On the coverings of closed orientable Euclidean manifolds $\mathcal{G}_2$ and $\mathcal{G}_4$.

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Abstract

There are only 10 Euclidean forms, that is flat closed three dimensional manifolds: six are orientable and four are non-orientable. The aim of this paper is to describe all types of n-fold coverings over orientable Euclidean manifolds $\mathcal{G}_2$ and $\mathcal{G}_4$, and calculate the numbers of non-equivalent coverings of each type. We classify subgroups in the fundamental groups $\pi_1(\mathcal{G}_2)$ and $\pi_1(\mathcal{G}_4)$ up to isomorphism and calculate the numbers of conjugated classes of each type of subgroups for index n. The manifolds $\mathcal{G}_2$ and $\mathcal{G}_4$ are uniquely determined among the others orientable forms by their homology groups $H_1(\mathcal{G}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$ and $H_1(\mathcal{G}_4) = \mathbb{Z}_2 \times \mathbb{Z}$.

Key words: Euclidean form, platycosm, flat 3-manifold, non-equivalent coverings, crystallographic group.

2010 Mathematics Subject Classification: 20H15, 57M10, 55R10.

Introduction

Let $\mathcal{M}$ be a manifold with fundamental group $\Gamma = \pi_1(\mathcal{M})$. Two coverings $p_1 : \mathcal{M}_1 \to \mathcal{M}$ and $p_2 : \mathcal{M}_2 \to \mathcal{M}$ are said to be equivalent if there exists a homeomorphism $h : \mathcal{M}_1 \to \mathcal{M}_2$ such that $p_1 = p_2 \circ h$. According to the general theory of covering spaces, any n-fold covering is uniquely determined by a subgroup of index n in the group Γ. The

*This work was supported by the Russian Foundation for Basic Research (grant $18-01-00036/18$).

†This work was supported by the Russian Foundation for Basic Research (grant 16-31-00138).

equivalence classes of n-fold covering of $\mathcal{M}$ are in one-to-one correspondence with the conjugacy classes of subgroups of index n in the fundamental group $\pi_1(\mathcal{M})$. (see, for example, [5], p. 67). A similar statement formulated in the language of orbifolds is valid for branched coverings.

In such a way the following two natural problems arise. The first one is to calculate the number of subgroups of given finite index n in $\pi_1(\mathcal{M})$. The second problem is to find the number of conjugacy classes of subgroups of index n in $\pi_1(\mathcal{M})$.

The problem of enumeration for nonequivalent coverings over a Riemann surface with given branch type goes back to the paper [7] by Hurwitz, in which the number of coverings over the Riemann sphere with given number of simple (of order two) branching points was determined. Later, in [8], it was found that this number has an adequate expression in terms of irreducible characters of symmetric groups, the theory of which was developed by Frobenius in the beginning of the twentieth century. The Hurwitz problem was considered by many authors. A detailed survey of the related results is contained in ([14], [10]). For closed Riemann surfaces, this problem was completely solved in [17]. However, of most interest is the case of unramified coverings. Let $s_\Gamma(n)$ denote the number of subgroups of index n in the group Γ, and let $c_\Gamma(n)$ be the number of conjugacy classes of such subgroups. According to what was said above, $c_\Gamma(n)$ coincides with the number of nonequivalent n-fold coverings over a manifold $\mathcal{M}$ with fundamental group Γ. If $\mathcal{M}$ is a compact surface with nonempty boundary of Euler characteristic $\chi(\mathcal{M}) = 1 - r$, where $r \geq 0$ (e.g., a disk with r holes), then its fundamental group $\Gamma = F_r$ is the free group of rank r. For this case, M. Hall [6] calculated the number $s_\Gamma(n)$ and V. A. Liskovets [11] found the number $c_\Gamma(n)$ by using his own method for calculating the number of conjugacy classes of subgroups in free groups. An alternative approach for counting conjugacy classes of subgroups in F_r was suggested by J. H. Kwak and Y. Lee [9]. The numbers $s_\Gamma(n)$ and $c_\Gamma(n)$ for the fundamental group of a closed surface (orientable or not) were calculated in ([15], [16], [18]). In the paper [19], a general method for calculating the number $c_\Gamma(n)$ of conjugacy classes of subgroups in an arbitrary finitely generated group Γ was given. Asymptotic formulas for $s_\Gamma(n)$ in many important cases were obtained by T. W. Müller and his collaborators ([20], [21], [22]).

In the three-dimensional case, for a large class of Seifert fibrations, the value of $s_\Gamma(n)$ was determined in [12] and [13]. In the previous paper by the authors [2], the numbers $s_\Gamma(n)$ and $c_\Gamma(n)$ were determined for the fundamental groups of non-orientable Euclidian manifolds $\mathcal{B}_1$ and $\mathcal{B}_2$ whose homologies are $H_1(\mathcal{B}_1) = \mathbb{Z}_2 \times \mathbb{Z}^2$ and $H_1(\mathcal{B}_2) = \mathbb{Z}^2$.

The aim of the present paper is to investigate n-fold coverings over orientable Euclidean three dimensional manifolds $\mathcal{G}_2$ and $\mathcal{G}_4$, whose homologies are $H_1(\mathcal{G}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$ and $H_1(\mathcal{G}_4) = \mathbb{Z}_2 \times \mathbb{Z}$. We classify subgroups of finite index in the fundamental groups of $\pi_1(\mathcal{G}_2)$ and $\pi_1(\mathcal{G}_4)$ up to isomorphism and calculate the numbers of conjugated classes of each type of subgroups for index n.

We note that numerical methods to solve these and similar problems for the three-dimensional crystallogical groups were developed by the Bilbao group [1]. The first homologies of such groups are determined in [23].

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Notations

According to Wolf notation, there are six orientable Euclidean 3-manifold $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{G}_6$, and four non-orientable ones $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$, see [24]. One can find the correspondence between Wolf and Conway-Rossetti notations of the Euclidean 3-manifold and its fundamental groups in Table 1. During the paper, we prefer to use Wolf notation.

name	Conway-Rosetti	other names	Wolf	fund.group (internatl. no name)	Homology group
torocosm	c_1	3-torus	$\mathcal{G}_1$	$1.P1$	$\mathbb{Z}^3$
dicosm	c_2	half turn space	$\mathcal{G}_2$	$4.P2_1$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}$
tricosm	c_3	one-third turn space	$\mathcal{G}_3$	$144.P3_1$ $145.P3_2$	$\mathbb{Z}_3 \oplus \mathbb{Z}$
tetracosm	c_4	quarter turn space	$\mathcal{G}_4$	$76.P4_1$ $78.P4_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}$
hexacosm	c_6	one-sixth turn space	$\mathcal{G}_5$	$169.P6_1$ $170.P6_5$	$\mathbb{Z}$
didicosm	c_{22}	Hantzsche-Wendt space	$\mathcal{G}_6$	$19.P2_12_12_1$	$\mathbb{Z}_4^2$
first amphicosm	$+a_1$	Klein bottle times circle	$\mathcal{B}_1$	$7.Pc$	$\mathbb{Z}_2 \oplus \mathbb{Z}^2$
second amphicosm	$-a_1$		$\mathcal{B}_2$	$9.Cc$	$\mathbb{Z}^2$
first amphidicosm	$+a_2$		$\mathcal{B}_3$	$29.Pca2_1$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}$
second amphidicosm	$-a_2$		$\mathcal{B}_4$	$33.Pa2_1$	$\mathbb{Z}_4 \oplus \mathbb{Z}$

Table 1

During this paper we will use the following notations: $s_{H,G}(n)$ is the number of subgroups of index n in the group G, isomorphic to the group H; $c_{H,G}(n)$ is the number conjugacy classes of subgroups of index n in the group G, isomorphic to the group H. Also we will need the following combinatorial functions:

$$\sigma_0(n) = \sum_{k|n} 1 \quad \text{if } n \text{ is natural, } \sigma_0(n) = 0 \text{ otherwise,}$$

$$\sigma_1(n) = \sum_{k|n} k \quad \text{if } n \text{ is natural, } \sigma_1(n) = 0 \text{ otherwise,}$$

$$\sigma_2(n) = \sum_{k|n} \sigma_1(k) \quad \text{if } n \text{ is natural, } \sigma_2(n) = 0 \text{ otherwise.}$$

$$\omega(n) = \sum_{k|n} k\sigma_1(k) \quad \text{if } n \text{ is natural, } \omega(n) = 0 \text{ otherwise.}$$

$$\tau(n) = |\{(s,t)|s,t \in \mathbb{Z}, s > 0, t \geq 0, s^2 + t^2 = n\}| \quad \text{if } n \text{ is natural, } \omega(n) = 0 \text{ otherwise.}$$

