ASYMPTOTICS OF PRINCIPAL EVALUATIONS OF SCHUBERT POLYNOMIALS FOR LAYERED PERMUTATIONS

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ABSTRACT. Denote by u(n) the largest principal specialization of the Schubert polynomial:

$$u(n) := \max_{w \in S_n} \mathfrak{S}_w(1, \dots, 1)$$

Stanley conjectured in [Sta] that there is a limit

$$\lim_{n \to \infty} \frac{1}{n^2} \log u(n),$$

and asked for a limiting description of permutations achieving the maximum u(n). Merzon and Smirnov conjectured in [MeS] that this maximum is achieved on layered permutations. We resolve both Stanley's problems restricted to layered permutations.

1. Introduction

Understanding the large-scale behavior of combinatorial objects is so fundamental to modern combinatorics, that it has become routine and no longer requires justification. However, in algebraic combinatorics, there are fewer results in this direction, as the objects tend to be have more structure and thus less approachable. This paper studies the asymptotic behavior of the principal evaluation of Schubert polynomials, partially resolving an open problem by Stanley [Sta]. As the reader shall see, the results are surprisingly precise.

Main results. Schubert polynomials $\mathfrak{S}_w(x_1,\ldots,x_n) \in \mathbb{N}[x_1,\ldots,x_n]$, $w \in S_n$, were introduced by Lascoux and Schützenberger [LS] to study Schubert varieties. They have been intensely studied in the last two decades and remain a central object in algebraic combinatorics. The principle evaluation of the Schubert polynomials can be defined via *Macdonald's identity* [Mac, Eq. 6.11]:

(1.1)
$$\Upsilon_w := \mathfrak{S}_w(1, \dots, 1) = \frac{1}{\ell!} \sum_{(a_1, \dots, a_\ell) \in R(w)} a_1 \cdots a_\ell.$$

Here $\ell = \ell(w)$ is the *length* of w (the number of inversions, and R(w) denotes the set of *reduced* words of $w \in S_n$: tuples (a_1, \ldots, a_ℓ) such that $s_{a_1} \cdots s_{a_\ell}$ is a reduced decomposition of w into simple transpositions $s_i = (i, i+1)$.

Note that Υ_w has a more direct (but less symmetric) combinatorial interpretation as the number of certain rc-graphs (also called *pipe dreams*), see e.g. [As]. In particular, we have $\Upsilon_w \in \mathbb{N}$, even though this is not immediately apparent from (1.1) (cf. §4.4).

Denote by u(n) the largest principal specialization of the Schubert polynomial:

$$u(n) := \max_{w \in S_n} \Upsilon_w$$

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Conjecture 1.1 (Stanley [Sta]). There is a limit

$$\lim_{n \to \infty} \frac{1}{n^2} \log u(n).$$

In addition, Stanley asked whether the permutations w in S_n achieving the maximum $\Upsilon_w = u(n)$ had a limiting description. There was some evidence in favor of this (see below), but before we turn to positive results let us put this conjecture into context.

One can think of Υ_w as a statistical sum of weighted random sorting networks of the permutation w. From a combinatorial point of view, this is a more natural notion, since e.g. $\Upsilon_{w_0} = 1$, where $w_0 = (n, n-1, \ldots, 1)$ is the permutation with maximal length $\ell(w_0) = \binom{n}{2}$. It is thus natural to expect u(n) to have nice asymptotic behavior. In fact, Stanley gave the first order of asymptotics for u(n):

Theorem 1.2 (Stanley [Sta]).

$$\frac{1}{4} \leq \liminf_{n \to \infty} \frac{\log_2 u(n)}{n^2} \leq \limsup_{n \to \infty} \frac{\log_2 u(n)}{n^2} \leq \frac{1}{2}.$$

Stanley's proof is nonconstructive and based on the *Cauchy identity* for Schubert polynomials, see [Man, Prop. 2.4.7]. The first constructive lower bound was given by the authors in [MPP1, §6], where the asymptotics of Υ_w was computed for several families of permutations. Notably, for a permutation

$$w(b, n-b) := (b, b-1, \dots, 1, n, n-1, \dots, b+1)$$
 where $b = \frac{n}{3}$,

we showed that

$$\frac{1}{n^2} \log_2 \Upsilon_{w(b,n-b)} \longrightarrow C \approx 0.25162 \text{ as } n \to \infty.$$

In fact, it is easy to see that the limit C is the largest limit value over all ratios 0 < b/n < 1. This also gives a small improvement on the lower bound in Stanley's theorem.

Layered permutations $w(b_k, \ldots, b_1)$ are defined as

$$w(b_k, b_{k-1}, \dots, b_1) := (b_k, b_k - 1, \dots, 1, b_k + b_{k-1}, b_k + b_{k-1} - 1, \dots, b_k + 1, \dots, n, \dots, n - b_1 + 1),$$

for integers $b_1 + \ldots + b_{k-1} + b_k = n$. They are also called *Richardson* and *pop-stack sortable* permutations in a different contexts, see e.g. [Kit, §2.1.4] and [MeS]. Denote by \mathcal{L}_n the set of layered permutations $w \in S_n$.

Theorem 1.3. Let

$$v(n) := \max_{w \in \mathcal{L}_n} \Upsilon_w.$$

Then there is a limit

$$\lim_{n \to \infty} \ \frac{1}{n^2} \, \log_2 v(n) \, = \, \frac{\gamma}{\log 2} \, \approx \, 0.2932362762,$$

where $\gamma \approx 0.2032558981$ is a universal constant. Moreover, the maximum value v(n) is achieved at a layered permutation

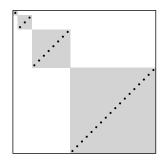
$$w(\ldots,b_2,b_1)$$
, where $b_i \sim \alpha^{i-1}(1-\alpha)n$ as $n \to \infty$,

for every fixed i, and where $\alpha \approx 0.4331818312$ is a universal constant.

In other words, the $runs\ b_i$ form a geometric distribution in the limit. See Figure 1 for examples of the permutation matrix of such w. A posteriori this is unsurprising, since the weights of reduced words are heavily skewed in favor of having many transpositions at the end.

The story behind the theorem is also quite interesting. Calculations for $n \leq 10$ reported in [MeS] and [Sta], prompted Merzon and Smirnov to make the following conjecture:

Conjecture 1.4 ([MeS, Conj. 5.7]). For every n, all permutations w attaining the maximum u(n) are layered permutations. In particular, u(n) = v(n).



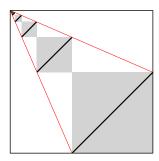


FIGURE 1. Shapes of optimal layered permutations w(1,3,8,18) and w(2,4,9,20,46,106,246,567), of size 30 and 1000, respectively.

In other words, if the Merzon–Smirnov conjecture holds, our Theorem proves Stanley's conjecture with the same limit value and limiting description, as suggested by Stanley (see §4.2 however). Unconditionally, Theorem 1.3 improves a the lower bound for the liminf in Theorem 1.2 to about 0.2932.

Remark 1.5. We learned about the Merzon–Smirnov conjecture from Hugh Thomas, who used it to compute v(n) and permutations attaining it up to n = 300 (see the Appendix). This data allowed us to make a conjecture on the limit shape, which we prove in the theorem.

Exact constants. The constants α and γ in Theorem 1.3 are defined as follows. Consider the function

$$(1.3) f(x) := x^2 \log x - \frac{1}{2} (1-x)^2 \log(1-x) - \frac{1}{2} (1+x)^2 \log(1+x) + 2x \log 2.$$

This function is obtained from a double integral that approximates the logarithm of the product formula of Proctor [Pro] for the number of certain plane partitions (Proposition 3.1). Then α is defined as the solution other than x = 1 of the equation

$$2xf(x) + (1 - x^2)f'(x) = 0,$$

see Figure 3 for plots of f(x) and the equation above. The constant γ is defined as

$$\gamma := \frac{f(\alpha)}{1 - \alpha^2} \, .$$

One can show that α is transcendental by using Baker's theorem, see [Ba, §2.1], but this goes beyond the scope of this paper. It would be interesting to see if existing technology allows to show that γ is also transcendental.

Outline of the paper. In Section 2 we give the necessary background on asymptotics and on the principal evaluation of Schubert polynomials of layered permutations. In Section 3 we prove Theorem 1.3. We conclude with final remarks and open problems in Section 4.

2. Background

2.1. **Permutations.** We write permutations of $\{1, 2, ..., n\}$ as $w = w_1 w_2 ... w_n \in S_n$, where w_i is the image of i. Given two permutations u in S_m and v in S_n we denote by $u \times v$ the following permutation of S_{m+n} :

$$u \times v := u_1 u_2 \dots u_m (m + v_1) (m + v_2) \dots (m + v_n).$$

Similarly, denote by $1^m \times w$ the permutation

$$1^m \times w := 12 \dots m (m + w_1) (m + w_2) \dots (m + w_n).$$

Finally, let $|b| = b_1 + \cdots + b_k$.

2.2. Product formulas for Υ_w for layered permutations. In this section we give a product formula for Υ_w when w is a layered permutation $w(b_k, \ldots, b_1)$.

Let w_0 be the longest permutation $(p, p-1, \ldots, 1)$ and let

$$F(m,p) := \Upsilon_{1^m \times w_0}.$$

Fomin–Kirillov [FK] showed that F(m, p) counts the number of plane partitions of shape (p - 1, p - 2, ..., 1) with entries at most m. This number of plane partitions has a product formula given by Proctor [Pro].

Theorem 2.1 ([FK, Pro]). In the notation above, we have:

$$F(m,p) = \prod_{1 \le i < j \le p} \frac{2m+i+j-1}{i+j-1}.$$

In notation of [MPP2], we have:

$$F(m,p) = \frac{\Lambda(2m+2p) \Lambda(2m+1) \Phi(p)}{\Phi(2m+p) \Lambda(2p)},$$

where $\Phi(n) := 1! \cdot 2! \cdots (n-1)!$ and $\Lambda(n) := (n-2)!(n-4)! \cdots$

Proposition 2.2. For nonnegative integers b_1, b_2, \ldots, b_k , let $w(b_k, \ldots, b_1)$ be the associated layered permutation then

$$\Upsilon_{w(b_k,...,b_1)} = \Upsilon_{w(b_k,...,b_2)} \cdot F(|b| - b_1, b_1),$$

where $|b| = b_1 + b_2 + \cdots + b_k$.

Proof. The permutation $w(b_k, \ldots, b_1)$ can be written as the product $w(b_k, \ldots, b_2) \times w_0(b_k)$. By properties of Schubert polynomials (e.g. see [Mac, (4.6)] or [Man, Cor. 2.4.6]) we have that

$$\mathfrak{S}_{w(b_k,\dots,b_1)} = \mathfrak{S}_{w(b_k,\dots,b_2)} \cdot \mathfrak{S}_{1^{|b|-b_1} \times w_0(b_k)},$$

and the result follows by doing a principal evaluation.

Remark 2.3. Equation (2.1) can be turned into a dynamic program to find layered permutations $w(b_k, \ldots, b_1)$ that achieves v(n), see the appendix.

3. Asymptotics of the largest v(n)

3.1. **The outline.** We will use (2.1) inductively to prove the main result. Let $p := b_1$ and m := n - p, so that $m = b_2 + \ldots + b_k$. By definition of v(n), we have that

$$v(n) = \max_{b:|b|=n} \Upsilon_{w(b)}.$$

Next, using (2.1), v(n) becomes

(3.1)
$$v(n) = \max_{1 \le p \le n} \{v(n-p)F(n-p,p)\}.$$

We will need very precise estimates on $\log F(m, n-m)$. Note that the exact asymptotic expansion for the *Barnes G-function*, which can be used to obtain the asymptotics of $\Phi(\cdot)$ and $\Lambda(\cdot)$, see e.g. [AR]. However, these bounds are insufficient as we also need sharp bounds for the error terms which hold for all m and n. We obtain these in the next subsection. These estimates are then combined with Proposition 2.2 to prove Theorem 1.3.

3.2. **Technical estimates.** Let f(x) be the function defined in (1.3). The next lemma gives bounds on $\log F(m, n-m)$ in terms of the function f(x).

Proposition 3.1. For all integers $n \ge m \ge 0$, we have:

$$-2n \le \log F(m, n - m) - n^2 f(m/n) \le 0.$$

We split the proof into two lemmas, one for the upper bound and the other for the lower bound.

Lemma 3.2. For all integers $n \ge m \ge 0$, we have:

$$\log F(m, n - m) - n^2 f(m/n) \le 0.$$

Proof. We use the product formula for F(m,p) in Theorem 2.1.

(3.2)
$$\log F(m,p) = \sum_{1 \le i < j \le p} \left(\log(2m+i+j-1) - \log(i+j-1) \right)$$
$$= \sum_{1 \le i \le j' \le p-1} \left(\log(2m+i+j') - \log(i+j') \right),$$

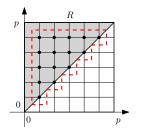
where we changed the index to j' = j - 1. Next, we approximate this sum using a double integral. Let

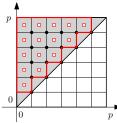
$$g(x,y) := \log(2m + x + y) - \log(x + y).$$

Notice that the function g(x,y) is constant along the lines x+y=k for constant k. Therefore, we can shift the terms of the sum in the RHS of (3.2) by $(i,j) \mapsto (i-1/\sqrt{2},j+1/\sqrt{2})$ without changing the sum (see center of Figure 2)

(3.3)
$$\log F(m,p) = \sum_{(i,j)\in S} \left(\log(2m+i+j') - \log(i+j')\right),$$

where $S = \{\mathbb{Z}^2 + (-1/\sqrt{2}, 1/\sqrt{2})\} \cap \{(x, y) : 0 \le x \le p, x < y \le p\}.$





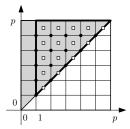


FIGURE 2. Illustration of the proof of the upper and lower bounds of Proposition 3.1 for $\log F(m,p)$ for p=5. The lattice points \bullet on the left are the support of the sum $\sum_{i\leq j} g(i,j)$. This sum remains the same if the support is shifted by $\left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, giving points \square in the middle. The original sum is bounded below by the sum over the support shifted by $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, giving points \square in the right.

