ON SZÁSZ-MIRAKYAN-JAIN OPERATORS PRESERVING EXPONENTIAL FUNCTIONS

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Abstract. In the present article we define the Jain type modification of the generalized Szász-Mirakjan operators that preserve constant and exponential mappings. Moments, recurrence formulas, and other identities are established for these operators. Approximation properties are also obtained with use of the Boham-Korovkin theorem.

Keywords. Szász-Mirakjan operators, Jain basis functions, Jain operators, Lambert W-function, Boham-Korovkin theorem.

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1. Introduction

In Approximation theory positive linear operatos have been studied with the test functions $\{1, x, x^2\}$ in order to determine the convergence of a function. Of interest are the Szász-Mirakjan operators, based on the Poisson distribution, which are useful in approximating functions on $[0, \infty)$ and are defined as, [10], [12],

$$S_n(f;x) = \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \, e^{-nx} \, f\left(\frac{k}{n}\right). \tag{1.1}$$

In 1972, Jain [9], used the Lagrange expansion formula

$$\phi(z) = \phi(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left[D^{k-1} \left(f^k(z)\, \phi'(z)\right)\right]_{z=0} \left(\frac{z}{f(z)}\right)^k \tag{1.2}$$

with $\phi(z) = e^{\alpha\, z}$ and $f(z) = e^{\beta\, z}$ to determined that

$$1 = \alpha \sum_{k=0}^{\infty} \frac{1}{k!} \, (\alpha + \beta k)^{k-1} \, z^k \, e^{-(\alpha + \beta\, k)\, z}. \tag{1.3}$$

Jain established the basis functions

$$L_{n,k}^{(\beta)}(x) = \frac{nx\,(nx + \beta k)^{k-1}}{k!} \, e^{-(nx + \beta\, k)} \tag{1.4}$$

with the normalization

$$\sum_{k=0}^{\infty} L_{n,k}^{(\beta)}(x) = 1$$

and considered the operators

$$B_n^{\beta}(f,x) = \sum_{k=0}^{\infty} L_{n,k}^{(\beta)}(x)\, f\left(\frac{k}{n}\right) \qquad x \in [0,\infty). \tag{1.5}$$

In the reduction of $\beta = 0$ the Jain operators reduce to the Szász-Mirakjan operators.

Recently Acar, Aral, and Gonska [1] considered the Szász-Mirakjan operators which preserve the test functions $\{1, e^{ax}\}$ and established the operators

$$R_n^*(f;x) = e^{-n\,\gamma_n(x)} \sum_{k=0}^{\infty} \frac{(n\,\gamma_n(x))^k}{k!}\, f\left(\frac{k}{n}\right) \tag{1.6}$$

for functions $f \in C[0,\infty)$, $x \geq 0$, and $n \in \mathbb{N}$ with the reservation property

$$R_n^*(e^{2at};x) = e^{2ax}. \tag{1.7}$$

Here the Jain basis is used to extend the the class of operators for the test functions $\{1, e^{-\lambda x}\}$ by defining Szász-Mirakyan-Jain operators which preserve the mapping of $e^{-\lambda x}$, for $\lambda, x > 0$. In the case of $\lambda = 0$ the Szász-Mirakyan-Jain operators are constant preserving operators. Moments, recurrence formulas, and other identities are established for these new operators. Approximation properties are also obtained with use of the Boham-Korovkin theorem. The Lambert W-function and related properties are used in the analaysis of the properties obtained for the Szász-Mirakyan-Jain operators.

2. Szász-Mirakyan-Jain Operators

The Szász-Mirakyan-Jain operators, (SMJ), which are a generalization of the Szász-Mirakyan operators, are defined by

$$R_n^{(\beta)}(f;x) = n\,\alpha_n(x) \sum_{k=0}^{\infty} \frac{1}{k!}\,(n\,\alpha_n(x) + \beta\,k)^{k-1}\, e^{-(n\,\alpha_n(x)+\beta\,k)}\, f\left(\frac{k}{n}\right) \tag{2.1}$$

for $f \in C[0,\infty)$. It is required that these operators preserve the mapping of $e^{-\lambda\,x}$, as given by

$$R_n^{(\beta)}(e^{-\lambda t};x) = e^{-\lambda x} \tag{2.2}$$

where $x \geq 0$ and $n \in \mathbb{N}$, and $\lambda \geq 0$. When $\beta = 0$ in (2.1) the operator reduces to that defined by Acar, Aral, and Gonska [1]. When $\beta = 0$ and $\alpha_n(x) = x$ the operator reduces to the well known Szász-Mirakyan operators given by (1.1). For $0 \leq \beta < 1$ and $\alpha_n(x) = x$ these operators reduce to the Szász-Mirakyan-Durrmeyer operators defined by Gupta and Greubel in [5].

Lemma 1. *For $x \geq 0, \lambda \geq 0$, we have*

$$\alpha_n(x) = \frac{-\lambda\,x}{n\,(z(\lambda/n,\beta) - 1)}, \tag{2.3}$$

where $-\beta\,z(t,\beta) = W(-\beta\,e^{-\beta-t})$ and $W(x)$ is the Lambert W-function.

Proof. Considering the mapping (2.2) it is required that

$$e^{-\lambda x} = n\,\alpha_n(x)\sum_{k=0}^{\infty}\frac{1}{k!}\,(n\,\alpha_n(x)+\beta\,k)^{k-1}\,e^{-(n\,\alpha_n(x)+\beta k)}\,e^{-\lambda k/n} \tag{2.4}$$

