GAMMA EXPANSIONS OF q-NARAYANA POLYNOMIALS, PATTERN AVOIDANCE AND THE (-1)-PHENOMENON

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Abstract. The aim of this paper is two-folded. We first prove several new interpretations of a kind of q-Narayana polynomials along with their corresponding γ-expansions using pattern avoiding permutations. Secondly, we give a complete characterization of certain (-1)-phenomenon for all Catalan subsets avoiding a single pattern of length three, and discuss their q-analogues utilizing the newly obtained q-γ-expansions, as well as the continued fraction of a quint-variate generating function due to Shin and the fourth author. Moreover, we enumerate the alternating permutations avoiding simultaneously two patterns, namely $(2413, 3142)$ and $(1342, 2431)$, of length four, and consider such (-1)-phenomenon for these two subsets as well.

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1. Introduction

A polynomial $f(x)=\sum_i a_ix^i\in\mathbb{R}[x]$ is called γ-positive if $f(x)=\sum_{i=0}^{\lfloor n/2\rfloor}\gamma_ix^i(1+x)^{n-2i}$ for $n\in\mathbb{N}$ and nonnegative reals $\gamma_0,\gamma_1,\ldots,\gamma_{\lfloor n/2\rfloor}$. The notion of γ-positivity appeared first in the work of Foata and Schützenberger [16], A recent survey on γ-positivity in combinatorics and geometry was given by Athanasiadis [2].

A permutation is said to be *alternating* (or *up-down*) if it starts with an ascent and then descents and ascents come in turn. This has been called *reverse alternating* in Stanley's survey [39] and some of the other literatures but we stick with this convention throughout the paper. We denote by $\mathfrak{S}_n$ (resp. $\mathfrak{A}_n$) the set of permutations (resp. alternating permutations) of length n. Given two permutations $\pi\in\mathfrak{S}_n$ and $p\in\mathfrak{S}_k$, we say that π *contains the pattern* p if there exists a set of indices $1\le i_1<i_2<\cdots<i_k\le n$ such that the subsequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ of π is order-isomorphic to p. Otherwise, π is said to *avoid* p. For example, 15324 contains 321 and avoids 231. The set of permutations (resp. alternating permutations) of length n that avoid patterns $p_1,p_2,\cdots,p_m$ is denoted as $\mathfrak{S}_n(p_1,p_2,\cdots,p_m)$ (resp. $\mathfrak{A}_n(p_1,p_2,\cdots,p_m)$). Recall the **exc**edance, **des**cent and **d**ouble-**d**escent statistics defined for any permutation $\pi=\pi(1)\pi(2)\cdots\pi(n)$:

$$\begin{aligned}\mathsf{exc}\,\pi&=|\{1\le i\le n:\pi(i)>i\}|,\\ \mathsf{des}\,\pi&=|\{1\le i\le n:\pi(i)>\pi(i+1)\}|,\\ \mathsf{dd}^*\,\pi&=|\{1\le i\le n:\pi(i-1)>\pi(i)>\pi(i+1)\}|,\end{aligned}$$

where we let $\pi(0)=\pi(n+1)=n+1$.

The following two γ-*expansions* set the stage for our investigation. Note that (1.1) is a classical result due to Foata and Schützenberger [16]. Foata and Strehl's celebrated valley-hopping [17] was a neat combinatorial argument that lead to both (1.1) and (1.2) (see also [32, Chapter 4] for a nice exposition and the references therein).

$$A_n(t):=\sum_{\pi\in\mathfrak{S}_n}t^{\mathsf{des}\,\pi}=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\gamma_{n,k}^{A}t^k(1+t)^{n-1-2k},\tag{1.1}$$

$$N_n(t):=\sum_{\pi\in\mathfrak{S}_n(231)}t^{\mathsf{des}\,\pi}=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\gamma_{n,k}^{N}t^k(1+t)^{n-1-2k}.\tag{1.2}$$

where

$$\gamma_{n,k}^{A}=\#\{\pi\in\mathfrak{S}_n:\mathsf{dd}^*\,\pi=0,\mathsf{des}\,\pi=k\},\tag{1.3}$$

$$\gamma_{n,k}^{N}=\#\{\pi\in\mathfrak{S}_n(231):\mathsf{dd}^*\,\pi=0,\mathsf{des}\,\pi=k\}.\tag{1.4}$$

There are explicit formulae (cf. [2, Eqs. (62) and (64)]) for both the Narayana polynomials $N_n(t)$ in (1.2) and the γ-coefficients $\gamma_{n,k}^N$ given by

$$N_n(t)=\sum_{i=0}^{n-1}\frac{1}{n}\binom{n}{i+1}\binom{n}{i}t^i=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{(n-1)!}{k!(k+1)!(n-1-2k)!}t^k(1+t)^{n-1-2k}.\tag{1.5}$$

There are several well-known q-Narayana polynomials in the litterature; see [21] and the references therein. In this paper we define the q-Narayana polynomials $N_n(t,q)$ as the Taylor coefficients in the following continued fraction expansion

$$\sum_{n=0}^{\infty} N_n(t,q) z^n = \cfrac{1}{1-\cfrac{c_1 z}{1-\cfrac{c_2 z}{1-\cfrac{c_3 z}{1-\cfrac{c_4 z}{\ddots}}}}}, \tag{1.6}$$

where $c_{2k-1}=q^{k-1}$ and $c_{2k}=tq^{k-1}$ for $k=1,2,\ldots$.

From Cheng, Elizalde, Kasraoui and Sagan [9, Theorem 7.3] (see Lemma 2.7) and the *contraction formula* (see Lemma 2.8) we derive immediately the following interpretation (see also [27, 28])

$$N_n(t,q) = \sum_{\pi\in\mathfrak{S}_n(321)} t^{\mathsf{exc}\,\pi} q^{\mathsf{inv}\,\pi-\mathsf{exc}\,\pi}. \tag{1.7}$$

On the other hand, Blanco and Petersen [4] defined a (q,t)-analog of Catalan numbers, i.e.,

$$\mathrm{Dyck}(n;t,q) = \sum_{p\in\mathrm{Dyck}(n)} t^{\mathsf{rank}\,p} q^{\mathsf{area}\,p},$$

where $\mathrm{Dyck}(n)$ denotes the set of Dyck paths of semilength n, $\mathsf{area}(p)$ is the area under the Dyck path p and $\mathsf{rank}(p)$ is the rank (in the noncrossing partition lattice) of the noncrossing partition corresponding to p via a bijection. By comparing the continued fraction (1.6) with that in Proposition 2.6 of [4], we have

$$N_n(tq,q^2) = \mathrm{Dyck}(n;t,q). \tag{1.8}$$

The first goal of this paper is to establish the following new combinatorial interpretations for $N_n(t,q)$, as well as their corresponding γ-expansions, using pattern avoiding permutations. Undefined statistics, sets and patterns will be given in the next section.

