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ON THE COMPLEXITY OF THE COGROWTH SEQUENCE

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Abstract. Given a finitely generated group with generating set S, we study the *cogrowth* sequence, which is the number of words of length n over the alphabet S that are equal to one. This is related to the probability of return for walks in a Cayley graph with steps from S. We prove that the cogrowth sequence is not P-recursive when G is an amenable group of superpolynomial growth, answering a question of Garrabant and Pak. In addition, we compute the cogrowth for certain infinite families of free products of finite groups and free groups, and prove that a gap theorem holds: if S is a finite symmetric generating set for a group G and if a_n denotes the number of words of length n over the alphabet S that are equal to 1 then either $\limsup_n a_n^{1/n} \leq 2$ or $\limsup_n a_n^{1/n} \geq 2\sqrt{2}$.

1. Introduction

One of the fundamental problems in the theory of finitely generated groups is the word problem, which asks whether there is an algorithm for determining when a word in a symmetric set of generators is equal to the identity. For many classes of groups the word problem is decidable (i.e., there exists a decision procedure for determining if a word on a set of generators is equal to the identity). This includes free groups, one-relator groups, polycyclic groups and fundamental groups of closed orientable two-manifolds of genus greater than one. On the other hand, there exist groups with unsolvable word problem, with the first such example being given by Novikov [27].

Algorithmically, given a finitely generated group G with a symmetric generating set $S = S^{-1}$, the word problem is to find the elements in the free monoid on S whose images in G are equal to the identity; this is a sublanguage of S^*, which we denote $\mathcal{L}(G; S)$. The complexity of this language gives a hint to the complexity of the word problem for G. From a computational point of view, there is a coarse hierarchy, formulated by Chomsky, which says that sublanguages of S^* can be divided into five nested classes: general languages (including those that lie outside of the realm of classical computation); recursively enumerable languages (those that are produced using Turing machines); context-sensitive languages (those produced using linear-bounded non-deterministic Turing machines); context-free languages (those produced using pushdown automata); and regular languages (those produced using finite-state automata).

Adopting this point of view, it is natural to ask which finitely generated groups G have the property that $\mathcal{L} := \mathcal{L}(G, S)$ lies in a given class of the Chomsky hierarchy. The answer is now understood for all classes except for context-sensitive. It is known that $\mathcal{L}$ is regular if and only if G is finite [5]; $\mathcal{L}$ is context-free if and only if G has a finite-index free subgroup (see Muller and Schupp [26] with a missing ingredient supplied by work of Dunwoody [10]); $\mathcal{L}$ is recursively enumerable if and only if G embeds in a finitely presented group [19]. As

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mentioned, groups with context-sensitive word problem have not been classified, but an important result in this direction is that finitely generated subgroups of automatic groups have context-sensitive word problem [32]. These classes provide a taxonomy of groups in terms of the complexity of their word problem.

Given a sublanguage $\mathcal{L}$ of a free monoid S^*, one can often also gain insight into $\mathcal{L}$ by looking at the generating function for the number of words in $\mathcal{L}$ of length n. Generating functions have their own parallel containment hierarchy. The simplest combinatorial power series are expansions of rational functions; they are contained in the algebraic power series (those that generate finite extensions of the field of rational functions); next there are D-finite series which satisfy a linear homogeneous differential equation with rational function coefficients; after this comes the differentiably algebraic, or ADE, series (series satsifying algebraic differential equations), which have the property that the function and all of its derivatives generate a finite transcendence degree extension of the field of rational functions; all of these are contained in the set of general power series.

Remarkably, these two hierachies correspond to some extent on the lower end. It is natural to expect that simpler languages (with simplicity being understood in terms of where the language fits into the Chomsky hierarchy) should have well behaved generating functions compared to more complex languages and this is generally the case, up until one hits context-sensitive languages. The simplest of these four classes in the Chomsky hierarchy is the collection of regular languages, which have rational generating series. Context-free languages have algebraic generating series by a theorem of Chomsky and Schützenberger [23, Chapter III]. At this point, the relationship between complexity of the language and complexity of the corresponding generating function breaks down. For example, there are context-sensitive languages with non differentiably algebraic generating function. At the same time, a standard example of a context-sensitive language, $\{a^n b^n c^n : n \in \mathbb{N}\}$, has a rational generating function.

Nonetheless, there are striking correspondences between finitely generated groups and the hierarchy of formal series. For a group G with generating set S we consider the ordinary generating function of the language $\mathcal{L}(G, S)$. In this setting, the number of words of length n in $\mathcal{L}(S, G)$ is called the *cogrowth* of G with respect to S, and we denote it $\mathrm{CL}(n; G, S)$. When G and S are understood, we will omit them and simply write $\mathrm{CL}(n)$. This is the *cogrowth sequence of* G, and its ordinary generating function, the series

$$F(t) := \sum \mathrm{CL}(n; G, S) t^n,$$

is the *cogrowth series of* G *with respect to the generating set* S.

Our first result concerns the cogrowth series for amenable groups (see §2 for a definition). Amenable groups are in some sense "small" and the class includes all solvable groups and groups of subexponential growth. On the other hand, groups containing a free subgroup of rank two are non-amenable. Our first main result is to show that the cogrowth series of an amenable group is never D-finite, except possibly when the group has polynomially bounded growth, answering a conjecture of Garrabant and Pak [16, Conjecture 13].

Theorem 1.1. *Let G be a finitely generated amenable group that is not nilpotent-by-finite and let S be a finite symmetric generating set for G. Then the cogrowth series*

$$\sum \mathrm{CL}(n; G, S) t^n$$

is not D-finite. Equivalently, if $p_{G,S}(n)$ denotes the probability of return, then this sequence is not P-recursive.

A celebrated theorem of Gromov says that finitely generated groups of polynomially bounded growth are precisely those groups that are virtually nilpotent. In particular, Theorem 1.1 says that amenable groups with superpolynomial growth do not have P-recursive cogrowth sequence. There are many examples of non-amenable groups with D-finite cogrowth generating series. Virtually free groups have algebraic cogrowth generating series [26] and a result of Elder, Janse van Rensburg, Rechnitzer, and Wong [12] shows that the cogrowth series for the Baumslag-Solitar group $BS(N,N)$, which is non-amenable, has D-finite cogrowth generating series. Elvey Price and Guttman [13] use sophisticated numerical techniques on the cogrowth sequence of Thompson's group F to demonstrate that the asymptotics are likely incompatible with the group being amenable.

We can combine some of these results. For finitely generated groups G with free subgroups of finite index (virtually free groups) the language $\mathcal{L}(G,S)$ is context free. It follows that the generating series

$$\sum \mathrm{CL}(n;G,S)t^n$$

is an algebraic power series. Using a combinatorial argument, we are able to directly determine a family of algebraic equations which yield the cogrowth series of a number of virtually free groups.

Theorem 1.2. *In the following examples, $F(t)$ denotes the cogrowth series for G with respect to S.*

(a) If $G = \langle x_1 \mid x_1^d\rangle \star \cdots \langle x_m \mid x_m^d\rangle \cong (\mathbb{Z}/d\mathbb{Z})^{\star m}$ and $S = \{x_1, \ldots, x_m\}$, then $F(t)$ is the unique solution to the equation

$$m^d t^d F(t)^d = (F(t)-1)(F(t)+m-1)^{d-1} \qquad \text{with } F(0)=1,$$

and $F(t)$ has radius of convergence $(d-1)^{(d-1)/d}/(d(m-1)^{1/d})$ for $d, m \geq 2$.

(b) If $G = \langle x \mid x^2\rangle \star \langle y\rangle \cong \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}$ and $S = \{x, y, y^{-1}\}$ then

$$F(t) = \frac{1}{2} \cdot \left(1 - 3\sqrt{1-8t^2}\right)/(1-9t^2),$$

and $F(t)$ has radius of convergence $1/2\sqrt{2}$.

(c) If $G = \langle x \mid x^2 = 1\rangle \star \langle y \mid y^n = 1\rangle \cong \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/n\mathbb{Z}$ and $S = \{x, y\}$ then

$$F(t) = (1-tD)/((1-tD)^2 - t^2),$$

where D is the unique power series solution to the equation

$$t^{n-1}(1-tD)^{n-1} = (1-tD-t^2)^{n-1}D$$

whose expansion begins $t^{n-1} + \text{higher degree terms}$. (We work out the solutions for some small values of n.)

The examples in Theorem 1.2 are worked out in §4. As a corollary of this theorem, we are able to prove a gap theorem of sorts.

