.281803
75	(1,1,3,8,19,43)	0.281969
76	(1,1,3,8,19,44)	0.282112
77	(1,1,4,8,19,44)	0.282210
78	(1,1,4,8,19,45)	0.282361
79	(1,1,4,8,20,45)	0.282488
80	(1,1,4,8,20,46)	0.282646
81	(1,1,4,8,20,47)	0.282755
82	(1,1,4,9,20,47)	0.282902
83	(1,1,4,9,20,48)	0.283019
84	(1,1,4,9,21,48)	0.283165
85	(1,1,4,9,21,49)	0.283288
86	(1,1,4,9,22,49)	0.283370
87	(1,1,4,9,22,50)	0.283501
88	(1,1,4,9,22,51)	0.283590
89	(1,1,4,10,22,51)	0.283715
90	(1,1,4,10,22,52)	0.283811
91	(1,1,4,10,23,52)	0.283914
92	(1,1,4,10,23,53)	0.284018
93	(1,2,4,10,23,53)	0.284090
94	(1,2,4,10,23,54)	0.284200
95	(1,2,4,10,24,54)	0.284279
96	(1,2,4,10,24,55)	0.284394
97	(1,2,4,10,24,56)	0.284475
98	(1,2,4,11,24,56)	0.284553
99	(1,2,4,11,24,57)	0.284641
100	(1,2,4,11,25,57)	0.284736

n	$(\ldots, b_2, b_1)$	$f(n)$
101	$(1, 2, 4, 11, 25, 58)$	0.284828
102	$(1, 2, 4, 11, 25, 59)$	0.284891
103	$(1, 2, 4, 11, 26, 59)$	0.284978
104	$(1, 2, 4, 11, 26, 60)$	0.285046
105	$(1, 2, 5, 11, 26, 60)$	0.285148
106	$(1, 2, 5, 11, 26, 61)$	0.285222
107	$(1, 2, 5, 11, 27, 61)$	0.285289
108	$(1, 2, 5, 11, 27, 62)$	0.285368
109	$(1, 2, 5, 12, 27, 62)$	0.285434
110	$(1, 2, 5, 12, 27, 63)$	0.285518
111	$(1, 2, 5, 12, 27, 64)$	0.285577
112	$(1, 2, 5, 12, 28, 64)$	0.285657
113	$(1, 2, 5, 12, 28, 65)$	0.285720
114	$(1, 2, 5, 12, 29, 65)$	0.285766
115	$(1, 2, 5, 12, 29, 66)$	0.285834
116	$(1, 2, 5, 13, 29, 66)$	0.285892
117	$(1, 2, 5, 13, 29, 67)$	0.285965
118	$(1, 2, 5, 13, 29, 68)$	0.286015
119	$(1, 2, 5, 13, 30, 68)$	0.286074
120	$(1, 2, 5, 13, 30, 69)$	0.286129
121	$(1, 2, 6, 13, 30, 69)$	0.286201
122	$(1, 2, 6, 13, 30, 70)$	0.286261
123	$(1, 2, 6, 13, 31, 70)$	0.286306
124	$(1, 2, 6, 13, 31, 71)$	0.286369
125	$(1, 2, 6, 13, 31, 72)$	0.286413
126	$(1, 2, 6, 14, 31, 72)$	0.286472
127	$(1, 2, 6, 14, 31, 73)$	0.286519
128	$(1, 2, 6, 14, 32, 73)$	0.286576
129	$(1, 2, 6, 14, 32, 74)$	0.286628
130	$(1, 3, 6, 14, 32, 74)$	0.286667
131	$(1, 3, 6, 14, 32, 75)$	0.286723
132	$(1, 3, 6, 14, 33, 75)$	0.286768
133	$(1, 3, 6, 14, 33, 76)$	0.286827
134	$(1, 3, 6, 14, 33, 77)$	0.286868
135	$(1, 3, 6, 14, 34, 77)$	0.286910
136	$(1, 3, 6, 14, 34, 78)$	0.286955