Theorem 1.1. *The q-Narayana polynomials $N_n(t,q)$ have the following ten interpretations:*

$$\begin{aligned} N_n(t,q) &= \sum_{\pi\in\mathfrak{S}_n(231)} t^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(231)} t^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(231)} t^{\mathsf{des}\,\pi} q^{\mathsf{adi}^*\,\pi} \\ &= \sum_{\pi\in\mathfrak{S}_n(312)} t^{\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(312)} t^{\mathsf{des}\,\pi} q^{(2\text{-}13)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(312)} t^{\mathsf{des}\,\pi} q^{\mathsf{adi}\,\pi} \\ &= \sum_{\pi\in\mathfrak{S}_n(213)} t^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(213)} t^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi} \\ &= \sum_{\pi\in\mathfrak{S}_n(132)} t^{\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(132)} t^{\mathsf{des}\,\pi} q^{(2\text{-}13)\,\pi}. \end{aligned}$$

Theorem 1.2. *For $n \geq 1$, the following γ-expansions formula holds true*

$$N_n(t,q) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}(q) t^k (1+t)^{n-1-2k}, \tag{1.9}$$

where

$$\gamma_{n,k}(q) = \sum_{\pi \in \widehat{\mathfrak{S}}_{n,k}(321)} q^{\mathsf{inv}\,\pi - \mathsf{exc}\,\pi} \tag{1.10}$$

$$= \sum_{\pi \in \widetilde{\mathfrak{S}}_{n,k}(213)} q^{(31\text{-}2)\,\pi} = \sum_{\pi \in \widetilde{\mathfrak{S}}_{n,k}(312)} q^{(2\text{-}13)\,\pi} \tag{1.11}$$

$$= \sum_{\pi \in \widetilde{\mathfrak{S}}_{n,k}(132)} q^{(2\text{-}31)\,\pi} = \sum_{\pi \in \widetilde{\mathfrak{S}}_{n,k}(231)} q^{(13\text{-}2)\,\pi}. \tag{1.12}$$

Remark 1.3. Eq. (1.10) is due to Lin and Fu [28]. Moreover, Blanco and Petersen [4] also obtained a γ-expansion formula for $N_n(tq, q^2)$, which sould yield another interpretation for the γ coefficients.

Theorem 1.4. *We have*

$$\sum_{\pi \in \mathfrak{S}_n(213)} t^{\mathsf{des}\,\pi} q^{\mathsf{adi}\,\pi} = \sum_{\pi \in \mathfrak{S}_n(132)} t^{\mathsf{des}\,\pi} q^{\mathsf{adi}^*\,\pi} \tag{1.13}$$

$$= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \Bigg(\sum_{\pi \in \widetilde{\mathfrak{S}}_{n,k}(213)} q^{\mathsf{adi}\,\pi} \Bigg) t^k (1+t)^{n-1-2k}, \tag{1.14}$$

$$= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \Bigg(\sum_{\pi \in \widetilde{\mathfrak{S}}_{n,k}(132)} q^{\mathsf{adi}^*\,\pi} \Bigg) t^k (1+t)^{n-1-2k}. \tag{1.15}$$

Thanks to the combinatorial interpretations in (1.3)–(1.4), taking $k = \lfloor \frac{n-1}{2} \rfloor$, we obtain precisely the number of alternating permutations in each class of permutations:

$$E_n := |\mathfrak{A}_n| = \gamma^A_{n,\lfloor \frac{n-1}{2} \rfloor}, \ |\mathfrak{A}_n(231)| = \gamma^N_{n,\lfloor \frac{n-1}{2} \rfloor}. \tag{1.16}$$

Moreover, we can take $t = -1$ in (1.1) and recover the following combinatorial interpretation of a classical identity involving the odd Euler number E_{2n+1} [14]:

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{des}\,\pi} = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathsf{exc}\,\pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} E_n & \text{if } n \text{ is odd,} \end{cases} \tag{1.17}$$

where the first equality needs the well-known fact that des and exc are equidistributed over $\mathfrak{S}_n$. A parallel result for the even Euler number E_{2n} can be obtained by the (-1)-evaluation of the Roselle polynomial [35]:

$$\sum_{\pi\in\mathfrak{D}_n^*}(-1)^{\mathsf{des}\,\pi}=\sum_{\pi\in\mathfrak{D}_n}(-1)^{\mathsf{exc}\,\pi}=\begin{cases}(-1)^{\frac{n}{2}}E_n & \text{if } n \text{ is even},\\ 0 & \text{if } n \text{ is odd},\end{cases} \tag{1.18}$$

where $\mathfrak{D}_n$ (resp. $\mathfrak{D}_n^*$) denotes the set of *derangements* (resp. *coderangements*, see Definition 2.3) of length n. The first equality in (1.18) follows from Lemma 2.4. In recent years, the q-analogs of (1.17) and (1.18) have attracted attentions of several authors [13,15,22,37].

The second goal of this paper is to consider such (-1)-evaluation with respect to (1.2), then derive results comparable to (1.17) and (1.18), as well as their various companions and q-analogues. More precisely, for a given subset $\mathfrak{S}_n(p_1,p_2,\cdots,p_m)\subset\mathfrak{S}_n$ arising from pattern avoidance, we do the following things.

(1) Enumerate $\mathfrak{A}_n(p_1,p_2,\cdots,p_m)$.
(2) Derive the generating function of des (resp. exc) over $\mathfrak{S}_n(p_1,p_2,\cdots,p_m)$, say $X_n(t)$, then evaluate $X_n(-1)$.
(3) Derive the generating function of des (resp. exc) over $\mathfrak{D}_n^*(p_1,p_2,\cdots,p_m)$ (resp. $\mathfrak{D}_n(p_1,p_2,\cdots,p_m)$), say $Y_n(t)$, then evaluate $Y_n(-1)$.

If the result of (1) (up to an index shift) matches with either that of (2) in the sense of (1.17), or that of (3) in the sense of (1.18), we say $\mathfrak{S}_n(p_1,p_2,\cdots,p_m)$ exhibits the (-1)-*phenomenon*. If we get a double match, then we call it the *strong* (-1)*-phenomenon*.

This approach was already used by Foata-Schützenberger [16, Chap. V] who first derived (1.17) and (1.18) via γ-expansions of the Eulerian polynomials.