Making use of (1.3) in the form

$$e^{n\,\alpha_n(x)\,z} = n\,\alpha_n(x)\sum_{k=0}^{\infty}\frac{1}{k!}\,(n\,\alpha_n(x)+\beta k)^{k-1}\,e^{-(\beta z-\ln(z))k} \tag{2.5}$$

and letting $\beta\,z-\ln(z)=\beta+\frac{\lambda}{n}$ then

$$e^{n\,\alpha_n(x)\,z} = n\,\alpha_n(x)\sum_{k=0}^{\infty}\frac{1}{k!}\,(n\,\alpha_n(x)+\beta k)^{k-1}\,e^{-(\beta+\lambda/n)\,k}$$

which provides

$$e^{-\lambda x} = e^{n\,\alpha_n(x)\,(z-1)}$$

or

$$\alpha_n(x) = -\frac{\lambda\,x}{n\,(z(\lambda/n,\beta)-1)}.$$

The value of z is determined by the equation $\beta\,z-\ln(z)=\beta+\frac{\lambda}{n}$ which can be seen in the form

$$z\,e^{-\beta\,z} = e^{-\beta-\lambda/n}$$

and has the solution

$$z(\lambda/n,\beta) = -\frac{1}{\beta}\,W(-\beta\,e^{-\beta-\lambda/n}), \tag{2.6}$$

where $W(x)$ is the Lambert W-function. □

Remark 1. *For the case $\lambda\to 0$ the resulting $\alpha_n(x)$ is*

$$\lim_{\lambda\to 0}\alpha_n(x) = (1-\beta)\,x.$$

Proof. For the case $\lambda\to 0$ the resulting $z=z(\lambda/n,\beta)$ of (2.6) yields $z(0,\beta)=1$. By considering

$$\frac{\partial z}{\partial\lambda} = -\frac{1}{\beta}\frac{\partial}{\partial\lambda}W(-\beta\,e^{-\beta-\lambda/n}) = \frac{W(-\beta\,e^{-\beta-\lambda/n})}{n\,\beta\,(1+W(-\beta\,e^{-\beta-\lambda/n}))}$$

and

$$\lim_{\lambda\to 0}\frac{\partial z}{\partial\lambda} = -\frac{1}{n\,(1-\beta)}.$$

Now, by use of L'Hospital's rule,

$$\lim_{\lambda\to 0}\alpha_n(x) = \frac{x}{n}\lim_{\lambda\to 0}\frac{\lambda}{z-1} = \frac{x}{n}\lim_{\lambda\to 0}\frac{1}{\frac{\partial z}{\partial\lambda}} = (1-\beta)\,x$$

as claimed. □

By taking the case of $\lambda \to 0$ the operators $R_n^{(\beta)}(f;x)$ reduce from exponential preserving to constant preserving operators. In this case the operators $R_n^{(\beta)}(f;x)|_{\lambda\to 0}$ are related to the Jain operators, (1.5), by $R_n^{(\beta)}(f;x) = B_n^{\beta}(f;(1-\beta)\,x)$.

The SMJ operators are now completely defined by

$$\begin{cases} R_n^{(\beta)}(f;x) = n\,\alpha_n(x) \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}\,(n\,\alpha_n(x)+\beta\,k)^{k-1}\,e^{-(n\,\alpha_n(x)+\beta\,k)}\,f\left(\frac{k}{n}\right), \\ \alpha_n(x) = -\dfrac{\lambda\,x}{n\,(z(\lambda/n,\beta)-1)}, \\ z(t,\beta) = -\dfrac{1}{\beta}\,W(-\beta\,e^{-\beta-t}) \end{cases} \tag{2.7}$$

and the requirement that $R_n^{(\beta)}(e^{-\lambda t};x) = e^{-\lambda x}$, for $x \geq 0$, $\lambda \geq 0$ and $n \in \mathbb{N}$.

3. Moment Estimations

Lemma 2. *The moments for the SMJ operators are given by:*

$$\begin{aligned} R_n^{(\beta)}(1;x) &= 1 \\ R_n^{(\beta)}(t;x) &= \frac{\alpha_n(x)}{1-\beta} \\ R_n^{(\beta)}(t^2;x) &= \frac{\alpha_n^2(x)}{(1-\beta)^2} + \frac{\alpha_n(x)}{n\,(1-\beta)^3} \\ R_n^{(\beta)}(t^3;x) &= \frac{\alpha_n^3(x)}{(1-\beta)^3} + \frac{3\,\alpha_n^2(x)}{n\,(1-\beta)^4} + (1+2\,\beta)\,\frac{\alpha_n(x)}{n^2\,(1-\beta)^5} \\ R_n^{(\beta)}(t^4;x) &= \frac{\alpha_n^4(x)}{(1-\beta)^4} + \frac{6\,\alpha_n^3(x)}{n\,(1-\beta)^5} + (7+8\,\beta)\,\frac{\alpha_n^2(x)}{n^2\,(1-\beta)^6} + (1+8\beta+6\beta^2)\,\frac{\alpha_n(x)}{n^3\,(1-\beta)^7} \\ R_n^{(\beta)}(t^5;x) &= \frac{\alpha_n^5(x)}{(1-\beta)^5} + \frac{10\,\alpha_n^4(x)}{n\,(1-\beta)^6} + 5\,(5+4\,\beta)\,\frac{\alpha_n^3(x)}{n^2\,(1-\beta)^7} \\ &\quad + 15\,(1+4\beta+2\beta^2)\,\frac{\alpha_n^2(x)}{n^3\,(1-\beta)^8} + (1+22\beta+58\beta^2+24\beta^3)\,\frac{\alpha_n(x)}{n^4\,(1-\beta)^9}. \end{aligned} \tag{3.1}$$

The proof follows directly from work of the author dealing with moment operators for the Jain basis, see [4, 5, 6].