Corollary 1.3. *Let G be a finitely generated group and let S be a symmetric generating set for G. Then if $\rho(G,S)$ denotes the radius of convergence of $\sum \mathrm{CL}(n;G,S)t^n$, then $\rho(G,S)^{-1} \in \{1,2\} \cup [2\sqrt{2},\infty)$.*

We suspect that all values in $[2\sqrt{2},\infty)$ can occur as the inverse of the radius of convergence of a cogrowth series, since the groups for which $2\sqrt{2}$ is realized as the inverse of a radius of convergence have uncountably many homomorphic images, although we have no evidence that all values in this interval can be realized; it is an interesting problem to determine the possible radii of convergence for a cogrowth generating series for a group with a symmetric generating set S.

The strategy behind the proof of Theorem 1.1 is to use the fact that if the cogrowth generating series is D-finite then it must be a G-function. Then, using work of Chudnovsky and Chudnovsky, Katz, and André [8], [21], [2] we get that the generating series must have a very specific expansion in an open set of the form $U \setminus L$, where U is an open neighbourhood of a singularity and L is a ray emanating from the singularity. This expression, combined with work of Kesten and Varopolous on the cogrowth of amenable groups of superpolynomial growth, is sufficient to prove our result.

Finite free products of finite groups and cyclic groups are virtually free, so there is a pushdown automaton that accepts the language of words on S equal to the identity. In theory, one can translate the automata description to a grammar description, and use this to find a system of equations. Kuksov [24] directly describes a recursive system which he solves to find generating series for some of the cases of Theorem 1.2 under the condition that one does not allow "doubling back" on the Cayley graph; that is, one does not allow a symbol x to appear immediately next to the symbol x^{-1} in the words. Nevertheless, it appears our systems resolve more cases than were previously known. Kuksov [24] also obtained the result of Theorem 1.2(a) in the case when $d = 2, 3$. These two counting sequences appear the Online Encyclopedia of Integer Sequences [33] as sequences A183135 and A265434. Alkauskas [1] worked out Theorem 1.2(c) in the case when $m = 2$ and $n = 3$, refining it to actually get the cubic equation for the cogrowth series (this is of special significance because this corresponds to the group $\mathrm{PSL}_2(\mathbb{Z})$). Corollary 4.3 was noticed as a curiosity coming from computations done in obtaining Theorem 1.2 and working out other examples.

The outline of this paper is as follows. In §2, we prove Theorem 1.1, including a more general, potentially more widely useful, criterion for showing certain power series are not D-finite. In §3, we give equations for computing the cogrowth of finite free groups and cyclic groups. We also prove the equations have a unique set of solutions in power series with a given initial condition and that the solutions are algebraic, although, as noted earlier, the algebraicity follows from the Chomsky-Schutznenberger theorem [7] and the work of Muller and Schupp [26]. In §4 we work out several examples, which are listed in the statement of Theorem 1.2; in §5 we prove the gap result for radii of convergence given in Corollary 4.3. Finally, in §6 we give some concluding remarks and pose some questions.

2. Cogrowth of amenable groups

2.1. **Amenable groups.** In this section we first recall what it means to be amenable, and then prove Theorem 1.1. Let G be a locally compact Hausdorff group. Then G is equipped with Haar measure and we let $L^\infty(G)$ be the essentially bounded measurable functions on G. Roughly speaking, amenable groups are groups for which the Banach-Tarski paradox does not occur when equipped with Haar measure. More formally, G is *amenable* if there is a linear functional $\Lambda \in \mathrm{Hom}(L^\infty(G), \mathbb{R})$ of norm 1 with the following properties. If $f \in L^\infty(g)$ is nonnegative almost everywhere then $\Lambda(f) \geq 0$ and $\Lambda(g \cdot f) = \Lambda(f)$ for $g \in L^\infty(G)$ and $g \in G$. The action of G on $L^\infty(G)$ is given by $g \cdot f(x) = f(g^{-1}x)$.

A remarkable result of Kesten [22] states that for finitely generated groups, amenability can be characterized in terms of cogrowth (or, equivalently, in terms of probability of return): a finitely generated group G with symmetric generating set S is amenable if and only if $\mathrm{CL}(n; G, S)^{1/n} \to |S|$ as $n \to \infty$, or equivalently, if the probability of return, $p_{G,S}(n)$, satisfies $p_{G,S}(n)^{1/n} \to 1$.

Pak and Garrabant [16, Conjecture 13] make the following conjecture: *"Let G be an amenable group of superpolynomial growth, and let S be a symmetric generating set. Then the probability of return sequence $p_{G,S}(n)$ is not P-recursive."* In addition, they prove their conjecture holds for the following classes of groups:

- virtually solvable groups of exponential growth with finite Prüfer rank;
- amenable linear groups of superpolynomial growth;
- groups of weakly exponential growth $A\exp(n^\alpha) < \gamma_{G,S}(n) < B\exp(n^\beta)$ with $A, B > 0$, and $0 < \alpha, \beta < 1$, where $\gamma_{G,S}(n)$ is the number of distinct elements of G that can be expressed as a product of n elements of S;
- the Baumslag-Solitar groups $BS(k, 1)$ with $k \geq 2$;
- the lamplighter groups $L(d, H) = \mathbb{Z}^d \wr H$, where H is a finite abelian group and $d \geq 1$.

To prove Theorem 1.1 we use basic results about singularities of D-finite power series. Observe that a cogrowth generating function has nonnegative integer coefficients and it has positive, finite radius of convergence; in particular, it is a G-function.

2.2. **The asymptotic growth of G-functions.** We recall some basic facts about G-functions. If F is a G-function then it is annihilated by a Fuchsian differential operator in the first Weyl algebra and consequently the singularities of F are all regular. Following work from Chudnovsky and Chudnovsky [8], Katz [21], André [2] (see André [3] for a discussion) we have that if ρ is a singularity of F then if L is a closed ray starting at ρ, then there is a simply connected open set U, containing 0 and ρ, such that F admits an analytic continuation on $U \setminus L$, and in some open neighbourhood of ρ intersected U we have an expression

$$F(z) = \sum_{i=1}^{s} \sum_{k=0}^{k_j} (\rho - z)^{\lambda_i} (\log(\rho - z))^k f_{i,k}(\rho - z),$$

with $\lambda_1, \ldots, \lambda_d$ rational numbers such that $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $i \neq j$ and such that $f_{i,k}(\rho - z)$ is analytic in a neighbourhood of $z = \rho$ and for each i there is some k such that $f_{i,k}(0) \neq 0$. Given such a decomposition of F, we define

$$\Lambda(F; \rho) = \sum_{j=1}^{s} \lambda_j. \tag{2.1}$$

In order to better understand the situation, we first focus on the case where there is only a single λ_i in the expression for F. This is the content of the following lemma.

Lemma 2.1. *Let λ be a positive rational number and let ρ be a complex number. Suppose that in some open set of the form $U \setminus L$ with U and open neighbourhood of $z = \rho$ and L a* *ray emanating from $z = \rho$, we have*

$$G(z) = \sum_{k=0}^{\ell} (\rho - z)^{\lambda} (\log(\rho - z))^k f_k(\rho - z),$$

where each f_k is analytic at $z = \rho$ and $f_k(0) \neq 0$ for some k if $\lambda \notin \mathbb{Z}_{\geq 0}$ and $f_0(\rho - z) = 0$ if $\lambda \in \mathbb{Z}_{\geq 0}$. Then $\Lambda(G'; \rho) = \Lambda(G; \rho) - 1$. Moreover, if λ is a positive integer then we can write $G' = -(\rho - z)^{\lambda-1} f_1(\rho - z) + H(z)$, where

$$H(z) = \sum_{k=1}^{\ell} (\rho - z)^{\lambda-1} (\log(\rho - z))^k h_k(\rho - z),$$

and each $h_k(\rho - z)$ analytic in a neighbourhood of $z = \rho$ and $h_k(0) \neq 0$ for some $k \geq 1$ (i.e., $\Lambda(H) = \lambda - 1$); if $\lambda = 0$ then we can write $G' = -(\rho - z)^{-1} f_1(\rho - z) + H(z)$, where either $f_1(0) = 0$ and $\Lambda(H) = -1$ or $f_1(0) \neq 0$.

Proof. First suppose that $\lambda \notin \mathbb{Z}_{\geq 0}$. We have

$$\begin{aligned} G'(z) &= -\sum_{k=0}^{\ell} \lambda (\rho - z)^{\lambda-1} (\log(\rho - z))^k f_k(\rho - z) \\ &- \sum_{k=0}^{\ell} k (\rho - z)^{\lambda-1} \log(\rho - z)^{k-1} f_k(\rho - z) \\ &- \sum_{k=0}^{\ell} (\rho - z)^{\lambda} (\log(\rho - z))^k f_k'(\rho - z). \end{aligned}$$

Then we immediately see that $\Lambda(G') \geq \lambda - 1$. By assumption, $f_k(0) \neq 0$ for some k, so pick the largest k for which it is nonzero. Then the coefficient of $(\rho - z)^{\lambda-1} (\log(\rho - z))^k$ in $G'(z)$ is $-\lambda f_k(\rho - z) - (k+1) f_{k+1}(\rho - z) - (\rho - z) f_k'(\rho - z)$, where we take $f_{\ell+1}(z) = 0$. Then we see that this coefficient is equal to $-\lambda f_k(0) \neq 0$ at $z = \rho$ and so we are done.