Next, compute the Hessian H of g(x, y). We have:

$$H = C \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
, where $C = \frac{1}{(x+y)^2} - \frac{1}{(2m+x+y)^2}$.

Matrix H has eigenvalues 0 and 2C that are nonnegative in $[0, p] \times [0, p]$. Thus g(x, y) is convex in this region. The modified sum in (3.3) is the sum of values of g(x, y) over centers of the unit

squares which fit entirely in R. By convexity, each such value of g(x, y) is less than the average value of g(x, y) over its square. Hence the sum in (3.3) is bounded above by the double integral,

$$\log F(m, p) \le \int_0^p \int_y^p (\log(2m + x + y) - \log(x + y)) \, dx \, dy.$$

Next, we compute this double integral and obtain

(3.4)
$$\int_0^p \int_y^p (\log(2m+x+y) - \log(x+y)) \ dx \, dy = (m+p)^2 f(m/(m+p)),$$

for f(x) defined in (1.3). This proves the upper bound.

Lemma 3.3. For all integers $n \ge m \ge 0$, we have:

$$\log F(m, n - m) - n^2 f(m/n) \ge -2n.$$

Proof. Since the function g(x,y) is decreasing along the x direction and y direction then each value g(i,j) in the sum is bigger than the average value of g(x,y) over the unit square with center $(i+1/\sqrt{2},j+1/\sqrt{2})$ (see right of Figure 2). Hence the original sum in (3.2) is bounded below by the double integral

(3.5)
$$\log F(m,p) = \sum_{1 \le i < j \le p} g(i,j) \ge \int_1^p \int_x^p g(x,y) \, dy \, dx.$$

This integral can be written in terms of the original integral, computed in (3.4), as follows

(3.6)
$$\int_{1}^{p} \int_{x}^{p} g(x,y) \, dy \, dx = \int_{0}^{p} \int_{x}^{p} g(x,y) \, dy \, dx - \int_{0}^{1} \int_{x}^{p} g(x,y) \, dy \, dx$$
$$= (m+p)^{2} f(m/(m+p)) - \int_{0}^{1} \int_{x}^{p} g(x,y) \, dy \, dx.$$

Since the function g(x, y) is decreasing in the x direction then the double integral in the RHS above is bounded by the following single integral

$$-\int_0^1 \int_x^p g(x,y) \, dy \, dx \ge -\int_0^p g(0,y) \, dy.$$

We evaluate this single integral and use Jensen's inequality to obtain

$$-\int_{0}^{p} g(0,y)dy = 2m\log(2m) + p\log(p) - (2m+p)\log(2m+p)$$

$$\geq (2m+p)(\log(2m+p) - \log 2) - (2m+p)\log(2m+p).$$
(3.8)

Combining (3.5),(3.6),(3.2), and (3.8) we have

$$\log F(m,p) \ge (m+p)^2 f(m/(m+p)) + (2m+p) (\log(2m+p) - \log(2)) - (2m+p) \log(2m+p).$$

The RHS is greater than or equal to $(m+p)^2 f(m/(m+p)) - 2(m+p)$, as desired.

3.3. Optimizing constants. Our goal is to show that $\lim_{n\to\infty} \log_2 v(n)/n^2$ is a constant. In the previous lemma we gave bounds on the error of approximating $\log F(m, n-m)$ by $n^2 f(x)$ where x = m/n in [0,1]. We now find a unique constant γ such that $f(x) + \gamma x^2$ has a unique maximum over $x \in [0,1)$.

Lemma 3.4. There exist a unique $\gamma > 0$ and $\alpha \in (0,1)$, such that:

(1)
$$2\gamma\alpha + f'(\alpha) = 0$$
,

(2)
$$\gamma \alpha^2 + f(\alpha) = \gamma$$
 with $2\gamma + r''(\alpha) \le 0$.

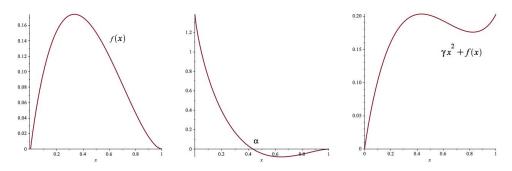


FIGURE 3. Graphs of the functions f(x), q(x) and $\gamma x^2 + f(x)$, on (0,1).

And for this γ , the maximum of $f(x) + \gamma x^2$ over $x \in [0,1)$ is achieved at the given α , and the value is precisely γ . That is,

$$\max_{x \in [0,1)} (f(x) + \gamma x^2) = f(\alpha) + \gamma \alpha^2 = \gamma.$$

Proof. First, it is straightforward to show that $\lim_{x\to 0} f(x) = \lim_{x\to 1} f(x) = 0$ and that f(x) > 0 for $x \in (0,1)$ (see plot of f(x) on the left of Figure 3).

Let α be a solution to the equation q(x) = 0 where

$$q(x) := f(x)2x + f'(x)(1 - x^2)$$

= $(1 - x)^2 \log(1 - x) - (1 + x)^2 \log(1 + x) + 2x \log(x) + 2(1 + x^2) \log(2)$.

This function on the RHS above has one root $\alpha = 0.4331818312$.. and the other is x = 1, as easily seen from the plot, but also can be shown analytically. Then we set

$$\gamma := \frac{f(\alpha)}{1 - \alpha^2} = -\frac{f'(\alpha)}{2\alpha},$$

so γ and α now satisfy conditions (1) and (2).

Next, we see that $\gamma = f(\alpha)/(1-\alpha^2) \approx 0.2032558981$. To prove that this is indeed a maximum for $f(x) + \gamma x^2$, we check that the second derivative, $d^2(\gamma x^2 + f(x))/dx^2 = 2\gamma + r''(x) < 0$ for $x = \alpha$. We have that $r''(x) = \log(x^2/(1-x^2))$. Since $\alpha \le 0.45$, we have that $x^2/(1-x^2) < 0.26$ and so $r''(\alpha) < -1.3 < -2\gamma$ and so the value is a local maximum and by condition (2) it is equal to γ .

3.4. **Proof of Theorem 1.3.** The theorem follows immediately from the following lemma.

Lemma 3.5. For all $n \geq 2$ we have:

$$\left|\log v(n) - \gamma n^2\right| \le 4n.$$

Conversely, suppose for a layered permutation $w(b) \in S_n$ we have

$$\left|\log \Upsilon_w - \gamma n^2\right| \le 4n.$$

Then
$$b = (..., b_2, b_1)$$
, s.t. $b_i \sim (1 - \alpha)\alpha^{i-1}n$ for all fixed $i \geq 1$.

Proof. We proceed by induction to show that $|\log v(n) - \gamma n^2| \le 4n$ holds for all $n \ge 2$. The base cases n = 2 can be checked directly (see exact values in the appendix).

We start with (3.1) and use the induction hypothesis and the upper bound of Proposition 3.1 to obtain

$$\log v(n) = \max_{m < n} \left(\log v(m) + \log F(m, n - m) \right)$$

$$\leq \max_{m < n} \left(\gamma m^2 + \log F(m, n - m) + 2m \right),$$

$$\leq n^2 \max_{x \in [0, 1)} \left(f(x) + \gamma x^2 \right) + 2n.$$

By Lemma 3.4, the maximum value of $f(x) + \gamma x^2$ is equal to γ . Thus, the above inequality becomes

$$\log v(n) < \gamma n^2 + 2n.$$

This maximum is achieved when $x = \alpha$, i.e. when $m = n\alpha$ and $p = b_1 = (1 - \alpha)n$. By the definition of v(n), for this value of m we have that

$$\log v(n) \ge \log v(n\alpha) + \log F(n\alpha, n - n\alpha).$$

By the induction hypothesis and the lower bound of Proposition 3.1, the above inequality becomes

$$\log v(n) \ge (\gamma n^2 \alpha^2 - 4n\alpha) + (n^2 f(\alpha) - 2n)$$
$$= \gamma n^2 - 2(1 + 2\alpha)n \ge \gamma n^2 - 4n.$$

Here we again used the fact that $f(\alpha) + \gamma \alpha^2 = \gamma$ and that $\alpha \le 1/2$. In summary,

$$\left|\log v(n) - \gamma n^2\right| \le 4n,$$

and this bound is attained when $b_1 \sim (1 - \alpha)n$. Recursively, we obtain $b_i \sim (1 - \alpha)\alpha^{i-1}n$ for every fixed $i = 2, 3, \ldots$

Remark 3.6. Note that the appendix shows rather slow rate of convergence for $h(n) := \frac{1}{n^2} \log_2 v(n)$, giving only $h(300) \approx 0.2904$. This suggests that $h(n) = \gamma/(\log 2) - 1/n - o(1/n)$, so that the bound in Lemma 3.5 is quite sharp.

4. Final remarks

4.1. Stanley's Conjecture 1.1 remains open but is very likely to hold. Denote by

$$a(n) = \sum_{w \in S_n} \Upsilon_w$$

the total number of rc-graphs (pipe dreams) of size n. Since

$$u(n) \le a(n) \le n! u(n),$$

we conclude that it suffices to prove the asymptotics result for a_n . This suggests connections to counting general tilings (see e.g. [AS]), as pipe dreams can be viewed as tilings of a staircase shape with two types of tiles, but with one global condition (strains can intersect at most once). The problem is especially similar to counting Knutson-Tao puzzles enumerating the Littlewood-Richardson coefficients, whose maximal asymptotics was recently studied in [PPY].

By analogy with the tilings, one can ask if u(n) satisfies some sort of super-multiplicativity property. Formally, let $w \otimes 1^c$ denote the *Kronecker product permutation* of size cn, whose permutation matrix equals the Kronecker product of the permutation matrix P_w and the identity I_c (see [MPP1]).

Conjecture 4.1. For $w \in S_n$, we have $\Upsilon_{w \otimes 1^2} \geq \Upsilon_w^4$.

We verified the conjecture for all $w \in S_n$ where $n \leq 5$, but perhaps more computational evidence would be helpful.

4.2. Similarly, the Merzon–Smirnov Conjecture 1.4 remains open. In our opinion, the numerical evidence in favor of the conjecture is insufficient, and it would be interesting to verify it for larger n. To speedup the computation, perhaps, there are large classes of permutations $u \in S_n$ which can be proved to be non-maximal, i.e. there exists $w \in S_n$, s.t. $\Upsilon_u \leq \Upsilon_w$. Such permutations can then be ignored in the exhaustive search.

In fact, Prop. 6.5 in [MPP1] gives explicit constructions of large families of permutations $w \in S_n$, for which $\log \Upsilon_w = \Theta(n)$. These permutations are very far from being layered (in the transposition distance), suggesting that if true, proving Conjecture 1.4 might not be easy.

4.3. In [Sta], Stanley also considered the case when Υ_w is small. It is well known that $\Upsilon_w = 1$ if and only if w is dominant [Man], i.e. 132-avoiding. Stanley conjectured that $\Upsilon_w = 2$ if and only if w has exactly one instance of the pattern 132. This was recently proved by Weigandt [Wei], who also showed that $\Upsilon_w - 1$ is greater than or equal the number of instances of the pattern 132 in w.

This suggests the problem of finding permutations where the number of patterns 132 is maximal. In the field of pattern avoidance, this problem can be rephrased as asking for permutations $w \in S_n$ with maximal packing density of the pattern 132, see [Kit, §8.3.1]. The solution due to Stromquist is extremely well understood, and has been both refined and generalized, see [A+, BSV, HSV], [Pri, §5.1] and [OEIS, A061061]. The maximal packing density is attained at a layered permutation $w(b_1, b_2, ...)$, where the runs b_i have a geometric distribution:

$$b_i \sim \rho (1 - \rho)^{i-1} n, \ i = 1, 2, \dots$$
 where $\rho = \frac{\sqrt{3} - 1}{2} \approx 0.366025$

While, of course, v(n) are attained at somewhat different layered permutations, the similarities to this work are rather striking and go beyond coincidences. They are rooted in the recursive nature of optimal permutations in both cases, which are solutions of similar (but different!) maximization problems.

4.4. The bounds for u(n) from Theorem 1.2 are obtained from the Cauchy identity of Schubert polynomials which gives

$$(4.1) \qquad \sum_{w_0=v^{-1}u} \Upsilon_u \Upsilon_v = 2^{\binom{n}{2}}.$$

One could then ask for large values of $\Upsilon_w\Upsilon_{ww_0^{-1}}$. Let $u'(n) := \max_{w \in S_n} \{\Upsilon_w \cdot \Upsilon_{ww_0^{-1}}\}$. The table below has the values of u'(n) for $n = 2, \ldots, 9$ and the permutations w (up to multiplying by w_0^{-1}) that achieve that value u'(n).

n	u'(n)	w
3	2	132
4	6	1423
5	33	15243
6	286	162534
7	4620	1736254
8	162360	18527364
9	9057090	195283746

Note that for a layered permutation w(b), the permutation $w(b)w_0^{-1}$ is dominant and so $\Upsilon_{w(b)w_0^{-1}} = 1$.