Lemma 3. *Let, $\phi = t - x$, then the central moments of the SMJ operators are:*

$$\begin{aligned} R_n^{(\beta)}(\phi^0;x) &= 1 \\ R_n^{(\beta)}(\phi^1;x) &= \frac{\alpha_n(x)}{1-\beta} - x \\ R_n^{(\beta)}(\phi^2;x) &= \left(\frac{\alpha_n(x)}{1-\beta} - x\right)^2 + \frac{\alpha_n(x)}{n\,(1-\beta)^3} \\ R_n^{(\beta)}(\phi^3;x) &= \left(\frac{\alpha_n(x)}{1-\beta} - x\right)^3 + \frac{3\,\alpha_n(x)}{n\,(1-\beta)^3}\left(\frac{\alpha_n(x)}{1-\beta} - x\right) + (1+2\,\beta)\,\frac{\alpha_n(x)}{n^2\,(1-\beta)^5} \end{aligned} \tag{3.2}$$

$$R_n^{(\beta)}(\phi^4;x) = \left(\frac{\alpha_n(x)}{1-\beta} - x\right)^4 + \frac{6\,\alpha_n(x)}{n\,(1-\beta)^3}\left(\frac{\alpha_n(x)}{1-\beta} - x\right)^2 + (7+8\,\beta)\,\frac{\alpha_n(x)}{n^2\,(1-\beta)^5}$$
$$\cdot\left(\frac{\alpha_n(x)}{1-\beta} - x\right) + (1+8\beta+6\beta^2)\,\frac{\alpha_n(x)}{n^3\,(1-\beta)^7} + \frac{3\,\alpha_n(x)}{n^2\,(1-\beta)^5}$$
$$R_n^{(\beta)}(\phi^5;x) = \left(\frac{\alpha_n(x)}{1-\beta} - x\right)^5 + \frac{10\,\alpha_n(x)}{n\,(1-\beta)^3}\left(\frac{\alpha_n(x)}{1-\beta} - x\right)^3 + \frac{5\,\alpha_n(x)}{n^2\,(1-\beta)^5}\left(\frac{\alpha_n(x)}{1-\beta} - x\right)\cdot\mu_1$$
$$+ \frac{5\,\alpha_n(x)}{n^3\,(1-\beta)^7}\cdot\mu_2 + (1+22\beta+58\beta^2+24\beta^3)\,\frac{\alpha_n(x)}{n^4\,(1-\beta)^9},$$

where

$$\mu_1 = (5+4\beta)\left(\frac{\alpha_n(x)}{1-\beta} - x\right) + 3\,x$$
$$\mu_2 = 3\,(1+4\beta+2\beta^2)\left(\frac{\alpha_n(x)}{1-\beta} - x\right) + 2\,(1+2\beta)\,x.$$

Proof. Utilizing the binomial expansion

$$\phi^m = (t-x)^m = \sum_{k=0}^{m}(-1)^k\binom{m}{k}t^{m-k}\,x^k$$

then

$$R_n^{(\beta)}(\phi^m;x) = \sum_{k=0}^{m}(-1)^k\binom{m}{k}x^k\,R_n^{(\beta)}(t^{m-k};x). \tag{3.3}$$

Wih the use of (3.1) the first few values of m are:

$$R_n^{(\beta)}(\phi^0;x) = R_n^{(\beta)}(t^0;x) = 1$$
$$R_n^{(\beta)}(\phi^1;x) = R_n^{(\beta)}(t;x) - x\,R_n^{(\beta)}(t^0;x) = \frac{\alpha_n(x)}{1-\beta} - x$$
$$R_n^{(\beta)}(\phi^2;x) = R_n^{(\beta)}(t^2;x) - 2x\,R_n^{(\beta)}(t;x) + x^2\,R_n^{(\beta)}(t^0;x)$$
$$= \frac{\alpha_n^2(x)}{(1-\beta)^2} + \frac{\alpha_n(x)}{n\,(1-\beta)^3} - 2x\,\frac{\alpha_n(x)}{1-\beta} + x^2$$
$$= \left(\frac{\alpha_n(x)}{1-\beta} - x\right)^2 + \frac{\alpha_n(x)}{n\,(1-\beta)^3}$$

The remainder of the central moments follow from (3.1) and (3.3). □

Lemma 4. *The central moments, given in Lemma 3, lead to the limits:*

$$\lim_{n\to\infty} n\,R_n^{(\beta)}(\phi;x) = \frac{\lambda\,x}{2!\,(1-\beta)^2}$$
$$\lim_{n\to\infty} n\,R_n^{(\beta)}(\phi^2;x) = \frac{x}{(1-\beta)^2} \tag{3.4}$$

Proof. By setting $t = \lambda/n$ in (6.4) then

$$\frac{(-\lambda)}{n\,(1-\beta)\,(z(\lambda/n,\beta)-1)} = 1 + \frac{v}{2!} + 2\,(1-4\beta)\,\frac{v^2}{4!} + 6\beta^2\,\frac{v^3}{4!}$$
$$-\,(1-8\beta+88\beta^2+144\beta^3)\,\frac{v^4}{6!} + 840\beta^2\,(1+12\beta+8\beta^2)\,\frac{v^5}{8!} + O(v^6),$$

where $n\,(1-\beta)^2\,v = \lambda$. This expansion may be placed into the form

$$\frac{\alpha_n(x)}{1-\beta} - x = \frac{v\,x}{2!}\,\left(1 + (1-4\beta)\,\frac{v}{3!} + 12\beta^2\,\frac{v^2}{4!} - O(v^3)\right).$$

Multiplying by n and taking the desired limit the resulting value is given by

$$\lim_{n\to\infty} n\,R_n^{(\beta)}(\phi;x) = \frac{\lambda\,x}{2!\,(1-\beta)^2}.$$

It is evident that

$$\left(\frac{\alpha_n(x)}{1-\beta} - x\right)^2 = \left(\frac{v\,x}{2!}\right)^2\,\left(1 + 2(1-4\beta)\,\frac{v}{3!} + 20\,(1-8\beta+52\beta^2)\,\frac{v^2}{6!} - O(v^3)\right)$$

for which

$$\left(\frac{\alpha_n(x)}{1-\beta} - x\right)^2 + \frac{\alpha_n(x)}{n\,(1-\beta)^3}$$
$$= \left(\frac{v\,x}{2!}\right)^2\,\left(1 + 2(1-4\beta)\,\frac{v}{3!} + 20\,(1-8\beta+52\beta^2)\,\frac{v^2}{6!} - O(v^3)\right)$$
$$+\,\frac{x}{n\,(1-\beta)^2}\,\left(1 + \frac{v}{2!} + 2\,(1-4\beta)\,\frac{v^2}{4!} - O(v^3)\right)$$

