

Number of Vertices of the Polytope of Integer Partitions and Factorization of the Partitioned Number

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Abstract

The polytope of integer partitions of n is the convex hull of the corresponding n -dimensional integer points. Its vertices are of importance because every partition is their convex combination. Computation shows intriguing features of $v(n)$, the number of the polytope vertices: its graph has a saw-toothed shape with the highest peaks at prime n 's. We explain the shape of $v(n)$ by the large number of partitions of even n 's that were counted by N. Metropolis and P. R. Stein. These partitions are convex combinations of two others. We reveal that divisibility of n by 3 also reduces the value of $v(n)$, which is caused by partitions that are convex combinations of three but not two others, and characterize convex representations of such integer points in arbitrary integral polytope. To approach the prime n phenomenon, we use a specific classification of integers and demonstrate that the graph of $v(n)$ is stratified to layers corresponding to resulting classes. Our main conjecture claims that $v(n)$ depends on collections of divisors of n . We also offer an initial argument for that the number of vertices of the Gomory's master corner polyhedron on the cyclic group has features similar to those of $v(n)$.

1 Introduction

Integer partitions are related to divergent problems in mathematics and statistical mechanics [1]. A partition of a positive integer n is any finite non-decreasing sequence ρ of positive integers n_1, n_2, \dots, n_r such that

$$\sum_{j=1}^r n_j = n.$$

The integers n_1, n_2, \dots, n_r are called parts of the partition ρ .

In this paper we develop the polyhedral approach to integer partitions proposed in [10]. It is based on the n -dimensional geometrical interpretation of partitions [17]. Every partition ρ is referred to as a non-negative integer point $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, a solution to the equation

$$x_1 + 2x_2 + \dots + nx_n = n, \tag{1}$$

with x_i , $i = 1, \dots, n$, being the number of parts i in ρ . For example, the partition $8 = 4 + 2 + 1 + 1$ with three distinct parts 1, 2, 4 is identified with $x = (2, 1, 0, 1, 0, 0, 0, 0) \in \mathbb{R}^8$. We keep on writing $x \vdash n$ to indicate that $x \in \mathbb{R}^n$ is a partition of n .

We consider the polytope $P_n \subset \mathbb{R}^n$ defined as the convex hull of the set $P(n)$ of partitions of n :

$$P_n := \text{conv } P(n) = \text{conv} \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x \vdash n\}.$$

The conversion from the set to a polytope reveals the geometrical structure of $P(n)$. As for every polytope, the key elements of P_n are its facets and vertices. The facets were characterized in [10] and the vertices were studied in [11, 12, 14].

Vertices of P_n and their number are of special importance since, by Carathéodory's theorem [2], every partition is a convex combination of some vertices. It is proved in [8] that the problem of recognizing vertices can be decided in polynomial time with the use of linear programming technique. However, no combinatorial characterization of vertices of P_n is available as yet. The only result in this direction is the criterion for a partition to be a convex combination of two others [12], see Theorem 2 below.

Let $\text{Vert } P_n$ denote the set of vertices of P_n and $v(n) := |\text{Vert } P_n|$ be the number of its vertices. We computed vertices of P_n for $n \leq 100$, see [16], and presented the values of $v(n)$ in the On-Line Encyclopedia of Integer Sequences (OEIS), sequence A203898. It turned out that the graph of $v(n)$ exhibits peculiar features, one can see this graph in Fig. 1. In contrast to $p(n) = |P(n)|$, the number of partitions of n , the function $v(n)$ does not increase monotonously. It drops down at every even n and its peaks at prime n 's seem to be higher than others. Inspired by these perplexing peculiarities, we concentrate on the asymptotic dependence of $v(n)$ on the multiplicative properties of n .

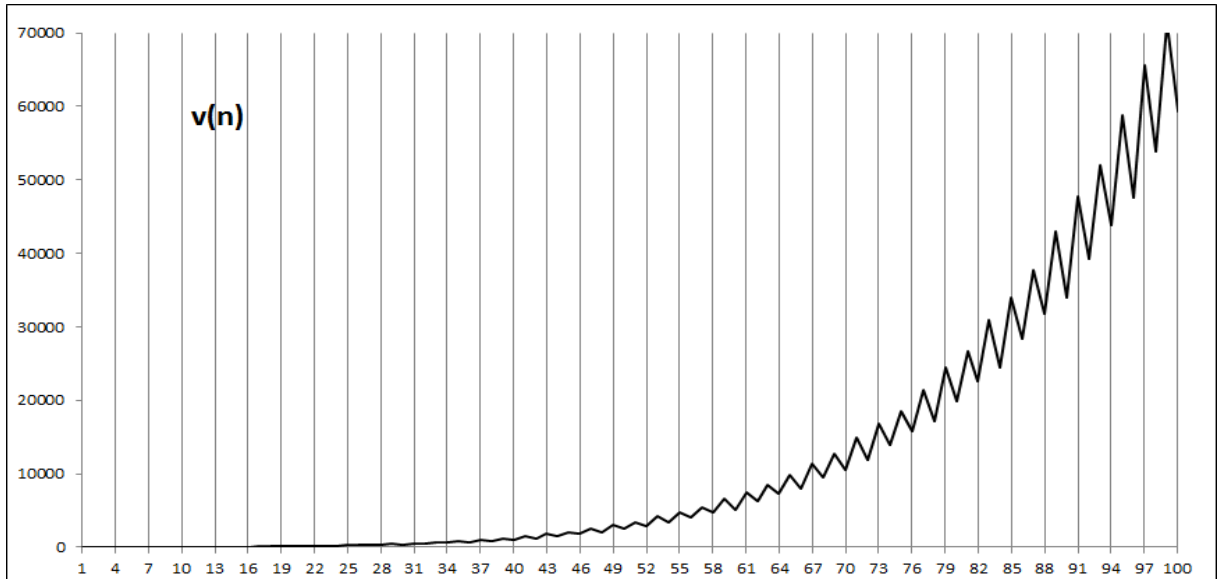


Fig. 1. Graph of the function $v(n)$, number of vertices of P_n .