Next assume that λ is a nonnegative integer. Then as before we get $\Lambda(G') \geq \Lambda(G) - 1$. Now $G'(z) = -(\rho - z)^{\lambda-1} f_1(\rho - z) + H(z)$, where $H(z)$ is of the form

$$\sum_{k=1}^{\ell} (\rho - z)^{\lambda-1} (\log(\rho - z))^k h_k(\rho - z)$$

with h_k analytic in a neighbourhood of $z = \rho$. Now there are two cases. If $\lambda > 0$ then there is some largest k for which $f_k(0) \neq 0$ and by assumption $k > 0$. Then the coefficient of $(\rho - z)^{\lambda-1} (\log(\rho - z))^k$ in G' is equal to $-\lambda f_k(\rho - z) - (k+1) f_{k+1}(\rho - z) - (\rho - z) f_k'(\rho - z)$, which is nonzero at $z = \rho$ and hence $h_k(0) \neq 0$ for some $k > 0$. If $\lambda = 0$ then we again have $G'(z) = -(\rho - z)^{\lambda-1} f_1(\rho - z) + H(z)$, where $H(z)$ is of the form

$$\sum_{k=1}^{\ell} (\rho - z)^{\lambda-1} (\log(\rho - z))^k h_k(\rho - z)$$

with h_k analytic in a neighbourhood of $z = \rho$. If $f_1(0) \neq 0$ then there is nothing to do. If $f_1(0) = 0$ then $f_k(0) \neq 0$ for some $k > 1$ and we pick the largest such $k > 1$ for which this holds. Then the coefficient of $(\rho - z)^{\lambda-1} (\log(\rho - z))^{k-1}$ in G' is equal to

$-kf_k(\rho - z) - (\rho - z)f'_{k-1}(\rho - z)$, which is nonzero at $z = \rho$. Thus $h_{k-1} \neq 0$ in this case. The result now follows. $\square$

Proposition 2.2. *Let ρ be a positive real number, let $s \geq 1$ and let $\lambda_1, \ldots, \lambda_s$ be distinct rational numbers with $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $i \neq j$. Suppose that*

$$F(z) = \sum_{i=1}^{s} \sum_{k=0}^{k_j} (\rho - z)^{\lambda_i} (\log(\rho - z))^k f_{i,k}(\rho - z),$$

in an open set of the form $U \setminus L$, where U is an open neighbourhood of $z = \rho$ and L is a closed ray emanating from $z = \rho$ that does not pass through the origin, where the $f_{i,k}$ are analytic in a neighbourhood of $z = \rho$. Suppose in addition that $f_{i,0}(\rho - z) = 0$ if λ_i is a nonnegative integer and that for each j there is some $k \in \{0, \ldots, k_j\}$ such that $f_{j,k}(0) \neq 0$. Then there is some $j \geq 0$ such that $\limsup_{t \to \rho^-} |F^{(j)}(t)| \to \infty$.

Proof. We may assume without loss of generality that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_s.$$

First suppose that $\lambda_1 < 0$ and that $\limsup_{t \to \rho^-} |F(t)| \not\to \infty$. Then $F(t)(\rho - t)^{-\lambda_1} \to 0$ as $t \to \rho^-$. But

$$F(t)(\rho - t)^{-\lambda_1} = \sum_{k=0}^{k_1} (\log(\rho - t))^k f_{1,k}(\rho - t) + \sum_{i=2}^{s} \sum_{k=0}^{k_i} (\rho - t)^{\lambda_i - \lambda_1} \log(\rho - t)^k f_{i,k}(\rho - t).$$

Now $\lambda_i - \lambda_1 > 0$ for $i = 2, \ldots, s$ and each $f_{i,k}$ is analytic in a neighbourhood of $z = \rho$ and so

$$(\rho - t)^{\lambda_i - \lambda_1} \log(\rho - t)^k f_{i,k}(\rho - t) \to 0$$

as $t \to \rho^-$ In particular, since $F(t)(\rho - t)^{-\lambda_1} \to 0$ as $t \to \rho^-$, we see that

$$\sum_{k=0}^{k_1} (\log(\rho - t))^k f_{1,k}(\rho - t) \to 0$$

as $t \to \rho^-$.

Now let $c_k = f_{1,k}(0)$. Then by assumption there is some k such that $c_k \neq 0$. Moreover, by L'Hôpital's rule we have $(\log(\rho - t))^k (f_{1,k}(\rho - t) - c_k) \to 0$ as $t \to \rho^-$ for all k. Hence the fact that

$$\sum_{k=0}^{k_1} (\log(\rho - t))^k f_{1,k}(\rho - t) \to 0$$

as $t \to \rho^-$ gives that

$$\sum_{k=0}^{k_1} c_k (\log(\rho - t))^k \to 0$$

as $t \to \rho^-$. Let $t = \rho - 1/n$. Then we have

$$\sum_{k=0}^{\infty} c_k (-1)^k (\log n)^k \to 0$$

as $n \to \infty$, which is impossible since $\log n \to \infty$ as $n \to \infty$ and $c_k \neq 0$ for some k.

So we may assume that $\lambda_1 \geq 0$. In particular, we may assume that $\Lambda(F)$ is a nonnegative rational number. So suppose that the conclusion to the statement of the lemma does not hold and let $\alpha \geq 0$ denote the infimum of all $\Lambda(F)$ such that F is of the form in the statement of the lemma and such that the conclusion to the statement of the lemma does not hold for F. Then we may pick an F for which the conclusion doesn't hold with $\Lambda(F) < \alpha + 1/2$. Then by Lemma 2.1 we have $\Lambda(F') = \Lambda(F) - s$ and moreover in the case that some λ_i is a nonegative integer we have that there is some function $g(\rho - z)$ that is analytic in a neighbourhood of $z = \rho$ such that $F'(z) - g(\rho - z)$ is of the form in the statement of the lemma and $\Lambda(F'(z)-g(\rho-z)) = \alpha - s$. Since $s \geq 1$, we see that $\Lambda(F'(z)-g(\rho-z)) < \alpha$ and so by our definition of α, there is some j such that $F^{(j+1)}(z)-g^{(j)}(\rho-z) = (F'(z)-g(\rho-z))^{(j)}$ has the property that $\limsup_{t\to\rho^-} |F^{(j+1)}(t) - g^{(j)}(\rho-t)| = \infty$. But $g(\rho - z)$ is analytic in a neighbourhood of $z = \rho$, so we then have $\limsup_{t\to\rho^-} |F^{(j+1)}(t)| = \infty$, a contradiction. The result follows. ☐

2.3. **A general non-D-finiteness criterion.** As a corollary, we have a criterion for non-D-finiteness, which is applicable to many series coming from combinatorics.

Corollary 2.3. *Let $F(t) \in \mathbb{Z}[[t]]$ be a power series with nonnegative integer coefficients and suppose that $F(t)$ has radius of convergence $\rho > 0$. If $F^{(j)}(t)$ absolutely converges on the closed disc of radius ρ for every $j \geq 1$ then F is not D-finite.*

Proof. Suppose that F is D-finite. Then F is a G-function and as a result there is an open neighbourhood U of ρ and a closed ray L emanating from ρ that doesn't pass through the origin such that on U we have

$$F(z) = \sum_{j=1}^{s}\sum_{k=0}^{k_j}(\rho - z)^{\lambda_i}(\log(\rho - z))^k f_{i,k}(\rho - z),$$

where $s \geq 1$ and $\lambda_1, \ldots, \lambda_s$ are distinct rational numbers with $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $i \neq j$, and the $f_{i,k}(\rho - z)$ are analytic in a neighbourhood of $z = \rho$. Moreover, we may assume that for each j there is some $k \in \{0, \ldots, k_j\}$ such that $f_{j,k}(0) \neq 0$. Now if $F^{(j)}(t)$ absolutely converges on the closed disc of radius ρ for every $j \geq 1$, then if there is some i such that λ_i is a nonnegative integer, we let $G(z) := F(z) - (\rho - z)^{\lambda_i} f_{i,0}(\rho - z)$ and otherwise we let $G(z) = F(z)$. If λ_i is a nonnegative integer, by modifying it by an integer, we may assume that $f_{i,k}(0) \neq 0$ for some $k > 0$. By construction the function G has the property that all of its derivatives converge absolutely at $z = \rho$ and is such that $\limsup_{t\to\rho^-} |G^{(j)}(t)| = |G^{(j)}(\rho)|$. But by Proposition 2.2 we have that there is some $j \geq 0$ such that $\limsup_{t\to\rho^-} |G^{(j)}(t)| \to \infty$, which is a contradiction. It follows that F is not D-finite and the theorem is established. ☐

2.4. **When is the growth series of an amenable group not D-finite?** To obtain the proof of Theorem 1.1, we need a remark.