There is a combinatorial proof of (4.1) by Bergeron and Billey [BB] involving taking a double rc-graph of w_0 ($2^{\binom{n}{2}}$ many) and reading from each half of it permutations u and v satisfying $w_0 = v^{-1}u$. All such double rc-graphs of w_0 can be obtained from an initial double rc-graph via certain local transformations (see [BB, Sec. 4]). One can use these local transformations in a Markov chain to obtain a random double rc-graph of w_0 and from it read off a permutation u;

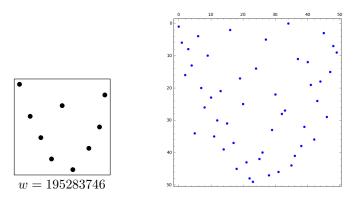


FIGURE 4. Permutation matrices of 195283746 and of a permutation $u \in S_{50}$ from the random double rc-graph.

see Figure 4. We conjecture that the permutation matrix of random permutations u has a parabolic frozen region.

The second permutation in Figure 4 is obtained by running a Markov chain for $5 \cdot 10^9$ local moves on a double rc-graph of $v^{-1}u = w_0 \in S_{50}$, described in [BB, Sec. 4]. Half of the resulting double rc-graph given in Figure 5 is then converted into a permutation $u \in S_{50}$.

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Table of exact values for $n \leq 300$. Below we present table of tuples b of layered permutations w(b) maximizing v(n). The third column is $f(n) := \frac{1}{n^2} \log_2 v(n)$.

	(f(n)		(1 1)	· · · · · · · · · · · · · · · · · · ·
$\frac{n}{1}$	$\frac{(\ldots,b_2,b_1)}{(1)}$	$\frac{f(n)}{0.000000}$	$\frac{n}{n}$	(\ldots,b_2,b_1)	$\frac{f(n)}{}$
2		0.000000	51	(1, 2, 5, 13, 30)	0.276896
3	(1,1)		52	(1, 2, 6, 13, 30)	0.277275
3 4	(1,2) $(1,3)$	0.111111 0.145121	53	(1, 2, 6, 13, 31)	0.277550
			54	(1, 2, 6, 14, 31)	0.277807
5 6	(1,1,3)	0.152294	55	(1, 2, 6, 14, 32)	0.278094
7	(1,1,4)	0.177564	56	(1, 3, 6, 14, 32)	0.278322
8	(1,2,4)	0.191149	57	(1, 3, 6, 14, 33)	0.278618
9	(1,2,5)	0.206317	58	(1,3,6,14,34)	0.278815
	(1,2,6)	0.213824	59	(1, 3, 6, 15, 34)	0.279103
10	(1,3,6)	0.220771	60	(1, 3, 6, 15, 35)	0.279313
11	(1,3,7)	0.227005	61	(1, 3, 7, 15, 35)	0.279525
12	(1,3,8)	0.229879	62	(1, 3, 7, 15, 36)	0.279747
13	(1,1,3,8)	0.233769	63	(1, 3, 7, 16, 36)	0.279962
14	(1, 1, 4, 8)	0.237048	64	(1, 3, 7, 16, 37)	0.280192
15	(1,1,4,9)	0.241677	65	(1, 3, 7, 16, 38)	0.280344
16	(1, 1, 4, 10)	0.244446	66	(1, 3, 7, 17, 38)	0.280532
17	(1, 2, 4, 10)	0.246954	67	(1, 3, 7, 17, 39)	0.280698
18	(1, 2, 4, 11)	0.249509	68	(1, 3, 8, 17, 39)	0.280862
19	(1, 2, 5, 11)	0.251966	69	(1, 3, 8, 17, 40)	0.281038
20	(1, 2, 5, 12)	0.254240	70	(1, 3, 8, 18, 40)	0.281178
21	(1, 2, 5, 13)	0.255575	71	(1,3,8,18,41)	0.281363
22	(1, 2, 6, 13)	0.257354	72	(1,3,8,18,42)	0.281486
23	(1, 2, 6, 14)	0.258685	73	(1, 1, 3, 8, 18, 42)	0.281670
24	(1, 3, 6, 14)	0.260063	74	(1, 1, 3, 8, 18, 43)	0.281803
25	(1, 3, 6, 15)	0.261360	75	(1, 1, 3, 8, 19, 43)	0.281969
26	(1, 3, 7, 15)	0.262425	76	(1, 1, 3, 8, 19, 44)	0.282112
27	(1, 3, 7, 16)	0.263673	77	(1, 1, 4, 8, 19, 44)	0.282210
28	(1, 3, 7, 17)	0.264435	78	(1, 1, 4, 8, 19, 45)	0.282361
29	(1,3,8,17)	0.265233	79	(1, 1, 4, 8, 20, 45)	0.282488
30	(1,3,8,18)	0.266034	80	(1, 1, 4, 8, 20, 46)	0.282646
31	(1, 1, 3, 8, 18)	0.266811	81	(1, 1, 4, 8, 20, 47)	0.282755
32	(1, 1, 3, 8, 19)	0.267619	82	(1, 1, 4, 9, 20, 47)	0.282902
33	(1, 1, 4, 8, 19)	0.268165	83	(1, 1, 4, 9, 20, 48)	0.283019
34	(1, 1, 4, 8, 20)	0.268973	84	(1, 1, 4, 9, 21, 48)	0.283165
35	(1, 1, 4, 9, 20)	0.269675	85	(1, 1, 4, 9, 21, 49)	0.283288
36	(1, 1, 4, 9, 21)	0.270460	86	(1, 1, 4, 9, 22, 49)	0.283370
37	(1, 1, 4, 9, 22)	0.270978	87	(1, 1, 4, 9, 22, 50)	0.283501
38	(1, 1, 4, 10, 22)	0.271548	88	(1, 1, 4, 9, 22, 51)	0.283590
39	(1, 1, 4, 10, 23)	0.272081	89	(1, 1, 4, 10, 22, 51)	0.283715
40	(1, 2, 4, 10, 23)	0.272523	90	(1, 1, 4, 10, 22, 52)	0.283811
41	(1, 2, 4, 10, 24)	0.273065	91	(1, 1, 4, 10, 23, 52)	0.283914
42	(1, 2, 4, 11, 24)	0.273453	92	(1, 1, 4, 10, 23, 53)	0.284018
43	(1, 2, 4, 11, 25)	0.273996	93	(1, 2, 4, 10, 23, 53)	0.284090
44	(1, 2, 4, 11, 26)	0.274357	94	(1, 2, 4, 10, 23, 54)	0.284200
45	(1, 2, 5, 11, 26)	0.274862	95	(1, 2, 4, 10, 24, 54)	0.284279
46	(1, 2, 5, 11, 27)	0.275235	96	(1, 2, 4, 10, 24, 55)	0.284394
47	(1, 2, 5, 12, 27)	0.275654	97	(1, 2, 4, 10, 24, 56)	0.284475
48	(1, 2, 5, 12, 28)	0.276036	98	(1, 2, 4, 11, 24, 56)	0.284553
49	(1, 2, 5, 12, 29)	0.276277	99	(1, 2, 4, 11, 24, 57)	0.284641
_50	(1, 2, 5, 13, 29)	0.276634	100	(1, 2, 4, 11, 25, 57)	0.284736