1 The brief overview of achieved results

Since the problem of enumeration of n-fold coverings reduces to the problem of enumeration of conjugacy classes of some subgroups, it is natural to expect that the enumeration of subgroups without respect of conjugacy would be helpful. The manifold$\mathcal{G}_1$ have the Abelian fundamental group $\mathbb{Z}^3$. Thus the number of subgroups of a given finite index n coincides with the number of conjugacy classes and well known:

$$s_{\mathbb{Z}^3}(n) = c_{\mathbb{Z}^3}(n) = \sum_{k|n} k\sigma_1(k).$$

In this paper we enumerate subgroups of a given finite index n and conjugacy classes of such subgroups with respect of their isomorphism class in groups $\pi_1(\mathcal{G}_2)$ and $\pi_1(\mathcal{G}_4)$. Similar results for manifolds $\mathcal{B}_1$ and $\mathcal{B}_2$ are achieved in [2]. Analogous results for other five Euclidean 3-manifolds are coming soon.

The next theorem provides the complete solution of the problem of enumeration of subgroups of a given index in $\pi_1(\mathcal{G}_2)$.

Theorem 1. *Every subgroup Δ of finite index n in $\pi_1(\mathcal{G}_2)$ is isomorphic to either $\pi_1(\mathcal{G}_2)$ or $\mathbb{Z}^3$. The respective numbers of subgroups are*

(i)
$$s_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_2)}(n) = \omega(n) - \omega(\frac{n}{2}),$$

(ii)
$$s_{\mathbb{Z}^3,\pi_1(\mathcal{G}_2)}(n) = \omega(\frac{n}{2});$$

where $\omega(n) = \sum_{k|n} k\sigma_1(k)$.

The next theorem provides the number of conjugacy classes of subgroups of index n in $\pi_1(\mathcal{G}_2)$ for each isomorphism type. That is the number of non-equivalent n-fold covering $\mathcal{G}_2$, which have a prescribe fundamental group.

Theorem 2. *Let $\mathcal{N} \to \mathcal{G}_2$ be an n-fold covering over $\mathcal{G}_2$. If n is odd then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_2$. If n is even then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_2$ or $\mathcal{G}_1$. The corresponding numbers of nonequivalent coverings are given by the following formulas:*

$$c_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_2)}(n) = \sigma_2(n) + 2\sigma_2(\frac{n}{2}) - 3\sigma_2(\frac{n}{4}). \quad (i)$$

$$c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_2)}(n) = \frac{1}{2}\Big(\sigma_2(\frac{n}{2}) + 3\sigma_2(\frac{n}{4}) + \omega(\frac{n}{2})\Big). \quad (ii)$$

Theorem 3 and Theorem 4 count the numbers of subgroups and the numbers of conjugacy classes of subgroups of index n in $\pi_1(\mathcal{G}_4)$. That is this theorems are analogues of Theorem 1 and Theorem 2 respectively for the manifold $\mathcal{G}_4$. We will need one more combinatorial function for the compact formulation of the results.

Notation. Denote $\tau(m) = |\{(s,t)|s,t \in \mathbb{Z}, s > 0, t \geq 0, s^2 + t^2 = m\}|$.

Theorem 3. *Every subgroup Δ of finite index n in $\pi_1(\mathcal{G}_4)$ is isomorphic to either $\pi_1(\mathcal{G}_4)$, or $\pi_1(\mathcal{G}_2)$, or $\mathbb{Z}^3$. The respective numbers of subgroups are*

(i) $$s_{\pi_1(\mathcal{G}_4),\pi_1(\mathcal{G}_4)}(n) = \sum_{a|n} a\tau(a) - \sum_{a|\frac{n}{2}} a\tau(a).$$

(ii) $$s_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_4)}(n) = \omega(\frac{n}{2}) - \omega(\frac{n}{4}),$$

(iii) $$s_{\mathbb{Z}^3,\pi_1(\mathcal{G}_4)}(n) = \omega(\frac{n}{4}).$$

Theorem 4. *Let $\mathcal{N} \to \mathcal{G}_4$ be an n-fold covering over $\mathcal{G}_4$. If n is odd then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_4$. If n is even but not divisible by 4 then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_4$ or $\mathcal{G}_2$. Finely, if n is divisible by 4 then $\mathcal{N}$ is homeomorphic to one of $\mathcal{G}_4$, $\mathcal{G}_2$ and $\mathcal{G}_1$. The corresponding numbers of nonequivalent coverings are given by the following formulas:*

(i) $$c_{\pi_1(\mathcal{G}_4),\pi_1(\mathcal{G}_4)} = \sum_{a|n} \tau(\frac{n}{a}) - \sum_{a|\frac{n}{4}} \tau(\frac{n}{4a})$$

(ii) $$c_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_4)} = \frac{1}{2}\Big(\sigma_2(\frac{n}{2}) + 2\sigma_2(\frac{n}{4}) - 3\sigma_2(\frac{n}{4}) + \sum_{a|\frac{n}{2}} \tau(a) - \sum_{a|\frac{n}{8}} \tau(a)\Big),$$

(iii) $$c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_4)} = \frac{1}{2}\sum_{a|\frac{n}{4}} \tau(a) + \frac{1}{2}\sum_{a|\frac{n}{8}} \tau(a) + \frac{1}{4}\sigma_2(\frac{n}{4}) + \frac{3}{4}\sigma_2(\frac{n}{8}) + \frac{1}{4}\omega(\frac{n}{4}).$$

2 Preliminaries

Further we use the following representations for the fundamental groups $\pi(\mathcal{G}_2)$ and $\pi(\mathcal{G}_4)$, see [24] or [3].

$$\pi_1(\mathcal{G}_2) = \langle x, y, z : xyx^{-1}y^{-1} = 1, zxz^{-1} = x^{-1}, zyz^{-1} = y^{-1}\rangle. \tag{2.1}$$

$$\pi_1(\mathcal{G}_4) = \langle \tilde{x}, \tilde{y}, \tilde{z} : \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} = 1, \tilde{z}\tilde{x}\tilde{z}^{-1} = \tilde{y}, \tilde{z}\tilde{y}\tilde{z}^{-1} = \tilde{x}^{-1}\rangle. \tag{2.2}$$

Further we will widely use the following statement.

Proposition 1. *The sublattices of index k in the 2-dimensional lattice $\mathbb{Z}^2$ are in one-to-one correspondence with the matrices $\begin{pmatrix} b & c \\ 0 & a \end{pmatrix}$, where $ab = k$, $0 \le c < b$. Consequently, the number of such sublattices is $\sigma_1(k)$.*

The sublattices of index k in the 3-dimensional lattice $\mathbb{Z}^3$ are in one-to-one correspondence with the integer matrices $\begin{pmatrix} c & e & f \\ 0 & b & d \\ 0 & 0 & a \end{pmatrix}$, where $a, b, c > 0$, $abc = k$, $0 \le d < b$ and $0 \le f, e < c$. Consequently, the number of such sublattices is $\omega(k)$.

Despite this statement is well-known we will quote here its proof, since we will need the technique for a number of more subtle questions.

Proof. We will prove the statement about 3-dimension lattice. The two dimensional case can be done similarly. Consider the group $\mathbb{Z}^3 = \{(x, y, z)| \ x, y, z \in \mathbb{Z}\}$, we use the additive notation. Let G be a subgroup of index k in $\mathbb{Z}^3$.