137	$(1, 3, 6, 15, 34, 78)$	0.287007
138	$(1, 3, 6, 15, 34, 79)$	0.287056
139	$(1, 3, 6, 15, 34, 80)$	0.287089
140	$(1, 3, 6, 15, 35, 80)$	0.287140
141	$(1, 3, 6, 15, 35, 81)$	0.287177
142	$(1, 3, 7, 15, 35, 81)$	0.287221
143	$(1, 3, 7, 15, 35, 82)$	0.287262
144	$(1, 3, 7, 15, 36, 82)$	0.287303
145	$(1, 3, 7, 15, 36, 83)$	0.287346
146	$(1, 3, 7, 16, 36, 83)$	0.287380
147	$(1, 3, 7, 16, 36, 84)$	0.287427
148	$(1, 3, 7, 16, 36, 85)$	0.287459
149	$(1, 3, 7, 16, 37, 85)$	0.287508
150	$(1, 3, 7, 16, 37, 86)$	0.287544

n	$(\ldots, b_2, b_1)$	$f(n)$
151	$(1, 3, 7, 16, 38, 86)$	0.287573
152	$(1, 3, 7, 16, 38, 87)$	0.287612
153	$(1, 3, 7, 17, 38, 87)$	0.287643
154	$(1, 3, 7, 17, 38, 88)$	0.287684
155	$(1, 3, 7, 17, 38, 89)$	0.287713
156	$(1, 3, 7, 17, 39, 89)$	0.287750
157	$(1, 3, 7, 17, 39, 90)$	0.287782
158	$(1, 3, 8, 17, 39, 90)$	0.287814
159	$(1, 3, 8, 17, 39, 91)$	0.287849
160	$(1, 3, 8, 17, 40, 91)$	0.287879
161	$(1, 3, 8, 17, 40, 92)$	0.287916
162	$(1, 3, 8, 17, 40, 93)$	0.287942
163	$(1, 3, 8, 18, 40, 93)$	0.287975
164	$(1, 3, 8, 18, 40, 94)$	0.288003
165	$(1, 3, 8, 18, 41, 94)$	0.288040
166	$(1, 3, 8, 18, 41, 95)$	0.288071
167	$(1, 3, 8, 18, 42, 95)$	0.288093
168	$(1, 3, 8, 18, 42, 96)$	0.288126
169	$(1, 1, 3, 8, 18, 42, 96)$	0.288155
170	$(1, 1, 3, 8, 18, 42, 97)$	0.288191
171	$(1, 1, 3, 8, 18, 42, 98)$	0.288216
172	$(1, 1, 3, 8, 18, 43, 98)$	0.288244
173	$(1, 1, 3, 8, 18, 43, 99)$	0.288272
174	$(1, 1, 3, 8, 19, 43, 99)$	0.288302
175	$(1, 1, 3, 8, 19, 43, 100)$	0.288332
176	$(1, 1, 3, 8, 19, 44, 100)$	0.288355
177	$(1, 1, 3, 8, 19, 44, 101)$	0.288387
178	$(1, 1, 3, 8, 19, 44, 102)$	0.288410
179	$(1, 1, 4, 8, 19, 44, 102)$	0.288432
180	$(1, 1, 4, 8, 19, 44, 103)$	0.288457
181	$(1, 1, 4, 8, 19, 45, 103)$	0.288486
182	$(1, 1, 4, 8, 19, 45, 104)$	0.288513
183	$(1, 1, 4, 8, 20, 45, 104)$	0.288533
184	$(1, 1, 4, 8, 20, 45, 105)$	0.288563
185	$(1, 1, 4, 8, 20, 46, 105)$	0.288586
186	$(1, 1, 4, 8, 20, 46, 106)$	0.288617
187	$(1, 1, 4, 8, 20, 46, 107)$	0.288640
188	$(1, 1, 4, 8, 20, 47, 107)$	0.288661
189	$(1, 1, 4, 8, 20, 47, 108)$	0.288686
190	$(1, 1, 4, 9, 20, 47, 108)$	0.288711