Remark 2.4. Let G be a finitely generated group that is generated by a finite symmetric generating set S, and let p be a positive integer. If $\mathrm{CL}(n; G, S) \geq |S|^n/n^p$ for infinitely many n, then G is virtually nilpotent.

Proof. Let $V(n)$ denote the number of distinct elements of G that can be expressed as a product of at most n elements of S. A theorem of Varopolous (see [35], [29, Theorem 2] shows that if $V(n) \geq cn^d$ for all n then we have that $p_{G,S}(n) \leq C_1 n^{-d/2}$ and so $\mathrm{CL}(n; G, S) \leq$

$C_1|S|^n/n^{d/2}$ for all n. $\mathrm{CL}(n;G,S) \geq |S|^n/n^p$ for infinitely many n then $V(n) > Cn^{2p+1}$ for all n then $\mathrm{CL}(n;G,S) \geq c|S|^n/n^{p+1/2}$ for all n and hence if $\mathrm{CL}(n;G,S) \geq |S|^n/n^p$ for infinitely many n then $\mathrm{V}(n) < n^{2p+2}$ for infinitely many n. A strengthening of Gromov's theorem due to Wilkie and van den Dries [9] shows that G must be virtually nilpotent. $\square$

Theorem 2.5. *Let G be a finitely generated amenable group that is generated by a finite symmetric subset S. If G is not virtually nilpotent then the cogrowth series of G with respect to S is not D-finite.*

Proof. Let
$$F(t) = \sum_{n\geq 0} \mathrm{CL}(n;G,S)t^n.$$
Then $F(t)$ has integer coefficients and by Kesten's criterion [22] we have that
$$\mathrm{CL}(n,G,S)^{1/n} \to |S|.$$
In particular, the radius of convergence of F is $\rho := 1/|S|$, F is analytic inside the disc of radius ρ, and F is a G-function. By Pringsheim's theorem, F has a singularity at $t = \rho$. We may assume that $\rho = 1$. Then by Proposition 2.2 we see that there is some $p \geq 0$ such that $\lim_{t\to\rho^-} F^{(p)}(t) \to \infty$. But by the result of Varopoulous mentioned above we have that $\mathrm{CL}(n;G,S) < |S|^n/n^{p+2}$ for all sufficiently large n. In particular, if a_n is the coefficient of z^n in $z^pF^{(p)}(z) = \sum_{n\geq p} n(n-1)\cdots(n-p+1)\mathrm{CL}(n,G,S)z^n$ then
$$a_n = n(n-1)\cdots(n-p+1)\mathrm{CL}(n,G,S) < |S|^n/n^2$$
for n sufficiently large and hence $z^pF^{(p)}(z)$ is absolutely convergent on the $\{z\colon |z| = \rho\}$. This is a contradiction. $\square$

3. Free products of finite groups and free groups

Computing the cogrowth of free products of free groups has been done in a number of cases [1], [24], [25]. We note that Kuksov's work is the most general, but he computes cogrowth using an altered definition. In particular, he only counts *reduced words* in the generating set S that are equal to 1; that is, if $x, x^{-1} \in S$ then he does not allow x to immediately follow x^{-1} or x^{-1} to immediately follow x in the words he considers. We prove an analogue of his result that allows "doubling back" on the Cayley graph. We give an explicit algebraic system satisfied by the generating function. Alkauskas [1] does a computation for $\mathrm{PSL}_2(\mathbb{Z})$, which is a free product of a cyclic group of order 2 and a cyclic group of order 3, using a non-symmetric generating set of size 2.

Throughout this section, we fix the following notation:

(1) We let m be a positive integer and we let $G_1, \ldots, G_m$ be groups;
(2) We let $S_i \subseteq G_i$ be a generating set for G_i for $i = 1, \ldots, m$;
(3) We let $S = \cup S_i \subseteq G_1 \star \cdots \star G_m$ be a generating set for the free product of $G_1, \ldots, G_m$, where we identify G_i with its image in the free product under the canonical inclusion when forming S;
(4) For each $i \in \{1, \ldots, m\}$, $g \in G_i$, and $X \subseteq G_i$, we let $F_{g,X}(t)$ denote the power series in which the coefficient of t^n is the number of words $s_1 \cdots s_n$ of length n over the alphabet S such that $s_1 \cdots s_n = g$ and such that for $1 \leq i < n$ we have $s_1 \cdots s_i \notin X$. In this case, we say that all proper prefixes of $s_1 \cdots s_n$ *avoid* X.

Lemma 3.1. *Adopt the notation above. Then for $i \in \{1, \ldots, m\}$, $g \in G_i$, and $X \subseteq G_i$, we have the following relations:*

(I) $F_{g,X}(t) = \delta_{g,1} + F_{1,X}(t)(F_{g,X\cup\{1\}}(t) - \delta_{g,1})$ *if* $1 \notin X$*;*

(II) $F_{g,X}(t) = \chi(g \in S_i)t + \sum_{s\in S_i\setminus X} tF_{s^{-1}g,s^{-1}X}(t)$ *if* $g \neq 1$ *and* $1 \in X$*;*

(III) $F_{1,X}(t) = 1 + \sum_{j\neq i} \sum_{s\in S_j} tF_{s^{-1},\{s^{-1}\}}(t) + \sum_{s\in S_i\setminus X} tF_{s^{-1},s^{-1}X}(t)$ *if* $1 \in X$*.*

Proof. Suppose first that $g \in G_i \setminus \{1\}$ for some $i \in \{1, \ldots, m\}$ and that $1 \in X$. Then if $s_1 \cdots s_n = g$ and $s_1 \in G_j \setminus \{1\}$ with $j \neq i$ then if $s_1 \cdots s_n = g$ then it is straightforward to deduce that for free products of finite groups give that some prefix of $s_1 \cdots s_n$ must be equal to 1. Hence every word of length n starting with $s_1 \notin G_i$ that is equal to g must pass through X. Thus if $s_1 \cdots s_n = g$ and every prefix avoids X then s_1 must be in G_i. Moreover if $n > 1$ then $s_1 \notin X$. Then $s_2 \cdots s_n = s_1^{-1}g$ and every prefix of $s_2 \cdots s_n$ avoids $s_1^{-1}X$. Moreover, if $s_2 \cdots s_n = s_1^{-1}g$ and every prefix of $s_2 \cdots s_n$ avoids $s_1^{-1}X$ then appending s_1 at the beginning gives a word of length n that is equal to g and such that every prefix avoids X. Hence we see that $F_{g,X}(t) = \chi(g \in S_i)t + \sum_{s\in S_i\setminus X} tF_{s^{-1}g,s^{-1}X}(t)$.

Next, if $g \in G_i$ for some $i \in \{1, \ldots, s\}$ and $1 \notin X$ then if $s_1 \cdots s_n = g$ then we let $i < n$ denote the largest index such that $s_1 \cdots s_i = 1$. (We note that $i = 0$ is possible.) In this case we have a decomposition of $s_1 \cdots s_n$ into a product ab with $a = s_1 \cdots s_i$ being equal to 1 and such that every prefix of a avoids X and the word $b = s_{i+1} \cdots s_n$, which is equal to g and such that every prefix avoids X and also 1. Thus we get

$$F_{g,X}(t) = F_{1,X}(t)F_{g,X\cup\{1\}}(t).$$

Finally, if $X \subseteq G_i$ and $1 \in X$ then for $s \in S$, if we count words $s_1 \cdots s_n$ that are equal to 1 and such that $s_1 = s \notin X$ and such that every prefix avoids X then for $n \geq 1$ there is a bijection between the collection of words of length n and words $s_2 \cdots s_n$ of length $n-1$ that are equal to s_1^{-1} and such that every prefix avoids $s_1^{-1}X$. Then

$$F_{1,X}(t) = 1 + \sum_{j=1}^{m} \sum_{s\in S_j\setminus X} tF_{s^{-1},s^{-1}X}(t).$$

□

Lemma 3.2. *Let $G_1, \ldots, G_m$ be finite groups and $S_1, \ldots, S_m$ be generating subsets for $G_1, \ldots, G_m$ respectively. There is a unique solution $(F_{g,X}(t))_{i\leq m, g\in G_i, X\subseteq G_i}$ to the finite system of equations in Lemma 3.1 with $F_{g,X}(t) = \sum_{n\geq 0} a_{g,X}(n)t^n \in \mathbb{Q}[[x]]$ and $F_{g,X}(0) = \delta_{g,1}$.*

Proof. For each $g \in G_i, X \subseteq G_i$ we create a variable $z_{g,X}$. Consider the ring $R := \mathbb{Q}[t, z_{g,X} \colon i \leq m, g \in G_i, X \subseteq G_i]/I$, where I is the ideal generated by the relations:

(i) $z_{g,X} = \chi(g \in S_i)t + \sum_{s\in S_i\setminus X} tz_{s^{-1}g,s^{-1}X}$ for $g \in G_i \setminus \{1\}$, $1 \in X \subseteq G_i$;
(ii) $z_{g,X} = z_{1,X}z_{g,X\cup\{1\}}$ for $g \in G_i$, $1 \notin X$;
(iii) $z_{1,X} = 1 + \sum_{j\neq i} \sum_{s\in S_j} tz_{s^{-1},\{s^{-1}\}} + \sum_{s\in S_i\setminus X} tz_{s^{-1},s^{-1}X}$ for $1 \notin X \subseteq G_i$.