Multiplying by n and taking the limit yields

$$\lim_{n\to\infty} n\,R_n^{(\beta)}(\phi^2;x) = \frac{x}{(1-\beta)^2}.$$

□

Remark 2. *Other limits may be determined by extending the work of Lemma 4, such as:*

$$\begin{aligned}
&\lim_{n\to\infty} R_n^{(\beta)}(\phi^m;x) = 0,\ \textit{for } m \geq 1\\
&\lim_{n\to\infty} n\,R_n^{(\beta)}(\phi^m;x) = 0,\ \textit{for } m \geq 3\\
&\lim_{n\to\infty} n^2\,R_n^{(\beta)}(\phi^3;x) = \frac{2(1+2\beta)\,x + 3\lambda\,x^2}{2!\,(1-\beta)^4}\\
&\lim_{n\to\infty} n^2\,R_n^{(\beta)}(\phi^4;x) = \frac{3\,x^2}{(1-\beta)^4}
\end{aligned} \tag{3.5}$$

Lemma 5. *Expansion on a general exponential weight is given by*

$$R_n^{(\beta)}(e^{-\mu\,t};x) = e^{n\,\alpha_n(x)\,(z(\mu/n,\beta)-1)},$$

or

$$R_n^{(\beta)}(e^{-\mu\,t};x) = Exp\left[-\lambda\,x\,\left(\frac{z(\mu/n,\beta)-1}{z(\lambda/n,\beta)-1}\right)\right] = Exp\left[-\mu\,x\cdot\frac{\lambda}{\mu}\,\frac{z(\mu/n,\beta)-1}{z(\lambda/n,\beta)-1}\right] \tag{3.6}$$

for $\mu \geq 0$ *and has the expansion*

$$\begin{aligned} R_n^{(\beta)}(e^{-\mu\,t};x) = e^{-\mu\,x}\,\Bigg(1 &+ \frac{\mu(\mu-\lambda)x}{2!\,n(1-\beta)^2} + ((3\mu x - 4 - 8\beta)\mu \\ &-(3\mu x - 2 + 8\beta)\lambda)\,\frac{\mu(\mu-\lambda)x}{4!\,n^2(1-\beta)^4} + \mathcal{O}\left(\frac{\mu(\mu-\lambda)x}{6!\,n^3(1-\beta)^6}\right)\Bigg) \end{aligned} \tag{3.7}$$

where $-\beta\,z(\mu/n,\beta) = W(-\beta\,e^{-\beta-\mu/n})$, $-\beta\,z(\lambda/n,\beta) = W(-\beta\,e^{-\beta-\lambda/n})$. *In the limit as* $n\to\infty$ *it is evident that*

$$\begin{aligned} &\lim_{n\to\infty} R_n^{(\beta)}(e^{-\mu t};x) = e^{-\mu x} \\ &\lim_{n\to\infty} n\,\left[R_n^{(\beta)}(e^{-\mu t};x) - e^{-\mu x}\right] = \frac{\mu(\mu-\lambda)\,x}{2!\,(1-\beta)^2}\,e^{-\mu x}. \end{aligned} \tag{3.8}$$

Proof. It is fairly evident that

$$R_n^{(\beta)}(e^{-\mu t};x) = n\alpha_n(x)\,\sum_{k=0}^{\infty}\frac{1}{k!}\,(n\,\alpha_n(x)+\beta k)^{k-1}\,e^{-n\,\alpha_n(x)-(\beta+\mu/n)k}$$

which, by comparison to (2.5), leads to

$$R_n^{(\beta)}(e^{-\mu t};x) = e^{-n\,\alpha_n(x)\,(z(\mu/n,\beta)-1)} = Exp\left[-\lambda\,x\,\left(\frac{z(\mu/n,\beta)-1}{z(\lambda/n,\beta)-1}\right)\right].$$

The expansion of (3.6), with use of (6.5), is given by

$$\begin{aligned} R_n^{(\beta)}(e^{-\mu t};x) &= \sum_{k=0}^{\infty}\frac{(-\mu x)^k}{k!}\,\left(\frac{\lambda}{\mu}\,\frac{z(\mu/n,\beta)-1}{z(\lambda/n,\beta)-1}\right)^k \\ &= \sum_{k=0}^{\infty}\frac{(-\mu x)^k}{k!}\,\Bigg(1 - \frac{k(\mu-\lambda)}{2!\,(1-\beta)^2} + k((3k+1+8\beta)\mu \\ &\quad +(3k-1-8\beta)\lambda)\,\frac{\mu-\lambda}{4!\,(1-\beta)^4} + \mathcal{O}\left(\frac{\mu-\lambda}{6!\,(1-\beta)^6}\right)\Bigg) \\ &= e^{-\mu x}\,\Bigg(1 + \frac{\mu(\mu-\lambda)x}{2!\,n(1-\beta)^2} + ((3\mu x - 4 - 8\beta)\mu \\ &\quad -(3\mu x - 2 + 8\beta)\lambda)\,\frac{\mu(\mu-\lambda)x}{4!\,n^2(1-\beta)^4} + \mathcal{O}\left(\frac{\mu(\mu-\lambda)x}{6!\,n^3(1-\beta)^6}\right)\Bigg). \end{aligned}$$

Taking the appropriate limits yields the desired results. □

Remark 3. *By use of Lemma 5 it may be stated that:*

$$\lim_{n\to\infty} n^2\,R_n^{(\beta)}((e^{-t}-e^{-x})^4;x) = \frac{3\,x^2\,e^{-4x}}{(1-\beta)^4}. \tag{3.9}$$

Proof. Since

$$R_n^{(\beta)}((e^{-t}-e^{-x})^4;x) = R_n^{(\beta)}(e^{-4t};x) - 4\,e^{-x}\,R_n^{(\beta)}(e^{-3t};x) + 6\,e^{-2x}\,R_n^{(\beta)}(e^{-2t};x) - 4\,e^{-3x}\,R_n^{(\beta)}(e^{-t};x) + e^{-4x}\,R_n^{(\beta)}(1;x)$$

then, by making use of (3.7), it becomes evident that

$$R_n^{(\beta)}((e^{-t}-e^{-x})^4;x) = \frac{3\,x^2\,e^{-4x}}{n^2\,(1-\beta)^4} + \mathcal{O}\left(\frac{1}{n^3\,(1-\beta)^6}\right).$$

Multiplying by n^2 and taking the limit as $n\to\infty$ yields the desired result. □

4. Analysis

Theorem 1. *Given the sequence $A_n : C^*[0,\infty) \to C^*[0,\infty)$ of positive linear operators which satisfies the conditions*

$$\lim_{n\to\infty} A_n(e^{-kt};x) = e^{-kx}, \qquad k = 0, 1, 2$$

uniformly in $[0,\infty)$ then

$$\lim_{n\to\infty} A_n(f;x) = f(x)$$

uniformly in $[0,\infty)$ for every $f \in C^*[0,\infty)$.