To analyse the observed phenomena we study the structure of the set $P(n) \setminus \text{Vert } P_n$ of partitions of n that 'fail' to be vertices of P_n . In section 2, we divide this set to subsets of partitions $x \vdash n$ according to the minimal number of partitions that are necessary to express x as their convex combination. We characterize the coefficients in such representations in the case when x needs three partitions. In fact, we do that for analogous integer points in an arbitrary integral polytope. In section 3, we establish that an important role in identifying vertices belongs to a class of partitions that were counted by N. Metropolis and P. R. Stein [7]. Gradually, from the obvious dependence of $v(n)$ on the evenness of n , it becomes clear that other divisors of n also affect the value of $v(n)$. In section 4, we use the known numbers

of knapsack partitions of n , that were introduced by R. Ehrenborg and M.A. Readdy [3], to disclose dependence of $v(n)$ on divisibility of n by 3.

We suggest several conjectures. Of primary interest is the Conjecture 4 in section 5. Grounded on the visual but quite convincing arguments, it claims that the value of $v(n)$ depends on factorization of n . The relatively largest values of $v(n)$ are attained at prime n 's. For a composite n , the smaller its every subsequent prime divisor is, the smaller the value of $v(n)$ gets.

In section 6, we offer an initial argument that the number of vertices of the master corner polyhedron on the cyclic group has features similar to those of $v(n)$. It is based on less than a dozen numbers of vertices computed by R. E. Gomory [4] and can be considered as no more than a hint. However, in view of the closeness of these two serial polyhedra revealed in [13], we believe that this argument deserves further examination.

Finally, in section 7, we outline the most promising directions for the future study.

2 Partitions that are not vertices and coefficients in their convex representations

Recall that for arbitrary polytope P , a point $x \in P$ is a vertex of P if it cannot be expressed as a convex combination $x = \sum_{j=1}^k \lambda_j y^j$, $\sum_{j=1}^k \lambda_j = 1$, $\lambda_j > 0$, of some other points $y^j \in P$, $j = 1, \dots, k$, in particular, of vertices. So, every partition $x \in P(n) \setminus \text{Vert} P_n$ is a convex combination of some partitions of n . Denote by $\xi(x)$ the minimal number of partitions of n , which are needed for such a representation of an $x \vdash n$, $x \notin \text{Vert} P_n$, and let $C_\xi(n)$ be the set of partitions $x \vdash n$, for which $\xi(x) = \xi$. It is easy to see that the sets $C_\xi(n)$, $\xi = 2, 3, 4, \dots$, are pairwise different and

$$\text{Vert} P_n = P(n) \setminus \bigcup_{\xi \geq 2} C_\xi(n). \quad (2)$$

While computing vertices of P_n , we saw that for all n the most of $x \vdash n$, $x \notin \text{Vert} P_n$, belong to $C_2(n)$. For $n < 15$ this is true for all $x \notin \text{Vert} P_n$. For $n = 15, 21, 24, 25, 27, 28$ and $n \geq 30$ there exist non-vertices of P_n that belong to $C_3(n)$. The minimal n for which some $x \vdash n$ belongs to $C_4(n)$ is $n = 36$. So, for n sufficiently large, $C_2(n)$, $C_3(n)$, $C_4(n) \neq \emptyset$. Non-emptiness of $C_\xi(n)$ for $\xi \geq 5$ is not confirmed yet but we dare to suggest the following conjecture.

Conjecture 1 *For every ξ , $C_\xi(n) \neq \emptyset$ for sufficiently large $n > n_0(\xi)$. For every n , $|C_\xi(n)|$ decreases when ξ grows.*

If this conjecture is true then the union in (2) can consist of arbitrarily large number of sets. However, the next theorem shows that for every fixed n , this union is finite.

Theorem 1 *For every $\xi > 2$ and $n < 2^{\xi-1} - 1$, $C_\xi(n) = \emptyset$.*

Proof. Let a partition $x \in C_\xi(n)$, $\xi > 2$, have m parts $\{n_1, n_2, \dots, n_m\}$. It is proved in [12] that if $m > \log_2(n+1)$ then $x \in C_2(n)$. Hence $m \leq \log_2(n+1)$ since $C_2(n) \cap C_\xi(n) = \emptyset$.

Let x be a convex combination of $y^1, y^2, \dots, y^\xi \vdash n$. Then $y_i^j = 0$ for $i \notin \{n_1, n_2, \dots, n_m\}$, $j = 1, 2, \dots, \xi$. Since $x \notin C_k(n)$ for $k < \xi$, then y^1, y^2, \dots, y^ξ are vertices of some $(\xi - 1)$ -dimensional simplex in \mathbb{R}^n and are affinely independent. Then the matrix with the rows $(y_{n_1}^j, y_{n_2}^j, \dots, y_{n_m}^j)$, $j = 1, 2, \dots, \xi - 1$, is of rank $\xi - 1$ and therefore $\xi - 1 \leq m$. The two inequalities imply that $\xi - 1 \leq \log_2(n+1)$. Thus $n \geq 2^{\xi-1} - 1$. \square

Figure 2 shows the structure of the set $P(n)$ provided Conjecture 1 holds. The whole rectangle corresponds to all partitions of n . Vertices of P_n form the utmost right rectangle. Other inner rectangles in order from left to right correspond to $C_2(n)$, $C_3(n)$, $C_4(n), \dots, C_k(n)$, where k depends on n . The set $M_2(n)$, whose exact definition will be given in Section 3, consists of two parts: one, a subset of $C_2(n)$ depicted as the large rectangle from the left edge to the dashed line, and the other part, a small subset of vertices forming a tiny rectangle at the bottom right of the picture. The set $K(n) = P(n) \setminus C_2(n)$ will be considered in Section 4.

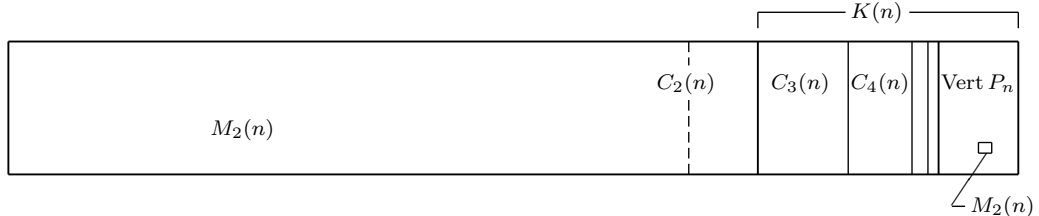


Fig. 2. Conjectured structure of the set of partitions of n .