191	$(1, 1, 4, 9, 20, 47, 109)$	0.288737
192	$(1, 1, 4, 9, 20, 47, 110)$	0.288756
193	$(1, 1, 4, 9, 20, 48, 110)$	0.288782
194	$(1, 1, 4, 9, 20, 48, 111)$	0.288803
195	$(1, 1, 4, 9, 21, 48, 111)$	0.288832
196	$(1, 1, 4, 9, 21, 48, 112)$	0.288854
197	$(1, 1, 4, 9, 21, 49, 112)$	0.288876
198	$(1, 1, 4, 9, 21, 49, 113)$	0.288900
199	$(1, 1, 4, 9, 21, 49, 114)$	0.288917
200	$(1, 1, 4, 9, 22, 49, 114)$	0.288937

n	$(\ldots, b_2, b_1)$	$f(n)$
201	$(1,1,4,9,22,49,115)$	0.288956
202	$(1,1,4,9,22,50,115)$	0.288982
203	$(1,1,4,9,22,50,116)$	0.289003
204	$(1,1,4,9,22,51,116)$	0.289019
205	$(1,1,4,9,22,51,117)$	0.289041
206	$(1,1,4,10,22,51,117)$	0.289061
207	$(1,1,4,10,22,51,118)$	0.289084
208	$(1,1,4,10,22,51,119)$	0.289101
209	$(1,1,4,10,22,52,119)$	0.289122
210	$(1,1,4,10,22,52,120)$	0.289141
211	$(1,1,4,10,23,52,120)$	0.289160
212	$(1,1,4,10,23,52,121)$	0.289180
213	$(1,1,4,10,23,53,121)$	0.289197
214	$(1,1,4,10,23,53,122)$	0.289219
215	$(1,1,4,10,23,53,123)$	0.289234
216	$(1,2,4,10,23,53,123)$	0.289251
217	$(1,2,4,10,23,53,124)$	0.289268
218	$(1,2,4,10,23,54,124)$	0.289289
219	$(1,2,4,10,23,54,125)$	0.289307
220	$(1,2,4,10,24,54,125)$	0.289320
221	$(1,2,4,10,24,54,126)$	0.289340
222	$(1,2,4,10,24,55,126)$	0.289358
223	$(1,2,4,10,24,55,127)$	0.289379
224	$(1,2,4,10,24,55,128)$	0.289394
225	$(1,2,4,10,24,56,128)$	0.289411
226	$(1,2,4,10,24,56,129)$	0.289428
227	$(1,2,4,11,24,56,129)$	0.289441
228	$(1,2,4,11,24,56,130)$	0.289460
229	$(1,2,4,11,24,57,130)$	0.289473
230	$(1,2,4,11,24,57,131)$	0.289492
231	$(1,2,4,11,24,57,132)$	0.289507
232	$(1,2,4,11,25,57,132)$	0.289526
233	$(1,2,4,11,25,57,133)$	0.289541
234	$(1,2,4,11,25,58,133)$	0.289558
235	$(1,2,4,11,25,58,134)$	0.289575
236	$(1,2,4,11,25,58,135)$	0.289587
237	$(1,2,4,11,25,59,135)$	0.289602
238	$(1,2,4,11,25,59,136)$	0.289615
239	$(1,2,4,11,26,59,136)$	0.289633
240	$(1,2,4,11,26,59,137)$	0.289648
241	$(1,2,4,11,26,60,137)$	0.289661
242	$(1,2,4,11,26,60,138)$	0.289676
243	$(1,2,5,11,26,60,138)$	0.289693
244	$(1,2,5,11,26,60,139)$	0.289710
245	$(1,2,5,11,26,60,140)$	0.289722
246	$(1,2,5,11,26,61,140)$	0.289738