The proof of this theorem 1 can be found in [2, 3, 8] and has, in essense, been demonstarted by (3.7) for $\mu \geq 0$. An estimate of the rate of convergence for the SMJ operators will require the use of the modulus of continuity

$$\omega(F,\delta) = \operatorname*{Sup}_{\mathrm{x,t}>0} |F(t) - F(x)|$$

and can be seen as, for the case where $F(e^{-t}) = f(t)$,

$$\omega^*(f;\delta) = \operatorname*{Sup}_{\substack{\mathrm{x,t}>0 \\ |e^{-t}-e^{-x}|\leq\delta}} |f(t) - f(x)|$$

and is well defined for $\delta \geq 0$ and all functions $f \in C^*[0,\infty)$. In the present case the modulus of continuity has the property

$$|f(t) - f(x)| \leq \left(1 + \frac{(e^{-x}-e^{-t})^2}{\delta^2}\right)\omega^*(f;\delta), \qquad \delta > 0. \tag{4.1}$$

Further properties and use of the modulus of continuity can be found in [3, 8]. The following theorem can also be found in the later.

Theorem 2. *If a sequence of positive linear operators $A_n : C^*[0,\infty) \to C^*[0,\infty)$ satisfy the equalities:*

$$\|A_n(1;x) - 1\|_{[0,\infty)} = a_n$$

$$\|A_n(e^{-t};x) - e^{-x}\|_{[0,\infty)} = b_n$$

$$\|A_n(e^{-2t};x) - e^{-2x}\|_{[0,\infty)} = c_n,$$

where a_n, b_n and c_n are bounded and finite, in the limit $n \to \infty$, then

$$\|A_n(f;x) - f(x)\|_{[0,\infty)} \leq a_n\,|f(x)| + (2 + a_n)\,\omega^*(f, \sqrt{a_n + 2\,b_n + c_n}),$$

for every function $f \in C^*[0,\infty)$, and satisfies

$$\|A_n(f;x) - f(x)\|_{[0,\infty)} \leq 2\,\omega^*(f, \sqrt{2\,b_n + c_n})$$

for constant preserving operators.

Proof. Since

$$A_n((e^{-t} - e^{-x})^2; x) = [A_n(e^{-2t};x) - e^{-2x}] - 2\,e^{-x}\,[A_n(e^{-t};x) - e^{-x}] + e^{-2x}\,[A_n(1;x) - 1]$$

then, by use of (4.1),

$$\begin{aligned} A_n(|f(t) - f(x)|; x) &\leq \left(A_n(1;x) + \frac{1}{\delta^2}\,A_n((e^{-t} - e^{-x})^2; x) \right)\,\omega^*(f,\delta) \\ &\leq \left(1 + a_n + \frac{a_n + 2\,b_n + c_n}{\delta^2} \right)\,\omega^*(f,\delta). \end{aligned}$$

By choosing $\delta = \sqrt{a_n + 2\,b_n + c_n}$ then

$$A_n(|f(t) - f(x)|; x) \leq (2 + a_n)\,\omega^*(f, \sqrt{a_n + 2\,b_n + c_n}).$$

Now, making use of

$$|A_n(f;x) - f(x)| \leq |f|\,|A_n(1;x) - 1| + A_n(|f(t) - f(x)|; x)$$

leads to the uniform estimation of convergence in the form

$$\|A_n(f;x) - f(x)\|_{[0,\infty)} \leq a_n\,|f(x)| + (2 + a_n)\,\omega^*(f, \sqrt{a_n + 2\,b_n + c_n}).$$

For constant preserving operators the property $\|A_n(1;x) - 1\|_{[0,\infty)} = a_n = 0$ holds and leads to

$$\|A_n(f;x) - f(x)\|_{[0,\infty)} \leq 2\,\omega^*(f, \sqrt{2\,b_n + c_n}).$$

□

Remark 4. *The SMJ operators satisfy*

$$\|R_n^{(\beta)}(f;x) - f(x)\|_{[0,\infty)} \leq 2\,\omega^*(f, \sqrt{2\,b_n + c_n}).$$

Proof. By using Lemma 2 it is evident that $R_n^{(\beta)}(1;x) = 1$ and yields $a_n = 0$. By using (3.7), of Lemma 5, it is seen that

$$R_n^{(\beta)}(e^{-\mu\,t}; x) - e^{-\mu x} = e^{-\mu\,x}\,\left(\frac{\mu(\mu - \lambda)x}{2!\,n\,(1-\beta)^2} - \frac{\Lambda(x,\mu,\lambda)\,\mu(\mu-\lambda)x}{4!\,n^2\,(1-\beta)^4} + \mathcal{O}\left(\frac{1}{n^3\,(1-\beta)^6} \right) \right),$$

where $\Lambda(x,\mu,\lambda) = (3\mu x - 4 - 8\beta)\,\mu - (3\mu x - 2 + 8\beta)\lambda$, and provides

$$\|R_n^{(\beta)}(e^{-\mu\,t}; x) - e^{-\mu x}\| = \left\| \frac{\mu(\mu - \lambda)\,x\,e^{-\mu\,x}}{2!\,n(1-\beta)^2}\,\left(1 + \frac{2\,\Lambda(x,\mu,\lambda)}{4!\,n\,(1-\beta)^2} + \mathcal{O}\left(\frac{1}{n^2\,(1-\beta)^4} \right) \right) \right\|$$

which, for $\mu \in \{1, 2\}$, the remaining limiting values, b_n and c_n can be seen to be bounded and finite. It is also evident that in the limiting case, $n \to \infty$, b_n and c_n tend to zero. By the resulting statements of Theorem 2 it is determined that

$$\|R_n^{(\beta)}(f; x) - f(x)\|_{[0,\infty)} \leq 2\,\omega^*(f, \sqrt{2\,b_n + c_n}).$$

as claimed. □

For the SMJ operators a quantitative Voronovskaya-type theorem can be defined in the following way.