Remark 1 $\xi(x)$ can be defined for any integer point in any integral polytope P . It could be called 'the index of convex embeddedness of x '. Then, in particular, vertices of P would be of index 1. However, we refrain from coining a special term. The common state, for example, in combinatorial optimization, is that when a polytope is generated by a set of integral points, each of these points is a vertex. In particular this is true for the travelling salesman polyhedron and other $(0, 1)$ -polytopes. Perhaps, this is a reason why the classes of points similar to C_ξ were not considered earlier.

A criterion for a partition $x \vdash n$ to belong to $C_2(n)$ is given in the following theorem.

Theorem 2 ([11]) *A partition $x \vdash n$ is a convex combination of two partitions of n if and only if there exist two different collections of parts of x with equal sums.*

No criterion for $x \in C_3(n)$ is known but the computations show that such an x always admits a representation $x = \sum_{j=1}^3 \lambda_j y^j$, $y^j \vdash n$, $\lambda_j \geq 0$, $\sum_{j=1}^3 \lambda_j = 1$, with all $\lambda_j = \frac{1}{3}$. The following theorem states that this holds for every integral polytope. Recall that a polytope is called integral if all its vertices are integer points.

Theorem 3 *If $P \in \mathbb{R}^n$ is an integral polytope and an integer point $x \in P$ is a convex combination of three integer points in P but is not a convex combination of any two integer points in P then there exist integer points $y^1, y^2, y^3 \in P$ such that*

$$x = \frac{1}{3}y_1 + \frac{1}{3}y_2 + \frac{1}{3}y_3 . \quad (3)$$

Proof. We begin with the general case of arbitrary integer $k > 2$ and an integer $x \in P$, which is a convex combination of k integer points in P and is not a convex combination of any less than k integer points in P . Then x is a strictly interior point in the $(k - 1)$ -dimensional simplex S with vertices in these k points. Assume there is one more integer point $z \in S$, $z \neq x$.

If z is strictly interior to S then it divides S to integral simplices S_1, S_2, \dots, S_k with vertices z and any $k - 1$ vertices of S . Since x is not a convex combination of any less than

k integer points in S , it does not lie in any facet of any S_j . Hence x lies strictly inside one of these simplices, say $x \in S_1$. If z lies on the boarder of S let q be the smallest number such that z is strictly interior to some q -dimensional face F of S . Then z divides F to $q + 1$ integral simplices F_1, F_2, \dots, F_{q+1} . This implies that the simplex S can be also divided to $q + 1$ integral simplices, each of whose vertices are vertices of some F_j and the vertices of S not belonging to F . As in the previous case, x lies strictly inside one of these simplices, denote it again by S_1 .

Applying the same reasoning to S_1 , if it contains an integer point $z_1 \neq x$, we come to a $(k - 1)$ -dimensional integral simplex $S_2 \subset S_1$ with analogous condition on $x \in S_2$. After repeating this procedure a finite number of times, we obtain a $(k - 1)$ -dimensional integral simplex $T \subset S_2 \subset S_1 \subset P$ with x as its single strictly interior integer point satisfying conditions of the theorem and no integer points on the boarder of T .

From here on, we consider that P is the triangle T and $k = 3$, as in the theorem statement. The rest of the proof can be carried with the help of the Pick's theorem as, for example, in [9]. We will continue using only elementary geometry. In Figure 3, we show the triangle T with vertices indicated by A, B, C and the point x denoted by O .

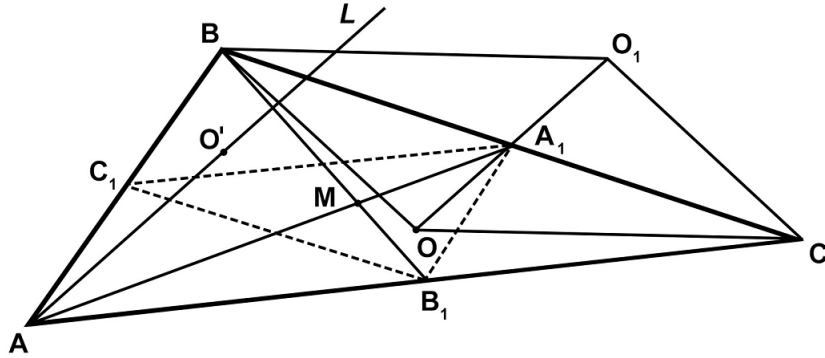


Fig. 3. To the proof of Theorem 3.

We will use the following property of the lattice H_I of integer points in the plain H that contains T : if for some $u_0 \in H_I$ and some n -dimensional vector \bar{c} , the point $u_0 + \bar{c}$ belongs to H_I then for every $u \in H_I$ the point $u \pm \bar{c}$ belongs to H_I .

Let A_1, B_1, C_1 be the midpoints of the sides of T and M be the barycenter of T . Assume $O \neq M$. Then O lies strictly inside the triangle $A_1B_1C_1$ since otherwise, if for example $O \in \triangle A_1B_1C$, we would have the point $O + \overline{CO} \neq O$ in $\in T \cap H_I$. (Here \overline{CO} is the vector from C to O .) Hence O lies in one of the triangles $A_1B_1M, A_1C_1M, B_1C_1M$ or on a common side of some two of them. Let $O \in \triangle A_1B_1M$. Draw the parallelogram $BOCO_1$ on the straight line segments OB and OC . By the above property, $O_1 \in H_I$. The diagonal OO_1 of the parallelogram passes through A_1 . Draw the ray \mathcal{L} parallel to OO_1 from A inside the triangle ABC . Since A_1O goes between A_1M and A_1B_1 , where A_1M is allowed but A_1B_1 is not, \mathcal{L} goes between AM and AB and can contain AM but not AB . Put the point O' on \mathcal{L} at the distance $|AO'| = |OO_1|$ from A . By the above property, O' is an integer point. The triangles ABM and A_1B_1M are congruent with the congruence coefficient 2 and $|AO'| = 2|A_1O|$. Hence $O \in \triangle A_1B_1M$ implies $O' \in \triangle ABM$. Note that O' can lie on AM or BM . In any case O' is in T and integrality of O' implies $O' = O$.