$\begin{array}{c ccccccccccccccccccccccccccccccccccc$						
102		(\ldots,b_2,b_1)	f(n)	n	(\ldots,b_2,b_1)	f(n)
103				151	(1, 3, 7, 16, 38, 86)	0.287573
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$				152	(1, 3, 7, 16, 38, 87)	0.287612
$\begin{array}{c} 105 (1,2,5,11,26,60) 0.285148 \\ 106 (1,2,5,11,26,61) 0.285222 \\ 107 (1,2,5,11,27,61) 0.285229 \\ 108 (1,2,5,11,27,62) 0.285368 \\ 109 (1,2,5,11,27,62) 0.285368 \\ 109 (1,2,5,12,27,62) 0.285368 \\ 109 (1,2,5,12,27,63) 0.285314 \\ 109 (1,2,5,12,27,63) 0.285518 \\ 110 (1,2,5,12,27,63) 0.285518 \\ 111 (1,2,5,12,27,64) 0.285577 \\ 111 (1,2,5,12,28,64) 0.285577 \\ 112 (1,2,5,12,28,64) 0.285577 \\ 113 (1,2,5,12,28,65) 0.285760 \\ 114 (1,2,5,12,29,65) 0.285760 \\ 115 (1,3,8,17,40,92) 0.287916 \\ 116 (1,2,5,12,29,66) 0.285766 \\ 164 (1,3,8,18,40,94) 0.288030 \\ 115 (1,2,5,13,29,66) 0.285834 \\ 165 (1,3,8,18,44,94) 0.288030 \\ 116 (1,2,5,13,29,66) 0.285892 \\ 118 (1,2,5,13,29,66) 0.285892 \\ 118 (1,2,5,13,30,68) 0.286015 \\ 119 (1,2,5,13,30,68) 0.286015 \\ 120 (1,2,5,13,30,69) 0.286219 \\ 121 (1,2,6,13,30,69) 0.286201 \\ 121 (1,2,6,13,30,70) 0.286261 \\ 122 (1,2,6,13,31,70) 0.286366 \\ 173 (1,1,3,8,18,42,96) 0.288126 \\ 122 (1,2,6,13,31,72) 0.286369 \\ 174 (1,1,3,8,18,43,99) 0.2882216 \\ 125 (1,2,6,13,31,72) 0.286413 \\ 175 (1,1,3,8,19,44,100) 0.288332 \\ 126 (1,2,6,14,32,73) 0.286672 \\ 180 (1,1,4,8,19,44,100) 0.288332 \\ 126 (1,2,6,14,32,73) 0.286678 \\ 181 (1,1,4,8,19,44,100) 0.288332 \\ 129 (1,2,6,14,32,73) 0.286678 \\ 181 (1,1,4,8,19,44,100) 0.2883515 \\ 131 (1,3,6,14,32,74) 0.286688 \\ 182 (1,1,4,8,19,44,100) 0.288332 \\ 133 (1,3,6,14,32,74) 0.286673 \\ 180 (1,1,4,8,19,44,100) 0.288533 \\ 134 (1,3,6,14,37,71) 0.286688 \\ 182 (1,1,4,8,19,44,100) 0.288533 \\ 134 (1,3,6,14,37,71) 0.286678 \\ 185 (1,1,4,8,19,44,100) 0.288533 \\ 146 (1,3,7,15,36,89) 0.287729 \\ 149 (1,2,6,13,37,71) 0.286688 \\ 180 (1,1,4,8,20,46,105) 0.288563 \\ 135 (1,3,6,14,37,71) 0.286679 \\ 180 (1,1,4,8,20,46,106) 0.288511 \\ 141 (1,3,6,15,34,89) 0.287077 \\ 187 (1,1,4,8,20,46,106) 0.288511 \\ 141 (1,3,6,15,34,89) 0.287070 \\ 188 (1,1,4,9,21,49,111) 0.288864 \\ 140 (1,3,7,15,3$			0.284978	153	(1, 3, 7, 17, 38, 87)	0.287643
$\begin{array}{c} 105 & (1,2,5,11,26,60) & 0.285148 \\ 106 & (1,2,5,11,27,61) & 0.285222 \\ 107 & (1,2,5,11,27,62) & 0.285289 \\ 108 & (1,2,5,11,27,62) & 0.285368 \\ 109 & (1,2,5,12,27,62) & 0.285368 \\ 109 & (1,2,5,12,27,63) & 0.285518 \\ 100 & (1,2,5,12,27,63) & 0.285518 \\ 101 & (1,2,5,12,27,63) & 0.285518 \\ 102 & (1,2,5,12,27,64) & 0.285577 \\ 103 & (1,2,5,12,27,64) & 0.285577 \\ 104 & (1,2,5,12,27,64) & 0.285667 \\ 105 & (1,3,8,17,40,91) & 0.287849 \\ 112 & (1,2,5,12,27,64) & 0.285667 \\ 112 & (1,2,5,12,28,65) & 0.285720 & 163 & (1,3,8,18,40,93) & 0.287942 \\ 113 & (1,2,5,12,29,65) & 0.285766 & 164 & (1,3,8,18,40,94) & 0.288040 \\ 115 & (1,2,5,12,29,66) & 0.285834 & 165 & (1,3,8,18,41,94) & 0.288040 \\ 116 & (1,2,5,12,29,66) & 0.285892 & 166 & (1,3,8,18,41,94) & 0.288041 \\ 116 & (1,2,5,13,29,66) & 0.285965 & 167 & (1,3,8,18,42,95) & 0.288071 \\ 117 & (1,2,5,13,29,66) & 0.285965 & 167 & (1,3,8,18,42,96) & 0.288126 \\ 119 & (1,2,5,13,30,69) & 0.286015 & 168 & (1,3,8,18,42,96) & 0.288126 \\ 120 & (1,2,5,13,30,69) & 0.286201 & 171 & (1,1,3,8,18,42,96) & 0.288155 \\ 120 & (1,2,6,13,30,70) & 0.286261 & 172 & (1,1,3,8,18,42,98) & 0.288216 \\ 121 & (1,2,6,13,31,70) & 0.286306 & 173 & (1,1,3,8,18,43,99) & 0.288224 \\ 123 & (1,2,6,13,31,71) & 0.286309 & 174 & (1,1,3,8,19,43,100) & 0.288332 \\ 126 & (1,2,6,13,31,72) & 0.286413 & 175 & (1,1,3,8,19,44,100) & 0.288332 \\ 126 & (1,2,6,14,32,74) & 0.286667 & 180 & (1,1,4,8,19,44,100) & 0.288352 \\ 127 & (1,2,6,14,32,74) & 0.286668 & 179 & (1,1,4,8,19,44,100) & 0.288353 \\ 129 & (1,2,6,14,32,74) & 0.286668 & 182 & (1,1,4,8,19,44,100) & 0.288363 \\ 131 & (1,3,6,14,32,75) & 0.286768 & 182 & (1,1,4,8,19,44,100) & 0.288361 \\ 132 & (1,3,6,14,32,75) & 0.286768 & 182 & (1,1,4,8,19,44,100) & 0.288513 \\ 133 & (1,3,6,14,33,75) & 0.286768 & 182 & (1,1,4,8,19,44,100) & 0.288513 \\ 134 & (1,3,6,14,33,75) & 0.286668 & 184 & (1,1,4,8,20,46,105) & 0.288664 \\ 135 & (1,3,6,14,33,75) & 0.286668 & 184 & (1,1,4,8,20,46,105) & 0.288566 \\ 140 & (1,3,6,15,34,89) & 0.287007 & 187 & (1,1,4,9,20,47,108) & 0.288711 \\ 141 & (1,3,6,15,34,8$	104		0.285046	154	(1, 3, 7, 17, 38, 88)	0.287684
107 (1, 2, 5, 11, 27, 61) 0.285289 157 (1, 3, 7, 17, 39, 90) 0.287782 108 (1, 2, 5, 11, 27, 62) 0.285368 158 (1, 3, 8, 17, 39, 90) 0.287849 110 (1, 2, 5, 12, 27, 62) 0.285318 159 (1, 3, 8, 17, 39, 91) 0.287849 110 (1, 2, 5, 12, 27, 64) 0.285577 161 (1, 3, 8, 17, 40, 92) 0.287912 111 (1, 2, 5, 12, 28, 64) 0.285677 162 (1, 3, 8, 17, 40, 93) 0.287942 113 (1, 2, 5, 12, 28, 65) 0.285720 163 (1, 3, 8, 18, 40, 93) 0.287975 114 (1, 2, 5, 12, 29, 66) 0.285866 164 (1, 3, 8, 18, 40, 93) 0.287975 114 (1, 2, 5, 13, 29, 66) 0.285864 165 (1, 3, 8, 18, 40, 93) 0.287975 114 (1, 2, 5, 13, 29, 66) 0.285894 166 (1, 3, 8, 18, 40, 93) 0.288040 116 (1, 2, 5, 13, 30, 68) 0.286015 167 (1, 3, 8, 18, 42, 95) 0.288041 119 (1, 2, 5, 13, 30, 68) 0.286015 167 (1, 3,	105			155		0.287713
$\begin{array}{c} 107 & (1,2,5,11,27,61) & 0.285289 & 157 & (1,3,7,17,39,90) & 0.287782 \\ 108 & (1,2,5,11,27,62) & 0.285368 & 158 & (1,3,8,17,39,90) & 0.287814 \\ 109 & (1,2,5,12,27,62) & 0.285434 & 159 & (1,3,8,17,39,91) & 0.287849 \\ 110 & (1,2,5,12,27,64) & 0.285517 & 161 & (1,3,8,17,40,92) & 0.287916 \\ 111 & (1,2,5,12,28,64) & 0.285557 & 162 & (1,3,8,17,40,93) & 0.287916 \\ 112 & (1,2,5,12,28,64) & 0.285567 & 162 & (1,3,8,17,40,93) & 0.287916 \\ 113 & (1,2,5,12,28,65) & 0.285766 & 164 & (1,3,8,18,40,93) & 0.287975 \\ 114 & (1,2,5,12,29,66) & 0.285866 & 164 & (1,3,8,18,40,94) & 0.288003 \\ 115 & (1,2,5,12,29,66) & 0.285892 & 166 & (1,3,8,18,40,94) & 0.288040 \\ 116 & (1,2,5,13,29,66) & 0.285892 & 166 & (1,3,8,18,40,95) & 0.288011 \\ 117 & (1,2,5,13,29,66) & 0.285695 & 167 & (1,3,8,18,42,95) & 0.288013 \\ 118 & (1,2,5,13,29,68) & 0.286015 & 168 & (1,3,8,18,42,96) & 0.288126 \\ 119 & (1,2,5,13,30,69) & 0.286015 & 168 & (1,3,8,18,42,96) & 0.288126 \\ 120 & (1,2,5,13,30,69) & 0.286201 & 170 & (1,1,3,8,18,42,97) & 0.288191 \\ 121 & (1,2,6,13,30,69) & 0.286201 & 171 & (1,1,3,8,18,42,97) & 0.288191 \\ 122 & (1,2,6,13,31,70) & 0.286366 & 173 & (1,1,3,8,18,43,99) & 0.288244 \\ 123 & (1,2,6,13,31,71) & 0.286369 & 174 & (1,1,3,8,19,43,100) & 0.288302 \\ 125 & (1,2,6,13,31,72) & 0.286413 & 175 & (1,1,3,8,19,43,100) & 0.288302 \\ 125 & (1,2,6,14,31,73) & 0.286676 & 178 & (1,1,3,8,19,44,100) & 0.288387 \\ 128 & (1,2,6,14,32,73) & 0.286676 & 178 & (1,1,4,8,19,44,100) & 0.288387 \\ 129 & (1,2,6,14,32,74) & 0.286668 & 179 & (1,1,4,8,19,45,104) & 0.288413 \\ 130 & (1,3,6,14,32,74) & 0.286688 & 179 & (1,1,4,8,20,46,107) & 0.288663 \\ 131 & (1,3,6,14,33,75) & 0.286723 & 181 & (1,1,4,8,20,46,107) & 0.288663 \\ 132 & (1,3,6,14,33,75) & 0.286768 & 182 & (1,1,4,8,20,46,107) & 0.288663 \\ 134 & (1,3,6,14,33,75) & 0.286768 & 184 & (1,1,4,8,20,46,107) & 0.288663 \\ 135 & (1,3,6,14,33,75) & 0.286768 & 184 & (1,1,4,8,20,46,107) & 0.288663 \\ 140 & (1,3,6,15,34,79) & 0.286867 & 180 & (1,1,4,9,20,47,109) & 0.288734 \\ 141 & (1,3,6,15,35,80) & 0.287300 & 196 & (1,1,4,9,20,47$	106	(1, 2, 5, 11, 26, 61)	0.285222	156	(1, 3, 7, 17, 39, 89)	0.287750
$\begin{array}{c} 108 & (1,2,5,11,27,62) & 0.285368 \\ 109 & (1,2,5,12,27,62) & 0.285434 \\ 109 & (1,2,5,12,27,63) & 0.285518 \\ 160 & (1,3,8,17,39,91) & 0.287814 \\ 110 & (1,2,5,12,27,63) & 0.285577 \\ 111 & (1,2,5,12,28,64) & 0.285577 \\ 112 & (1,2,5,12,28,65) & 0.285657 \\ 113 & (1,2,5,12,28,65) & 0.285766 \\ 114 & (1,2,5,12,29,65) & 0.285766 \\ 115 & (1,2,5,12,29,65) & 0.285766 \\ 116 & (1,3,8,18,40,93) & 0.287975 \\ 114 & (1,2,5,12,29,65) & 0.285766 \\ 115 & (1,2,5,13,29,66) & 0.285834 \\ 116 & (1,2,5,13,29,66) & 0.285892 \\ 117 & (1,2,5,13,29,66) & 0.285892 \\ 118 & (1,2,5,13,29,66) & 0.285895 \\ 119 & (1,2,5,13,30,68) & 0.286015 \\ 119 & (1,2,5,13,30,68) & 0.286015 \\ 119 & (1,2,5,13,30,68) & 0.286015 \\ 120 & (1,2,5,13,30,69) & 0.286201 \\ 121 & (1,2,6,13,30,70) & 0.286201 \\ 122 & (1,2,6,13,31,70) & 0.286306 \\ 123 & (1,2,6,13,31,70) & 0.286306 \\ 124 & (1,2,6,13,31,70) & 0.286309 \\ 125 & (1,2,6,13,31,72) & 0.286309 \\ 126 & (1,2,6,14,31,72) & 0.286413 \\ 127 & (1,2,6,14,31,72) & 0.286413 \\ 129 & (1,2,6,14,32,74) & 0.286672 \\ 129 & (1,2,6,14,32,74) & 0.286667 \\ 120 & (1,2,6,14,32,74) & 0.286667 \\ 120 & (1,2,6,14,32,74) & 0.286667 \\ 120 & (1,2,6,14,32,74) & 0.286667 \\ 130 & (1,1,4,8,19,44,100) & 0.288332 \\ 126 & (1,2,6,14,37,7) & 0.286667 \\ 130 & (1,3,8,19,44,100) & 0.288332 \\ 126 & (1,2,6,14,37,70) & 0.286667 \\ 130 & (1,3,6,14,32,74) & 0.286667 \\ 130 & (1,3,6,14,32,74) & 0.286667 \\ 130 & (1,3,6,14,32,74) & 0.286667 \\ 130 & (1,3,6,14,33,77) & 0.286688 \\ 140 & (1,1,4,8,19,44,100) & 0.288353 \\ 131 & (1,3,6,14,3,77) & 0.286688 \\ 142 & (1,1,4,8,19,45,103) & 0.288513 \\ 133 & (1,3,6,14,3,78) & 0.286795 \\ 134 & (1,1,4,8,20,45,104) & 0.288533 \\ 145 & (1,3,6,15,35,80) & 0.287140 & 190 & (1,1,4,9,20,47,108) & 0.288513 \\ 134 & (1,3,6,15,35,80) & 0.287140 & 190 & (1,1,4,9,20,47,108) & 0.288514 \\ 141 & (1,3,6,15,36,83) & 0.287140 & 190 & (1,1,4,9,20,47,108) & 0.288716 \\ 143 & (1,3,7,15,36,82) & 0.287380 & 196 & (1,1,4,9,21,49,111) & 0.288890 \\ 144 & (1,3,7,15,36,83) & 0.287380 & 196 & (1,1,4,9,21,49,111) & 0.288891 \\ 144 & (1,3,7,15,36,85) & 0.287380 $		(1, 2, 5, 11, 27, 61)	0.285289	157		0.287782
$\begin{array}{c} 109 & (1,2,5,12,27,62) & 0.285434 & 159 & (1,3,8,17,30,91) & 0.287849 \\ 110 & (1,2,5,12,27,64) & 0.285518 & 160 & (1,3,8,17,40,91) & 0.287879 \\ 111 & (1,2,5,12,27,64) & 0.285567 & 162 & (1,3,8,17,40,93) & 0.287916 \\ 112 & (1,2,5,12,28,64) & 0.285620 & 163 & (1,3,8,18,40,93) & 0.287942 \\ 113 & (1,2,5,12,28,65) & 0.285720 & 163 & (1,3,8,18,40,93) & 0.287942 \\ 114 & (1,2,5,12,29,65) & 0.285766 & 164 & (1,3,8,18,40,94) & 0.288003 \\ 115 & (1,2,5,12,29,66) & 0.285892 & 166 & (1,3,8,18,41,95) & 0.288040 \\ 116 & (1,2,5,13,29,66) & 0.285892 & 166 & (1,3,8,18,42,95) & 0.288093 \\ 118 & (1,2,5,13,29,68) & 0.285095 & 167 & (1,3,8,18,42,95) & 0.288093 \\ 118 & (1,2,5,13,30,68) & 0.286015 & 168 & (1,3,8,18,42,96) & 0.288126 \\ 119 & (1,2,5,13,30,69) & 0.286201 & 170 & (1,1,3,8,18,42,96) & 0.288155 \\ 120 & (1,2,5,13,30,69) & 0.286201 & 171 & (1,1,3,8,18,42,97) & 0.288191 \\ 121 & (1,2,6,13,30,69) & 0.286201 & 171 & (1,1,3,8,18,43,99) & 0.288216 \\ 122 & (1,2,6,13,31,70) & 0.286306 & 173 & (1,1,3,8,19,43,99) & 0.288244 \\ 123 & (1,2,6,13,31,71) & 0.286306 & 173 & (1,1,3,8,19,43,99) & 0.288232 \\ 125 & (1,2,6,13,31,72) & 0.286413 & 175 & (1,1,3,8,19,43,100) & 0.288332 \\ 126 & (1,2,6,14,31,73) & 0.286519 & 177 & (1,1,3,8,19,43,100) & 0.288332 \\ 126 & (1,2,6,14,31,73) & 0.286576 & 178 & (1,1,3,8,19,44,100) & 0.288335 \\ 129 & (1,2,6,14,32,73) & 0.286677 & 180 & (1,1,4,8,19,44,101) & 0.288357 \\ 129 & (1,2,6,14,32,73) & 0.286678 & 178 & (1,1,4,8,19,44,102) & 0.288410 \\ 129 & (1,2,6,14,32,73) & 0.286678 & 182 & (1,1,4,8,19,45,104) & 0.288533 \\ 130 & (1,3,6,14,32,75) & 0.286688 & 149 & (1,1,4,8,20,46,106) & 0.288533 \\ 131 & (1,3,6,14,3,77) & 0.2866867 & 188 & (1,1,4,8,20,46,106) & 0.288533 \\ 132 & (1,3,6,14,3,77) & 0.286678 & 182 & (1,1,4,8,20,46,106) & 0.288531 \\ 133 & (1,3,6,14,3,77) & 0.286686 & 184 & (1,1,4,8,20,46,106) & 0.288561 \\ 134 & (1,3,7,15,36,80) & 0.287089 & 189 & (1,1,4,9,20,47,108) & 0.288661 \\ 139 & (1,3,6,15,34,80) & 0.287089 & 189 & (1,1,4,9,20,47,108) & 0.288737 \\ 142 & (1,3,7,15,36,83) & 0.287308 & 199 & (1,1,4,9,$	108		0.285368	158	(1, 3, 8, 17, 39, 90)	0.287814
$\begin{array}{c} 110 & (1,2,5,12,27,63) & 0.285518 \\ 111 & (1,2,5,12,27,64) & 0.285577 \\ 112 & (1,2,5,12,28,64) & 0.285657 \\ 162 & (1,3,8,17,40,92) & 0.287916 \\ 113 & (1,2,5,12,28,65) & 0.285766 \\ 164 & (1,3,8,18,40,94) & 0.288003 \\ 115 & (1,2,5,12,29,66) & 0.285766 \\ 164 & (1,3,8,18,40,94) & 0.288003 \\ 115 & (1,2,5,12,29,66) & 0.285892 & 166 \\ (1,3,8,18,41,95) & 0.288001 \\ 116 & (1,2,5,13,29,66) & 0.285982 & 166 \\ (1,3,8,18,41,95) & 0.288011 \\ 117 & (1,2,5,13,29,66) & 0.285965 & 167 & (1,3,8,18,42,95) & 0.288093 \\ 118 & (1,2,5,13,29,68) & 0.286015 & 168 & (1,3,8,18,42,96) & 0.288126 \\ 119 & (1,2,5,13,30,68) & 0.286074 & 169 & (1,1,3,8,18,42,96) & 0.288126 \\ 119 & (1,2,5,13,30,69) & 0.28629 & 170 & (1,1,3,8,18,42,97) & 0.288191 \\ 121 & (1,2,6,13,30,70) & 0.286201 & 171 & (1,1,3,8,18,42,98) & 0.288216 \\ 122 & (1,2,6,13,30,70) & 0.286201 & 171 & (1,1,3,8,18,43,99) & 0.288216 \\ 123 & (1,2,6,13,31,70) & 0.286369 & 174 & (1,1,3,8,19,43,99) & 0.288227 \\ 124 & (1,2,6,13,31,72) & 0.286413 & 175 & (1,1,3,8,19,43,99) & 0.288302 \\ 125 & (1,2,6,13,31,72) & 0.286413 & 175 & (1,1,3,8,19,44,100) & 0.288352 \\ 127 & (1,2,6,14,31,73) & 0.286576 & 178 & (1,1,3,8,19,44,100) & 0.288351 \\ 129 & (1,2,6,14,32,74) & 0.286628 & 179 & (1,1,4,8,19,44,102) & 0.288410 \\ 129 & (1,2,6,14,32,74) & 0.286667 & 180 & (1,1,4,8,19,44,102) & 0.288410 \\ 129 & (1,2,6,14,32,75) & 0.286768 & 182 & (1,1,4,8,19,44,102) & 0.288451 \\ 131 & (1,3,6,14,32,75) & 0.286768 & 182 & (1,1,4,8,19,44,102) & 0.288451 \\ 133 & (1,3,6,14,33,75) & 0.286768 & 182 & (1,1,4,8,19,44,102) & 0.288531 \\ 134 & (1,3,6,14,34,77) & 0.2866910 & 185 & (1,1,4,8,20,45,104) & 0.288531 \\ 135 & (1,3,6,14,33,76) & 0.286768 & 182 & (1,1,4,8,19,44,102) & 0.288561 \\ 136 & (1,3,6,14,34,77) & 0.2866910 & 185 & (1,1,4,8,20,45,104) & 0.288531 \\ 134 & (1,3,6,14,34,77) & 0.2866910 & 185 & (1,1,4,8,20,45,104) & 0.288531 \\ 135 & (1,3,6,14,34,77) & 0.2866910 & 185 & (1,1,4,8,20,45,104) & 0.288561 \\ 139 & (1,3,6,15,34,80) & 0.287089 & 189 & (1,1,4,9,20,47,1109) & 0.288561 \\ 140 & (1,3,7,15,36,80) & 0.287303 & 194 $	109		0.285434	159		0.287849