Theorem 3. *Let $f, f', f'' \in C^*[0, \infty)$ then*

$$\left| n\,[R_n^{(\beta)}(f; x) - f(x)] - \frac{\lambda\, x}{2!\,(1-\beta)^2}\, f'(x) - \frac{x}{n\,(1-\beta)^2}\, f''(x) \right|$$
$$\leq |\mu_n(x, \beta)|\,|f'(x)| + |\nu_n(x, \beta)|\,|f''(x)|$$
$$+ 2\,(2\,\nu_n(x, \beta) + \frac{x}{(1-\beta)^2} + \zeta_n(x, \beta))\,\omega^*\left(f''; \frac{1}{\sqrt{n}}\right)$$

where

$$\mu_n(x, \beta) = n\,R_n^{(\beta)}(\phi; x) - \frac{\lambda\, x}{2!\,(1-\beta)^2}$$
$$\nu_n(x, \beta) = \frac{1}{2!}\left(n\,R_n^{(\beta)}(\phi^2; x) - \frac{x}{(1-\beta)^2}\right)$$
$$\zeta_n(x, \beta) = n^2\sqrt{R_n^{(\beta)}((e^{-x} - e^{-t})^4; x)}\,\sqrt{R_n^{(\beta)}(\phi^4; x)}.$$

Proof. The Taylor expansion for the function $f(x)$ is seen by

$$f(t) = f(x) + f'(x)\,(t - x) + \frac{f''(x)}{2!}\,(t - x)^2 + \theta(t, x)\,(t - x)^2 \tag{4.2}$$

where $2\,\theta(t, x) = f''(\eta) - f''(x)$ for $x \leq \eta \leq t$. Applying the SMJ operator to the Taylor expansion it is determined that

$$|R_n^{(\beta)}(f(t); x) - f(x)\,R_n^{(\beta)}(1; x) - f'(x)\,R_n^{(\beta)}(\phi; x) - \frac{f''(x)}{2!}\,R_n^{(\beta)}(\phi^2; x)|$$
$$\leq |R_n^{(\beta)}(\theta(t, x)\,\phi^2; x)|.$$

Using the results of lemma 4 and 5 this can be seen by

$$\left| n\,\left(R_n^{(\beta)}(f; x) - f(x)\right) - \frac{\lambda\, x}{2!\,(1-\beta)^2}\, f'(x) - \frac{x}{2!\,(1-\beta)^2}\, f''(x) \right|$$
$$\leq \left| n\,R_n^{(\beta)}(\phi; x) - \frac{\lambda\, x}{2!\,(1-\beta)^2} \right|\,|f'(x)| + \frac{1}{2!}\left| n\,R_n^{(\beta)}(\phi^2; x) - \frac{x}{(1-\beta)^2} \right|\,|f''(x)|$$
$$+ |n\,R_n^{(\beta)}(\theta(t, x)\,\phi^2; x)|$$

or

$$\left| n\,\left(R_n^{(\beta)}(f;x)-f(x)\right)-\frac{\lambda\,x}{2!\,(1-\beta)^2}\,f'(x)-\frac{x}{2!\,(1-\beta)^2}\,f''(x)\right| \\ \leq |\mu_n(x,\beta)|\,|f'(x)|+|\nu_n(x,\beta)|\,|f''(x)|+|n\,R_n(\theta(t,x)\,\phi^2;x)|$$

where

$$\mu_n(x,\beta)=n\,R_n^{(\beta)}(\phi;x)-\frac{\lambda\,x}{2!\,(1-\beta)^2}$$
$$\nu_n(x,\beta)=\frac{1}{2!}\left(n\,R_n^{(\beta)}(\phi^2;x)-\frac{x}{(1-\beta)^2}\right).$$

By using (3.8) it is given that

$$|\theta(t,x)|\leq\left(1+\frac{(e^{-t}-e^{-x})^2}{\delta^2}\right)\,\omega^*(f'';\delta)$$

which becomes, when $|e^{-t}-e^{-x}|\leq\delta$ is taken into consideration, $|\theta(t,x)|\leq 2\,\omega^*(f'';\delta)$. If $|e^{-t}-e^{-x}|>\delta$ then $|\theta(t,x)|\leq(2/\delta^2)\,(e^{-t}-e^{-x})^2\,\omega^*(f'';\delta)$. Therefore, it can be concluded that

$$|\theta(t,x)|\leq 2\,\left(1+\frac{(e^{-t}-e^{-x})^2}{\delta^2}\right)\,\omega^*(f'';\delta).$$

The term $n\,R_n^{(\beta)}(\theta(t,x)\,\phi^2;x)$ becomes

$$n\,R_n^{(\beta)}(\theta(t,x)\,\phi^2;x)\leq 2\,n\,\left(R_n^{(\beta)}(\phi^2;x)+\frac{1}{\delta^2}\,R_n^{(\beta)}((e^{-t}-e^{-x})^2\,\phi^2;x)\right)\,\omega^*(f'';\delta)$$

which, by applying the Cauchy-Swarz inequality, becomes

$$n\,R_n^{(\beta)}(\theta(t,x)\,\phi^2;x)\leq 2\,n\,\left(R_n^{(\beta)}(\phi^2;x)+\frac{1}{\delta^2}\,\zeta_n(x,\beta)\right)\,\omega^*(f'';\delta),$$

where

$$\zeta_n(x,\beta)=n^2\,\sqrt{R_n^{(\beta)}((e^{-x}-e^{-t})^4;x)}\,\sqrt{R_n^{(\beta)}(\phi^4;x)}.$$