Since M is the single common point in $\triangle A_1B_1M \cap \triangle ABM$ the assumption $O \neq M$ implies $O' \neq O$. The contradiction proves that O is the barycenter of T and satisfies (3). \square

Since all integer points in P_n are partitions of n [10], Theorem 3 implies the following corollary.

Corollary 1 *A partition $x \vdash n$ belongs to $C_3(n)$ if and only if x is a convex combination of some three partitions $y^1, y^2, y^3 \vdash n$ such that the equality (3) holds.*

All known partitions $x \in C_4(n)$ admit convex representations with coefficients $\frac{1}{4}$. However the analogue of Theorem 3 does not hold for such an x . This follows from the results of B. Reznik [9]: in the case of an integral simplex P with 4 vertices (a 3-dimensional tetrahedron) there are 7 variants for the values of coefficients in a convex representation of a single integer point in P via its vertices. It is interesting that in each variant all denominators are simultaneously equal to one of the numbers 4, 5, 7, 11, 13, 17, 19. Nothing is known about the coefficients in convex representations via 5 integer points but we expect that $\frac{1}{5}$ is far from being the only one possible value.

3 Numbers of partitions of even, odd, and prime n and Metropolis partitions

Let us return to Figure 1 that presents the graph of the function $v(n)$ for $n \leq 100$. One immediately sees that the value of $v(n)$ depends on the evenness of n :

$$v(2r - 1) > v(2r) \tag{4}$$

except small r . So, we can refer to the $v(n)$ graph as consisting of two subgraphs: for odd and even n 's, the latter lying below the former. This radically differs from the monotone increasing of $p(n)$, the number of partitions of n .

Upon careful examination of Figure 1 we noticed that some points $(n, v(n))$ with n odd are disposed slightly higher than the main line. It turned out that they correspond to prime n 's. Comparison of their heights $v(n)$ with the half-sums $\frac{1}{2}(v(n - 2) + v(n + 2))$ confirmed this observation for all prime $n \geq 43$ except $n = 61$. The observed tooth-shaped form of the $v(n)$ graph and special role of prime numbers raised the question of what multiplicative property of n affects the value of $v(n)$.

We know from the computation that for every n , the majority of partitions that are not vertices belong to $C_2(n)$. By Theorem 2, these partitions have two collections of parts with equal sums. In particular, for even $n = 2r$, $C_2(n)$ contains partitions of the form

$$[\text{partition}_1 \text{ of } r] + [\text{partition}_2 \text{ of } r], \tag{5}$$

where

$$\text{partition}_1 \neq \text{partition}_2. \tag{6}$$

Denote the number of partitions (5), disregarding condition (6), by $m_2(2r)$. It is not hard to see that

$$m_2(2r) = \frac{1}{2} (p(r)^2 + p(r)) - [\text{number of duplicates in (5)}], \tag{7}$$

but it is far from clear how to count the duplicates. Note that if a partition of the form (5) satisfies (6) it can be a vertex. The partition $(0, 2, 2, 0^7) \vdash 10$ (with 7 zero parts) is an instance. The number of such vertices is less than $p(r)$, which is a rough estimate. Thus, when we are interested in the asymptotics of $C_2(n)$ and $v(n)$, we can ignore vertices of the form (5). Note that these vertices are shown in Figure 1 by the small rectangle in the Vert P_n area. The following conjecture may explain inequality (4) and the tooth-shaped form of the $v(n)$ graph.

Conjecture 2 For n even, $m_2(n)$ is large relative to $v(n)$.

After having searched in the OEIS by the sequence of the first values of $m_2(n)$, we got to the sequence A002219 and the work of Metropolis and Stein [7], where the authors had counted partitions of n that can be obtained by joining r , r divides n , not necessarily different partitions of $\frac{n}{r}$ (for convenience, we slightly changed the original notation in [7]). We will call these partitions Metropolis r -partitions. For n multiple of r , denote the set of Metropolis r -partitions of n by $M_r(n)$ and set $m_r(n) := |M_r(n)|$. Note that Metropolis 2-partitions coincide with partitions (5). The main result of [7] is the formula for $m_r(n)$ in the form of a finite series of binomial coefficients multiplied by certain integer coefficients, which depend only on r . For $m_2(n)$ this formula reads

$$m_2(n) = \binom{g+2}{2} + (g+2)c_1 + c_2, \quad g = \left\lfloor \frac{\frac{n}{2} + 1}{2} \right\rfloor, \quad \frac{n}{2} > 5, \quad (8)$$

where c_1 and c_2 'must be determined by direct calculation' [7]. The values of $m_2(n)$ are presented in the sequence A002219 for even $n \leq 178$. Using (8), we obtain an upper bound $b(n)$ for the number of vertices of P_n .

Theorem 4

$$v(n) \leq b(n) := \begin{cases} p(n) - m_2(n), & n \text{ even,} \\ p(n) - m_2(n-1), & n \text{ odd,} \end{cases} \quad (9)$$

where values of $m_2(\cdot)$ are calculated with the use of (8).

Proof. The proof follows from the inclusion $M_2(n) \subset C_2(n)$, if we ignore the small number of vertices belonging to $M_2(n)$, and the fact that adding the part 1 to every partition in $M_2(n-1)$, n odd, results in a partition in $C_2(n)$. \square

Disregarding the duplicates, we can obtain from (7) an upper bound on $m_2(n)$. However, Metropolis and Stein pointed that, for large n , much better is the bound $m_2(n) < p(n, \frac{n}{2})$, where $p(n, k)$ is the number of partitions of n with no part greater than k . It is not hard to show that $p(n, \frac{n}{2})$ is asymptotically equal to $p(n)$. An anonymous author under the nickname 'joriki' presented the following proof of this fact in Stackexchange [15]. Every partition of n has at most one part m larger than $\frac{n}{2}$, and the remaining parts form a partition of $n - m$. Thus

$$p\left(n, \frac{n}{2}\right) = p(n) - \sum_{i=0}^{\frac{n}{2}-1} p(i).$$

For large n , the terms in the sum are exponentially smaller than $p(n)$, so asymptotically

$$p\left(n, \frac{n}{2}\right) \sim p(n).$$

One can draw the graph of $m_2(n)$ over the known values and see that the equivalence

$$m_2(n) \sim p(n) \quad (10)$$

is also very likely to be true.