After preparing ourselves with preliminary works in section 2, we prove Theorems 1.1, 1.2 and 1.4, among other things, in section 3. We consider a variation involving the *weak excedance* in section 4. Next in section 5, we completely determine the existence of (-1)-phenomenon for $\mathfrak{S}_n(\tau)$, where τ runs through all permutations in $\mathfrak{S}_3$. For example, we have the following q-version of the strong (-1)-phenomenon on $\mathfrak{S}_n(321)$ concerning exc. Recall [4] that Carlitz's q-Catalan numbers $C_n(q)$ are defined by

$$C_n(q):=N_n(q,q^2). \tag{1.19}$$

It is easy to see that $C_n(q)$ is a polynomial of degree $\binom{n}{2}$. For instance,

$$\begin{aligned}C_0(q)&=C_1(q)=1,\\ C_2(q)&=q+1,\\ C_3(q)&=q^3+q^2+2q+1,\\ C_4(q)&=q^6+q^5+2q^4+3q^3+3q^2+3q+1.\end{aligned}$$

Theorem 1.5. *For any $n\geq 1$,*

$$N_n(-1,q)=\sum_{\pi\in\mathfrak{S}_n(321)}(-1)^{\mathsf{exc}\,\pi}q^{\mathsf{inv}\,\pi-\mathsf{exc}\,\pi}=\begin{cases}0 & \text{if } n \text{ is even},\\ (-q)^{\frac{n-1}{2}}C_{\frac{n-1}{2}}(q^2) & \text{if } n \text{ is odd},\end{cases} \tag{1.20}$$

$$\sum_{\pi\in\mathfrak{D}_n(321)} (-1)^{\mathsf{exc}\,\pi} q^{\mathsf{inv}\,\pi} = \begin{cases} (-q)^{\frac{n}{2}} C_{\frac{n}{2}}(q^2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{1.21}$$

We continue our exploration in section 6 with $\mathfrak{S}_n(2413, 3142)$ and $\mathfrak{S}_n(1342, 2431)$, and close with some remarks to motivate further study along this line.

2. Definitions and preliminaries

2.1. **Permutation statistics and a proof of Theorem 1.5.** We follow [10], [37] and [38] for notations and the nomenclature of various permutation statistics. First we recall four classical involutions defined on $\mathfrak{S}_n$, namely, the inverse, reverse, complement and the composition of the latter two. For $\pi \in \mathfrak{S}_n$,

$$\begin{aligned} \pi \mapsto \pi^{-1} &:= \pi^{-1}(1)\pi^{-1}(2)\cdots\pi^{-1}(n), \\ \pi \mapsto \pi^{r} &:= \pi(n)\cdots\pi(2)\pi(1), \\ \pi \mapsto \pi^{c} &:= (n+1-\pi(1))(n+1-\pi(2))\cdots(n+1-\pi(n)), \\ \pi \mapsto \pi^{rc} &:= (n+1-\pi(n))\cdots(n+1-\pi(2))(n+1-\pi(1)). \end{aligned}$$

There are four statistics concerning three consecutive letters in π. Note that the dd below and the dd^* mentioned in the introduction have different initial conditions concerning $\pi(0)$ and $\pi(n+1)$, hence are indeed different. We emphasize here to avoid any future confusion, that whenever two versions of the same type of statistic exist, we use the $*$ version to indicate the initial condition $\pi(0) = \pi(n+1) = n+1$, while the non-$*$ version means $\pi(0) = \pi(n+1) = 0$, with the only exception being Lemma 4.2.

Definition 2.1. For $\pi \in \mathfrak{S}_n$, let $\pi(0) = \pi(n+1) = 0$. Then any entry $\pi(i)$ $(i \in [n])$ can be classified according to one of the four cases:

- a *peak* if $\pi(i-1) < \pi(i)$ and $\pi(i) > \pi(i+1)$;
- a *valley* if $\pi(i-1) > \pi(i)$ and $\pi(i) < \pi(i+1)$;
- a *double ascent* if $\pi(i-1) < \pi(i)$ and $\pi(i) < \pi(i+1)$;
- a *double descent* if $\pi(i-1) > \pi(i)$ and $\pi(i) > \pi(i+1)$.

Let $\mathsf{peak}\,\pi$ (resp. $\mathsf{valley}\,\pi$, $\mathsf{da}\,\pi$, $\mathsf{dd}\,\pi$) count the number of peaks (resp. valleys, double ascents, double descents) in π, and define

$$\begin{aligned} \widetilde{\mathfrak{S}}_{n,k}(213) &:= \{\pi \in \mathfrak{S}_n(213) : \mathsf{dd}\,\pi = 0, \mathsf{des}\,\pi = k\}, \\ \widetilde{\mathfrak{S}}_{n,k}(312) &:= \{\pi \in \mathfrak{S}_n(312) : \mathsf{dd}\,\pi = 0, \mathsf{des}\,\pi = k\}, \\ \widetilde{\mathfrak{S}}_{n,k}(132) &:= \{\pi \in \mathfrak{S}_n(132) : \mathsf{dd}^*\,\pi = 0, \mathsf{des}\,\pi = k\}, \\ \widetilde{\mathfrak{S}}_{n,k}(231) &:= \{\pi \in \mathfrak{S}_n(231) : \mathsf{dd}^*\,\pi = 0, \mathsf{des}\,\pi = k\}. \end{aligned}$$

Besides the patterns mentioned in the introduction, we shall also consider the so-called vincular patterns [3]. The number of occurrences of vincular patterns 31-2, 2-31, 2-13 and 13-2 in $\pi \in \mathfrak{S}_n$ are defined by

$$(31\text{-}2)\,\pi = \#\{(i,j) : i+1 < j \le n \text{ and } \pi(i+1) < \pi(j) < \pi(i)\},$$

$$(2\text{-}31)\,\pi = \#\{(i,j) : j < i < n \text{ and } \pi(i+1) < \pi(j) < \pi(i)\},$$
$$(2\text{-}13)\,\pi = \#\{(i,j) : j < i < n \text{ and } \pi(i) < \pi(j) < \pi(i+1)\},$$
$$(13\text{-}2)\,\pi = \#\{(i,j) : i+1 < j \le n \text{ and } \pi(i) < \pi(j) < \pi(i+1)\}.$$

Definition 2.2. The statistic MAD, the number of **fix**ed points, **w**eak **ex**cedances, the **inv**ersion number, **cros**sing number and **i**nverse **cr**ossing number, **nest**ing number and **i**nverse **ne**sting number of $\pi \in \mathfrak{S}_n$ are defined by

$$\begin{aligned}
\mathsf{MAD}\,\pi &= \mathsf{des}\,\pi + (31\text{-}2)\,\pi + 2(2\text{-}31)\,\pi,\\
\mathsf{fix}\,\pi &= \sum_{1\le i\le n} \chi\Big(\pi(i) = i\Big),\\
\mathsf{wex}\,\pi &= \mathsf{exc}\,\pi + \mathsf{fix}\,\pi,\\
\mathsf{inv}\,\pi &= \sum_{1\le i<j\le n} \chi\Big(\pi(i) > \pi(j)\Big),\\
\mathsf{cros}\,\pi &= \#\{(i,j) :\ i < j \le \pi(i) < \pi(j) \quad \text{or} \quad \pi(i) < \pi(j) < i < j\},\\
\mathsf{nest}\,\pi &= \#\{(i,j) :\ i < j \le \pi(j) < \pi(i) \quad \text{or} \quad \pi(j) < \pi(i) < i < j\},\\
\mathsf{icr}\,\pi &= \mathsf{cros}\,\pi^{-1},\\
\mathsf{ine}\,\pi &= \mathsf{nest}\,\pi^{-1},
\end{aligned}$$

where $\chi(A) = 1$ if A is true and 0 otherwise.