$\begin{array}{c} 1111 & (1,2,5,12,27,64) & 0.285577 \\ 112 & (1,2,5,12,28,64) & 0.285657 \\ 113 & (1,2,5,12,28,65) & 0.285720 \\ 114 & (1,2,5,12,29,65) & 0.285726 \\ 115 & (1,2,5,12,29,66) & 0.285834 \\ 115 & (1,2,5,12,29,66) & 0.285834 \\ 116 & (1,2,5,12,29,66) & 0.285834 \\ 116 & (1,2,5,13,29,66) & 0.285892 \\ 117 & (1,2,5,13,29,67) & 0.285965 \\ 118 & (1,2,5,13,29,67) & 0.285965 \\ 119 & (1,2,5,13,30,68) & 0.286015 \\ 119 & (1,2,5,13,30,69) & 0.286074 \\ 119 & (1,2,5,13,30,69) & 0.286129 \\ 110 & (1,2,5,13,30,69) & 0.286291 \\ 110 & (1,2,6,13,30,69) & 0.286201 \\ 111 & (1,2,6,13,30,70) & 0.286261 \\ 112 & (1,2,6,13,170) & 0.286306 \\ 123 & (1,2,6,13,31,71) & 0.286306 \\ 124 & (1,2,6,13,31,72) & 0.286413 \\ 125 & (1,2,6,13,31,72) & 0.286413 \\ 126 & (1,2,6,14,31,73) & 0.286519 \\ 127 & (1,2,6,14,31,73) & 0.286519 \\ 127 & (1,2,6,14,31,73) & 0.286519 \\ 127 & (1,2,6,14,32,74) & 0.286676 \\ 128 & (1,2,6,14,32,74) & 0.286678 \\ 130 & (1,3,6,14,32,75) & 0.286723 \\ 131 & (1,3,6,14,33,75) & 0.286678 \\ 132 & (1,3,6,14,33,75) & 0.286678 \\ 133 & (1,3,6,14,33,75) & 0.286678 \\ 134 & (1,3,6,14,33,77) & 0.286688 \\ 135 & (1,3,6,14,33,77) & 0.286678 \\ 136 & (1,3,6,14,33,77) & 0.286688 \\ 137 & (1,1,4,8,19,44,102) & 0.288432 \\ 139 & (1,3,6,14,33,75) & 0.286723 \\ 181 & (1,1,4,8,19,45,104) & 0.288531 \\ 133 & (1,3,6,14,33,77) & 0.286678 \\ 134 & (1,3,6,14,33,77) & 0.286678 \\ 135 & (1,3,6,14,33,77) & 0.286678 \\ 136 & (1,3,6,14,33,77) & 0.286678 \\ 137 & (1,1,4,8,19,45,104) & 0.288531 \\ 138 & (1,3,6,14,33,77) & 0.286688 \\ 140 & (1,3,6,15,34,80) & 0.287087 \\ 187 & (1,1,4,8,20,46,105) & 0.288541 \\ 140 & (1,3,6,15,35,80) & 0.287140 \\ 190 & (1,1,4,9,20,47,100) & 0.288781 \\ 144 & (1,3,7,15,35,81) & 0.287221 \\ 192 & (1,1,4,9,20,47,110) & 0.288782 \\ 144 & (1,3,7,15,36,83) & 0.287380 & 196 & (1,1,4,9,21,49,114) & 0.288917 \\ 148 & (1,3,7,16,36,85) & 0.287308 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 149 & (1,3,7,16,36,85) & 0.287308 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 149 & (1,3,7,16,36,85) & 0.287308 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 149 & (1,3,7,16,36,85) & 0.287$	110		0.285518	160	(1, 3, 8, 17, 40, 91)	0.287879
$\begin{array}{c} 113 & (1,2,5,12,28,65) & 0.285720 \\ 114 & (1,2,5,12,29,65) & 0.285766 \\ 115 & (1,2,5,12,29,66) & 0.285834 \\ 116 & (1,3,8,18,40,94) & 0.288031 \\ 115 & (1,2,5,13,29,66) & 0.285892 & 166 \\ (1,3,8,18,41,94) & 0.288040 \\ 116 & (1,2,5,13,29,67) & 0.285965 & 167 \\ (1,3,8,18,42,95) & 0.288071 \\ 117 & (1,2,5,13,29,68) & 0.286015 & 168 \\ (1,3,8,18,42,95) & 0.288093 \\ 118 & (1,2,5,13,30,68) & 0.286074 & 169 \\ (1,2,5,13,30,69) & 0.28629 & 170 \\ (1,2,5,13,30,69) & 0.28629 & 170 \\ (1,1,3,8,18,42,97) & 0.288191 \\ 120 & (1,2,6,13,30,70) & 0.286261 & 171 \\ (1,1,3,8,18,42,98) & 0.288216 \\ 122 & (1,2,6,13,31,70) & 0.286306 & 173 \\ (1,2,6,13,31,70) & 0.286306 & 173 \\ (1,2,6,13,31,71) & 0.286369 & 174 \\ (1,2,6,13,31,72) & 0.286413 & 175 \\ (1,1,3,8,19,43,190) & 0.288302 \\ 125 & (1,2,6,14,31,72) & 0.286413 & 175 \\ (1,2,6,14,32,73) & 0.286519 & 177 \\ (1,1,1,3,8,19,44,100) & 0.288355 \\ 127 & (1,2,6,14,32,73) & 0.286519 & 177 \\ (1,1,1,3,8,19,44,101) & 0.288355 \\ 128 & (1,2,6,14,32,74) & 0.286628 & 179 \\ (1,1,4,8,19,44,102) & 0.288410 \\ 129 & (1,2,6,14,32,74) & 0.286628 & 179 \\ (1,1,4,8,19,44,102) & 0.288431 \\ 130 & (1,3,6,14,32,75) & 0.286723 & 181 \\ (1,1,4,8,19,44,102) & 0.288451 \\ 131 & (1,3,6,14,33,77) & 0.286868 & 182 \\ (1,1,4,8,19,44,102) & 0.288451 \\ 132 & (1,3,6,14,33,77) & 0.286688 & 182 \\ (1,1,4,8,19,45,104) & 0.288531 \\ 133 & (1,3,6,14,33,77) & 0.286686 & 180 \\ (1,1,4,8,19,45,104) & 0.288531 \\ 134 & (1,3,6,14,33,77) & 0.286686 & 180 \\ (1,1,4,8,20,45,104) & 0.288531 \\ 135 & (1,3,6,14,33,77) & 0.286955 & 186 \\ (1,1,4,8,20,45,104) & 0.288561 \\ 137 & (1,1,3,8,19,44,102) & 0.288561 \\ 137 & (1,1,3,8,19,44,102) & 0.288513 \\ 138 & (1,3,6,14,33,76) & 0.286955 & 186 \\ (1,1,4,8,20,45,104) & 0.288561 \\ 137 & (1,3,6,14,34,77) & 0.286910 & 185 \\ (1,1,4,8,20,45,104) & 0.288561 \\ 139 & (1,3,6,15,34,80) & 0.287140 & 190 \\ (1,1,4,9,20,47,108) & 0.288711 \\ 141 & (1,3,6,15,35,81) & 0.287177 & 191 \\ (1,1,4,9,20,47,100) & 0.288782 \\ 144 & (1,3,7,15,36,82) & 0.287380 & 196 \\ (1,1,4,9,21,48,111) & 0.288872 \\ 146 & (1,3,7,16,36,85) & 0.287427 &$		(1, 2, 5, 12, 27, 64)	0.285577	161		0.287916
$\begin{array}{c} 114 & (1,2,5,12,29,65) & 0.285766 \\ 115 & (1,2,5,12,29,66) & 0.285834 \\ 116 & (1,2,5,13,29,66) & 0.285892 \\ 116 & (1,3,8,18,41,94) & 0.288040 \\ 117 & (1,2,5,13,29,66) & 0.285995 \\ 118 & (1,2,5,13,29,68) & 0.286015 \\ 119 & (1,2,5,13,29,68) & 0.286015 \\ 119 & (1,2,5,13,30,68) & 0.286074 \\ 119 & (1,2,5,13,30,68) & 0.286074 \\ 119 & (1,2,5,13,30,69) & 0.286129 \\ 110 & (1,2,5,13,30,69) & 0.286201 \\ 111 & (1,2,5,13,30,69) & 0.286201 \\ 112 & (1,2,6,13,30,70) & 0.286261 \\ 122 & (1,2,6,13,31,70) & 0.286306 \\ 123 & (1,2,6,13,31,71) & 0.286369 \\ 124 & (1,2,6,13,31,72) & 0.286413 \\ 125 & (1,2,6,14,31,72) & 0.286413 \\ 126 & (1,2,6,14,31,73) & 0.286519 \\ 127 & (1,2,6,14,31,73) & 0.286519 \\ 127 & (1,2,6,14,32,73) & 0.286628 \\ 129 & (1,2,6,14,32,73) & 0.286628 \\ 119 & (1,3,6,14,32,74) & 0.286667 \\ 110 & (1,4,8,19,44,100) & 0.288352 \\ 110 & (1,2,6,14,31,73) & 0.286628 \\ 111 & (1,4,8,19,44,102) & 0.288410 \\ 112 & (1,2,6,14,32,73) & 0.286576 \\ 113 & (1,3,6,14,32,74) & 0.286628 \\ 117 & (1,1,4,8,19,44,102) & 0.288410 \\ 129 & (1,2,6,14,31,73) & 0.286672 \\ 130 & (1,3,6,14,32,74) & 0.286667 \\ 180 & (1,1,4,8,19,44,102) & 0.288432 \\ 130 & (1,3,6,14,32,74) & 0.286668 \\ 181 & (1,1,4,8,19,44,102) & 0.288432 \\ 131 & (1,3,6,14,33,75) & 0.286723 \\ 181 & (1,1,4,8,19,44,102) & 0.288432 \\ 132 & (1,3,6,14,33,76) & 0.286723 \\ 183 & (1,1,4,8,19,44,102) & 0.288432 \\ 134 & (1,3,6,14,33,77) & 0.286686 \\ 184 & (1,1,4,8,20,45,104) & 0.288533 \\ 134 & (1,3,6,14,34,78) & 0.286955 \\ 186 & (1,1,4,8,20,45,104) & 0.288533 \\ 134 & (1,3,6,15,34,89) & 0.287007 \\ 187 & (1,1,4,8,20,47,109) & 0.288561 \\ 139 & (1,3,5,15,35,81) & 0.287177 \\ 191 & (1,1,4,9,20,47,109) & 0.288761 \\ 141 & (1,3,7,15,35,82) & 0.287366 \\ 143 & (1,3,7,15,36,83) & 0.287380 & 196 & (1,1,4,9,21,49,114) & 0.288873 \\ 144 & (1,3,7,15,36,83) & 0.287380 & 196 & (1,1,4,9,21,49,114) & 0.288871 \\ 144 & (1,3,7,15,36,83) & 0.287380 & 196 & (1,1,4,9,21,49,114) & 0.288891 \\ 149 & (1,3,7,16,36,83) & 0.287369 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 140 & (1,3,7,16,36,83) & 0.287360 & 199 & (1,1,4,9,21,$	112		0.285657	162	(1, 3, 8, 17, 40, 93)	0.287942
$\begin{array}{c} 115 & (1,2,5,12,29,66) \\ 116 & (1,2,5,13,29,66) \\ 0.285892 \\ 116 & (1,3,8,18,41,95) \\ 0.288093 \\ 118 & (1,2,5,13,29,68) \\ 0.286015 \\ 119 & (1,2,5,13,29,68) \\ 0.286015 \\ 119 & (1,2,5,13,30,68) \\ 0.286074 \\ 119 & (1,2,5,13,30,68) \\ 0.286074 \\ 119 & (1,2,5,13,30,69) \\ 0.286129 \\ 110 & (1,1,3,8,18,42,96) \\ 0.288155 \\ 120 & (1,2,5,13,30,69) \\ 0.286201 \\ 171 & (1,1,3,8,18,42,97) \\ 0.288216 \\ 122 & (1,2,6,13,30,70) \\ 0.286261 \\ 172 & (1,1,3,8,18,42,98) \\ 0.288214 \\ 123 & (1,2,6,13,31,70) \\ 0.286306 \\ 173 & (1,1,3,8,18,43,98) \\ 0.288244 \\ 123 & (1,2,6,13,31,70) \\ 0.286306 \\ 173 & (1,1,3,8,18,43,99) \\ 0.288212 \\ 124 & (1,2,6,13,31,70) \\ 0.286369 \\ 174 & (1,1,3,8,19,43,100) \\ 0.288332 \\ 126 & (1,2,6,14,31,72) \\ 0.286413 \\ 175 & (1,1,3,8,19,43,100) \\ 0.288332 \\ 126 & (1,2,6,14,31,73) \\ 0.286519 \\ 177 & (1,1,3,8,19,44,101) \\ 0.288387 \\ 128 & (1,2,6,14,32,74) \\ 0.286628 \\ 179 & (1,1,4,8,19,44,102) \\ 0.288432 \\ 130 & (1,3,6,14,32,74) \\ 0.286667 \\ 180 & (1,1,4,8,19,44,102) \\ 0.288432 \\ 131 & (1,3,6,14,32,75) \\ 0.286723 \\ 181 & (1,1,4,8,19,45,103) \\ 0.288533 \\ 134 & (1,3,6,14,33,75) \\ 0.286688 \\ 184 & (1,1,4,8,19,45,104) \\ 0.288533 \\ 134 & (1,3,6,14,33,77) \\ 0.286668 \\ 184 & (1,1,4,8,20,45,104) \\ 0.288533 \\ 134 & (1,3,6,14,34,77) \\ 0.286688 \\ 184 & (1,1,4,8,20,45,104) \\ 0.288533 \\ 134 & (1,3,6,14,34,77) \\ 0.286668 \\ 184 & (1,1,4,8,20,45,104) \\ 0.288533 \\ 135 & (1,3,6,14,34,77) \\ 0.286668 \\ 184 & (1,1,4,8,20,45,104) \\ 0.288533 \\ 134 & (1,3,6,15,34,80) \\ 0.287067 \\ 185 & (1,1,4,8,20,45,104) \\ 0.288533 \\ 134 & (1,3,6,15,34,80) \\ 0.287067 \\ 185 & (1,1,4,8,20,45,104) \\ 0.288566 \\ 136 & (1,1,4,8,20,45,104) \\ 0.288533 \\ 146 & (1,3,7,15,35,81) \\ 0.287069 \\ 189 & (1,1,4,9,20,47,100) \\ 0.288737 \\ 142 & (1,3,7,15,36,83) \\ 0.287380 \\ 0.287380 \\ 199 & (1,1,4,9,21,49,114) \\ 0.288917 \\ 149 & (1,3,7,16,36,83) \\ 0.287459 \\ 199 & (1,1,4,9,21,49,114) \\ 0.288917 \\ 149 & (1,3,7,16,36,85) \\ 0.287459 \\ 199 & (1,1,4,9,21,49,114) \\ 0.288917 \\ 149 & (1,3,7,16,36,85) \\ 0.287508 \\ 199 & (1,1,4,9,21,49,114) \\ 0.288917 \\ 149 & (1,3,7,16,36,85) \\ 0.2$			0.285720	163	(1, 3, 8, 18, 40, 93)	0.287975
$\begin{array}{c} 116 & (1,2,5,13,29,66) & 0.285892 \\ 117 & (1,2,5,13,29,67) & 0.285965 \\ 118 & (1,2,5,13,29,68) & 0.286015 \\ 119 & (1,2,5,13,30,68) & 0.286015 \\ 119 & (1,2,5,13,30,68) & 0.286074 \\ 169 & (1,1,3,8,18,42,96) & 0.288155 \\ 120 & (1,2,5,13,30,69) & 0.286129 \\ 170 & (1,1,3,8,18,42,96) & 0.288191 \\ 121 & (1,2,6,13,30,69) & 0.286201 \\ 171 & (1,1,3,8,18,42,98) & 0.288216 \\ 122 & (1,2,6,13,30,70) & 0.286261 \\ 172 & (1,1,3,8,18,43,98) & 0.288244 \\ 123 & (1,2,6,13,31,70) & 0.286366 \\ 173 & (1,1,3,8,18,43,99) & 0.288272 \\ 124 & (1,2,6,13,31,71) & 0.286369 & 174 & (1,1,3,8,19,43,99) & 0.288302 \\ 125 & (1,2,6,13,31,72) & 0.286413 & 175 & (1,1,3,8,19,43,100) & 0.288332 \\ 126 & (1,2,6,14,31,72) & 0.286472 & 176 & (1,1,3,8,19,44,100) & 0.288355 \\ 127 & (1,2,6,14,31,73) & 0.286519 & 177 & (1,1,3,8,19,44,101) & 0.288355 \\ 128 & (1,2,6,14,32,73) & 0.286576 & 178 & (1,1,3,8,19,44,102) & 0.288410 \\ 129 & (1,2,6,14,32,74) & 0.286628 & 179 & (1,1,4,8,19,44,102) & 0.288412 \\ 130 & (1,3,6,14,32,74) & 0.286667 & 180 & (1,1,4,8,19,44,102) & 0.288432 \\ 130 & (1,3,6,14,33,75) & 0.286768 & 182 & (1,1,4,8,19,45,104) & 0.288513 \\ 133 & (1,3,6,14,33,76) & 0.286788 & 182 & (1,1,4,8,19,45,104) & 0.288513 \\ 134 & (1,3,6,14,34,77) & 0.286688 & 184 & (1,1,4,8,20,45,104) & 0.288533 \\ 134 & (1,3,6,14,34,78) & 0.286955 & 186 & (1,1,4,8,20,46,105) & 0.288561 \\ 137 & (1,3,6,14,34,78) & 0.287007 & 187 & (1,1,4,8,20,46,105) & 0.288561 \\ 139 & (1,3,6,15,34,80) & 0.287177 & 191 & (1,1,4,9,20,47,109) & 0.288737 \\ 142 & (1,3,7,15,35,81) & 0.287177 & 191 & (1,1,4,9,20,47,109) & 0.288737 \\ 142 & (1,3,7,15,36,83) & 0.287346 & 195 & (1,1,4,9,21,48,111) & 0.288832 \\ 144 & (1,3,7,15,36,83) & 0.287346 & 195 & (1,1,4,9,21,48,111) & 0.288832 \\ 145 & (1,3,7,16,36,85) & 0.287508 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 149 & (1,3,7,16,36,85) & 0.287508 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 149 & (1,3,7,16,36,85) & 0.287508 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 149 & (1,3,7,16,36,85) & 0.287508 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 140 & (1,3,7,16,36,85) & 0$	114			164	(1, 3, 8, 18, 40, 94)	0.288003
$\begin{array}{c} 116 & (1,2,5,13,29,66) \\ 117 & (1,2,5,13,29,67) \\ 12,2,5,13,29,68) \\ 12,2,5,13,29,68) \\ 12,2,5,13,30,68) \\ 12,2,5,13,30,69) \\ 12,2,5,13,30,69) \\ 12,2,5,13,30,69) \\ 12,2,5,13,30,69) \\ 12,2,6,13,30,69) \\ 12,2,6,13,30,69) \\ 12,2,6,13,30,69) \\ 12,2,6,13,30,70) \\ 12,2,6,13,31,70) \\ 12,2,6,13,31,70) \\ 12,2,6,13,31,70) \\ 12,2,6,13,31,71) \\ 12,2,6,13,31,72) \\ 12,2,6,13,31,72) \\ 12,2,6,13,31,72) \\ 12,2,6,13,31,72) \\ 12,2,6,13,31,72) \\ 12,2,6,13,31,72) \\ 12,2,6,13,31,72) \\ 12,2,6,13,31,72) \\ 12,2,6,13,31,72) \\ 12,2,6,13,31,72) \\ 12,2,6,13,31,72) \\ 12,2,6,13,31,72) \\ 12,2,2,13,13,172) \\ 12,2,2,2,13,13,172) \\ 12,2,2,2,13,13,172) \\ 12,2,2,2,13,13,172) \\ 12,2,2,2,13,13,172) \\ 12,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,2,13,13,172) \\ 12,2,2,2,13,13,172) \\ 12,2,2,2,13,13,172) \\ 12,2,2,2,13,13,173,173) \\ 12,2,2,2,13,13,173,173,173,173,173,173,173,173,1$			0.285834	165	(1, 3, 8, 18, 41, 94)	0.288040