Let us turn to the relations between $m_2(n)$, $p(n)$, and $v(n)$, whose values are known for $n \leq 100$. Table 1 presents some data for a few values of n . We see that the ratios $v(n)/p(n)$ and $v(n)/m_2(n)$ are small and very close, and the ratio $(p(n) - m_2(n))/p(n)$

n	$p(n)$	$v(n)$	$m_2(n)$	$\frac{v(n)}{p(n)}$	$\frac{v(n)}{m_2(n)}$	$p(n) - m_2(n)$	$\frac{p(n)-m_2(n)}{p(n)}$
60	966467	5148	924522	0.005327	0.005568	41945	0.0434
78	12132164	17089	11850304	0.001409	0.001442	281860	0.0232
100	190569292	59294	188735609	0.000311	0.000314	1833683	0.0096

Table 1: Relations between $p(n)$, $v(n)$, and $m_2(n)$

rapidly decreases when n grows, though the difference $p(n) - m_2(n)$ also increases. This corroborates (10) and gives an additional argument in favor of Conjecture 2.

The expression (9) for $b(n)$, the upper bound on the number of vertices of P_n , may help to clarify, though not prove, the cause of the tooth-shaped form of the graph of $v(n)$ under Conjecture 2. For n odd, it yields

$$b(n) - \frac{1}{2}(b(n-1) + b(n+1)) = \left(p(n) - \frac{1}{2}(p(n-1) + p(n+1))\right) + \frac{1}{2}(m(n+1) - m(n-1)),$$

where the first term is asymptotically zero and the second term is positive. This means that $b(n)$ has a peak at every large odd n and the graph of $b(n)$ is of the tooth-shaped form, similar to that in Figure 1 for $v(n)$.

Let us consider two examples to see what happens when n is even. For $n = 78$, we have $b(78) = p(78) - m_2(78) = 281\,860$, while $b(77) = p(77) - m_2(76) = 1\,549\,719$. So, the bound for $v(78)$ is less than $0.19b(77)$. In the same way we have $b(100) < 0.09b(99)$! Hence it is more than likely that, for n even, $v(n)$ is not only less than $\frac{1}{2}(v(n-1) + v(n+1))$ but $v(n) < v(n-1)$. Thus, Conjecture 2 and the asymptotic equivalence (10), as its stronger form observed from the numerical data, reasonably justify the inequality (4) and the gap between the values of $v(n)$ for even and odd n 's.

The next theorem provides a supplemental indication of the importance of Metropolis 2-partitions for recognizing vertices of P_n . Call a partition $x \vdash n$ an extension of a partition $y \vdash m$, $m < n$, if every part of y is a part of x .

Theorem 5 *For every n , every partition $x \in C_2(n)$ is either a Metropolis 2-partition or an extension of some Metropolis 2-partition $y \vdash m$, $m < n$.*

Proof. Consider arbitrary n and $x \in C_2(n)$, $x \notin M_2(n)$ if n is even. By Theorem 2, there exist two collections of parts of x with the same sum. Let s be the minimal possible value of such a sum. Clearly, $s \leq \frac{n}{2}$. The corresponding collections are disjoint and their union is a Metropolis 2-partition y of $m = 2s \leq n$. Hence $x = y$ if $m = n$ or x is an extension of y if $m < n$. \square

4 Vertices and knapsack partitions

R. Ehrenborg and M. A. Readdy [3] called a partition x a knapsack partition if for every integer, there is utmost one way to represent it as a sum of some parts of x . Denote the set of knapsack partitions of n by $K(n)$ and set $k(n) := |K(n)|$. Theorem 2 implies relations

$$K(n) = P(n) \setminus C_2(n),$$

$$C_\xi(n) \subset K(n), \quad \xi > 2,$$

$$\text{Vert } P_n \subset K(n).$$

The smallness of $|\text{Vert}P_n \cap M_2(n)|$ implies that for large n ,

$$v(n) < k(n) < p(n) - m_2(n).$$

Ehrenborg and Readdy computed the values $k(n)$ for $n \leq 50$ and exhibited them in the OEIS, sequence A108917. We extended this sequence till $n = 165$ as a by-product of our computation of vertices of P_n . Table 2 enhances Table 1 by the $k(n)$ values. Consider its first three rows with even n . Looking at the columns $v(n)/(p(n) - m_2(n))$ and $k(n) - v(n)$ and comparing the columns $k(n)$ and $p(n) - m_2(n)$, we see that there are quite many partitions of n that are not vertices of P_n and do not belong to the class of Metropolis 2-partitions. They are convex combinations of 2, 3, or more partitions of n . Note that $k(n)$ is a much better upper bound on $v(n)$ than $b(n)$ in (9) but no formula for $k(n)$ is known.

n	$p(n)$	$v(n)$	$m_2(n)$	$p(n) - m_2(n)$	$\frac{v(n)}{p(n) - m_2(n)}$	$k(n)$	$\frac{v(n)}{k(n)}$	$k(n) - v(n)$
60	966 467	5 148	924 522	41 945	0.12	5 341	0.964	193
78	12 132 164	17 089	11 850 304	281 860	0.06	17 871	0.956	782
100	190 569 292	59 294	188 735 609	1 833 683	0.03	61 692	0.967	2 398
77	10 619 863	21 393				22 128	0.967	735

Table 2: Relations between $p(n)$, $v(n)$, $m_2(n)$, and $k(n)$

One can check that the graph of $k(n)$, like that of $v(n)$, disintegrates into two graphs, for n odd and n even. However in the column $v(n)/k(n)$ we see that for $n = 78$ this ratio is less than those for even $n = 60$ and $n = 100$. Moreover, this ratio is approximately the same for odd $n = 77$ and even $n = 100$, which are rather far from each other. Figure 4 presents the graph of $v(n)/k(n)$. We obviously see that the n 's multiple of 3 are the local minima of $v(n)/k(n)$. This means that such n 's have more partitions that are not vertices of P_n and do not belong to $C_2(n)$ than the n 's not multiple of 3.