Then observe that if we let J denote the ideal of $\mathbb{Q}[t, z_{g,X} \colon i \leq m, g \in G_i, X \subseteq G_i]$ generated by I and t then items (i) shows that $z_{g,X} \in J$ for $g \neq 1$ and $1 \in X$. Then item (ii) gives $z_{g,X} \in J$ for $g \neq 1$ and $1 \notin X$. Finally, item (iii) shows that $z_{1,X} - 1 \in J$ for $1 \notin X$ and item (ii) gives $z_{1,X} - 1 \in J$ for $1 \in X$. Thus $\mathbb{Q}[t, z_{g,X} : i \leq m, g \in G_i, X \subseteq G_i]/J \cong \mathbb{Q}$ and thus $R/(\bar{t}) = \mathbb{Q}$. It follows from Krull's principal ideal theorem [11, Theorem 10.1] that R has Krull dimension at most one and since the image of t in R is not algebraic (this follows since we know there is a solution to the system when we localize at $\mathbb{Q}[t] \setminus \{0\}$), we then see that there is a non-trivial polynomial relation between t and $F_{g,X}(t)$ for each $g \in G_i$, $X \subseteq G_i$. Hence $F_{g,X}(t)$ is algebraic. Now R has Krull dimension one and the image of J in R is a height one prime ideal with $R/J \cong \mathbb{Q}$. If we localize at R at J we obtain a local ring and since the image of J in R is generated by the image of t, we see that J is principal and hence we obtain a regular local ring. Then if we complete this local ring at the ideal J, by Cohen's Structure theorem we obtain $\mathbb{Q}[[t]]$ and since our system has a unique solution mod J a straightforward argument using Hensel's lemma shows we have a unique solution in the power series ring subject to the initial conditions coming from looking at the solution mod J.

□

Remark 3.3. Although the system in Lemma 3.1 is not a priori finite when the groups are not finite, one can easily adapt this construction to handle the case where some of the G_i are allowed to be infinite cyclic groups and $S_i = \{x, x^{-1}\}$ with x a generator for G_i. The reason for this is that if a word $s_1 \cdots s_n$ has some proper prefix equal to x^i with $i > 0$ then it has a proper prefix equal to x^j for $0 < j < i$; similarly, if it has a proper prefix equal to x^i with $i < 0$ then it has a proper prefix equal to x^j for $j < 0$ with $j > i$. Thus we only need to consider X with at most three elements (potentially one positive exponent, one negative exponent, and the identity), since if x^i and x^j are in X with $i > j > 0$ then $F_{g,X} = F_{g,X\setminus\{x^i\}}$ and an analogous result holds for negative exponents. Also, if $i > j > 0$ then and $x^j \in X$ then $F_{x^i,X} = 0$ and similarly in the negative case. Thus if one uses these facts and looks at the dependency tree that arises when looking at equations from Lemma, then we see that in the case that G_i is an infinite cyclic group with generator x, we only need to consider $F_{g,X}$ for $g \in G_i$ and $X \subseteq G$ with $g \in \{x^{-1}, 1, x\}$ and $X \subseteq \{x^{-1}, 1, x\}$. See Example 4.5 for a case where this is implemented.

4. Examples

For the following examples, we generated the system given by the equations, and applied simplifications. The solvable examples generally posess an exploitable symmetry, so although in theory one might have to manipulate $\sum_{i=1}^{s} |G_i| 2^{|G_i|}$ equations, in practice there are far fewer. We incrementally eliminated the variables[1] to determine the algebraic equation satisfied by $F_{1,\emptyset}$. When listed, the OEIS numbers refer to the Online Encyclopedia of Integer Sequences [33]. Table 1 summarizes the sequences.

We start with a straightforward infinite family.

Example 4.1. Let $d, m \geq 2$ and let $G = (\mathbb{Z}/d\mathbb{Z})^{\star m} = \langle x_1 \mid x_1^d = 1\rangle \star \cdots \star \langle x_m \mid x_m^d = 1\rangle$ and let $S = \{x_1, \ldots, x_m\}$. If $Z(t)$ is the generating series for $\mathrm{CL}(n; G, S)$ is the unique power series satisfying

$$m^d t^d Z^d = (Z-1)(Z+m-1)^{d-1}$$

with initial condition $Z(0) = 1$.

[1] Specifically, we have used the eliminate command of Maple 2018.

Proof. It is straightforward to solve the system from Lemma 3.1 with an additional remark. We let

$$A := F_{1,\{x_1\}}(t), B = F_{1,\{1,x_1\}}(t), C = F_{x_1^{-1},\{x_1^{-1}\}}(t). \tag{4.2}$$

Notice that a word $s_1 \cdots s_n \in S^n$ that is equal to x_1^{-1} and has no proper prefix equal to x_1^{-1} can be decomposed uniquely in the form $w_1 x_1 w_2 x_1 \cdots w_{d-1} x_1$, where each w_i is a word in $s_1, \ldots, s_m$ that is equal to 1 with no proper prefix equal to x_1. This then gives us the equation

$$t^{d-1} A^{d-1} = C. \tag{4.3}$$

Lemma 3.1(I) gives us that $A = 1 + A(B-1)$; that is, $A = 1/(2-B)$. Finally, remark that the symmetry of the groups means we can rewrite Lemma 3.1(III) into

$$B = 1 + (m-1)tC. \tag{4.4}$$

Using the fact that $A = 1/(2-B)$ then gives that $A = (1-(m-1)tC)^{-1}$ and so Equation (4.3) yields

$$t^{d-1} = C(1-(m-1)tC)^{d-1}. \tag{4.5}$$

Let $W = F_{1,\{1\}}(t)$ and $Z = F_{1,\emptyset}(t)$. Then Z is the generating series for the cogrowth. Lemma 3.1(I) gives

$$Z = 1 + Z(W-1) \tag{4.6}$$

and Lemma 3.1(III), again using symmetry, gives

$$W = 1 + mtC. \tag{4.7}$$

Using Equations (4.6) and (4.7) that $Z = (1-mtC)^{-1}$, so substituting $C = (Z-1)/(mtZ)$ into Equation 4.5 gives

$$mt^d Z = (Z-1)(1-(m-1)(Z-1)/mZ)^{d-1},$$

or equivalently

$$m^d t^d Z^d = (Z-1)(Z+m-1)^{d-1},$$

as claimed. To see uniqueness of the solution once we impose the initial condition $Z(0) = 1$, note that if there is a unique polynomial solution of degree $n-1$ to this equation mod (t^n) for $n \geq 1$ then we get a unique polynomial solution of degree n to this equation mod (t^{n+1}) by Hensel's lemma and so by induction there is a unique power series solution with this initial condition. $\square$

To determine the equation, in fact we did the computation for d, $d \leq 9$, and symbolic m and we were able to guess the general form using the Maple package `gfun` [30] to guess the general form of the algebraic equation satisfied by the cogrowth. This then suggested a way of proving this fact. Given this equation, we can determine dominant singularity of the cogrowth generating function.

Lemma 4.2. *Let $\beta > 0$. Then if the polynomial*

$$m^d \beta^d Z^d - (Z-1)(Z+m-1)^{d-1}$$

has a repeated root, it must be that $\beta = (d-1)^{(d-1)/d}/(d(m-1)^{1/d})$.