For all $1 \le i \le n$, the entry $\pi(i)$ is called a *nondescent top* (resp. *nonexcedance top*) of π, if $\pi(i) < \pi(i+1)$ (resp. $\pi(i) \le i$), where $\pi(n+1) = n+1$. $\pi(i)$ is called a *left-to-right maximum* if $\pi(i) = \max\{\pi(1), \pi(2), \cdots, \pi(i)\}$. A nondescent top $\pi(i)$ $(i = 1, \cdots, n)$ is called a *foremaximum* of π if it is at the same time a left-to-right maximum. Denote the number of foremaximum of π by $\mathsf{fmax}\,\pi$.

Definition 2.3 (Shin-Zeng)**.** A permutation π is called *coderangement* if $\mathsf{fmax}\,\pi = 0$. Let $\mathfrak{D}_n^*$ be the subset of $\mathfrak{S}_n$ of coderangements.

For the rest of this subsection, we collect all the lemmas that will be useful in later sections, and prove Theorem 1.5.

The Clarke-Steingrímsson-Zeng bijection [10] linking des based statistics with exc based ones is crucial for our ensuing derivation. It is the composition of the Françon-Viennot bijection $\Psi_{FV} :\ \mathfrak{S}_n \to \mathfrak{L}_n$ in [19] and the inverse of the Foata-Zeilberger bijection $\Psi_{FZ} : \mathfrak{S}_n \to \mathfrak{L}_n$ in [18]. See [10] for a direct description of this composition $\Phi := \Psi_{FZ}^{-1} \circ \Psi_{FV}$ and further details. The following equidistribution result relies on Φ and is equivalent to Theorem 8 in [37] modulo one application of the inverse map: $\pi \mapsto \pi^{-1}$.

Lemma 2.4 (Shin-Zeng)**.** *For $n \ge 1$, there is a bijection Φ on $\mathfrak{S}_n$ such that*

$$(\mathsf{des}, \mathsf{fmax}, 31\text{-}2, 2\text{-}31, \mathsf{MAD})\,\pi = (\mathsf{exc}, \mathsf{fix}, \mathsf{icr}, \mathsf{ine}, \mathsf{inv})\,\Phi(\pi) \quad \text{for all } \pi \in \mathfrak{S}_n.$$

Using *Laguerre history* of the Motzkin path, Shin and the fourth author [37] deduced the continued fraction expansion for the quint-variate generating function of $\mathfrak{S}_n$ with respect to the above statistics.

Lemma 2.5 (Shin-Zeng)**.** *Let*

$$A_n(x,y,q,p,s) := \sum_{\pi\in\mathfrak{S}_n} x^{\mathsf{des}\,\pi} y^{\mathsf{fmax}\,\pi} q^{(31\text{-}2)\,\pi} p^{(2\text{-}31)\,\pi} s^{\mathsf{MAD}\,\pi}$$
$$= \sum_{\pi\in\mathfrak{S}_n} x^{\mathsf{exc}\,\pi} y^{\mathsf{fix}\,\pi} q^{\mathsf{icr}\,\pi} p^{\mathsf{ine}\,\pi} s^{\mathsf{inv}\,\pi}. \tag{2.1}$$

Then we have

$$1+\sum_{n=1}^{\infty} A_n(x,y,q,p,s)z^n = \cfrac{1}{1-b_0z-\cfrac{a_0c_1z^2}{1-b_1z-\cfrac{a_1c_2z^2}{\ddots}}}, \tag{2.2}$$

where, for $h\geq 0$,

$$a_h = s^{2h+1}[h+1]_{q,ps}, \quad b_h = yp^hs^{2h}+(x+q)s^h[h]_{q,ps},$$

and

$$c_h = x[h]_{q,ps}, \quad [h]_{u,v} := (u^h-v^h)/(u-v).$$

In order to make (2.1) suitable for the Catalan case, we have to make the following observation.

Lemma 2.6. *For any $n\geq 1$,*

$$\mathfrak{S}_n(2\text{-}13) = \mathfrak{S}_n(213), \tag{2.3}$$
$$\mathfrak{S}_n(31\text{-}2) = \mathfrak{S}_n(312), \tag{2.4}$$
$$\mathfrak{S}_n(13\text{-}2) = \mathfrak{S}_n(132), \tag{2.5}$$
$$\mathfrak{S}_n(2\text{-}31) = \mathfrak{S}_n(231). \tag{2.6}$$

Moreover, the mapping Φ has the property that $\Phi(\mathfrak{S}_n(231)) = \mathfrak{S}_n(321)$. Consequently, $\pi\in\mathfrak{S}_n(321)$ if and only if $\mathsf{nest}\,\pi = \mathsf{ine}\,\pi = 0$.

Proof. By definition we have $\mathfrak{S}_n(213)\subset\mathfrak{S}_n(2\text{-}13)$, Conversely, if $\pi\notin\mathfrak{S}_n(213)$, then π has the pattern 213, that is $k<i<j$, $\pi(i)<\pi(k)<\pi(j)$, then there must be some i', $i\leq i'<j$, and $\pi(i')<\pi(k)<\pi(i'+1)$, then $\pi\notin\mathfrak{S}_n(2\text{-}13)$. This proves (2.3). The proofs for (2.4)–(2.6) are similar.

For the second claim, since we already know Φ is a bijection and that $|\mathfrak{S}_n(231)| = |\mathfrak{S}_n(321)| = C_n$, it will suffice to show that for any $\sigma\in\mathfrak{S}_n(231)$, we have $\pi := \Phi(\sigma)\in\mathfrak{S}_n(321)$. Suppose on the contrary that $\pi\notin\mathfrak{S}_n(321)$, and we have $\pi(i)>\pi(j)>\pi(k)$ with $1\leq i<j<k\leq n$. We discuss by two cases:

- if $\pi(j)\leq j$, then $\pi(k)<\pi(j)\leq j<k$ form an inverse nesting of π;

- if $\pi(j) > j$, then $i < j < \pi(j) < \pi(i)$ form an inverse nesting of π.

Therefore in either case, we have (2-31) $\sigma = \mathsf{ine}\,\pi > 0$, which implies that $\sigma \notin \mathfrak{S}_n(231)$, a contradiction. For the final claim, note that $\pi \in \mathfrak{S}_n(321)$ if and only if $\pi^{-1} \in \mathfrak{S}_n(321)$, and the above argument indicates that $\pi \in \mathfrak{S}_n(321)$ if and only if $\mathsf{ine}\,\pi = 0$. Combine these two equivalences to finish the proof. □

By the above observation, the special $p = 0$, $q = 1$ case of Lemma 2.5 yields a result of Cheng et al. [9, Theorem 7.3].