$\begin{array}{c} 118 & (1,2,5,13,29,68) & 0.286015 \\ 119 & (1,2,5,13,30,68) & 0.286074 \\ 120 & (1,2,5,13,30,68) & 0.286074 \\ 121 & (1,2,6,13,30,69) & 0.286129 \\ 122 & (1,2,6,13,30,69) & 0.286201 \\ 121 & (1,2,6,13,30,70) & 0.286261 \\ 122 & (1,2,6,13,31,70) & 0.286306 \\ 123 & (1,2,6,13,31,70) & 0.286306 \\ 124 & (1,2,6,13,31,71) & 0.286306 \\ 125 & (1,2,6,13,31,71) & 0.286369 \\ 126 & (1,2,6,13,31,71) & 0.286369 \\ 127 & (1,1,3,8,18,43,98) & 0.288212 \\ 128 & (1,2,6,13,31,71) & 0.286369 \\ 129 & (1,2,6,13,31,72) & 0.286413 \\ 120 & (1,2,6,14,31,72) & 0.286413 \\ 121 & (1,2,6,14,31,73) & 0.286519 \\ 121 & (1,2,6,14,32,74) & 0.286576 \\ 122 & (1,2,6,14,32,74) & 0.286628 \\ 123 & (1,3,6,14,32,75) & 0.286723 \\ 131 & (1,3,6,14,32,75) & 0.286723 \\ 132 & (1,3,6,14,33,75) & 0.286786 \\ 132 & (1,3,6,14,33,76) & 0.286827 \\ 133 & (1,3,6,14,33,76) & 0.286868 \\ 134 & (1,3,6,14,34,77) & 0.286688 \\ 135 & (1,3,6,14,34,77) & 0.286678 \\ 186 & (1,1,4,8,19,45,103) & 0.288531 \\ 137 & (1,3,6,14,33,76) & 0.286786 \\ 182 & (1,1,4,8,19,45,104) & 0.288531 \\ 133 & (1,3,6,14,34,77) & 0.286688 \\ 140 & (1,3,6,14,34,78) & 0.286955 \\ 140 & (1,3,6,14,34,78) & 0.287007 \\ 187 & (1,1,4,8,20,46,105) & 0.288566 \\ 140 & (1,3,6,15,34,78) & 0.287007 \\ 187 & (1,1,4,8,20,46,105) & 0.288566 \\ 140 & (1,3,6,15,34,80) & 0.287177 \\ 191 & (1,1,4,8,20,47,108) & 0.288737 \\ 141 & (1,3,6,15,35,81) & 0.287177 \\ 191 & (1,1,4,9,20,47,108) & 0.288737 \\ 142 & (1,3,7,15,35,81) & 0.287177 \\ 191 & (1,1,4,9,20,47,108) & 0.288737 \\ 143 & (1,3,7,15,36,82) & 0.287366 & 199 & (1,1,4,9,24,8111) & 0.288832 \\ 146 & (1,3,7,15,36,83) & 0.287360 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 141 & (1,3,6,16,36,85) & 0.287508 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 141 & (1,3,7,16,36,85) & 0.287508 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 141 & (1,3,7,16,36,85) & 0.287508 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 142 & (1,3,7,16,36,85) & 0.287508 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 143 & (1,3,7,16,36,85) & 0.287508 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 144 & (1,3,7,16,36,85) & 0.287508 & 199 & (1,1,4,9,21$			0.285892	166	(1, 3, 8, 18, 41, 95)	0.288071
$\begin{array}{c} 119 & (1,2,5,13,30,68) \\ 120 & (1,2,5,13,30,69) \\ 0.286129 \\ 121 & (1,2,6,13,30,69) \\ 0.286201 \\ 122 & (1,2,6,13,30,70) \\ 0.286261 \\ 122 & (1,2,6,13,30,70) \\ 0.286306 \\ 123 & (1,2,6,13,31,70) \\ 0.286306 \\ 124 & (1,2,6,13,31,71) \\ 0.286306 \\ 125 & (1,2,6,13,31,72) \\ 0.286309 \\ 126 & (1,2,6,13,31,72) \\ 0.286413 \\ 127 & (1,1,3,8,19,43,99) \\ 0.288212 \\ 124 & (1,2,6,13,31,72) \\ 0.286413 \\ 125 & (1,2,6,13,31,72) \\ 0.286472 \\ 126 & (1,2,6,14,31,72) \\ 0.286472 \\ 127 & (1,2,6,14,31,73) \\ 0.286519 \\ 127 & (1,2,6,14,31,73) \\ 0.286519 \\ 127 & (1,2,6,14,32,73) \\ 0.286576 \\ 128 & (1,2,6,14,32,74) \\ 0.286628 \\ 129 & (1,2,6,14,32,74) \\ 0.286628 \\ 130 & (1,3,6,14,32,74) \\ 0.286628 \\ 131 & (1,3,6,14,32,74) \\ 0.286667 \\ 180 & (1,1,4,8,19,44,102) \\ 0.288432 \\ 130 & (1,3,6,14,32,75) \\ 0.286768 \\ 132 & (1,3,6,14,33,75) \\ 0.286768 \\ 133 & (1,3,6,14,33,77) \\ 0.286827 \\ 134 & (1,3,6,14,34,77) \\ 0.286868 \\ 135 & (1,3,6,14,34,77) \\ 0.286868 \\ 136 & (1,3,6,14,34,77) \\ 0.286868 \\ 136 & (1,1,4,8,20,45,104) \\ 0.288531 \\ 135 & (1,3,6,14,34,77) \\ 0.286910 \\ 185 & (1,1,4,8,20,45,104) \\ 0.288531 \\ 136 & (1,3,6,14,34,77) \\ 0.2869695 \\ 186 & (1,1,4,8,20,45,105) \\ 0.288561 \\ 139 & (1,3,6,15,34,80) \\ 0.287089 \\ 189 & (1,1,4,8,20,47,108) \\ 0.288737 \\ 142 & (1,3,7,15,35,81) \\ 0.2877221 \\ 191 & (1,1,4,9,20,47,108) \\ 0.288737 \\ 142 & (1,3,7,15,36,82) \\ 0.287380 \\ 194 & (1,1,4,9,20,48,111) \\ 0.288832 \\ 146 & (1,3,7,16,36,83) \\ 0.287380 \\ 196 & (1,1,4,9,21,49,114) \\ 0.288817 \\ 149 & (1,3,7,16,36,83) \\ 0.287580 \\ 199 & (1,1,4,9,21,49,114) \\ 0.288917 \\ 149 & (1,3,7,16,36,85) \\ 0.287580 \\ 199 & (1,1,4,9,21,49,114) \\ 0.288917 \\ 149 & (1,3,7,16,36,85) \\ 0.287580 \\ 199 & (1,1,4,9,21,49,114) \\ 0.288917 \\ 149 & (1,3,7,16,36,85) \\ 0.287580 \\ 199 & (1,1,4,9,21,49,114) \\ 0.288917 \\ 149 & (1,3,7,16,37,85) \\ 0.287580 \\ 199 & (1,1,4,9,21,49,114) \\ 0.288917 \\ 149 & (1,3,7,16,37,85) \\ 0.287580 \\ 199 & (1,1,4,9,21,49,114) \\ 0.288917 \\ 140 & (1,3,7,16,37,85) \\ 0.287580 \\ 199 & (1,1,4,9,21,49,114) \\ 0.288917 \\ 140 & (1,3,7,16,37,85) \\ 0.287580 \\ 199 & (1,1,4,9,21,49,$	117		0.285965	167	(1, 3, 8, 18, 42, 95)	0.288093
$\begin{array}{c} 120 (1,2,5,13,30,69) 0.286129 \\ 121 (1,2,6,13,30,69) 0.286201 \\ 122 (1,2,6,13,30,69) 0.286261 \\ 122 (1,2,6,13,31,70) 0.286306 \\ 123 (1,2,6,13,31,71) 0.286306 \\ 124 (1,2,6,13,31,71) 0.286306 \\ 125 (1,2,6,13,31,71) 0.286309 \\ 126 (1,2,6,13,31,71) 0.286309 \\ 127 (1,1,3,8,19,43,99) 0.288272 \\ 128 (1,2,6,13,31,72) 0.286413 175 (1,1,3,8,19,43,100) 0.288302 \\ 129 (1,2,6,14,31,72) 0.286472 176 (1,1,3,8,19,44,100) 0.288352 \\ 127 (1,2,6,14,31,73) 0.286519 177 (1,1,3,8,19,44,101) 0.288387 \\ 128 (1,2,6,14,32,74) 0.286628 179 (1,1,4,8,19,44,102) 0.288410 \\ 129 (1,2,6,14,32,74) 0.286628 179 (1,1,4,8,19,44,102) 0.288410 \\ 129 (1,2,6,14,32,74) 0.286667 180 (1,1,4,8,19,44,103) 0.288451 \\ 130 (1,3,6,14,32,74) 0.286667 180 (1,1,4,8,19,44,103) 0.288451 \\ 131 (1,3,6,14,33,75) 0.286768 182 (1,1,4,8,19,45,103) 0.288486 \\ 132 (1,3,6,14,33,75) 0.286768 182 (1,1,4,8,19,45,104) 0.288533 \\ 134 (1,3,6,14,33,77) 0.286868 184 (1,1,4,8,20,45,104) 0.288533 \\ 134 (1,3,6,14,34,77) 0.286868 184 (1,1,4,8,20,46,105) 0.288563 \\ 135 (1,3,6,14,34,78) 0.286955 186 (1,1,4,8,20,46,105) 0.288561 \\ 136 (1,3,6,15,34,80) 0.287007 187 (1,1,4,8,20,47,108) 0.288561 \\ 139 (1,3,6,15,34,80) 0.287056 188 (1,1,4,8,20,47,108) 0.288561 \\ 130 (1,3,6,15,34,80) 0.287140 190 (1,1,4,9,20,47,108) 0.288736 \\ 140 (1,3,6,15,35,80) 0.287140 190 (1,1,4,9,20,47,109) 0.288736 \\ 141 (1,3,6,15,35,80) 0.287140 190 (1,1,4,9,20,47,109) 0.288756 \\ 143 (1,3,7,15,36,82) 0.287262 193 (1,1,4,9,20,47,109) 0.288756 \\ 143 (1,3,7,15,36,83) 0.287303 194 (1,1,4,9,21,48,111) 0.288832 \\ 146 (1,3,7,16,36,84) 0.287380 196 (1,1,4,9,21,48,111) 0.288832 \\ 149 (1,3,7,16,36,85) 0.287589 198 (1,1,4,9,21,49,114) 0.288917 \\ 149 (1,3,7,16,36,85) 0.287589 199 (1,1,4,9,21,49,114) 0.288917 \\ 149 (1,3,7,16,36,85) 0.287589 199 (1,1,4,9,21,49,114) $	118	(1, 2, 5, 13, 29, 68)		168	(1, 3, 8, 18, 42, 96)	0.288126
$\begin{array}{c} 121 & (1,2,6,13,30,69) & 0.286201 & 171 & (1,1,3,8,18,42,98) & 0.288216 \\ 122 & (1,2,6,13,30,70) & 0.286261 & 172 & (1,1,3,8,18,43,98) & 0.288244 \\ 123 & (1,2,6,13,31,70) & 0.286369 & 173 & (1,1,3,8,18,43,99) & 0.288272 \\ 124 & (1,2,6,13,31,71) & 0.286369 & 174 & (1,1,3,8,19,43,99) & 0.288302 \\ 125 & (1,2,6,13,31,72) & 0.286413 & 175 & (1,1,3,8,19,43,90) & 0.288332 \\ 126 & (1,2,6,14,31,72) & 0.286472 & 176 & (1,1,3,8,19,44,100) & 0.288355 \\ 127 & (1,2,6,14,32,73) & 0.286519 & 177 & (1,1,3,8,19,44,101) & 0.288387 \\ 128 & (1,2,6,14,32,73) & 0.286576 & 178 & (1,1,3,8,19,44,102) & 0.288410 \\ 129 & (1,2,6,14,32,74) & 0.286628 & 179 & (1,1,4,8,19,44,102) & 0.288432 \\ 130 & (1,3,6,14,32,74) & 0.286667 & 180 & (1,1,4,8,19,44,103) & 0.288457 \\ 131 & (1,3,6,14,32,75) & 0.286723 & 181 & (1,1,4,8,19,45,103) & 0.288456 \\ 132 & (1,3,6,14,33,75) & 0.286768 & 182 & (1,1,4,8,19,45,104) & 0.288513 \\ 133 & (1,3,6,14,33,76) & 0.286827 & 183 & (1,1,4,8,20,45,104) & 0.288513 \\ 134 & (1,3,6,14,34,77) & 0.2866910 & 185 & (1,1,4,8,20,45,104) & 0.288533 \\ 135 & (1,3,6,14,34,78) & 0.286910 & 185 & (1,1,4,8,20,46,105) & 0.288566 \\ 136 & (1,3,6,14,34,78) & 0.287007 & 187 & (1,1,4,8,20,46,105) & 0.288640 \\ 138 & (1,3,6,15,34,80) & 0.287089 & 189 & (1,1,4,8,20,47,108) & 0.288617 \\ 139 & (1,3,6,15,35,80) & 0.287140 & 190 & (1,1,4,9,20,47,108) & 0.288737 \\ 142 & (1,3,7,15,35,81) & 0.287221 & 192 & (1,1,4,9,20,47,109) & 0.288737 \\ 144 & (1,3,7,15,35,81) & 0.287221 & 193 & (1,1,4,9,20,48,111) & 0.288736 \\ 143 & (1,3,7,15,36,82) & 0.287380 & 196 & (1,1,4,9,21,48,111) & 0.288876 \\ 144 & (1,3,7,16,36,83) & 0.287380 & 196 & (1,1,4,9,21,49,114) & 0.288917 \\ 145 & (1,3,7,16,36,85) & 0.287459 & 198 & (1,1,4,9,21,49,114) & 0.288917 \\ 149 & (1,3,7,16,36,85) & 0.287566 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 140 & (1,3,7,16,36,85) & 0.287459 & 198 & (1,1,4,9,21,49,114) & 0.288917 \\ 144 & (1,3,7,16,36,85) & 0.287589 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 144 & (1,3,7,16,36,85) & 0.287589 & 199 & (1,1,4,9,21,49,114) & 0.288917 \\ 144 & (1,3,7,16,36$		(1, 2, 5, 13, 30, 68)	0.286074	169	(1, 1, 3, 8, 18, 42, 96)	0.288155
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				170	(1, 1, 3, 8, 18, 42, 97)	0.288191
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	121			171	(1, 1, 3, 8, 18, 42, 98)	0.288216
$\begin{array}{cccccccccccccccccccccccccccccccccccc$			0.286261	172	(1, 1, 3, 8, 18, 43, 98)	0.288244
$\begin{array}{cccccccccccccccccccccccccccccccccccc$		(1, 2, 6, 13, 31, 70)		173	(1, 1, 3, 8, 18, 43, 99)	0.288272
$\begin{array}{c} 126 (1,2,6,14,31,72) 0.286472 \\ 127 (1,2,6,14,31,73) 0.286519 \\ 127 (1,2,6,14,31,73) 0.286519 \\ 128 (1,2,6,14,32,73) 0.286576 \\ 129 (1,2,6,14,32,74) 0.286628 \\ 130 (1,3,6,14,32,74) 0.286667 \\ 130 (1,3,6,14,32,74) 0.286667 \\ 131 (1,3,6,14,32,75) 0.286723 \\ 132 (1,3,6,14,33,75) 0.286723 \\ 133 (1,3,6,14,33,75) 0.286768 \\ 132 (1,3,6,14,33,76) 0.286827 \\ 133 (1,3,6,14,33,77) 0.286868 \\ 134 (1,3,6,14,34,77) 0.286868 \\ 135 (1,3,6,14,34,77) 0.286910 \\ 136 (1,3,6,14,34,78) 0.286955 \\ 136 (1,3,6,14,34,78) 0.287007 \\ 137 (1,3,6,15,34,79) 0.287056 \\ 138 (1,1,4,8,20,46,105) 0.288661 \\ 139 (1,3,6,15,35,80) 0.287140 \\ 190 (1,1,4,8,20,47,108) 0.288737 \\ 142 (1,3,7,15,35,81) 0.287262 \\ 193 (1,3,7,15,36,82) 0.287380 \\ 146 (1,3,7,16,36,83) 0.287380 \\ 146 (1,3,7,16,36,83) 0.287380 \\ 146 (1,3,7,16,36,83) 0.287380 \\ 146 (1,3,7,16,36,83) 0.287380 \\ 149 (1,3,7,16,36,83) 0.287380 \\ 140 (1,3,7,16,36,83) 0.287380 \\ 140 (1,3,7,16,36,83) 0.287380 \\ 141 (1,3,7,16,36,83) 0.287380 \\ 142 (1,3,7,16,36,85) 0.287459 \\ 143 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287459 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287459 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287459 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,36,85) 0.287558 \\ 144 (1,3,7,16,3$			0.286369	174		0.288302
$\begin{array}{c} 127 (1,2,6,14,31,73) \\ 128 (1,2,6,14,32,73) \\ 128 (1,2,6,14,32,73) \\ 129 (1,2,6,14,32,74) \\ 129 (1,2,6,14,32,74) \\ 129 (1,2,6,14,32,74) \\ 129 (1,3,6,14,32,74) \\ 129 (1,3,6,14,32,74) \\ 129 (1,3,6,14,32,74) \\ 129 (1,3,6,14,32,74) \\ 129 (1,2,6,14,32,74) \\ 129 (1,2,6,14,32,74) \\ 129 (1,2,6,14,32,74) \\ 129 (1,2,6,14,32,74) \\ 129 (1,2,6,14,32,74) \\ 129 (1,2,6,14,32,74) \\ 129 (1,2,6,14,32,74) \\ 129 (1,2,6,14,32,74) \\ 129 (1,2,6,14,32,74) \\ 120 (2,286428) \\ 130 (1,3,6,14,32,75) \\ 121 (1,3,6,14,33,75) \\ 121 (1,3,6,14,33,75) \\ 121 (1,3,6,14,33,77) \\ 122 (1,3,6,14,34,77) \\ 123 (1,3,6,14,34,77) \\ 124 (1,3,6,14,34,78) \\ 125 (1,3,6,14,34,78) \\ 127 (1,3,6,15,34,78) \\ 128 (1,3,6,15,34,79) \\ 128 (1,3,6,15,34,80) \\ 128 (1,3,6,15,34,80) \\ 128 (1,3,6,15,35,81) \\ 128 (1,3,6,15,35,81) \\ 128 (1,3,7,15,35,82) \\ 128 (1,3,7,15,36,82) \\ 128 (1,3,7,15,36,83) \\ 128 (1,3,7,16,36,83) \\ 128 (1,3,7,16,36,83) \\ 128 (1,3,7,16,36,83) \\ 128 (1,1,4,9,21,49,111) \\ 122 (1,3,7,16,36,83) \\ 122 (1,3,7,16,36,83) \\ 122 (1,3,7,16,36,83) \\ 122 (1,3,7,16,36,83) \\ 122 (1,3,7,16,36,85) \\ 122 (1,3,7,16,36,85) \\ 122 (1,3,7,16,37,85) \\ 122 (1,1,4,9,21,49,113) \\ 123 (1,3,7,16,37,85) \\ 124 (1,3,7,16,37,85) \\ 124 (1,3,7,16,36,83) \\ 124 (1,3,7,16,36,83) \\ 124 (1,3,7,16,36,83) \\ 124 (1,3,7,16,36,83) \\ 124 (1,3,7,16,36,83) \\ 124 (1,3,7,16,36,84) \\ 124 (1,3,7,16,36,85) \\ 124 (1,3,7,1$				175	(1, 1, 3, 8, 19, 43, 100)	0.288332
$\begin{array}{cccccccccccccccccccccccccccccccccccc$			0.286472	176	(1, 1, 3, 8, 19, 44, 100)	0.288355
$\begin{array}{cccccccccccccccccccccccccccccccccccc$			0.286519	177	(1, 1, 3, 8, 19, 44, 101)	0.288387
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				178	(1, 1, 3, 8, 19, 44, 102)	0.288410
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				179	(1, 1, 4, 8, 19, 44, 102)	0.288432
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				180	(1, 1, 4, 8, 19, 44, 103)	0.288457
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				181	(1, 1, 4, 8, 19, 45, 103)	0.288486
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				182	(1, 1, 4, 8, 19, 45, 104)	0.288513
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				183	(1, 1, 4, 8, 20, 45, 104)	0.288533
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				184	(1, 1, 4, 8, 20, 45, 105)	0.288563
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				185		0.288586
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				186		0.288617
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				187	(1, 1, 4, 8, 20, 46, 107)	0.288640
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				188	(1, 1, 4, 8, 20, 47, 107)	0.288661
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				189	(1, 1, 4, 8, 20, 47, 108)	0.288686
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	140			190	(1, 1, 4, 9, 20, 47, 108)	0.288711
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				191		0.288737
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				192	(1, 1, 4, 9, 20, 47, 110)	0.288756
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				193	(1, 1, 4, 9, 20, 48, 110)	0.288782
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				194	(1, 1, 4, 9, 20, 48, 111)	0.288803
$\begin{array}{cccccccccccccccccccccccccccccccccccc$				195	(1, 1, 4, 9, 21, 48, 111)	
$\begin{array}{cccccccccccccccccccccccccccccccccccc$						0.288854
$149 (1,3,7,16,37,85) \qquad 0.287508 \qquad 199 (1,1,4,9,21,49,114) \qquad 0.288917$						
(1,1,1,0,21,10,111)						
	150	(1, 3, 7, 16, 37, 86)	0.287544	200	(1, 1, 4, 9, 22, 49, 114)	0.288937