Let $\bar{w} = (f, d, a) \in G$ be an element with the minimal positive third coordinate among all elements of G. Such element exists, otherwise the index $|\mathbb{Z}^3 : G|$ is infinite. Denote $\bar{h}(G) = (f, d, 0)$ and $H(G) = G \bigcap (\mathbb{Z}, \mathbb{Z}, 0)$.

Let $\bar{v} = (e, b, 0) \in H(G)$ be an element with the minimal positive second coordinate among all elements of $H(G)$. Such element exists, otherwise the index $|\mathbb{Z}^2 : H(G)|$ is infinite. Let $\bar{u} = (c, 0, 0) \in H(G)$ be the element with the minimal possible positive c. Replacing $\bar{w}$ with $\bar{w} + i\bar{v} + j\bar{u}$ $i, j \in \mathbb{Z}$ we may assume $0 \leq d < b$ and $0 \leq f < c$. Similarly we achieve $0 \leq e < c$.

Thus we got the map form the subgroups to matrices. Now we prove its injectivity. Indeed, the number a and the subgroup $H(G)$ are uniquely defined by G. In its turn, $H(G)$ uniquely defines b, c, e. Finally, the set $(\mathbb{Z}, \mathbb{Z}, a) \bigcap G$ is uniquely defined by G. But $(\mathbb{Z}, \mathbb{Z}, a) \bigcap G = h + H(G)$, that means that the coset of the element h in the coset decomposition $(\mathbb{Z}, \mathbb{Z}, a)/H(G)$ is uniquely defined. Keeping in mind that $H(G) = \langle (c, 0, 0), (e, b, 0) \rangle$, we get that the pair (d, f) where $0 \leq d < b$ and $0 \leq f < c$ is unique.

The fact that each matrix corresponds to some subgroup of index k is obvious.

□

3 On the coverings of $\mathcal{G}_2$

3.1 The structure of the group $\pi_1(\mathcal{G}_2)$

The following proposition provides the canonical form of an element in $\pi_1(\mathcal{G}_2)$.

Proposition 2. *(i) Each element of $\pi_1(\mathcal{G}_2)$ can be represented in the canonical form $x^a y^b z^c$ for some integer a, b, c.*

(ii) The product of two canonical forms is given by the formula

$$x^a y^b z^c \cdot x^d y^e z^f = x^{a+(-1)^c d} y^{b+(-1)^c e} z^{c+f}. \tag{3.3}$$

(iii) The canonical epimorphism $\phi_{\mathcal{G}2} : \pi_1(\mathcal{G}_2) \to \pi_1(\mathcal{G}_2)/\langle x, y \rangle \cong \mathbb{Z}$, given by the formula $x^a y^b z^c \to c$ is well-defined.

(iv) The representation in the canonical form $g = x^a y^b z^c$ for each element $g \in \pi_1(\mathcal{G}_2)$ is unique.

Proof. Part (ii) can be verified directly using the relations 2.1. To prove part (i) consider an arbitrary word in generators, say $s_1^{d_1} s_2^{d_2} \cdots s_k^{d_k}$, where $s_1, s_2, \ldots, s_k \in \{x, y, z\}$. We prove by induction on k that such a word can be represented in the form $x^a y^b z^c$. Indeed, for $k = 1$ the word already have the desired representation; in this case two numbers among a, b, c equal zero. Suppose for some k the statement is proved, and consider $s_1^{d_1} s_2^{d_2} \cdots s_{i+1}^{d_{i+1}} = x^a y^b z^c \cdot s_{i+1}^{d_{i+1}}$. Since $s_{i+1}^{d_{i+1}} = x^d y^e z^f$ (two among d, e, f equal 0) the use of part (ii) finishes the proof.

Part (ii) implies that $\langle x, y \rangle$ is a normal subgroup in $\pi_1(\mathcal{G}_2)$. Substituting $x = y = 1$ to (2.1) we get that there is no relations containing z, this proves (iii). To prove (iv) consider an arbitrary element $g \in \pi_1(\mathcal{G}_2)$ and an arbitrary representation of this element in the canonical form $g = x^a y^b z^c$. The value of c is uniquely defined by (iii), indeed $c = \phi_{\mathcal{G}2}(g)$. The numbers a and b are also uniquely defined, since x and y are generators of the free abelian group $\langle x, y \rangle = \mathbb{Z}^2$. □

Notation. Set $\Gamma = \langle x, y \rangle$.

In the next definition we introduce the invariants, similar to one used in Proposition 1.

Definition 1. *Suppose Δ is a subgroup of finite index n in $\pi_1(\mathcal{G}_2)$. Put $H(\Delta) = \Delta \bigcap \Gamma$. We consider all elements of $\pi_1(\mathcal{G}_2)$ represented in the canonical form, given by Proposition 2. By $a(\Delta)$ denote the minimal positive degree at z among the elements of Δ. Let $Z(\Delta)$ be some element with such degree at z, put $Z(\Delta) = hz^{a(\Delta)}$, where $h \in \Gamma$. By $\nu(\Delta) = hH(\Delta)$ denote the coset in coset decomposition $\Gamma/H(\Delta)$. For the coset decomposition $\Gamma/H(\Delta)$ we will use the additive notation. By $Y(\Delta)$ and $X(\Delta)$ denote a pair of generators of $H(\Delta)$ of the form, provided by Proposition 1, that is $Y(\Delta) = x^{e(\Delta)} y^{b(\Delta)}$, $X(\Delta) = x^{c(\Delta)}$ where $0 \leq e(\Delta) < c(\Delta)$. Further we will omit Δ for $X(\Delta), Y(\Delta), Z(\Delta)$.*

It is worth noting that, despite the choice of Z and h is not unique, the choice of $\nu(\Delta)$, X and Y is unique. More precisely:

Lemma 1. *The number $a(\Delta)$, the subgroup $H(\Delta)$ and the coset $\nu(\Delta)$ are well-defined. Furthermore $a(\Delta)b(\Delta)c(\Delta) = a(\Delta)[\Gamma : H(\Delta)] = [\pi_1(\mathcal{G}_2) : \Delta]$.*

Proof. The number $a(\Delta)$ exists, otherwise the elements $z, z^2, z^3, \ldots$ belong to mutually different cosets of Δ in $\pi_1(\mathcal{G}_2)$. Thus the index of Δ is infinite, which is a contradiction.

The number $a(\Delta)$ and the subgroup $H(\Delta)$ are unique by definition. Let $s_1 H(\Delta), \ldots, s_m H(\Delta)$ be a complete system of cosets of $H(\Delta)$ in Γ. Then $z^i s_j H(\Delta)$, $0 \leq i < a(\Delta)$, $1 \leq j \leq |\Gamma : H(\Delta)|$ is a complete system of cosets of $H(\Delta)$ in $\pi_1(\mathcal{G}_2)$, thus $k(\Delta) \cdot \Big|\Gamma : H(\Delta)\Big| = \Big|\pi_1(\mathcal{G}_2) : \Delta\Big| = n$. The equality $b(\Delta)c(\Delta) = [\Gamma : H(\Delta)]$ is given by Proposition 1.

To prove that $\nu(\Delta)$ does not depends upon a choice of Z consider $Z_1 = h_1 z^{a(\Delta)} \in \Delta$ and $Z_2 = h_2 z^{a(\Delta)} \in \Delta$, where $h_1, h_2 \in \Gamma$. Since $Z_1 Z_2^{-1} \in H(\Delta)$, we have $Z_1 Z_2^{-1} = h_1 z^{a(\Delta)} z^{-a(\Delta)} h_2^{-1} = h_1 h_2^{-1} \in H(\Delta)$. That is h_1 and h_2 belongs to the same coset of $H(\Delta)$ in Γ. □

Definition 2. *A 3-plet (a, H, ν) is called* n-essential *if the following conditions holds:*

(i) a is a positive divisor of n,

(ii) H is a subgroup of index n/a in Γ

(iii) ν is an element of Γ/H.

Lemma 2. *For arbitrary n-essential 3-plet (a, H, ν) there exists a subgroup Δ in the group $\pi_1(\mathcal{G}_2)$ such that $(a, H, \nu) = (k(\Delta), H(\Delta), \nu(\Delta))$.*

Proof. Let h be a representative of the coset ν. In case a is odd consider the set

$$\{hz^{(2l+1)a}H | l \in \mathbb{Z}\} \bigcup \{z^{2la}H | l \in \mathbb{Z}\}.$$

This set is a subgroup of index n in $\pi_1(\mathcal{G}_2)$, which fact can be proven directly.