Lemma 2.7 (Cheng et al.)**.** *We have*

$$\sum_{n=0}^{\infty}\Bigg(\sum_{\pi\in\mathfrak{S}_n(321)} q^{\mathsf{inv}\,\pi}t^{\mathsf{exc}\,\pi}y^{\mathsf{fix}\,\pi}\Bigg)z^n = \cfrac{1}{1-yz-\cfrac{tqz^2}{1-(1+t)qz-\cfrac{tq^3z^2}{1-(1+t)q^2z-\cfrac{tq^5z^2}{\ddots}}}}. \tag{2.7}$$

We also need a standard contraction formula for continued fractions, see [37, Eqs. (43) and (44)].

Lemma 2.8 (Contraction formula)**.** *There holds*

$$\cfrac{1}{1-\cfrac{c_1z}{1-\cfrac{c_2z}{1-\cfrac{c_3z}{1-\cfrac{c_4z}{\ddots}}}}} = \cfrac{1}{1-c_1z-\cfrac{c_1c_2z^2}{1-(c_2+c_3)z-\cfrac{c_3c_4z^2}{\ddots}}}$$

$$= 1+\cfrac{c_1z}{1-(c_1+c_2)z-\cfrac{c_2c_3z^2}{1-(c_3+c_4)z-\cfrac{c_4c_5z^2}{\ddots}}}.$$

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Letting $(t, y) = (-1/q, 1)$ in (2.7), we have by applying the contraction formula

$$1+\sum_{n=1}^{\infty}\Bigg(\sum_{\pi\in\mathfrak{S}_n(321)} q^{\mathsf{inv}\,\pi-\mathsf{exc}\,\pi}(-1)^{\mathsf{exc}\,\pi}\Bigg)z^n = 1+\cfrac{z}{1+\cfrac{qz^2}{1+\cfrac{q^3z^2}{1+\cfrac{q^5z^2}{\ddots}}}}.$$

We derive (1.20) by comparing this with (1.6).

In the same vein, by setting $(t, y) = (-1, 0)$ in (2.7), we obtain

$$\sum_{n=0}^{\infty}\Bigg(\sum_{\pi\in\mathfrak{D}_n(321)}(-1)^{\mathsf{exc}\,\pi}q^{\mathsf{inv}\,\pi}\Bigg)z^n = \cfrac{1}{1-\cfrac{(-q)z^2}{1-\cfrac{(-q^3)z^2}{1-\cfrac{(-q^5)z^2}{\ddots}}}}.$$

Comparing with (1.6), we readily get (1.21). □

2.2. **Other combinatorial interpretations of $C_n(q)$.** We can derive several pattern avoiding interpretations for our q-Catalan numbers $C_n(q)$ from γ-expansions due in [28] and [27]. Let

$$\widehat{\mathfrak{S}}_{n,k}(321) := \{\pi\in\mathfrak{S}_n(321) : \mathsf{exc}\,\pi = k \text{ and if } i<\pi(i), \text{ then } i+1 \text{ is a nonexcedance top}\}.$$

According to this definition, for any $\pi\in\widehat{\mathfrak{S}}_{n,k}(321)$, each occurrence of excedance is uniquely linked to an occurrence of nonexcedance. So when n is odd, the maximum for $\mathsf{exc}\,\pi$ is achieved at $k = \frac{n-1}{2}$, and in this case, the "if" condition becomes "if and only if". More precisely, take any $\pi\in\widehat{\mathfrak{S}}_{n,\frac{n-1}{2}}(321)$, we have for $1\le i\le n-1$,

- $i<\pi(i)$, if and only if
- $i+1$ is a nonexcedance top, if and only if
- $\pi(i)-1$ is a nonexecedance bottom.

This analysis shows that $\pi\in\widehat{\mathfrak{S}}_{2n+1,n}(321)$ is enumerated by C_n (see excercise 145 in [40]). The first alternative interpretation is (1.10) that we have seen in the introduction. Interestingly, we find yet another two q-γ-expansions in Lin's work [27, Theorems 1.2 and 1.4], that are amenable for such (-1)-evaluation as well.

Lemma 2.9 (Lin)**.** *For any $n\ge 1$,*

$$\sum_{\pi\in\mathfrak{S}_n(321)} t^{\mathsf{wex}\,\pi}q^{\mathsf{inv}\,\pi} = \sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}\Bigg(\sum_{\pi\in\mathrm{NDW}_{n,k}(321)} q^{\mathsf{inv}\,\pi}\Bigg)t^k(1+t/q)^{n+1-2k}, \tag{2.8}$$

$$\sum_{\pi\in\mathfrak{D}_n(321)} t^{\mathsf{exc}\,\pi}q^{\mathsf{inv}\,\pi} = \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\Bigg(\sum_{\pi\in\mathrm{NDE}_{n,k}(321)} q^{\mathsf{inv}\,\pi}\Bigg)t^k(1+t)^{n-2k}, \tag{2.9}$$

where

$$\mathrm{NDW}_{n,k}(321) := \{\pi\in\mathfrak{S}_n(321) : \mathsf{wex}\,\pi = k,\ \text{no } i \text{ such that } \pi(i+1)\ge i+1, i\ge\pi^{-1}(i)\},$$
$$\mathrm{NDE}_{n,k}(321) := \{\pi\in\mathfrak{D}_n(321) : \mathsf{exc}\,\pi = k,\ \mathsf{cda}\,\pi = 0\}.$$

See Definition 4.1 for the definition of cda. Now if we plug in $t=-1$ in (1.10) (resp. $t=-q$ in (2.8), $t=-1$ in (2.9)), and compare the result with (1.20) (resp. with (1.21)), we discover the following relations.

Proposition 2.10. *For any $n\geq 1$,*

$$C_n(q^2)=q^{-2n}\sum_{\pi\in\widehat{\mathfrak{S}}_{2n+1,n}(321)}q^{\mathsf{inv}\,\pi}=q^{-2n}\sum_{\pi\in\mathrm{NDW}_{2n+1,n+1}(321)}q^{\mathsf{inv}\,\pi}=q^{-n}\sum_{\pi\in\mathrm{NDE}_{2n,n}(321)}q^{\mathsf{inv}\,\pi}.$$

Remark 2.11. Two remarks on Proposition 2.10 are in order. First, as a by-product we note that $\mathsf{inv}\,\pi$ is even for any $\pi\in\widehat{\mathfrak{S}}_{2n+1,n}(321)$ (resp. $\pi\in\mathrm{NDW}_{2n+1,n+1}(321)$), and $\mathsf{inv}\,\pi$ has the parity of n for any $\pi\in\mathrm{NDE}_{2n,n}(321)$. A direct combinatorial explanation of this might be interesting. On the other hand, from a bijective point of view, we note that the second equality above is a natural result of the inverse map $\pi\mapsto\pi^{-1}$, while a bijection deducing the third equality is possible via the two colored Motzkin path [27, 28]. We leave the details as exercises for motivated readers. Moreover, we note by passing that $|\mathrm{NDE}_{2n,n}(321)|=C_n$ is equivalent to Exercise 151 in [40].