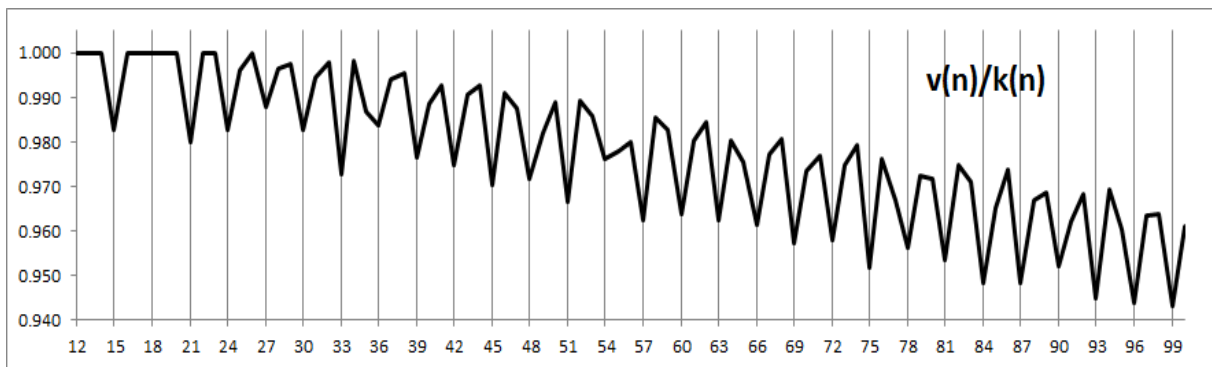


Fig. 4. Ratio $v(n)/k(n)$ of the number of vertices to the number of knapsack partitions.

The following conjecture naturally explains this phenomenon.

Conjecture 3 *The majority of partitions in $K(n)$ that are not vertices of P_n are convex combinations of three partitions of n .*

Conjecture 3 is consistent with our computation experience, from which we know that for n multiple of 3, most partitions in $C_3(n)$ have a part $\frac{n}{3}$ or are extensions of some partitions in $C_3(q)$, where $q < n$, q multiple of 3, with the additional part $n - q$. In any case one

of the partitions involved in the convex combination has three parts $\frac{n}{3}$ or $\frac{q}{3}$. For example, the non-vertex $x = (1^3, 9, 17, 22) \vdash 51$ has a part $17 = \frac{51}{3}$, and its convex representation is $\frac{1}{3}(1^7, 22^2) + \frac{1}{3}(1^2, 9^3, 22) + \frac{1}{3}(17^3)$.

As for the obvious tendency of $v(n)/k(n)$ to decrease, we see its explanation in the increase of the number of partitions in $C_\xi(n)$, $\xi > 3$, with the growth of n .

5 Stratification of the numbers of vertices

To reveal the discovered dependence of the number of vertices of the polytope P_n on multiplicative properties of n and examine it in more details, we consider the classes of integers

$$N_k := \{n \mid n = pk, p \text{ prime}\}, \quad k = 2, 3, 4, 5, \dots$$

and the corresponding numbers of vertices

$$v_k(n) := v(n), \quad n \in N_k.$$

Remark 2 *It would be better to add the condition $k \leq p$ in the definition of N_k . We do not do this because of the lack of known values of $v(n)$.*

Figure 5 demonstrates the graphs of the functions $v_k(n)$ for $k = 1, 7, 5, 3, 2, 4, 6$ in order from top to bottom. They are generated with the use of the FindFit method of Wolfram Mathematica. We approximated the known values of $v_k(n)$ by the functions of the form $Ae^{B\sqrt{n}}$ with parameters A and B . The segment $n \in [60, 70]$ is chosen to split the graphs $v_k(n)$ visually. It also lies in the most interesting part of the segment $[1, 100]$, where we can expect our approximations to reveal a reliable picture of what happens. We do not include the graphs of $v_k(n)$, $k > 7$, in Figure 5 because they are little informative. For these k , there are too few prime numbers $p \geq k$ such that $kp \in [1, 100]$.

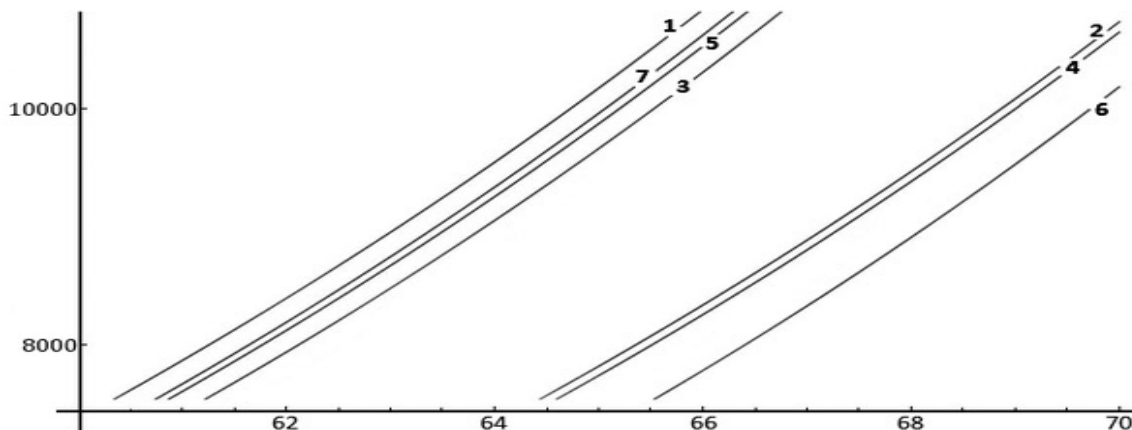


Fig. 5. Stratification of the number of vertices of the polytope P_n function to the $v_k(n)$ functions, $k = 1, 2, \dots, 7$.