\overline{n}	(\ldots,b_2,b_1)	f(n)	\overline{n}	(\ldots,b_2,b_1)	f(n)
201	(1, 1, 4, 9, 22, 49, 115)	0.288956	251	(1, 2, 5, 11, 27, 62, 143)	0.289807
202	(1, 1, 4, 9, 22, 50, 115)	0.288982	252	(1, 2, 5, 11, 27, 62, 144)	0.289818
203	(1, 1, 4, 9, 22, 50, 116)	0.289003	253	(1, 2, 5, 12, 27, 62, 144)	0.289833
204	(1, 1, 4, 9, 22, 51, 116)	0.289019	254	(1, 2, 5, 12, 27, 62, 145)	0.289845
205	(1, 1, 4, 9, 22, 51, 117)	0.289041	255	(1, 2, 5, 12, 27, 63, 145)	0.289862
206	(1, 1, 4, 10, 22, 51, 117)	0.289061	256	(1, 2, 5, 12, 27, 63, 146)	0.289875
207	(1, 1, 4, 10, 22, 51, 118)	0.289084	257	(1, 2, 5, 12, 27, 64, 146)	0.289885
208	(1, 1, 4, 10, 22, 51, 119)	0.289101	258	(1, 2, 5, 12, 27, 64, 147)	0.289899
209	(1, 1, 4, 10, 22, 52, 119)	0.289122	259	(1, 2, 5, 12, 28, 64, 147)	0.289912
210	(1, 1, 4, 10, 22, 52, 120)	0.289141	260	(1, 2, 5, 12, 28, 64, 148)	0.289927
211	(1, 1, 4, 10, 23, 52, 120)	0.289160	261	(1, 2, 5, 12, 28, 64, 149)	0.289938
212	(1, 1, 4, 10, 23, 52, 121)	0.289180	262	(1, 2, 5, 12, 28, 65, 149)	0.289951
213	(1, 1, 4, 10, 23, 53, 121)	0.289197	263	(1, 2, 5, 12, 28, 65, 150)	0.289964
214	(1, 1, 4, 10, 23, 53, 122)	0.289219	264	(1, 2, 5, 12, 29, 65, 150)	0.289972
215	(1, 1, 4, 10, 23, 53, 123)	0.289234	265	(1, 2, 5, 12, 29, 65, 151)	0.289985
216	(1, 2, 4, 10, 23, 53, 123)	0.289251	266	(1, 2, 5, 12, 29, 66, 151)	0.289996
217	(1, 2, 4, 10, 23, 53, 124)	0.289268	267	(1, 2, 5, 12, 29, 66, 152)	0.290010
218	(1, 2, 4, 10, 23, 54, 124)	0.289289	268	(1, 2, 5, 12, 29, 66, 153)	0.290020
219	(1, 2, 4, 10, 23, 54, 125)	0.289307	269	(1, 2, 5, 13, 29, 66, 153)	0.290033
220	(1, 2, 4, 10, 24, 54, 125)	0.289320	270	(1, 2, 5, 13, 29, 66, 154)	0.290044
221	(1, 2, 4, 10, 24, 54, 126)	0.289340	271	(1, 2, 5, 13, 29, 67, 154)	0.290058
222	(1, 2, 4, 10, 24, 55, 126)	0.289358	272	(1, 2, 5, 13, 29, 67, 155)	0.290070
223	(1, 2, 4, 10, 24, 55, 127)	0.289379	273	(1, 2, 5, 13, 29, 67, 156)	0.290078
224	(1, 2, 4, 10, 24, 55, 128)	0.289394	274	(1, 2, 5, 13, 29, 68, 156)	0.290091
225	(1, 2, 4, 10, 24, 56, 128)	0.289411	275	(1, 2, 5, 13, 29, 68, 157)	0.290100
226	(1, 2, 4, 10, 24, 56, 129)	0.289428	276	(1, 2, 5, 13, 30, 68, 157)	0.290113
227	(1, 2, 4, 11, 24, 56, 129)	0.289441	277	(1, 2, 5, 13, 30, 68, 158)	0.290124
228	(1, 2, 4, 11, 24, 56, 130)	0.289460	278	(1, 2, 5, 13, 30, 69, 158)	0.290134
229	(1, 2, 4, 11, 24, 57, 130)	0.289473	279	(1, 2, 5, 13, 30, 69, 159)	0.290146
230	(1, 2, 4, 11, 24, 57, 131)	0.289492	280	(1, 2, 6, 13, 30, 69, 159)	0.290158
231	(1, 2, 4, 11, 24, 57, 132)	0.289507	281	(1, 2, 6, 13, 30, 69, 160)	0.290171
232	(1, 2, 4, 11, 25, 57, 132)	0.289526	282	(1, 2, 6, 13, 30, 69, 161)	0.290179
233	(1, 2, 4, 11, 25, 57, 133)	0.289541	283	(1, 2, 6, 13, 30, 70, 161)	0.290192
234	(1, 2, 4, 11, 25, 58, 133)	0.289558	284	(1, 2, 6, 13, 30, 70, 162)	0.290202
235	(1, 2, 4, 11, 25, 58, 134)	0.289575	285	(1, 2, 6, 13, 31, 70, 162)	0.290211
236	(1, 2, 4, 11, 25, 58, 135)	0.289587	286	(1, 2, 6, 13, 31, 70, 163)	0.290222
237	(1, 2, 4, 11, 25, 59, 135)	0.289602	287	(1, 2, 6, 13, 31, 71, 163)	0.290233
238	(1, 2, 4, 11, 25, 59, 136)	0.289615	288	(1, 2, 6, 13, 31, 71, 164)	0.290245
239	(1, 2, 4, 11, 26, 59, 136)	0.289633	289	(1, 2, 6, 13, 31, 71, 165)	0.290253
	(1, 2, 4, 11, 26, 59, 137)	0.289648		(1, 2, 6, 13, 31, 72, 165)	0.290263
241	(1, 2, 4, 11, 26, 60, 137)	0.289661	291	(1, 2, 6, 13, 31, 72, 166)	0.290272
242	(1, 2, 4, 11, 26, 60, 138)	0.289676	292	(1, 2, 6, 14, 31, 72, 166)	0.290284
243	(1, 2, 5, 11, 26, 60, 138)	0.289693	293	(1, 2, 6, 14, 31, 72, 167)	0.290294
244	(1, 2, 5, 11, 26, 60, 139)	0.289710	294	(1, 2, 6, 14, 31, 73, 167)	0.290302
245	(1, 2, 5, 11, 26, 60, 140)	0.289722	295	(1, 2, 6, 14, 31, 73, 168)	0.290313
246	(1, 2, 5, 11, 26, 61, 140)	0.289738	296	(1, 2, 6, 14, 32, 73, 168)	0.290322
247	(1, 2, 5, 11, 26, 61, 141)	0.289751	297	(1, 2, 6, 14, 32, 73, 169)	0.290334
248	(1, 2, 5, 11, 27, 61, 141)	0.289764	298	(1, 2, 6, 14, 32, 73, 170)	0.290342
249	(1, 2, 5, 11, 27, 61, 142)	0.289778	299	(1, 2, 6, 14, 32, 74, 170)	0.290353
250	(1, 2, 5, 11, 27, 62, 142)	0.289792	_300	(1, 2, 6, 14, 32, 74, 171)	0.290362