Similarly, in case a is even the set

$$\{h^l z^{la} H | l \in \mathbb{Z}\}$$

form a subgroup of index n in $\pi_1(\mathcal{G}_2)$. □

Proposition 3. *There is a bijection between the set of n-essential 3-plets (a, H, ν) and the set of subgroups Δ of index n in $\pi_1(\mathcal{G}_2)$, given by the correspondence $\Delta \leftrightarrow (a(\Delta), H(\Delta), \nu(\Delta))$. Moreover, $\Delta \cong \pi_1(\mathcal{G}_1)$ if $a(\Delta)$ is even and $\Delta \cong \pi_1(\mathcal{G}_2)$ if $a(\Delta)$ is odd.*

Proof. Consider the family of subgroups Δ of index n in $\pi_1(\mathcal{G}_2)$.Lemma 1 builds the map of the family of subgroups Δ to n-essential 3-plets. Lemma 2 shows that this map is a bijection. Now we describe the isomorphism type of a subgroup.

If $a(\Delta)$ is even Lemma 2 implies that Δ is a subgroup of $\langle x, y, z^2 \rangle$. Substituting the canonical representations with even degrees at z into (3.3) one gets that $\langle x, y, z^2 \rangle \cong \mathbb{Z}^3$, thus Δ is a subgroup of finite index in $\mathbb{Z}^3$. As a result Δ is isomorphic to $\mathbb{Z}^3$.

Consider the case $a(\Delta)$ is odd. For the sake of brevity, we write $X = X(\Delta)$, $Y = Y(\Delta)$ and $Z = Z(\Delta)$. Recall Δ is generated by X, Y, Z. Direct verification shows that the relations $XYX^{-1}Y^{-1} = 1$, $ZXZ^{-1} = X^{-1}$, and $ZYZ^{-1} = Y^{-1}$ holds. Further we call this relations *the proper relations of the subgroup* Δ. Thus the map $x \to X$, $y \to Y$, $z \to Z$ can be extended to an epimorphism $\pi_1(\mathcal{G}_2) \to \Delta$. To prove that this epimorphism is really an isomorphism we need to show that each relation in Δ is a corollary of proper relations. We call a relation, that is not a corollary of proper relations an *improper relation*.

Assume the contrary, i.e. there are some improper relations in Δ. Since in Δ the proper relations holds, each element can be represented in the canonical form, given by Proposition 2 in terms of X, Y, Z, by using just the proper relations. That is each element g can be represented as

$$g = X^r Y^s Z^t.$$

If there is an improper relation then there is an equality

$$X^r Y^s Z^t = X^{r'} Y^{s'} Z^{t'}, \tag{3.4}$$

where at least one of the inequalities $r \neq r'$, $s \neq s'$, $t \neq t'$ holds. Applying $\phi_{\mathcal{G}2}$ to both parts we get $ta(\Delta) = t'a(\Delta)$, thus $t = t'$. Then $X^r Y^s = X^{r'} Y^{s'}$, that means

$$\begin{cases} c(\Delta)r + e(\Delta)s = c(\Delta)r' + e(\Delta)s' \\ b(\Delta)s = b(\Delta)s' \end{cases} \tag{3.5}$$

Keep in mind that $b(\Delta)c(\Delta) \neq 0$ since $a(\Delta)b(\Delta)c(\Delta) = n$. The contradiction of equations (3.5) with $(r,s) \neq (r',s')$ proves that $\Delta \cong \pi_1(\mathcal{G}_2)$. $\square$

3.2 The proof of Theorem 1

Proceed to the proof of Theorem 1. Proposition 3 claims that each subgroup Δ of finite index n is isomorphic to $\pi_1(\mathcal{G}_2)$ or $\mathbb{Z}^3$, depending upon whether $a(\Delta)$ is odd or even. Consider these two cases separately.

Case (i). To find the number of subgroups isomorphic to $\pi_1(\mathcal{G}_2)$, by Proposition 3 we need to calculate the cardinality of the set of n-essential 3-plets with odd a.

For each odd $a \mid n$ there are $\sigma_1(\frac{n}{a})$ subgroups H in Γ, such that $\big|\Gamma : H\big| = \frac{n}{a}$. Also there are $\frac{n}{a}$ different choices of a coset ν. Thus, for each odd a the number of n-essential 3-plets is $\frac{n}{a}\sigma_1(\frac{n}{a})$. So, the total number of subgroups is given by

$$s_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_2)}(n) = \sum_{a|n,\, 2\nmid a} \frac{n}{a}\sigma_1(\frac{n}{a}).$$

Equivalently,

$$s_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_2)}(n) = \sum_{a|n} \frac{n}{a}\sigma_1(\frac{n}{a}) - \sum_{2a|n} \frac{n}{2a}\sigma_1(\frac{n}{2a}) = \omega(n) - \omega(\frac{n}{2}).$$

Case (ii). Similarly to the previous case, we get the formula

$$s_{\mathbb{Z}^3,\pi_1(\mathcal{G}_2)}(n) = \sum_{2a|n} \frac{n}{2a}\sigma_1(\frac{n}{2a}) = \omega(\frac{n}{2}).$$

3.3 The proof of Theorem 2

The isomorphism types of subgroups are already provided by Proposition 3. Thus we just have to calculate the number of conjugacy classes for each type.

Notation. By ξ denote the canonical homomorphism $\Gamma \to \Gamma/H(\Delta)$.

To prove case (ii) consider a subgroup Δ of index n in $\pi_1(\mathcal{G}_2)$ isomorphic to $\mathbb{Z}^3$. By Proposition 3, a subgroup Δ is uniquely defined by the n-essential 3-plet $(a(\Delta), H(\Delta), \nu(\Delta))$, where a is even. Consider the conjugacy class of subgroups Δ^g, $g \in \pi_1(\mathcal{G}_2)$. It consists of subgroups corresponding to 3-plets $\big(a(\Delta^g), H(\Delta^g), \nu(\Delta^g)\big)$, $g \in \pi_1(\mathcal{G}_2)$.

Obviously, $a(\Delta^g) = a(\Delta)$ and $H(\Delta^g) = H(\Delta)$. Furthermore, $\nu(\Delta) = \nu(\Delta^x) = \nu(\Delta^y) = \nu(\Delta^{z^2})$ and $\nu(\Delta) = -\nu(\Delta^z)$, thus $\nu(\Delta^g) = \pm\nu(\Delta)$. So, the conjugacy class of the subgroup Δ consists of one or two subgroups, depending on does the condition $2\nu(\Delta) = 0$ holds or not. In the former case, Δ is normal in $\pi_1(\mathcal{G}_2)$.

By $\mathfrak{M}_1$ and $\mathfrak{M}_2$ denote the set of normal subgroups, and the set of subgroups having exactly two subgroups in their conjugacy class, respectively. The obvious formula for the number of conjugacy classes is $c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_2)} = |\mathfrak{M}_1| + \frac{|\mathfrak{M}_2|}{2}$, we rewrite it in the form $c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_3)} = \frac{|M_1|}{2} + \frac{|\mathfrak{M}_1|+|\mathfrak{M}_2|}{2}$. Note that $\mathfrak{M}_1 \bigcup \mathfrak{M}_2$ is the set of all subgroups, thus $|\mathfrak{M}_1| + |\mathfrak{M}_2|$ is given by Theorem 1. So we have to find $|\mathfrak{M}_1|$.

Lemma 3. *Let a be an even divisor of n. Then the number of normal subgroups Δ of index n in $\pi_1(\mathcal{G}_2)$ isomorphic to $\mathbb{Z}^3$ and satisfying $a(\Delta) = a$ is given by*

$$\sigma_1(\frac{n}{a}) + 3\sigma_1(\frac{n}{2a}).$$

Proof. For the given even $a(\Delta) = a$ we need to calculate the number of pairs $(H(\Delta), \nu(\Delta))$, such that $2\nu(\Delta) = 0$. By Proposition 1, subgroups $H(\Delta)$ bijectively correspond to pairs of generators $X = x^c$, $Y = x^e y^b$, that is $H(\Delta)$ is uniquely defined by integers b, c, e such that $b, c > 0, bc = \frac{n}{a}$ and $0 \leq e < c$.