Now, by choosing $\delta=1/\sqrt{n}$, the desired result is obtained. □

Remark 5. *By use of Lemma 4 it is clear that $\mu_n(x,\beta)\to 0$ and $\nu_n(x,\beta)\to 0$ as $n\to\infty$. Using* (3.5) *and* (3.9) *the limit of $\zeta_n(x,\beta)$ becomes*

$$\lim_{n\to\infty}\zeta_n(x,\beta)=\frac{3\,x^2\,e^{-2x}}{(1-\beta)^4}$$

and yields

$$\lim_{n\to\infty}\left(2\,\nu_n(x,\beta)+\frac{x}{(1-\beta)^2}+\zeta_n(x,\beta)\right)=\frac{x}{(1-\beta)^2}+\frac{3\,x^2\,e^{-2x}}{(1-\beta)^4}.$$

Corollary 1. *Let $f, f', f'' \in C^*[0,\infty)$ then the inequality*

$$\lim_{n\to\infty} n\left|R_n^{(\beta)}(f;x) - f(x)\right| = \frac{\lambda\, x}{2!\,(1-\beta)^2}\, f'(x) + \frac{x}{(1-\beta)^2}\, f''(x)$$

holds for all $x \in [0,\infty)$.

5. Further Considerations

Having established several results for the Szász-Mirakyan-Jain operators further considerations can be considered. One such consideration could be an application of a theorem found in a recent work of Gupta and Tachev, [7]. In order to do so the following results are required.

Lemma 6. *Let $z_\mu = z(\mu/n, \beta)$, $\phi = t - x$, and $f = Exp[n\,\alpha_n(x)\,(z_\mu - 1)]$. The exponentially weighted moments are then given by:*

$$\begin{aligned}
R_n^{(\beta)}(e^{-\mu x}\,\phi^0;x) &= f\\
R_n^{(\beta)}(e^{-\mu x}\,\phi^1;x) &= \left[\frac{\alpha_n(x)\,z_\mu}{1-\beta\,z_\mu} - x\right] f\\
R_n^{(\beta)}(e^{-\mu x}\,\phi^2;x) &= \left[\left(\frac{\alpha_n(x)\,z_\mu}{1-\beta\,z_\mu} - x\right)^2 + \frac{\alpha_n(x)\,z_\mu}{n\,(1-\beta\,z_\mu)^3}\right] f\\
R_n^{(\beta)}(e^{-\mu x}\,\phi^3;x) &= \left[\left(\frac{\alpha_n(x)\,z_\mu}{1-\beta\,z_\mu} - x\right)^3 + \frac{3\,\alpha_n(x)\,z_\mu}{n\,(1-\beta\,z_\mu)^3}\left(\frac{\alpha_n(x)\,z_\mu}{1-\beta\,z_\mu} - x\right)\right.\\
&\qquad \left. +(1+2\,\beta\,z_\mu)\,\frac{\alpha_n(x)\,z_\mu}{n^2\,(1-\beta\,z_\mu)^5}\right] f\\
R_n^{(\beta)}(e^{-\mu t}\,\phi^4;x) &= \left[\left(\frac{\alpha_n(x)\,z_\mu}{1-\beta\,z_\mu} - x\right)^4 + \frac{6\,\alpha_n(x)\,z_\mu}{n\,(1-\beta\,z_\mu)^3}\left(\frac{\alpha_n(x)\,z_\mu}{1-\beta\,z_\mu} - x\right)^2\right.\\
&\qquad +(7+8\,\beta\,z_\mu)\,\frac{\alpha_n(x)\,z_\mu}{n^2\,(1-\beta\,z_\mu)^5}\cdot\left(\frac{\alpha_n(x)\,z_\mu}{1-\beta\,z_\mu} - x\right) + (1+8\beta\,z_\mu+6\beta^2\,z_\mu)\\
&\qquad \left.\cdot\frac{\alpha_n(x)\,z_\mu}{n^3\,(1-\beta\,z_\mu)^7} + \frac{3\,\alpha_n(x)\,z_\mu}{n^2\,(1-\beta\,z_\mu)^5}\right] f
\end{aligned} \tag{5.1}$$

Proof. By using (2.7) then

$$\begin{aligned}
R_n^{(\beta)}(e^{-\mu t}\phi^m;x) &= n\,\alpha_n \sum_{k=0}^{\infty} \frac{1}{k!}\,(n\alpha_n + \beta k)^{k-1}\, e^{-(n\alpha_n+\beta k)}\, e^{-\mu k/n}\left(\frac{k}{n} - x\right)^m\\
&= (-1)^m \left(\frac{d}{d\mu} + x\right)^m e^{n\alpha_n(x)\,(z_\mu - 1)}.
\end{aligned}$$

For the case $m = 1$ it is given that

$$R_n^{(\beta)}(e^{-\mu t}\,\phi; x) = -\left(\frac{d}{d\mu} + x\right)\, e^{n\alpha_n(x)\,(z_\mu - 1)} = \left[\frac{\alpha_n(x)\, z_\mu}{1 - \beta\, z_\mu} - x\right]\, e^{n\alpha_n(x)\,(z_\mu - 1)}.$$

The remainder of the moments follow. □

Remark 6. *The ratio of $R_n^{(\beta)}(e^{-\mu t}\,\phi^4; x)$ and $R_n^{(\beta)}(e^{-\mu t}\,\phi^2; x)$ as $n \to \infty$ is*

$$\lim_{n\to\infty} \frac{R_n^{(\beta)}(e^{-\mu t}\,\phi^4; x)}{R_n^{(\beta)}(e^{-\mu t}\,\phi^2; x)} = 0, \tag{5.2}$$

with order of convergence $\mathcal{O}(n^{-2})$.