247	$(1,2,5,11,26,61,141)$	0.289751
248	$(1,2,5,11,27,61,141)$	0.289764
249	$(1,2,5,11,27,61,142)$	0.289778
250	$(1,2,5,11,27,62,142)$	0.289792

n	$(\ldots, b_2, b_1)$	$f(n)$
251	$(1,2,5,11,27,62,143)$	0.289807
252	$(1,2,5,11,27,62,144)$	0.289818
253	$(1,2,5,12,27,62,144)$	0.289833
254	$(1,2,5,12,27,62,145)$	0.289845
255	$(1,2,5,12,27,63,145)$	0.289862
256	$(1,2,5,12,27,63,146)$	0.289875
257	$(1,2,5,12,27,64,146)$	0.289885
258	$(1,2,5,12,27,64,147)$	0.289899
259	$(1,2,5,12,28,64,147)$	0.289912
260	$(1,2,5,12,28,64,148)$	0.289927
261	$(1,2,5,12,28,64,149)$	0.289938
262	$(1,2,5,12,28,65,149)$	0.289951
263	$(1,2,5,12,28,65,150)$	0.289964
264	$(1,2,5,12,29,65,150)$	0.289972
265	$(1,2,5,12,29,65,151)$	0.289985
266	$(1,2,5,12,29,66,151)$	0.289996
267	$(1,2,5,12,29,66,152)$	0.290010
268	$(1,2,5,12,29,66,153)$	0.290020
269	$(1,2,5,13,29,66,153)$	0.290033
270	$(1,2,5,13,29,66,154)$	0.290044
271	$(1,2,5,13,29,67,154)$	0.290058
272	$(1,2,5,13,29,67,155)$	0.290070
273	$(1,2,5,13,29,67,156)$	0.290078
274	$(1,2,5,13,29,68,156)$	0.290091
275	$(1,2,5,13,29,68,157)$	0.290100
276	$(1,2,5,13,30,68,157)$	0.290113
277	$(1,2,5,13,30,68,158)$	0.290124
278	$(1,2,5,13,30,69,158)$	0.290134
279	$(1,2,5,13,30,69,159)$	0.290146
280	$(1,2,6,13,30,69,159)$	0.290158
281	$(1,2,6,13,30,69,160)$	0.290171
282	$(1,2,6,13,30,69,161)$	0.290179
283	$(1,2,6,13,30,70,161)$	0.290192
284	$(1,2,6,13,30,70,162)$	0.290202
285	$(1,2,6,13,31,70,162)$	0.290211
286	$(1,2,6,13,31,70,163)$	0.290222
287	$(1,2,6,13,31,71,163)$	0.290233
288	$(1,2,6,13,31,71,164)$	0.290245
289	$(1,2,6,13,31,71,165)$	0.290253
290	$(1,2,6,13,31,72,165)$	0.290263
291	$(1,2,6,13,31,72,166)$	0.290272
292	$(1,2,6,14,31,72,166)$	0.290284
293	$(1,2,6,14,31,72,167)$	0.290294
294	$(1,2,6,14,31,73,167)$	0.290302
295	$(1,2,6,14,31,73,168)$	0.290313
296	$(1,2,6,14,32,73,168)$	0.290322
297	$(1,2,6,14,32,73,169)$	0.290334
298	$(1,2,6,14,32,73,170)$	0.290342
299	$(1,2,6,14,32,74,170)$	0.290353
300	$(1,2,6,14,32,74,171)$	0.290362

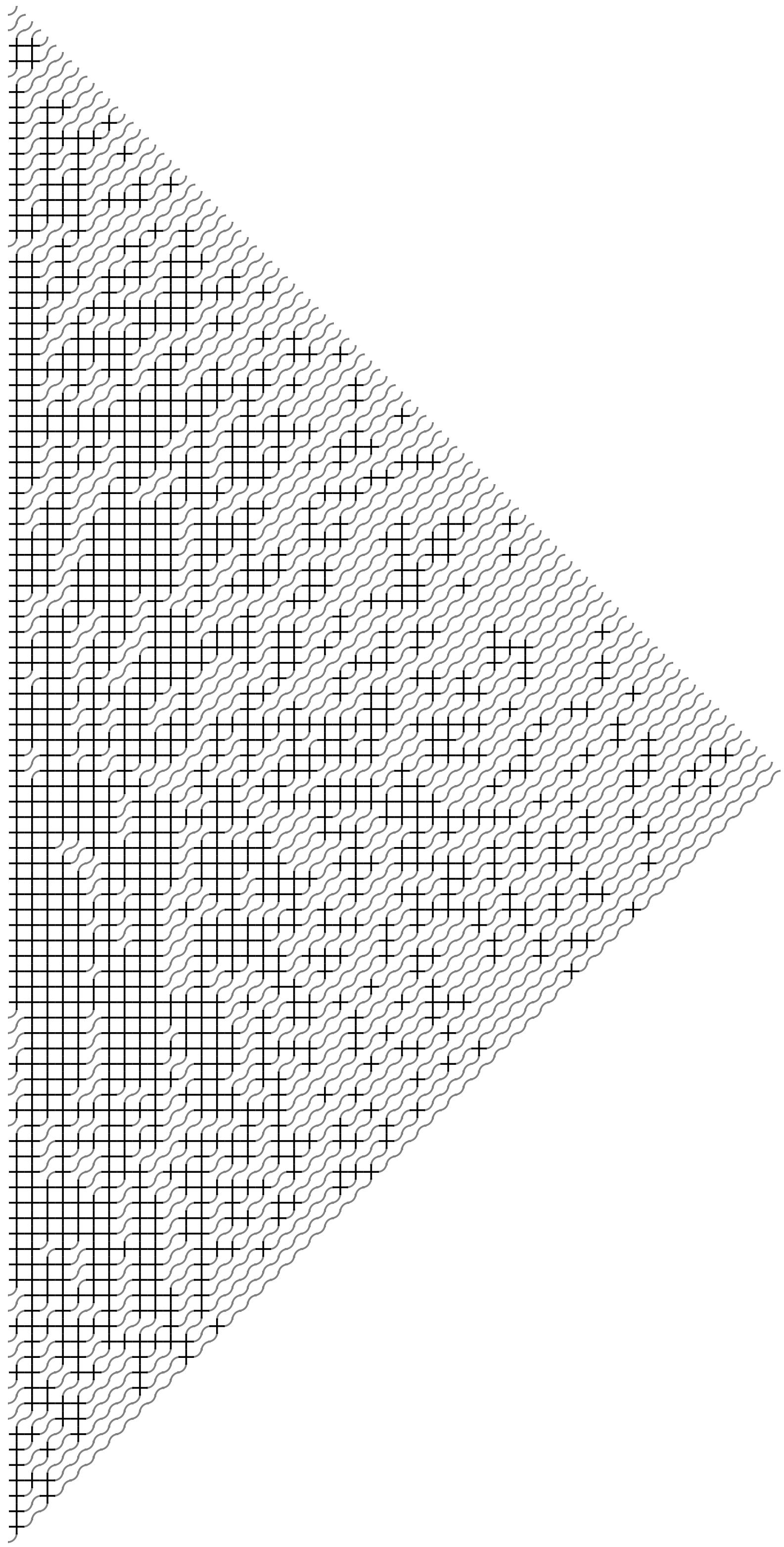

FIGURE 5. Random double rc-graph corresponding to a permutation in Figure 4.