The cosets of $\Gamma/H(\Delta)$ bijectively correspond to elements of the set $F = \{x^i y^j | \, 0 \leq i < c, \, 0 \leq j < b\}$. The condition $2\nu(\Delta) = 0$ means $(2i, 2j) \in \langle (c, 0), (e, b) \rangle$.

Fix a subgroup $H(\Delta)$. If both b and c are odd then there is only one element ν with $2\nu = 0$, namely $\nu = \xi(x^0 y^0) = 0$. If c is even and b is odd then there are two elements: $\nu = 0$ and $\nu = \xi(x^{c/2})$. If c is odd and b is even then among the numbers $\frac{e}{2}$ and $\frac{e+c}{2}$ exactly one is integer, thus there are two elements ν, satisfying $2\nu = 0$: namely 0 and one of $\xi(x^{\frac{e}{2}} y^{\frac{b}{2}})$ or $\xi(x^{\frac{e+c}{2}} y^{\frac{b}{2}})$.

Finally, let both c and b are even. For odd e there are only two different ν: $\nu = 0$ and $\nu = \xi(x^{c/2})$; for even e there are four different choices: $\nu = 0$, $\nu = \xi(x^{c/2})$, $\nu = \xi(x^{e/2} y^{b/2})$ and $\nu = \xi(x^{(e+c)/2} y^{(b)/2})$. So, if one fix a pair of even (b, c), the subgroups $H(\Delta)$ with $(b(\Delta), c(\Delta)) = (b, c)$ bijectively corresponds to the values of $0 \leq e(\Delta) < c$. For $\frac{c}{2}$ values $0 \leq e < c$, $2 \nmid e$, there are 2 choices of ν, for other $\frac{c}{2}$ values of $0 \leq e < c$, $2 \mid e$ there are 4 choices of ν.

Summarizing, for a fixed pair (b, c) the number of pairs $(H(\Delta), \nu(\Delta))$, such that $(b(\Delta), c(\Delta)) = (b, c)$ and $2\nu = 0$ is

- c if both b and c are odd
- $2c$ if exactly one of b and c is even
- $3c$ if both b and c are even.

Summing over all possible values of c we get the required number of pairs equals

$$|\{(H(\Delta), \nu(\Delta)\}| = \sum_{c | \frac{n}{a}} c + \sum_{c | \frac{n}{2a}} 2c + \sum_{c | \frac{n}{2a}} c = \sigma_1(\frac{n}{a}) + 3\sigma_1(\frac{n}{2a}).$$

□

Now, using the value of $|\mathfrak{M}_1| + |\mathfrak{M}_2|$, provided by Theorem 1 and summing over all possible values of $a(\Delta)$ one gets the proof of case (ii) of Theorem 2

$$c_{\mathbb{Z}^3, \pi_1(\mathcal{G}_2)}(n) = \frac{1}{2}\Big(\sigma_2(\frac{n}{2}) + 3\sigma_2(\frac{n}{4}) + \omega(\frac{n}{2})\Big).$$

This finishes the proof of case (ii) of Theorem 2.

The proof of case (i) resembles the proof of (ii). Consider a subgroup Δ of index n in $\pi_1(\mathcal{G}_2)$ isomorphic to $\pi_1(\mathcal{G}_2)$. By Proposition 3, a subgroup Δ is uniquely defined by the

n-essential 3-plet $(a(\Delta), H(\Delta), \nu(\Delta))$ with odd $a(\Delta)$. Consider the conjugacy class of subgroups Δ^g, $g \in \pi_1(\mathcal{G}_2)$. It consists of subgroups, bijectively corresponding to 3-plets $(a(\Delta^g), H(\Delta^g), \nu(\Delta^g))$.

Obviously, $a(\Delta^g) = a(\Delta)$ and $H(\Delta^g) = H(\Delta)$. Furthermore, $\nu(\Delta^x) = \nu(\Delta) + 2\xi(x)$, $\nu(\Delta^y) = \nu(\Delta) + 2\xi(y)$ and $\nu(\Delta) = -\nu(\Delta^z)$. So, $\nu(\Delta^g) = \nu(\Delta) + 2r\xi(x) + 2s\xi(y)$ for integer r and s. Thus, the number of conjugacy classes of subgroups Δ with the given $a(\Delta)$ and $H(\Delta)$ is equal to the index $\Big|\big(\Gamma/H(\Delta)\big) : \langle 2\xi(x), 2\xi(y)\rangle\Big|$, or in terms of Proposition 1, to the number $\Big|\big(\mathbb{Z}^2/\langle (c(\Delta), 0), (e(\Delta), b(\Delta)), (2, 0), (0, 2)\rangle\Big|$.

Similarly to the previous case, we get that $\Big|\big(\Gamma/H(\Delta)\big) : \langle 2\xi(x), 2\xi(y)\rangle\Big|$ is equal to

- 1 if both $b(\Delta)$ and $c(\Delta)$ are odd
- 2 if exactly one of $b(\Delta)$ and $c(\Delta)$ is even
- 2 if both $b(\Delta)$ and $c(\Delta)$ are even and $e(\Delta)$ is odd
- 4 if $b(\Delta)$, $c(\Delta)$ and $e(\Delta)$ are even.

Fix some value $a(\Delta) = a$. Counting all integer triplets $b(\Delta), c(\Delta), e(\Delta) : b(\Delta)c(\Delta) = \frac{n}{a}$, $0 \le e(\Delta) < c(\Delta)$ one gets $|\{(H(\Delta), \nu(\Delta)\}| = \sigma_1(\frac{n}{a}) + 3\sigma_1(\frac{n}{2a})$. Summing this over all possible values of a, which are odd divisors of n, we get the final formula

$$c_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_2)}(n) = \sum_{k|n\ 2\nmid k} \big(\sigma_1(\frac{n}{k}) + 3\sigma_1(\frac{n}{2k})\big) = \sigma_2(n) + 2\sigma_2(\frac{n}{2}) - 3\sigma_2(\frac{n}{4}).$$

4 On the coverings of $\mathcal{G}_4$

4.1 The structure of the group $\pi_1(\mathcal{G}_4)$

Recall that $\pi_1(\mathcal{G}_4)$ is given by generators and relations by $\pi_1(\mathcal{G}_4) = \langle \tilde{x}, \tilde{y}, \tilde{z} : \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} = 1, \tilde{z}\tilde{x}\tilde{z}^{-1} = \tilde{y}, \tilde{z}\tilde{y}\tilde{z}^{-1} = \tilde{x}^{-1}\rangle$ The following proposition provides the canonical form of an element in $\pi_1(\mathcal{G}_4)$.

Proposition 4. *(i) Each element of $\pi_1(\mathcal{G}_4)$ can be represented in the canonical form $\tilde{x}^a\tilde{y}^b\tilde{z}^c$ for some integer a, b, c.*

(ii) The product of two canonical forms is given by the formula

$$\tilde{x}^a\tilde{y}^b\tilde{z}^c \cdot \tilde{x}^d\tilde{y}^e\tilde{z}^f = \begin{cases} \tilde{x}^{a+d}\tilde{y}^{b+e}\tilde{z}^{c+f} & \textit{if} \quad c \equiv 0 \mod 4 \\ \tilde{x}^{a-e}\tilde{y}^{b+d}\tilde{z}^{c+f} & \textit{if} \quad c \equiv 1 \mod 4 \\ \tilde{x}^{a-d}\tilde{y}^{b-e}\tilde{z}^{c+f} & \textit{if} \quad c \equiv 2 \mod 4 \\ \tilde{x}^{a+e}\tilde{y}^{b-d}\tilde{z}^{c+f} & \textit{if} \quad c \equiv 3 \mod 4 \end{cases} \tag{4.6}$$

(iii) The canonical epimorphism $\phi_{\mathcal{G}4} : \pi_1(\mathcal{G}_4) \to \pi_1(\mathcal{G}_4)/\langle \tilde{x}, \tilde{y}\rangle \cong \mathbb{Z}$, given by the formula $\tilde{x}^a\tilde{y}^b\tilde{z}^c \to c$ is well-defined.