Proof. Let

$$P(z) = m^d\beta^d Z^d - (Z-1)(Z+m-1)^{d-1}.$$

Suppose that $P(z)$ has a repeated root. Then the system $P(z) = P'(z) = 0$ has a solution. Notice that

$$P'(z) = 0 \iff dm^d\beta^d Z^{d-1} = (Z+m-1)^{d-1} + (d-1)(Z-1)(Z+m-1)^{d-2}.$$

It is easy to see that if $P'(z) = 0$ then $Z \neq 0, 1, -m+1$. If $P(Z) = 0$ then we have $m^d\beta^d Z^d = (Z-1)(Z+m-1)^{d-1}$, so dividing the equation

$$dm^d\beta^d Z^{d-1} = (Z+m-1)^{d-1} + (d-1)(Z-1)(Z+m-1)^{d-2}$$

on the left by $m^d\beta^d Z^d$ and on the right by $(Z-1)(Z+m-1)^{d-1}$, we see that if $P(Z)$ and $P'(Z)$ are both equal to zero, then we must have

$$d/Z = 1/(Z-1) + (d-1)/(Z+m-1).$$

Multiplying by $Z(Z-1)(Z+m-1)$ then gives the equation

$$d(Z-1)(Z+m-1) = Z(Z+m-1) + (d-1)Z(Z-1),$$

which has the unique solution $Z = (m-1)d/(dm-d-m)$. So now we substitute $Z = (m-1)d/(dm-d-m)$ into $P(Z)$ and we get that

$$m^d\beta^d(m-1)^d/(d-2)^d - (m-d+1)/(d-2) \cdot (m-1)^{d-1}(d-1)^{d-1}/(d-2)^{d-1},$$

which has the solution

$$\beta^d = (d-1)^{d-1}/((m-1)d^d),$$

which has the unique positive solution

$$\beta = (d-1)^{(d-1)/d}/(d(m-1)^{1/d}),$$

and we have shown that $P(Z)$ has a root at this β at

$$Z = (m-1)d/(dm-d-m),$$

and we also get that $P'(Z) = 0$ at this β for $Z = (m-1)d/(dm-d-m)$, and so the result follows. $\square$

Corollary 4.3. *The radius of convergence of the cogrowth generating function for* $G = \langle x_1 \mid x_1^d = 1\rangle \cdots \langle x_m \mid x_m^d = 1\rangle$ *with respect to* $S = \{x_1, \ldots, x_m\}$ *is* $(d-1)^{(d-1)/d}/(d(m-1)^{1/d})$.

Proof. The singularities of an algebraic power series $F(t)$ satisfying a polynomial equation $P(t, F(t)) = 0$ for some polynomial $P(t, X) \in \mathbb{C}[t, X]$ are in the set T, where T is the set of zeros of the leading coefficient of $P(t, X)$ as a polynomial in X and the zeros of the discriminant of $P(t, X)$ with respect to X (see Flajolet and Sedgewick [15, §7.36]). In the case that $F(t)$ is the cogrowth generating function of G with respect to X, we have $F(t)$ is a root of $P(t, X)$, where

$$P(t, X) = m^d t^d X^d - (X-1)(X+m-1)^{d-1},$$

which has leading coefficient $m^d t^d - 1$. We showed that $P(t, X)$ can only have repeated roots for $t \geq 0$ at $t = (d-1)^{(d-1)/d}/(d(m-1)^{1/d})$. Thus the only positive singularities of $F(t)$ are in $T \cap (0, \infty) = \{(d-1)^{(d-1)/d}/(d(m-1)^{1/d}), 1/m\}$. Since $F(t)$ has nonnegative coefficients, it has a singularity at $t = \rho$, where $\rho > 0$ is the radius of convergence. If $d, m \geq 2$ and

$(d, m) \neq (2, 2)$, then G is nonamenable, but we cannot invoke Kesten's criterion to show the radius of convergence is $> 1/m$ since the generating set is not symmetric. A strengthening of Kesten's criterion due to Gray and Kambites [17, Corollary 6.6] applies, however, and we get that the radius of convergence is less than $1/|S| = 1/m$. Thus the radius of convergence is $(d-1)^{(d-1)/d}/(d(m-1)^{1/d})$ for $(d, m) \neq (2, 2)$. When $d = m = 2$, G is amenable and $(d-1)^{(d-1)/d}/(d(m-1)^{1/d}) = 1/m$, so the result follows. □

Remark 4.4. Kuksov [24] did this computation when $d = 2$ and $d = 3$, but did not do the general case. The case $d = 2$ is classical, as it can be interpreted in terms of rooted closed walks of length $2n$ on the infinite rooted m-ary tree. The cases $d = 2$ and $d = 3$ appear in the OEIS as entries A126869 and A265434, respectively.

Example 4.5. Let $G = \langle x \mid x^2 = 1\rangle \star \langle y\rangle$ and let $S = \{x, y, y^{-1}\}$. Then the cogrowth series for G with respect to S is equal to

$$Z = \frac{1}{2} \cdot \left(1 - 3\sqrt{1 - 8t^2}\right) / (1 - 9t^2).$$

Proof. Using the fact that $F_{y,Y} = F_{y^{-1},Y^{-1}}$, which follows from the obvious symmetry and using Lemma 3.1(I)–(III) along with Remark 3.3 about how to apply them in the infinite cyclic case, we get the equations:

(1) $F_{x,\{1,x\}}(t) = t$;
(2) $F_{x,\{x\}}(t) = tF_{1,\{x\}}(t)$;
(3) $F_{1,\{x\}}(t) = 1 + F_{1,\{x\}}(t)(F_{1,\{1,x\}}(t) - 1)$;
(4) $F_{1,\{1,x\}} = 1 + 2tF_{y,y}$;
(5) $F_{1,\{y\}}(t) = 1 + F_{1,\{y\}}(t)(F_{1,\{1,y\}}(t) - 1)$;
(6) $F_{y,\{y\}}(t) = F_{1,\{y\}}(t)F_{y,\{1,y\}}(t)$;
(7) $F_{y,\{1,y\}}(t) = t$;
(8) $F_{1,\{1,y\}}(t) = 1 + tF_{x,\{x\}}(t) + tF_{y,\{y\}}(t)$;
(9) $F_{1,\{1\}}(t) = 1 + tF_{x,\{x\}} + 2tF_{y,\{y\}}(t)$;
(10) $F_{1,\emptyset}(t) = 1 + F_{1,\emptyset}(t)(F_{1,\{1\}}(t) - 1)$.

□

Then we solve this equation using Maple and find that $Z = F_{1,\emptyset}(t)$ satisfies the polynomial equation

$$(3t-1)(3t+1)(t-1)(t+1)Z^3 + (-10t^2 + 2)Z^2 + (2t^2 - 1)Z - 2 = 0,$$

which factors as

$$((9t^2 - 1)Z^2 - Z + 2)((t^2 - 1)Z - 1) = 0.$$

Now Z is a power series whose initial terms are $1 + 3t^2 + \cdots$ and so we see that it is a root of the first factor, which we can solve:

$$Z = \frac{1}{2} \cdot \left(1 - 3\sqrt{1 - 8t^2}\right) / (1 - 9t^2).$$

The dominant singularity comes from the branch cut, and the radius of convergence is $1/2\sqrt{2}$.

Remark 4.6. We note that the cogrowth series for $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}$ given above is the same as the series for $d = 2$, $m = 3$ in Example 4.1; that is, for the free product of three copies of the cyclic group of order 2. The reason for this can be seen by the fact that if u, v are elements of order two that generate an infinite dihedral group and if x, x^{-1} generate an infinite cyclic group, then if we let $f_i(u) = x$, $f_i(v) = x^{-1}$ for i odd and let $f_i(u) = x^{-1}$ and $f_i(u) = x$ for i even then we have a map $g : \{u, v\}^* \to \{x, x^{-1}\}^*$ given by $g(a_1 \cdots a_n) = f_1(a_1)f_2(a_2)\cdots f_n(a_n)$ for $a_1, \ldots, a_n \in \{u, v\}$ and a straightforward induction shows that this map gives that these two groups have the same cogrowth.

Example 4.7. Let $G = \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/n\mathbb{Z} = \langle x \mid x^2 = 1\rangle \star \langle y \mid y^n = 1\rangle$ and let $S = \{x, y\}$ and let $F(t)$ denote the cogrowth generating series for G with respect to S. Then

$$F(t) = (1 - tD)/((1 - tD)^2 - t^2),$$

where D is the unique power series solution to the equation

$$t^{n-1}(1 - tD)^{n-1} = (1 - tD - t^2)^{n-1}D$$

whose expansion begins t^{n-1} + higher degree terms.