We see that the graph of $v(n)$ is neither a single line nor a conjunction of two lines, for odd and even n , as in Figure 1. It is stratified into layers corresponding to the classes N_k and resembles a layered cake. Its layers are of the same shape but are disposed at different levels. The topmost line corresponds to N_1 , the class of primes. The graph of $v_3(n)$ is disposed

below it. Between them, one below another, are disposed the graphs of $v_7(n)$ and $v_5(n)$, and the graphs of $v_k(n)$, k even, are disposed below $v_3(n)$.

The levels of the graphs of $v_1(n)$, $v_2(n)$, $v_3(n)$ agree with Conjectures 2 and 3. The intermediate position of $v_7(n)$ and $v_5(n)$ and Conjecture 1 move us to suggest a more general conjecture that for prime k , k divides n , the determining influence on the level of $v_k(n)$ is exerted by the number of partitions of $n \in N_k$ that belong to $C_k(n)$. If k_1, k_2 are two primes, $k_1 > k_2$, then, in accordance with Conjecture 1, for large and sufficiently close to each other $n_1 \in N_{k_1}$ and $n_2 \in N_{k_2}$, the inequality $|C_{k_1}(n_1)| < |C_{k_2}(n_2)|$ holds and therefore the $v_{k_1}(n)$ graph is disposed above the $v_{k_2}(n)$ graph.

The case of k , a composite divisor of n , can be explained using the graphs of $v_6(n)$ and $v_4(n)$. $v_6(n)$ is disposed below $v_2(n)$ and $v_3(n)$ because the level of $v_6(n)$ is affected by partitions in $C_2(n)$ and partitions in $C_3(n)$. Similarly, $v_4(n)$ goes between $v_2(n)$ and $v_6(n)$ because 4 is an additional (to 2) divisor of n and $|C_4(n)| < |C_3(n)|$.

We summarize the above in the final conjecture.

Conjecture 4 *For $n = kp$, $k \leq p$, p prime, the level of the graph of $v_k(n)$ depends on factorization of k . If k_1, k_2, k_3, \dots are divisors of k sorted in ascending order, then the major influence on the level of the $v_k(n)$ graph is rendered by the divisor k_1 . Every successive divisor makes its additional contribution to lowering the level of $v_k(n)$.*

Generalizing, we might say that the value of $v(n)$ is determined by the proximity of n to its greatest prime divisor, which is defined by the lexicographic order on the set of sequences of divisors of n . For example, $38 = 2 \cdot 19$ would be more prime than $39 = 3 \cdot 13$, hence the layer $v_2(n)$, that contains $v(38)$, is disposed lower than the layer $v_3(n)$ containing $v(39)$. The same holds for $78 = 2 \cdot 3 \cdot 13$ and $70 = 2 \cdot 5 \cdot 7$. If we extend this speculation, we might come to a fractal structure of the graph of $v(n)$. For example, the graph of $v_5(n)$ together with $v_{10}(n), v_{15}(n), v_{20}(n), \dots$ may have a structure similar to that of $v(n)$. However, it is too early to foresee so far ahead — more numerical data of $v(n)$ is needed. Then the Conjecture 4 might be further detailed.

6 Remark on the Gomory's corner polyhedron

There is another serial integral polyhedron that is a close relative of the polytope of integer partitions. This is the well-known master corner polyhedron, the atom of integer linear programming [4, 5]. It is defined as follows. Let G be a finite Abelian group and some $g_0 \in G$. The master corner polyhedron $P(G, g_0)$ is the convex hull of solutions

$$t = (t(g); g \in G^+) \in \mathbb{R}^{|G^+|}, \quad t(g) \text{ integer, } t(g) \geq 0,$$

to the equation

$$\sum_{g \in G^+} t(g)g = g_0, \tag{11}$$

where G^+ is the set of nonzero group elements. For $G_{n+1} := \mathbb{Z}/(n+1)\mathbb{Z}$, the cyclic group of order $n+1$, and $g_0 = n$, the equation (11) reads

$$t_1 + 2t_2 + \dots + nt_n \equiv n \pmod{(n+1)},$$

which differs from (1) only in that the addition here is modulo $n+1$.

Our experience in studying both polyhedra showed that the vertex structure of P_n is more transparent and easy for understanding than that of the $P(G, g_0)$, even in the case of the cyclic group. In our opinion, this is because the standard addition on the segment of integers $[1, n]$, albeit defined only partially, is much easier to comprehend than the group addition. Most results on vertices of P_n were successfully transferred to vertices of the master corner polyhedron [13].

Statistics on vertices of $P(G, g_0)$ is unbelievably poor. All that we know about their numbers comes from the R. E. Gomory's seminal paper [4]. Gomory computed vertices of $P(G, g_0)$ for all groups G of the order up to 11 and all $g_0 \in G$.¹ The numbers of vertices of the corner polyhedra $P(G_{n+1}, n)$, $n = 1, 2, \dots, 10$, are 1, 2, 3, 5, 7, 10, 16, 19, 31, 32, the sequence A300795 in the OEIS. Figure 6 exhibits the graph of this sequence.

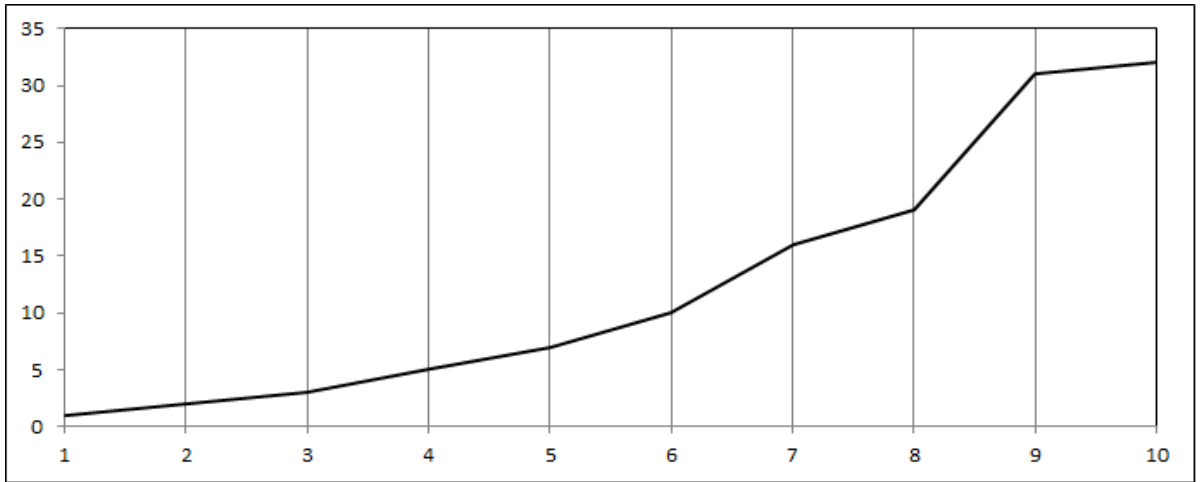


Fig. 6. Graph of the number of vertices function for the master corner polyhedron $P(G_{n+1}, n)$ on the cyclic group G_{n+1} , $1 \leq n \leq 10$.