(iv) The representation in the canonical form $g = \tilde{x}^a\tilde{y}^b\tilde{z}^c$ for each element $g \in \pi_1(\mathcal{G}_2)$ is unique.

Proof. The proof is similar to the proof of Proposition 2. □

Notation. Set $\Gamma = \langle \tilde{x}, \tilde{y} \rangle$.

In the next definition we introduce the invariants, similar to one used in Proposition 1.

Definition 3. *Suppose Δ is a subgroup of finite index n in $\pi_1(\mathcal{G}_4)$. Put $H(\Delta) = \Delta \bigcap \Gamma$. We consider all elements of $\pi_1(\mathcal{G}_4)$ represented in the canonical form, given by Proposition 4. By $a(\Delta)$ denote the minimal positive degree at z among the elements of Δ. Let $Z(\Delta)$ be some element with such degree at z, put $Z(\Delta) = hz^{a(\Delta)}$, where $h \in \Gamma$. By $\nu(\Delta) = hH(\Delta)$ denote the coset in coset decomposition $\Gamma/H(\Delta)$. For the coset decomposition $\Gamma/H(\Delta)$ we will use the additive notation.*

It is worth noting that, despite the choice of Z and h is not unique, the choice of $\nu(\Delta)$ is.

Lemma 4. *The number $a(\Delta)$, the subgroup $H(\Delta)$ and the coset $\nu(\Delta)$ are well-defined. Furthermore $a(\Delta)[\Gamma : H(\Delta)] = [\pi_1(\mathcal{G}_4) : \Delta]$.*

The proof is similar to Lemma 1.

Lemma 5. *If $a(\Delta)$ is odd then $H(\Delta) \lhd \pi_1(\mathcal{G}_4)$.*

Proof. Recall that $Z = h\tilde{z}^{a(\Delta)} \in \Delta$, where $a(\Delta)$ is odd and $h \in \langle \tilde{x}, \tilde{y} \rangle$. First $H(\Delta)^Z = H(\Delta)$. Also $H(\Delta)^{\tilde{x}} = H(\Delta)^{\tilde{y}} = H(\Delta)^{\tilde{z}^2} = H(\Delta)$. The former fact means that $H(\Delta)^g = H(\Delta)$, $g \in \pi_1(\mathcal{G}_4)$, hence $\langle \tilde{x}, \tilde{y}, \tilde{z}^2, Z \rangle = \pi_1(\mathcal{G}_4)$ in the case of odd $a(\Delta)$. □

Lemma 6. *Let G be a subgroup in Γ. Then $G \lhd \pi_1(\mathcal{G}_4)$ if and only if there exist a pair of generators of G of the form $(\tilde{x}^p\tilde{y}^q, \tilde{x}^{-q}\tilde{y}^p)$. In this case $[\Gamma : G] = p^2 + q^2$.*

Proof. Obvious. □

Lemma 7. *The number of subgroups of index m in Γ normal in $\pi_1(\mathcal{G}_4)$ is $\tau(m)$, where $\tau(m) = |\{(s,t) | s, t \in \mathbb{Z}, s > 0, t \geq 0, s^2 + t^2 = m\}|$.*

Proof. The trivial corollary of the previous lemma. □

Definition 4. *A 3-plet (a, H, ν) is called n-*essential *if the following conditions holds:*

(i) a is a positive divisor of n,

(ii) H is a subgroup of index n/a in Γ also if a is odd then $H \lhd \pi_1(\mathcal{G}_4)$,

(iii) ν is an element of Γ/H.

Lemma 8. *For an arbitrary n-essential 3-plet (a, H, ν) there exists a subgroup Δ of $\pi_1(\mathcal{G}_4)$ such that $(a, H, \nu) = (a(\Delta), H(\Delta), \nu(\Delta))$.*

Proof. Consider some n-essential 3-plet (a, H, ν) and let h be a representative of the coset ν.

If $a \equiv 0 \mod 4$ consider the set

$$\Delta = \{h^l \tilde{z}^{la} H | l \in \mathbb{Z}\}.$$

Direct verification shows that Δ is a subgroup of index n in $\pi_1(\mathcal{G}_4)$ with required characteristics $a(\Delta)$, $H(\Delta)$ and $\nu(\Delta)$.

In three remained cases the structure of subgroup is defined the following way: in case $a \equiv 2 \mod 4$ put

$$\Delta = \{h\tilde{z}^{(2l+1)a} H | l \in \mathbb{Z}\} \bigcup \{\tilde{z}^{2la} H | l \in \mathbb{Z}\}.$$

In case $a \equiv 1 \mod 4$ put

$$\Delta = \{h\tilde{z}^{(4l+1)a} H | l \in \mathbb{Z}\} \bigcup \{hh^{\tilde{z}} \tilde{z}^{(4l+2)a} H | l \in \mathbb{Z}\} \bigcup$$
$$\{h^{\tilde{z}} \tilde{z}^{(4l+3)a} H | l \in \mathbb{Z}\} \bigcup \{\tilde{z}^{4la(\Delta)} H | l \in \mathbb{Z}\},$$

keep in mind that $gg^{\tilde{z}^2} = 1$ for all $g \in \Gamma$.

In the same way, if $k \equiv 3 \mod 4$ put

$$\Delta = \{h\tilde{z}^{(4l+1)k} H | l \in \mathbb{Z}\} \bigcup \{hh^{\tilde{z}^3} \tilde{z}^{4lk+2} H | l \in \mathbb{Z}\} \bigcup$$
$$\{h^{\tilde{z}^3} \tilde{z}^{(4l+3)k} H | l \in \mathbb{Z}\} \bigcup \{\tilde{z}^{4lk(\Delta)} H | l \in \mathbb{Z}\}.$$

□

Proposition 5. *There is a bijection between the set of n-essential 3-plets (a, H, ν) and the set of subgroups of index n in $\pi_1(\mathcal{G}_4)$ given by the correspondence $\Delta \leftrightarrow (a(\Delta), H(\Delta), \nu(\Delta))$. Moreover, $\Delta \cong \mathbb{Z}^3$ if $a(\Delta) \equiv 0 \mod 4$, $\Delta \cong \pi_1(\mathcal{G}_2)$ if $a(\Delta) \equiv 2 \mod 4$ and $\Delta \cong \pi_1(\mathcal{G}_4)$ if $a(\Delta) \equiv 1 \mod 2$.*

Proof. Consider the family of subgroups Δ of index n in $\pi_1(\mathcal{G}_4)$. Lemma 4 builds the map of the family of subgroups Δ to n-essential 3-plets $\Delta \leftrightarrow (a(\Delta), H(\Delta), \nu(\Delta))$. Lemma 8 shows that this map is a bijection.

The proof of isomorphism part or the statement is similar to Proposition 3. Consider canonical forms of all elements in Δ. In case $a(\Delta) \equiv 0 \mod 4$ equation 4.6 provides the commutativity, thus $\Delta \cong \mathbb{Z}^3$. The case $a(\Delta) \equiv 2 \mod 4$ is proven in Proposition 3. The case $a(\Delta) \equiv 1 \mod 2$ follows the same way as the proofs of Proposition 3: one fixes the suitable generators $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$, for which all required relations holds, and prove that any unnecessary relation implies a relation in Γ, which contradiction completes the proof. □

4.2 The proof of Theorem 3

The isomorphism types of finite index subgroups in $\pi_1(\mathcal{G}_4)$ are provided by Proposition 5. The cases of subgroups $\pi_1(\mathcal{G}_2)$ and $\mathbb{Z}^3$ in the group $\pi_1(\mathcal{G}_4)$ are similar to the cases of subgroups $\pi_1(\mathcal{G}_2)$ and $\mathbb{Z}^3$ in the group $\pi_1(\mathcal{G}_2)$ respectively.