Proof. We let

$$A_1 := F_{1,\{x\}}(t), B_1 = F_{1,\{1,x\}}(t), C = F_{x^{-1},\{x^{-1}\}}(t) \tag{4.8}$$

and

$$A_2 := F_{1,\{y\}}(t), B_2 = F_{1,\{1,y\}}(t), D = F_{y^{-1},\{y^{-1}\}}(t). \tag{4.9}$$

Notice that a word $s_1 \cdots s_n \in S^n$ that is equal to x^{-1} and has no proper prefix equal to x^{-1} can be decomposed uniquely in the form w_1x, where each w_i is a word in $s_1, \ldots, s_m$ that is equal to 1 with no proper prefix equal to x. This then gives us the equation

$$tA_1 = C. \tag{4.10}$$

Similarly, we have

$$t^{n-1}A_2^{n-1} = D. \tag{4.11}$$

We now make use of Lemma 3.1(I)–(III). Equation (I) gives us that $A_1 = 1/(2 - B_1)$ and $A_2 = 1/(2 - B_2)$. Equation (I) gives

$$B_1 = 1 + tD \qquad \text{and} \qquad B_2 = 1 + tC. \tag{4.12}$$

Combining these equations then gives that $t = (1-tD)C$ and we have $t^{n-1} = (1-tC)^{n-1}D$. Thus we see that D is a solution to the equation

$$t^{n-1}(1 - tD)^{n-1} = (1 - tD - t^2)^{n-1}D.$$

Finally, Equation (I) gives $F_{1,\{1\}}(t) = 1 + t(C + D)$ and $F(t) = F_{1,\emptyset}(t) = 1/(1 - tC - tD)$, and so if we use the fact that $C = t/(1 - tD)$, we see that

$$F(t) = (1 - tD)/((1 - tD)^2 - t^2) = \frac{1}{2}\left(1/(1 - t - tD) + 1/(1 + t - tD)\right).$$

Uniqueness of D after imposing the initial conditions follows from a standard application of Hensel's lemma.

For $n = 3, 4, 5$ we get the following expressions for the minimal polynomial of $F(t)$ using Maple. (Note that the case $n = 2$ is done is the case $m = d = 2$ in Example 4.1.)

$n = 3$**:**

$$((t - 1)^3 + t^3)((t + 1)^3 - t^3)Z^3 + (t^5 - t^4 + t^3 + 2t^2 - 1)Z^2 + (t^3 - t^2 + 1)Z + 1$$

$n=4$**:**

$((t^4-(t-1)^4)((t+1)^4-t^4)Z^4+2(4t^6-2t^4+3t^2-1)Z^3+t^4(t^2+3)Z^2+(t^4-2t^2+2)Z+1$

$n=5$**:**

$$((t-1)^5+t^5)((t+1)^5-t^5)\,Z^5+A\,Z^4+B\,Z^3+C\,Z^2+Z+1.$$

Here

$$A=3(t^9-t^8+6t^7+4t^6+t^5-6t^4+4t^2-1) \qquad B=2(4t^7+t^6+3t^5-3t^4+3t^2-1)$$
$$C=(t^7+4t^5+2t^4-4t^2+2) \qquad D=(t^5-3t^2+3)$$

We computed the minimal polynomials for some higher n too, but the expressions became increasingly unwieldy and we could not discern any obvious patterns governing the coefficients of the annihilating polynomials. One exception is the leading term which is predicted to be:

$$-\left((t+1)^n-t^n\right)\left((1-t)^n-t^n\right)Z^n$$

□

The case when $n=3$ was previously worked out by Alkauskas [1] and our formula appears in his Theorem 1. It corresponds to the cogrowth of $\mathrm{PSL}_2(\mathbb{Z})$ as a semigroup generated by two elements, one of order 2 and another of order 3. Again, we apply standard techniques to this algebraic equation to deduce that the singularities of F are contained in the set of zeros of the leading coefficient and the discriminant, in particular, those in the range $[1/2,\infty)$. In the first case, it is a solution to $((t-1)^3+t^3)$, that is, $1/2$. In the second case, the discriminant,

$$\left(t^{13}-8t^{12}-4t^{11}+164t^{10}-392t^9+404t^8-752t^7+260t^6-512t^5-128t^4-160t^3-64t^2+64\right)t^3,$$

has a solution in that we numerically estimate to be .5072330945.... The radius of convergence is one of these two values. Since the cogrowth function is bounded above by 2^n and since $F(t)$ has a singularity at its radius of convergence by Pringsheim's thoerem. We again invoke [17, Corollary 6.6] to get that the radius of convergence is strictly greater than $1/2$, and so it is $0.50723\cdots$. For similar reasons, for general n we predict it will come from the discriminant, and not the leading coefficient.

5. A gap result for radii of convergence

In this section we prove the gap theorem mentioned in the introduction. We first make a remark

Remark 5.1. Let $\phi : G \to H$ be a group homomorphism and let S be a symmetric generating set for G. If the restriction of ϕ is injective on S then $\mathrm{CL}(n; G, S) \le \mathrm{CL}(n; H, \phi(S))$.

Proof. Observe that if $s_1,\ldots,s_n \in S$ and $s_1\cdots s_n = 1$ then $\phi(s_1)\cdots\phi(s_n)=1$ and so the inequality is immediate. □

Theorem 5.2. *Let G be a finitely generated group with finite symmetric generating set S and let $\rho_{G,S}$ denote the radius of convergence of the cogrowth generating series of G with respect to S. Then $\rho_{G,S}^{-1} \in \{1,2\}\cup[2\sqrt{2},\infty)$.*

G	ρ_G	OEIS	Initial terms of $CL(n; G, S)$
$\mathbb{Z}_2 \star \mathbb{Z}_2$	$1/2$	A126869	$1, 0, 2, 0, 6, 0, 20, 0, 70, 0, 252, 0, 924, 0, 3432, 0, 12870, 0, 48620, 0, 184756$
$\mathbb{Z}_3 \star \mathbb{Z}_3$	$\frac{2^{2/3}}{3}$	A047098	$1, 0, 0, 2, 0, 0, 8, 0, 0, 38, 0, 0, 196, 0, 0, 1062, 0, 0, 5948, 0, 0, 34120$
$\mathbb{Z}_4 \star \mathbb{Z}_4$	$\frac{3^{3/4}}{4}$	A107026	$1, 0, 0, 0, 2, 0, 0, 0, 10, 0, 0, 0, 62, 0, 0, 0, 426, 0, 0, 0, 3112, 0, 0, 0, 23686$
$\mathbb{Z}_5 \star \mathbb{Z}_5$	$\frac{4^{4/5}}{5}$	NEW	$1, 0, 0, 0, 0, 2, 0, 0, 0, 0, 12, 0, 0, 0, 0, 92, 0, 0, 0, 0, 792, 0, 0, 0, 0, 7302$
$\mathbb{Z}_2 \star \mathbb{Z}_3$	.5072330945	A265434	$1, 0, 1, 1, 1, 5, 2, 14, 13, 31, 66, 77, 240, 286, 722, 1226, 2141, 4760, 7268, 16473$
$\mathbb{Z}_2 \star \mathbb{Z}_4$	.5171996045	NEW	$1, 0, 1, 0, 2, 0, 7, 0, 22, 0, 66, 0, 209, 0, 687, 0, 2278, 0, 7612, 0$
$\mathbb{Z}_2 \star \mathbb{Z}_5$	.5259851993	NEW	$1, 0, 1, 0, 1, 1, 1, 7, 1, 27, 2, 77, 19, 182, 148, 379, 793, 748, 3268, 1729$
$\mathbb{Z}_2 \star \mathbb{Z}_6$	.5333879707	NEW	$1, 0, 1, 0, 1, 0, 2, 0, 9, 0, 36, 0, 114, 0, 316, 0, 873, 0, 2636, 0$
$\mathbb{Z}_2 \star \mathbb{Z}_7$	.5396278153	NEW	$1, 0, 1, 0, 1, 0, 1, 1, 1, 9, 1, 44, 1, 156, 2, 450, 25, 1122, 262, 2508, 1851, 5149$
$\mathbb{Z}_2 \star \mathbb{Z}$	$(2\sqrt{2})^{-1}$	A089022	$1, 3, 15, 87, 543, 3543, 23823, 163719, 1143999, 8099511, 57959535, 418441191$
$\mathbb{Z}_2^{\star m}$	$\frac{1}{2\sqrt[2]{m-1}}$		$1, 0, m, 0, 2m^2 - m, 0, 5m^3 - 6m^2 + 2m, 0, 14m^4 - 28m^3 + 20m^2 - 5m$
$\mathbb{Z}_3^{\star m}$	$\frac{2^{2/3}}{3\sqrt[3]{m-1}}$		$1, 0, 0, m, 0, 0, m(3m-2), 0, 0, m\left(12m^2 - 18m + 7\right), 0, 0$
$\mathbb{Z}_4^{\star m}$	$\frac{3^{3/4}}{4\sqrt[4]{m-1}}$		$1, 0, 0, 0, m, 0, 0, 0, m(4m-3), 0, 0, 0, m\left(22m^2 - 36m + 15\right), 0, 0, 0$
$\mathbb{Z}_5^{\star m}$	$\frac{4^{4/5}}{5\sqrt[5]{m-1}}$		$1, 0, 0, 0, 0, m, 0, 0, 0, 0, m(5m-4), 0, 0, 0, 0, m\left(35m^2 - 60m + 26\right), 0, 0, 0, 0$

TABLE 1. The examples considered in Section 4. The algebraic equations satisfied by the generating functions are found in that section. Here, we use $\{x\}$ as a generating set for $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \langle x \mid x^n = 1\rangle$. If $S_i \subseteq G_i$ is the generating set for G_i, the above cogrowth series is given with respect to $S = \cup S_i \subseteq G_1 \star \cdots \star G_m$

Proof. Suppose that H is a group with symmetric generating set S and let $s_1, \ldots, s_p$ be the elements of order 2 in S and let $u_1^{\pm 1}, \ldots, u_q^{\pm 1}$ be the remaining elements of S. Then $p + 2q = |S|$. We claim that if either $p \geq 3$, $q \geq 2$ or $p, q \geq 1$ then $\rho_{H,S}^{-1} \geq 2\sqrt{2}$. To do this, we deal with a few cases.