Proof. Consider the expansion of

$$\frac{\alpha_n(x)\, z_\mu}{1 - \beta\, z_\mu} = z_\mu \cdot \frac{1-\beta}{1 - \beta\, z_\mu} \cdot \frac{\alpha_n(x)}{1-\beta}$$

by making use of the expansion used in the proof of Lemma 4, (6.3), and by

$$\frac{1-\beta}{1-\beta\, z_\mu} = 1 - \frac{\beta\,\mu}{n(1-\beta)^2} + \frac{3\,\beta^2\,\mu^2}{2!\,n^2(1-\beta)^4} - \frac{(\beta + 14\beta^2)\,\mu^3}{3!\,n^3(1-\beta)^6} + \mathcal{O}\left(\frac{\mu^4}{n^4(1-\beta)^8}\right)$$

then

$$\frac{\alpha_n(x)\, z_\mu}{1 - \beta\, z_\mu} - x = \frac{x}{2\,n(1-\beta)^2}\,\left((\lambda - 2\mu) + \frac{\sigma(\lambda,\mu)}{3!\,n(1-\beta)^2} + \mathcal{O}\left(\frac{1}{n^2(1-\beta)^4}\right)\right). \tag{5.3}$$

where $\sigma(\lambda,\mu) = (1-4\beta)\lambda - 6\lambda\mu + 6(1 - 2\beta + 3\beta^2)\mu^2$. By squaring this result and taking the limit it is determined that

$$\lim_{n\to\infty} \frac{R_n^{(\beta)}(e^{-\mu t}\,\phi^4; x)}{R_n^{(\beta)}(e^{-\mu t}\,\phi^2; x)} = \lim_{n\to\infty} \frac{(\lambda - 2\mu)^2\, x^2}{4\,n^2(1-\beta)^4}\,\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \to 0.$$

□

With Lemma 6 and Remark 6 use could be made of Theorem 5 of Gupta and Tachev, [7], which can be stated as

Theorem 4. *Let E be a subspace of $C[0,\infty)$ which contains the polynomials and suppose $L_n : E \to C[0,\infty)$ is a sequence of linear positive operators preserving linear functions. Suppose that for each constant $\mu > 0$, and fixed $x \in [0,\infty)$, the operators L_n satisfy*

$$L_n(e^{-\mu t}\,(t-x)^2; x) \le Q(\mu, x)\, R_n^{(\beta)}(e^{-\mu t}(t-x)^2; x).$$

Additionally, if $f \in C^2[0,\infty) \bigcap E$ and $f^n \in Lip(\alpha,\mu)$, for $0 < \alpha \le 1$, then, for $x \in [0,\infty)$,

$$\left| L_n(f; x) - f(x) - \frac{f''(x)}{2}\,\mu_{n,2}^{R^{(\beta)}} \right|$$

$$\le \left[e^{-\mu x} + \frac{Q(\mu,x)}{2} + \sqrt{\frac{Q(2\mu,x)}{4}}\right]\,\mu_{n,2}^{R^{(\beta)}} \cdot \omega_1\left(f^n, \sqrt{\frac{\mu_{n,4}^{R^{(\beta)}}}{\mu_{n,2}^{R^{(\beta)}}}}, \mu\right)$$

where $\mu_{n,2}^{R^{(\beta)}} = R_n^{(\beta)}(e^{-\mu t}(t-x)^2; x)$.

6. Appendix

Expansion of the function $f(ae^t)$ in powers of t is is given by

$$f(ae^t) = \sum_{k=0}^{\infty} \left[D_t^k \, f(ae^t)\right]_{t=0} \, \frac{t^k}{k!} = f(a) + \sum_{k=1}^{\infty} p_k(a) \, \frac{t^k}{k!}, \tag{6.1}$$

where

$$p_n(a) = \left[D_t^n \, f(ae^t)\right]_{t=0} = \sum_{r=1}^{n} S(n, n-r) \, a^r \, f^{(r)}(a), \tag{6.2}$$

with $S(n,m)$ being the Stirling numbers of the second kind. Applying this expansion to the Lambert W-function the formula $W(xe^x) = x$ and the n^{th}-derivative coefficients, Oeis A042977, [11, 13] are required to obtain

$$-\frac{1}{\beta} \, W(-\beta \, e^{-\beta+t}) = 1 + (1-\beta) \sum_{n=1}^{\infty} \frac{B_{n-1}(\beta) \, u^n}{n!}, \tag{6.3}$$

where $(1-\beta)^2 \, u = t$ and $B_n(x)$ are the Eulerian polynomials of the second kind. Let $z(t)$ be the left-hand side of (6.3), $-\beta \, z(t) = W(-\beta \, e^{-\beta+t})$, to obtain

$$\begin{aligned} \frac{t}{(1-\beta)\,(z(t)-1)} &= 1 - \frac{u}{2!} + 2\,(1-4\beta)\,\frac{u^2}{4!} - 6\beta^2\,\frac{u^3}{4!} - (1 - 8\beta + 88\beta^2 + 144\beta^3)\,\frac{u^4}{6!} \\ &\quad - 840\beta^2\,(1 + 12\beta + 8\beta^2)\,\frac{u^5}{8!} + O(u^6). \end{aligned} \tag{6.4}$$

The ratio of $z(x) - 1$ to $z(t) - 1$ is given by

$$\frac{t}{x} \, \frac{z(x)-1}{z(t)-1} = 1 + \frac{(x-t)}{2!\,(1-\beta)^2} + \delta_1 \, \frac{(x-t)}{4!\,(1-\beta)^4} + \delta_2 \, \frac{(x-t)}{4!\,(1-\beta)^6} + \mathcal{O}\left(\frac{(x-t)}{8!\,(1-\beta)^8}\right), \tag{6.5}$$

where

$$\begin{aligned} \delta_1 &= 4(1+2\beta)x - 2(1-4\beta)\,t \\ \delta_2 &= (1 + 8\beta + 6\beta^2)\,x^2 - (1 - 4\beta - 6\beta^2)\,xt + 6\beta^2\,t^2 \end{aligned}$$

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