In case of a subgroup, isomorphic to $\pi_1(\mathcal{G}_4)$, for each fixed $k \equiv 1 \mod 2$ there are $\tau(\frac{n}{k})$ different H, and $\frac{n}{k}$ different ν for each fixed H, thus the final value is

$$s_{\pi_1(\mathcal{G}_4),\pi_1(\mathcal{G}_4)}(n) = \sum_{k|n} \frac{n}{k}\tau(\frac{n}{k}) - \sum_{2k|n} \frac{n}{2k}\tau(\frac{n}{2k}) = \sum_{k|n} k\tau(k) - \sum_{k|\frac{n}{2}} k\tau(k).$$

4.3 The proof of Theorem 4

The isomorphism types of subgroups are already provided by Proposition 5. Thus we just have to calculate the number of conjugacy classes for each type.

Notation. By ξ denote the canonical homomorphism $\Gamma \to \Gamma/H(\Delta)$.

4.3.1 Case (iii)

Let Δ be a subgroup of index n in $\pi_1(\mathcal{G}_4)$, isomorphic to $\mathbb{Z}^3$.

Lemma 9. *The conjugacy class of Δ consists of 1, 2 or 4 subgroups.*

Proof. In virtue of Proposition 5 the subgroup Δ is uniquely determined by its n-essential 3-plet. Also, by Proposition 5 if $\tilde{x}^a\tilde{y}^b\tilde{z}^c \in \Delta$ then $4 \mid c$.

Again, $a(\Delta^g) = a(\Delta)$, $g \in \pi_1(\mathcal{G}_4)$ and $H(\Delta^{\tilde{x}}) = H(\Delta^{\tilde{y}}) = H(\Delta^{\tilde{z}^2}) = H(\Delta)$. Also $\nu(\Delta)$: $\nu(\Delta^{\tilde{x}}) = \nu(\Delta^{\tilde{y}}) = \nu(\Delta^{\tilde{z}^4}) = \nu(\Delta)$, here it is important that $4 \mid c$.

Thus the conjugacy class of Δ contains at most 4 groups: Δ, $\Delta^{\tilde{z}}$, $\Delta^{\tilde{z}^2}$ and $\Delta^{\tilde{z}^3}$. If $\Delta \neq \Delta^{\tilde{z}^2}$ then it contains exactly 4 groups, otherwise it contains two or one. In the latter case Δ is normal in $\pi_1(\mathcal{G}_4)$. □

Definition 5. *By $\mathfrak{M}_1$, $\mathfrak{M}_2$ and $\mathfrak{M}_4$ denote the respective sets of subgroups $\Delta \cong \mathbb{Z}^3$ of index n: which are normal in $\pi_1(\mathcal{G}_4)$, which belong to a conjugacy class of exactly two subgroups, and which belong to a conjugacy class of exactly four subgroups.*

The obvious formula for the number of conjugacy classes is $c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_4)} = |\mathfrak{M}_1| + \frac{|\mathfrak{M}_2|}{2} + \frac{|\mathfrak{M}_4|}{4}$, we rewrite it in the form $c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_4)} = \frac{|\mathfrak{M}_1|}{2} + \frac{|\mathfrak{M}_1|+|\mathfrak{M}_2|}{4} + \frac{|\mathfrak{M}_1|+|\mathfrak{M}_2|+|\mathfrak{M}_4|}{4}$. Note that $\mathfrak{M}_1 \bigcup \mathfrak{M}_2 \bigcup \mathfrak{M}_4$ is the set of all subgroups, and $\mathfrak{M}_1 \bigcup \mathfrak{M}_2$ is the set of all subgroups, which conjugacy class contains at most two subgroups.

Theorem 3 claims $|\mathfrak{M}_1| + |\mathfrak{M}_2| + |\mathfrak{M}_4| = \sum_{4k|n} \frac{n}{4k}\sigma_1(\frac{n}{4k})$.

Lemma 10.

$$|\mathfrak{M}_1| + |\mathfrak{M}_2| = \sigma_2(\frac{n}{4}) + 3\sigma_2(\frac{n}{8}).$$

Proof. We have to calculate the amount of subgroups $\Delta \cong \mathbb{Z}^3$ of index n, which satisfy $\Delta = \Delta^{\tilde{z}^2}$. This is done exactly similar to the proof of Lemma 3. □

Lemma 11.

$$|\mathfrak{M}_1| = \sum_{4k|n} \tau(\frac{n}{4k}) + \sum_{8k|n} \tau(\frac{n}{8k}).$$

Proof. We have to calculate the amount of subgroups $\Delta \cong \mathbb{Z}^3$ of index n and such that $\Delta = \Delta^{\tilde{z}}$. Proposition 5 claims $4 \mid a(\Delta)$. Since $H(\Delta) = H(\Delta^{\tilde{z}})$, $H(\Delta) \lhd \pi_1(\mathcal{G}_4)$. Then by Lemma 7 the number of choices of $H(\Delta)$ is $\tau(\frac{n}{a(\Delta)})$. Finally, $\nu(\Delta) = \nu(\Delta^{\tilde{z}})$, which is possible for one value if $\frac{n}{a(\Delta)}$ is odd and for two values if $\frac{n}{a(\Delta)}$ is even. Since $\tau(\frac{n}{2a(\Delta)}) = \tau(\frac{n}{a(\Delta)})$ if $\frac{n}{a(\Delta)}$ is even, and $\tau(\frac{n}{2a(\Delta)}) = 0$ otherwise, the number of pairs $\big(H(\Delta), \nu(\Delta)\big)$ equals $\tau(\frac{n}{a(\Delta)}) + \tau(\frac{n}{2a(\Delta)})$. We finish the prove summing the respective number of pairs over all possible values of $a(\Delta)$. Keep in mind that $4 \mid a(\Delta)$, so

$$|\mathfrak{M}_1| = \sum_{4a|n} \tau(\frac{n}{4a}) + \sum_{8a|n} \tau(\frac{n}{8a}).$$

□

Summarizing the results of Theorem 3 (iii), Lemma 10 and Lemma 11 one gets

$$c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_4)} = \frac{1}{2}\sum_{4a|n} \tau(\frac{n}{4a}) + \frac{1}{2}\sum_{8a|n} \tau(\frac{n}{8a}) + \frac{1}{4}\sigma_2(\frac{n}{4}) + \frac{3}{4}\sigma_2(\frac{n}{8}) + \frac{1}{4}\omega(\frac{n}{4}) =$$
$$\frac{1}{2}\sum_{a|\frac{n}{4}} \tau(a) + \frac{1}{2}\sum_{a|\frac{n}{8}} \tau(a) + \frac{1}{4}\sigma_2(\frac{n}{4}) + \frac{3}{4}\sigma_2(\frac{n}{8}) + \frac{1}{4}\omega(\frac{n}{4}).$$

4.3.2 Case (ii)

Let Δ be a subgroup of index n in $\pi_1(\mathcal{G}_4)$ isomorphic to $\pi_1(\mathcal{G}_2)$.

Recall that the subgroup is uniquely defined by an n-essential 3-plet. First we have to describe the triplets of all subgroups, which belongs to the conjugacy class of Δ. Again, $a(\Delta^g) = a(\Delta)$, $g \in \pi_1(\mathcal{G}_4)$. Also, $H(\Delta^{\tilde{x}}) = H(\Delta^{\tilde{y}}) = H(\Delta^{\tilde{z}^2}) = H(\Delta)$, thus for arbitrary $g \in \pi_1(\mathcal{G}_4)$ either $H(\Delta^g) = H(\Delta)$ or $H(\Delta^g) = H(\Delta^{\tilde{z}})$. $\nu(\Delta^{\tilde{x}}) = \nu(\Delta) + 2\xi(\tilde{x})$, $\nu(\Delta^{\tilde{y}}) = \nu(\Delta) + 2\xi(\tilde{y})$, $\nu(\Delta^{\tilde{z}^2}) = -\nu(\Delta)$. Then the conjugacy class of Δ consists of all subgroups, corresponding to 3-plets $\big(a(\Delta), H(\Delta), \nu(\Delta) + \langle 2\xi(\tilde{x}), 2\xi(\tilde{y})\rangle\big)$ and $\big(a(\Delta), H(\Delta^{\tilde{z}}), \nu(\Delta^{\tilde{z}}) + \langle 2\xi(\tilde{x}), 2\xi(\tilde{y})\rangle\big)$.