Case I: $p \geq 3$. Let G be the free product of 3 copies of $\mathbb{Z}/2\mathbb{Z}$ with generators x_1, x_2, x_3 Then we have a group homomorphism $\phi : G \to H$ sending $x_i \to s_i$ for $i = 1, 2, 3$ and this is injective on $T := \{x_1, \ldots, x_3\}$. Thus $\mathrm{CL}(n; G, T) \leq \mathrm{CL}(n; H, S)$ and hence $1/\rho_{G,T} \leq 1/\rho_{H,S}$. In Example 4.1 with $d = 2, m = 3$, we compute the cogrowth generating function for G with respect to T to be

$$4/\left(1 + 3\sqrt{1 - 8t^2}\right),$$

which has radius of convergence $2\sqrt{2}^{-1}$ and so $\rho_{H,S}^{-1} \geq 2\sqrt{2}$ in this case.

Case II: $q \geq 2$. In this case, we let G be the free product of two copies of $\mathbb{Z}$ with generating set $T = \{y_1, y_1^{-1}, y_2, y_2^{-1}\}$. Then we have a homomorphism $\phi : G \to H$ sending $y_i \to u_i$ for $i = 1, 2$. Then we again have $\mathrm{CL}(n; G, T) \leq \mathrm{CL}(n; H, S)$ and hence $1/\rho_{G,T} \leq 1/\rho_{H,S}$. One can show that the cogrowth of G is given by the series $3/(1 + 2\sqrt{1 - 12x^2})$ using the work of Chomsky and Schützenberger [7] (see also OEIS A035610). This series has $1/\rho_{H,S} \geq 1/\rho_{G,T} = \sqrt{12} > 2\sqrt{2}$ and so we get the result in this case.

Case III: $p, q \geq 1$. In this case, we let G be the free product of $\mathbb{Z}/2\mathbb{Z}$ (with generator x) with $\mathbb{Z}$ (with generating set y, y^{-1}. We let T be the symmetric generating set $\{x, y, y^{-1}\}$ and we have a homomorphism from $G \to H$ sending x to s_1, $y \mapsto u_1$. Then this is injective on T and sends T into S, so $\mathrm{C}(n; H, S) \geq \mathrm{CL}(n; G, T)$, and in Example 4.5 we showed that the cogrowth generating series for G has radius of convergence $1/2\sqrt{2}$, so we get the result in this case.

Thus we see that it suffices to consider the case when $p \leq 2$, $q \leq 1$, and $pq = 0$. Hence $(p,q) \in \{(2,0),(1,0),(0,1)\}$. But then we see that H is a homomorphic image of either D_∞ or $\mathbb{Z}$, and hence it is amenable and so by Kesten's criterion $\rho_{H,S}^{-1} = |S| \in \{1,2\}$. The result follows. $\square$

We pose the following question.

Question 5.3. *Does there exist $\alpha \in [2\sqrt{2}, \infty)$ that cannot be realized as $1/\rho_{G,S}$ for some finitely generated group G and finite symmetric generating set S?*

6. Concluding remarks

We showed that finitely generated amenable groups that are not virtually nilpotent have non-P-recursive associated cogrowth sequences. It is natural to ask what happens in the virtually nilpotent case. It is not difficult to show that virtually abelian groups have P-recursive cogrowth sequences. The reason for this is that in the torsion free abelian case one can interpret the cogrowth generating function as a diagonal of a multivariate rational power series. Such a series is known to be D-finite. Dealing with the virtually abelian case presents minor additional difficulties and can be dealt with by first fixing a free abelian subgroup H of finite index and then, given a generating set S, determining the regular sublanguage of S^* consisting of words in S that are in H—this is relatively simple to compute. Once one has this, it is not difficult to express the cogrowth generating function as a sum of diagonals.

In the non-virtually abelian case we do not know whether the cogrowth sequence for a virtually nilpotent group can be P-recursive. In particular, the case of the Heisenberg group, of unipotent upper-triangular integer matrices is an interesting case to work out. A question related to Stanley's conjecture [34] concerning whether the generating function for $\binom{2n}{n}^d$ (as a function of n) is transcendental for $d \geq 2$ (which was solved by Flajolet [14] and Sharif and Woodcock [31]), is the question of whether a virtually nilpotent group that is not virtually cyclic must have transcendental cogrowth generating series. The connection is the observation that $\binom{2n}{n}^d$ is precisely the cogrowth sequence for the group $\mathbb{Z}^d = \langle x_1, \ldots, x_d \mid x_i x_j = x_j x_i \rangle$ with $S = \{x_1^{\pm 1}, \ldots, x_d^{\pm 1}\}$. In general, for a finitely generated virtually nilpotent group G, one can find a (finitely generated) normal nilpotent subgroup N of finite index. If we let d_i denote the rank of the i-th group in the descending central series, then if

$$\mathrm{GKdim}(N) := \sum_{i \geq 1} i d_i$$

is even, then work of Bass [6], along with work of Varopoulous [35], and asymptotic results concerning coefficients of algebraic power series (see, for example, [20]) show that the cogrowth generating function is not algebraic. This applies, in particular to the Heisenberg group H, which has $\mathrm{GKdim}(H) = 4$. Kuksov [25] showed that groups whose cogrowth generating function is rational are exactly the finite groups (although he uses a slightly different definition of cogrowth, the proof is easily modified), and these are precisely the groups where the language consisting of words on the generating set is equal to the identity forms a regular language. Thus, by analogy, one might guess that groups whose associated cogrowth series are all algebraic are necessarily virtually free, since these are the groups for which the word problem is context-free. We thus pose the following question.

Question 6.1. *Let G be a finitely generated group with finite symmetric generating set S. If $\sum \mathrm{CL}(n; G, S)t^n$ is algebraic, is G virtually free?*

The second theme of this project deals with free products of finite groups and free groups. For general free products, it becomes increasingly difficult to compute the cogrowth series. In the case that G is a free product of m groups of orders $d_1, \ldots, d_m$, and $F(t)$ is the cogrowth generating set for some generating set S from the form given in Lemma 3.1, it would be interesting to get good upper bounds on the degree of the extension $D_F := [\mathbb{C}(t, F(t)) : \mathbb{C}(t)]$. We note that upper bounds can be obtained using a theorem of Heinz [18] (see, for example, the remarks before and the proof of Fact 6.4 in [4]). In particular, since each equation in Lemma 3.1 is of degree 2 and we have at most $2^{d_1}d_1 + \cdots + 2^{d_m}d_m$ equations, we get that $D_F \leq 2^{2^{d_1}d_1 + \cdots + 2^{d_m}d_m}$. This appears to be far from optimal. In particular, we found that the dependency tree for functions needed to compute the cogrowth was relatively small and that D_F, was nowhere near the size of the upper bound provided above.

Another curiosity that arises in the paper is the fact that if

$$G_1 = \langle x_1 \mid x_1^2 = 1\rangle \star \langle x_2 \mid x_2^2 = 1\rangle \star \langle x_3 \mid x_3^2 = 1\rangle$$

and $S_1 = \{x_1, x_2, x_3\}$ and if $G_2 = \langle x \; x^2 = 1\rangle \star \langle y\rangle$ and $S_2 = \{x, y, y^{-1}\}$ then

$$\mathrm{CL}(n; G_1, S_1) = \mathrm{CL}(n; G_2, S_2)$$

for all n. It would be interesting to know whether one can find all pairs of non-isomorphic groups with symmetric generating sets whose cogrowth series are the same.

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