3. Proofs of Theorems 1.1, 1.2 and 1.4

The statistic *admissible inversion* was first introduced by Shareshian and Wachs [36].

Definition 3.1. Let $\pi=\pi(1)\pi(2)\cdots\pi(n)$ be a permutation of $\mathfrak{S}_n$ and $\pi(0)=\pi(n+1)=0$. An *admissible inversion* of π is an inversion pair $(\pi(i),\pi(j))$, i.e., $1\leq i<j\leq n$ and $\pi(i)>\pi(j)$, satisfying either of the following conditions:

- $\pi(j)<\pi(j+1)$ or
- there is some l such that $i<l<j$ and $\pi(j)>\pi(l)$.

We need also a variant of the above definition introduced in [29, Definition 1].

Definition 3.2. Let $\pi=\pi(1)\pi(2)\cdots\pi(n)$ be a permutation of $\mathfrak{S}_n$ and $\pi(0)=\pi(n+1)=n+1$. A *star admissible inversion* of π is a pair $(\pi(i),\pi(j))$ such that $1\leq i<j\leq n$ and $\pi(i)>\pi(j)$ and satisfies either of the following conditions:

- $\pi(i-1)<\pi(i)$ or
- there is some l such that $i<l<j$ and $\pi(i)<\pi(l)$.

Let $\mathsf{adi}\,\pi$ and $\mathsf{adi}^*\,\pi$ be the numbers of admissible inversions and star admissible inversions of $\pi\in\mathfrak{S}_n$, respectively. For example, if $\pi=231$, then $\mathsf{adi}\,\pi=0$ while $\mathsf{adi}^*\,\pi=2$.

Lemma 3.3. *We have*

$$\mathsf{adi}\,\pi=(2\text{-}13)\,\pi,\ \text{ if } \pi\in\mathfrak{S}_n(312), \tag{3.1}$$
$$\mathsf{adi}^*\,\pi=(13\text{-}2)\,\pi,\ \text{ if } \pi\in\mathfrak{S}_n(231). \tag{3.2}$$

Proof. By Definition 3.1 an inversion pair $(\pi(i),\pi(j))$ of a permutation $\pi\in\mathfrak{S}_n$ is admissible if and only if either of the following conditions holds

- the triple $(\pi(i),\pi(j),\pi(j+1))$ forms a pattern 2-13 or 3-12;

- the triple $(\pi(i), \pi(l), \pi(j))$ with $i < l < j$ forms a pattern 312.

Thus, if $\pi \in \mathfrak{S}_n(312)$, by (2.4) the permutation π avoids both 312 and 3-12. This proves (3.1). The proof of (3.2) is similar. □

Let $\pi \in \mathfrak{S}_n$, for any $x \in [n]$, the *x-factorization* of π reads $\pi = w_1 w_2 x w_3 w_4$, where w_2 (resp. w_3) is the maximal contiguous subword immediately to the left (resp. right) of x whose letters are all larger than x. Following Foata and Strehl [17] we define the action φ_x by

$$\varphi_x(\pi) = w_1 w_3 x w_2 w_4.$$

For instance, if $x = 3$ and $\pi = 28531746 \in \mathfrak{S}_7$, then $w_1 = 2, w_2 = 85, w_3 = \varnothing$ and $w_4 = 1746$. Thus $\varphi_x(\pi) = 23851746$. Clearly, φ_x is an involution acting on $\mathfrak{S}_n$ and it is not hard to see that φ_x and φ_y commute for all $x, y \in [n]$. Brändén [6] modified φ_x to be

$$\varphi'_x(\pi) := \begin{cases} \varphi_x(\pi), & \text{if } x \text{ is a double ascent or double descent of } \pi; \\ \pi, & \text{if } x \text{ is a valley or a peak of } \pi. \end{cases}$$

Again it is clear that φ'_x's are involutions and commute. For any subset $S \subseteq [n]$ we can then define the function $\varphi'_S : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\varphi'_S(\pi) = \prod_{x \in S} \varphi'_x(\pi).$$

Hence the group $\mathbb{Z}_2^n$ acts on $\mathfrak{S}_n$ via the functions φ'_S, $S \subseteq [n]$. This action will be called the *Modified Foata–Strehl action* (*MFS-action* for short) as depicted in Fig. 1 (recall the initial condition $\pi(0) = \pi(n+1) = 0$).

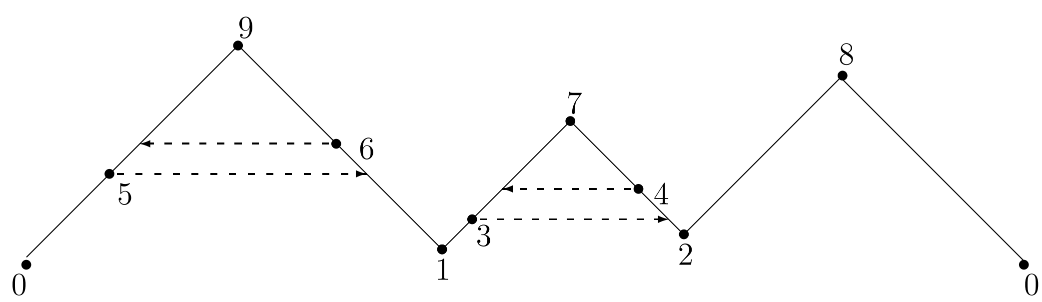

FIGURE 1. MFS-actions on 596137428

Remark 3.4. The initial condition $\pi(0) = \pi(n+1) = 0$, the definition of adi, and the construction of the MSF-action, are all dual to those used by Lin-Zeng in [29]. When patterns $\{231, 132, 2\text{-}31, 13\text{-}2\}$ are concerned, we use Lin-Zeng's version, while for patterns $\{213, 312, 2\text{-}13, 31\text{-}2\}$ we use our current version. We include all the constructions here to make this paper self-contained.

Lemma 3.5. *Let $\pi \in \mathfrak{S}_n$. For each $x \in [n]$, we have* $\mathsf{adi}\,\pi = \mathsf{adi}\,\varphi'_x(\pi)$.

Proof. If x is a peak or a valley of π, then $\varphi'_x(\pi) = \pi$ and the result is true. If x is a double descent of π, then $\pi = w_1 w_2 x w_4$ with $w_3 = \varnothing$, and there are no admissible inversions of π formed by x and one letter in w_2. As $\varphi'_x(\pi) = w_1 x w_2 w_4$, there are no inversions of $\varphi'_x(\pi)$ between w_2 and x. Let $(\pi(i), \pi(j)) \notin \{(y, x) : y \text{ is a letter in } w_2\}$ be a pair in π such that $i < j$. We claim that $(\pi(i), \pi(j))$ is an admissible inversion of π if and only if it is an admissible inversion of $\varphi'_x(\pi)$, from which the result follows.