We perceive this picture as an embryo of Figure 1. In our view, the tooth-shaped form of the $|\text{Vert } P(G_{n+1}, n)|$ graph is foreshadowed in this initial part. Some of the above features of $v(n)$ may also become visible when the sequence of numbers of vertices of $P(G_{n+1}, n)$ will be extended.

7 Concluding remarks

In order to study the number of vertices $v(n)$ of the polytope P_n of integer partitions of n we investigated the structure of the set of partitions that are not vertices. We divided this set to disjoint subsets $C_\xi(n)$ according to the minimum number ξ of partitions necessary to represent a partition $x \notin \text{Vert } P_n$ as their convex combination. Using the available numerical data, we demonstrated that $M_2(n)$, the set of Metropolis 2-partitions of n , constitutes a larger part of partitions that are not vertices of P_n . As a consequence, vertices of P_n form a small subset of partitions of n . We proved that an integer point in an arbitrary integral polytope P , which belongs to the subset of integer points in P analogous to $C_3(n)$, admits a convex representation via three integer points with all coefficients equal to $\frac{1}{3}$.

Thorough analysis of the computed values of $v(n)$ revealed intriguing properties of this function. Comparing this data with available numbers of knapsack and Metropolis 2-partitions moved us to suggest several conjectures that explain observed peculiarities. The

¹One extra point $t \in P(G_{11}, 10)$, with $t(5) = 1$, $t(9) = 3$ and all other $t(i) = 0$, indicated in [4] as a vertex was excluded in [13].

main conjecture claims that $v(n)$ depends on factorization of n . We presented visual but, in our view, convincing arguments in its favor. We showed that the graph of $v(n)$ is stratified into layers, the subgraphs corresponding to the classes of integers that are determined by factorization of n . The upper layer corresponds to prime numbers and the others correspond to collections of small divisors of n . Every prime divisor makes its own contribution to lowering the level of the layer. The smaller the divisor the more significant its effect.

We provided an embryonic argument in favor of a similar dependence for the number of vertices of the master corner polyhedron on the cyclic group. The master corner polyhedron is of key importance in integer programming, though, regrettably, after the Gomory's pioneering work [4] its vertices fell out of research for about 40 years in pursuit of efficient cuts for integer programs.

This work draws forth new questions. Formal proof and detailed study of the dependence of the number of vertices of P_n on factorization of n remain open problems for the future research. Further computation of $v(n)$ would be of great help. One of the most important problems is to find a combinatorial criterion for vertices of P_n . More specific problems are concerned with the nature of partitions in $C_\xi(n)$. Counting knapsack partitions does not look unworkable. This problem looks easier than enumerating vertices. Its solution will provide a rather good estimate for $v(n)$. We also hope that this work will give an impetus to further study of vertices of the corner polyhedron.

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References

- [1] G.E. Andrews, The Theory of Partitions, Addison Wesley, Reading, Mass., 1976.
- [2] C. Carathéodory, Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen. *Rend. Circ. Mat. Palermo* 32, 1911, 193-217.
- [3] R. Ehrenborg and M.A. Readdy, The Möbius function of partitions with restricted block sizes. *Adv. in Appl. Math.*, 2007, 39, 283-292.
- [4] R.E. Gomory, Some Polyhedra Related to Combinatorial Problems. *Linear Algebra Appl.*, 1969, 2(4), 451-558.
- [5] R.E. Gomory, The atoms of integer programming. *Ann. Oper. Res.*, 2007, 149(1), 99-102.
- [6] R.E. Gomory, Origin and early evolution of corner polyhedra. *European J. Oper. Res.*, 2016, 253(3), 543-556.
- [7] N. Metropolis and P.R. Stein, An elementary solution to a problem in restricted partitions. *J. of Combinatorial Theory*, 1970, 9, 365-376.

- [8] S. Onn and V.A. Shlyk, Some Efficiently solvable problems over integer partition polytopes. *Discrete Appl. Math.*, 2015, 180, 135-140.
- [9] B. Reznik, Lattice point simplices. *Discrete Math.*, 1986, 60, 219-242.
- [10] V.A. Shlyk, Polytopes of partitions of numbers. *European J. Combin.*, 2005, 26, 1139-1153.
- [11] V.A. Shlyk, On the vertices of the polytopes of partitions of numbers. *Dokl. Nats. Akad. Nauk Belarusi*, 2008, 52/3, 5-10 (in Russian).
- [12] V.A. Shlyk, Integer partitions from the polyhedral point of view. *Electron. Notes Discrete Math.*, 2013, 43, 319-327.
- [13] V.A. Shlyk, Master Corner Polyhedron: Vertices. *European J. Oper. Res.*, 2013, 226/2, 203-210.
- [14] V.A. Shlyk, Polyhedral approach to integer partitions, *J. Combin. Math. Combin. Comput.*, 2014, 89, 113-128.
- [15] Number of partitions of $2n$ with no element greater than n . <https://math.stackexchange.com/questions/96085/number-of-partitions-of-2n-with-no-element-greater-than-n>.
- [16] A.S. Vroublevski and V.A. Shlyk, Computing vertices of integer partition polytopes. *Informatics*, 2015, 4(48), 34-48 (in Russian).
- [17] E.W. Weisstein, Partition. From *MathWorld* – A Wolfram Web Resource. <http://mathworld.wolfram.com/Partition.html>.