So, to calculate the number of conjugacy classes we need to calculate the number of pairs consisting of a subgroup H and an element of the factor $\Gamma/(\langle \tilde{x}^2, \tilde{y}^2, H\rangle$.

Fix some $a(\Delta) = a$. Each conjugacy class corresponds to two pairs of a subgroup and an element of the factor, unless this two pairs coincide. Analogous to the previous case, let $\mathfrak{L}_1(a)$ be the family of defined above pairs, such that $a(\Delta) = a$ and one pair form a conjugacy class, and $\mathfrak{L}_2(a)$ be the family of pairs, such that $a(\Delta) = a$ and two pairs form a conjugacy class. By $\mathfrak{L}_1$ and $\mathfrak{L}_2$ denote the union of $\mathfrak{L}_1(a)$ and $\mathfrak{L}_2(a)$ over all values of a respectively. Certainly,

$$c_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_4)} = |\mathfrak{L}_1| + \frac{|\mathfrak{L}_2|}{2} = \frac{|\mathfrak{L}_1|}{2} + \frac{|\mathfrak{L}_1| + |\mathfrak{L}_2|}{2}. \quad (4.7)$$

Lemma 12.

$$|\mathfrak{L}_1| + |\mathfrak{L}_2| = \sigma_2(\frac{n}{2}) + 2\sigma_2(\frac{n}{4}) - 3\sigma_2(\frac{n}{4}).$$

Proof. Consider integer a, such that $a \equiv 2 \mod 4$ and $a \mid n$. We claim $|\mathfrak{L}_1(a)|+|\mathfrak{L}_2(a)| = \sigma_1(\frac{n}{a}) + 3\sigma_1(\frac{n}{2a})$. Exactly this was shown in the proof of Lemma 3. If a does not satisfy the above condition, then $|\mathfrak{L}_1(a)| = |\mathfrak{L}_2(a)| = 0$.

Summing over all values of a one gets $|\mathfrak{L}_1| + |\mathfrak{L}_2| = \sum_{2a|n, 4\nmid 2a} \Big(\sigma_1(\frac{n}{2a}) + 3\sigma_1(\frac{n}{4a})\Big) = \sum_{2a|n} \Big(\sigma_1(\frac{n}{2a}) + 3\sigma_1(\frac{n}{4a})\Big) - \sum_{4a|n} \Big(\sigma_1(\frac{n}{4a}) + 3\sigma_1(\frac{n}{8a})\Big) = \sigma_2(\frac{n}{2}) + 2\sigma_2(\frac{n}{4}) - 3\sigma_2(\frac{n}{4})$. □

Lemma 13.

$$|\mathfrak{L}_1| = \sum_{a|\frac{n}{2}} \tau(\frac{n}{2a}) - \sum_{a|\frac{n}{8}} \tau(\frac{n}{8a}).$$

Proof. We claim $|\mathfrak{L}_1(a)| = \tau(\frac{n}{a})+\tau(\frac{n}{2a})$ if $a \equiv 2 \mod 4$, $|\mathfrak{L}_1(a)| = 0$ otherwise. The proof is similar to Lemma 11. Summing over all values of a we get $|\mathfrak{L}_1| = \sum_{2a|n,4\nmid 2a} \Big(\tau(\frac{n}{2a}) + \tau(\frac{n}{4a})\Big) = \sum_{2a|n} \Big(\tau(\frac{n}{2a}) + \tau(\frac{n}{4a})\Big) - \sum_{4a|n} \Big(\tau(\frac{n}{4a}) + \tau(\frac{n}{8a})\Big) = \sum_{a|\frac{n}{2}} \tau(\frac{n}{2a}) - \sum_{a|\frac{n}{8}} \tau(\frac{n}{8a})$. □

Substituting Lemma 12 and Lemma 13 to (4.7) one gets

$$c_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_4)} = \frac{1}{2}\Big(\sigma_2(\frac{n}{2}) + 2\sigma_2(\frac{n}{4}) - 3\sigma_2(\frac{n}{4}) + \sum_{a|\frac{n}{2}} \tau(\frac{n}{2a}) - \sum_{a|\frac{n}{8}} \tau(\frac{n}{8a})\Big) = \frac{1}{2}\Big(\sigma_2(\frac{n}{2}) + 2\sigma_2(\frac{n}{4}) - 3\sigma_2(\frac{n}{4}) + \sum_{a|\frac{n}{2}} \tau(a) - \sum_{a|\frac{n}{8}} \tau(a)\Big).$$

4.3.3 Case (i)

Let Δ be a subgroup of index n in $\pi_1(\mathcal{G}_4)$ isomorphic to $\pi_1(\mathcal{G}_4)$. The proof is analogous to Case (ii).

For an odd $a \mid n$ the number of conjugacy classes of subgroups Δ, such that $a(\Delta) = a$ equals $\tau(\frac{n}{a})$ if $\frac{n}{a}$ is odd and equals $2\tau(\frac{n}{a})$ if $\frac{n}{a}$ is even. Since $\tau(\frac{n}{2a}) = \tau(\frac{n}{a})$ if $\frac{n}{a}$ is even and $\tau(\frac{n}{2a}) = 0$ if $\frac{n}{a}$ is odd, we get

$$c_{\pi_1(\mathcal{G}_4),\pi_1(\mathcal{G}_4)} = \sum_{a|n,2\nmid a} \Big(\tau(\frac{n}{a}) + \tau(\frac{n}{2a})\Big) = \sum_{a|n} \Big(\tau(\frac{n}{a}) + \tau(\frac{n}{2a})\Big) - \sum_{2a|n} \Big(\tau(\frac{n}{2a}) + \tau(\frac{n}{4a})\Big) = \sum_{a|n} \tau(\frac{n}{a}) - \sum_{a|\frac{n}{4}} \tau(\frac{n}{4a}).$$

5 Appendix

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$s_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_2)}(n)$	1	6	13	28	31	78	57	120	130	186	133	364	183	342	403	496
$c_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_2)}(n)$	1	6	5	16	7	30	9	36	18	42	13	80	15	54	35	76
$s_{\mathbb{Z}^3,\pi_1(\mathcal{G}_2)}(n)$		1		7		13		35		31		91		57		155
$c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_2)}(n)$		1		7		9		29		19		63		33		107
$s_{\pi_1(\mathcal{G}_4),\pi_1(\mathcal{G}_4)}(n)$	1	2	1	4	11	2	1	8	10	22	1	4	27	2	11	16
$c_{\pi_1(\mathcal{G}_4),\pi_1(\mathcal{G}_4)}(n)$	1	2	1	2	3	2	1	2	2	6	1	2	3	2	3	2
$s_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_4)}(n)$		1		6		13		28		31		78		57		120
$c_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_4)}(n)$		1		4		3		9		5		16		5		19
$s_{\mathbb{Z}^3,\pi_1(\mathcal{G}_4)}(n)$				1				7				13				35
$c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_4)}(n)$				1				5				5				17

Table 2

We note some properties of functions, achieved in theorems 1, 2, 3 and 4. The proofs follows by direct calculation, based on the explicit formulas for above functions.

A function $f(n)$ is called *multiplicative* if $f(kl) = f(k)f(l)$ for coprime integers k, l. The functions $s_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_2)}(n)$, $c_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_2)}(n)$, $s_{\pi_1(\mathcal{G}_4),\pi_1(\mathcal{G}_4)}(n)$ and $c_{\pi_1(\mathcal{G}_4),\pi_1(\mathcal{G}_4)}(n)$ are multiplicative. For other mentioned functions some close relations holds: the functions $n \to s_{\mathbb{Z}^3,\pi_1(\mathcal{G}_2)}(s_{\mathbb{Z}^3,\pi_1(\mathcal{G}_2)}(2n))$, $n \to c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_2)}(2n)$, $n \to s_{\mathbb{Z}^3,\pi_1(\mathcal{G}_4)}(4n)$ and $n \to c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_4)}(4n)$ are multiplicative.

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