For a word w, we write $a \in w$ if a is a letter in w. To check the claim, there are six cases to be considered: (1) $\pi(i) \in w_1$ and $\pi(j) \in w_1$; (2) $\pi(i) \in w_1$ and $\pi(j) \in w_2 x$; (3) $\pi(i) \in w_1$ and $\pi(j) \in w_4$; (4) $\pi(i) \in w_2$ and $\pi(j) \in w_2$; (5) $\pi(i) \in w_2 x$ and $\pi(j) \in w_4$; (6) $\pi(i) \in w_4$ and $\pi(j) \in w_4$. We will only show case (5), other cases are similar. If $(\pi(i), \pi(j))$ is an admissible inversion of π, then $\pi(i) > \pi(j) < \pi(j+1)$ or $\pi(i) > \pi(j) > \pi(k)$ for some $i < k < j$. Clearly, $(\pi(i), \pi(j))$ is an admissible of $\varphi'_x(\pi)$ if $\pi(k) \neq x$. Otherwise $\pi(k) = x$, then we denote x' the first letter of w_4 and consider the triple $(\pi(i), x', \pi(j))$ in $\varphi'_x(\pi)$. This indicates that $(\pi(i), \pi(j))$ is an admissible inversion of $\varphi'_x(\pi)$, since $x' < x < \pi(j) < \pi(i)$. To show that, if $(\pi(i), \pi(j))$ is an admissible inversion of $\varphi'(\pi)$ then $(\pi(i), \pi(j))$ is an admissible inversion of π, is similar and we omit. This finishes the proof of our claim in case (5). □

Lemma 3.6. *The statistics* (2-31), (13-2), (2-13) *and* (31-2) *are constant on any orbit under the MFS-action.*

Proof. For $\pi \in \mathfrak{S}_n$, when $\pi(0) = \pi(n+1) = n+1$, the cases (2-31) and (13-2) were proved by Brändén [6, Theorem 5.1]. For the case (2-13), let $\pi(0) = \pi(n+1) = 0$, and note that (2-13) π is the number of triples $(\pi(i), \pi(j), \pi(k))$ such that $1 \leq i < j < k \leq n$ and $\pi(j) < \pi(i) < \pi(k)$, where $(\pi(j), \pi(k))$ is a pair of consecutive valley and peak, that is, there are no other peaks and valleys in between $\pi(j)$ and $\pi(k)$. The number of such triples is invariant under the action since $\pi(j)$ and $\pi(k)$ cannot move and neither can $\pi(i)$ hop over the valley $\pi(j)$. A similar argument leads to the case (31-2). □

Lemma 3.7. *The MFS-action preserves the pattern* 213, 312, 132 *and* 231*, i.e., the map φ'_S is closed on the subsets $\mathfrak{S}_n(\tau)$, for $\tau = 213, 312, 132, 231$.*

Proof. Suppose $\pi \notin \mathfrak{S}_n(213)$, so that there is a triple of indices $i < j < k$ with $\pi(j) < \pi(i) < \pi(k)$. Then without loss of generality, we may assume $\pi(j)$ is a valley. (Otherwise, there is a valley $\pi(j')$ with $i < j' < j$ or $j < j' < k$, and $\pi(j') < \pi(j)$.) Under the MFS-action, the relative positions of the letters $\pi(i), \pi(j), \pi(k)$ are preserved, since neither $\pi(i)$ nor $\pi(k)$ can hop past $\pi(j)$. See Fig. 2 for an illustration. □

Proof of Theorem 1.2. Thanks to Lemma 3.6, the statistics tracked by the power of q remain constant inside each orbit under the MFS-action. We prove the 213-avoiding case in (1.11) here and omit the details for the remaining ones. For any permutation $\pi \in \mathfrak{S}_n$, let $\mathsf{Orb}(\pi) = \{g(\pi) : g \in \mathbb{Z}_2^n\}$ be the orbit of π under the MFS-action. The MFS-action divides the set $\mathfrak{S}_n$ into disjoint orbits. Moreover, for $\pi \in \mathfrak{S}_n$, x is a double descent of π if and only if x is a double ascent of $\varphi'_x(\pi)$. Hence, there is a unique permutation in

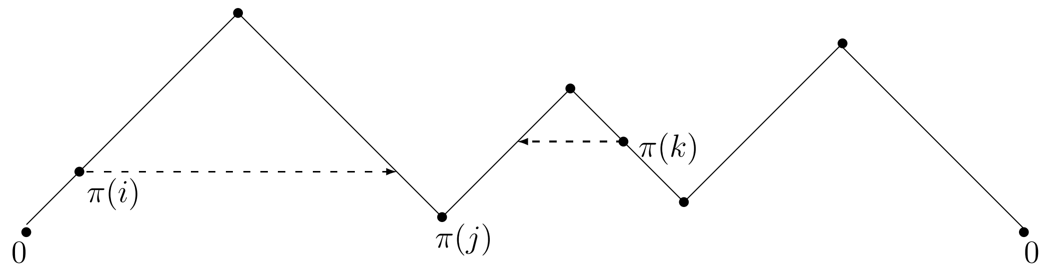

FIGURE 2. MFS-actions on pattern avoidance 213

each orbit which has no double descent. Now, let $\bar\pi$ be this unique element in $\mathsf{Orb}(\pi)$, then $\mathsf{da}\,\bar\pi = n - \mathsf{peak}\,\bar\pi - \mathsf{valley}\,\bar\pi$ and $\mathsf{des}\,\bar\pi = \mathsf{peak}\,\bar\pi - 1 = \mathsf{valley}\,\bar\pi$. Thus

$$\sum_{\sigma\in\mathsf{Orb}\,\pi} q^{(31\text{-}2)\,\sigma} t^{\mathsf{des}\,\sigma} = q^{(31\text{-}2)\,\bar\pi} t^{\mathsf{des}\,\bar\pi}(1+t)^{\mathsf{da}\,\bar\pi} = q^{(31\text{-}2)\,\bar\pi} t^{\mathsf{des}\,\bar\pi}(1+t)^{n-2\mathsf{des}\,\bar\pi-1}.$$

According to Lemma 3.7, by summing over all the orbits that compose together to form $\mathfrak{S}_n(213)$, we obtain the 213-avoiding case (1.11) immediately. □

Proof of Theorem 1.4. Clearly the reverse-complement transformation $\pi \mapsto \pi^{rc}$ satisfies $(\mathsf{des}, 213, \mathsf{adi})\,\pi = (\mathsf{des}, 132, \mathsf{adi}^*)\,\pi^{rc}$, which yields (1.13) directly. With Lemma 3.5 and Lemma 3.7, we obtain (1.14) and (1.15) via the MFS-action in a similar fashion as in the proof of Theorem 1.2. □

Remark 3.8. When $q = 1$, Theorem 1.2 reduces to

$$N_n(t) = \sum_{\pi\in\mathfrak{S}_n(\tau)} t^{\mathsf{des}\,\pi} = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \left|\widetilde{\mathfrak{S}}_{n,k}(\tau)\right| t^k(1+t)^{n-1-2k}, \tag{3.3}$$

where $\tau \in \{213, 312, 132, 231\}$. We note that the 231-case is exactly (1.2), and the 312-case then follows via the reverse-complement map. We have not found the 213-case or 132-case in the literature.