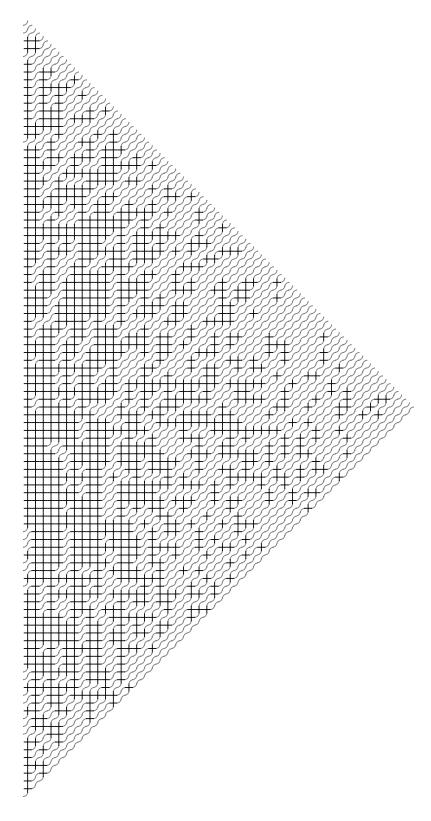


Figure 5. Random double rc-graph corresponding to a permutation in Figure 4.