The following result follows from [38, Eq. (39)].

Lemma 3.9 (Shin-Zeng)**.** *The following four polynomials are equal*

$$\begin{aligned}\sum_{\pi\in\mathfrak{S}_n} t^{\mathsf{des}\,\pi} p^{(2\text{-}13)\,\pi} q^{(31\text{-}2)\,\pi} &= \sum_{\pi\in\mathfrak{S}_n} t^{\mathsf{des}\,\pi} p^{(31\text{-}2)\,\pi} q^{(2\text{-}13)\,\pi}\\ = \sum_{\pi\in\mathfrak{S}_n} t^{\mathsf{des}\,\pi} p^{(2\text{-}31)\,\pi} q^{(31\text{-}2)\,\pi} &= \sum_{\pi\in\mathfrak{S}_n} t^{\mathsf{des}\,\pi} p^{(31\text{-}2)\,\pi} q^{(2\text{-}31)\,\pi}.\end{aligned}$$

Proof. Indeed, the equation (39) in [38] reads:

$$\sum_{n=0}^{\infty}\Bigg(\sum_{\pi\in\mathfrak{S}_n} t^{\mathsf{des}\,\pi} p^{(2\text{-}13)\,\pi} q^{(31\text{-}2)\,\pi}\Bigg) z^n = \sum_{n=0}^{\infty}\Bigg(\sum_{\pi\in\mathfrak{S}_n} t^{\mathsf{des}\,\pi} p^{(2\text{-}31)\,\pi} q^{(31\text{-}2)\,\pi}\Bigg) z^n$$

$$= \cfrac{1}{1-\cfrac{c_1 z}{1-\cfrac{c_2 z}{1-\cfrac{c_3 z}{\ddots}}}}$$

with $c_{2i} = t[i]_{p,q}$ and $c_{2i-1} = [i]_{p,q}$ for $i \geq 1$, where two misprints in [38] are corrected. The continued fraction shows clearly that the generating function is symmetric in p and q. □

Proof of Theorem 1.1. With (2.1) and Lemma 2.6, we obtain

$$\sum_{\pi\in\mathfrak{S}_n(321)} t^{\mathsf{exc}\,\pi} q^{\mathsf{inv}\,\pi-\mathsf{exc}\,\pi} = \sum_{\pi\in\mathfrak{S}_n(231)} t^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi}.$$

With Lemma 2.6 and Lemma 3.9, we have

$$\sum_{\pi\in\mathfrak{S}_n(213)} t^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(312)} t^{\mathsf{des}\,\pi} q^{(2\text{-}13)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(231)} t^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(312)} t^{\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi}.$$

By Lemma 3.3, we have

$$\sum_{\pi\in\mathfrak{S}_n(312)} t^{\mathsf{des}\,\pi} q^{(2\text{-}13)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(312)} t^{\mathsf{des}\,\pi} q^{\mathsf{adi}\,\pi},$$

$$\sum_{\pi\in\mathfrak{S}_n(231)} t^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(231)} t^{\mathsf{des}\,\pi} q^{\mathsf{adi}^*\,\pi}.$$

Clearly the reverse-complement transformation $\pi \mapsto \pi^{rc}$ provides us with

$$\sum_{\pi\in\mathfrak{S}_n(213)} t^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(132)} t^{\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi}.$$

Finally the reverse map $\pi \mapsto \pi^r$ together with (1.9) give us

$$\sum_{\pi\in\mathfrak{S}_n(312)} t^{\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(312)} t^{n-1-\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(213)} t^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi},$$

$$\sum_{\pi\in\mathfrak{S}_n(231)} t^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(231)} t^{n-1-\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(132)} t^{\mathsf{des}\,\pi} q^{(2\text{-}13)\,\pi},$$

$$\sum_{\pi\in\mathfrak{S}_n(132)} t^{\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(132)} t^{n-1-\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(231)} t^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi}.$$

By gathering all the equalities above, we complete the proof. □

Lemma 3.10. *For* $\pi \in \widetilde{\mathfrak{S}}_{2n+1,n}(213)$, *we have* $\mathsf{adi}\,\pi = 2(3\text{-}12)\,\pi$, *and* $\mathsf{adi}\,\pi^r = (31\text{-}2)\,\pi$.

Proof. For $\pi \in \widetilde{\mathfrak{S}}_{2n+1,n}(213)$, that is π is a down-up permutation with the first letter being $2n+1$. For $\pi(i) > \pi(j)$ with $1 \leq i < j \leq 2n+1$, if $\pi(j)$ is a valley, we have $(i, j, j+1)$ such that $\pi(i) > \pi(j+1) > \pi(j)$, thus a (3-12) pattern. In the meantime, by Definition 3.1, this triple produces two admissible inversion pairs, namely $(\pi(i), \pi(j))$ and $(\pi(i), \pi(j+1))$.

if $\pi(j)$ is a peak, we have instead $(i, j-1, j)$ such that $\pi(i) > \pi(j) > \pi(j-1)$, and the two admissible inversion paris are $(\pi(i), \pi(j-1))$ and $(\pi(i), \pi(j))$. For the second equality, simply note that if $\pi \in \mathfrak{S}_n(213)$, then $\pi^r \in \mathfrak{S}_n(312)$, and by Lemma 3.3 we get $\mathsf{adi}\,\pi^r = (2\text{-}13)\,\pi^r = (31\text{-}2)\,\pi$ and the proof is completed. □

Lemma 3.11. *If* $\pi \in \widetilde{\mathfrak{S}}_{2n+1,n}(213)$, *then* $\mathsf{adi}\,\pi + \mathsf{adi}\,\pi^r = 2n^2 + n$.

Proof. Take any $\pi = \pi(1)\pi(2)\cdots\pi(2n+1) \in \widetilde{\mathfrak{S}}_{2n+1,n}(213)$, the right-hand side is the number of ways to choose a pair $(\pi(i), \pi(j))$ with $1 \le i < j \le 2n+1$. It will suffice now to show that any such pair contributes to either $\mathsf{adi}\,\pi$ or $\mathsf{adi}\,\pi^r$. If $\pi(i) > \pi(j)$, we have seen in the proof of the last lemma, that no matter $\pi(j)$ is a valley or a peak, $(\pi(i), \pi(j))$ always forms an admissible inversion pair. Otherwise we have $\pi(i) < \pi(j)$, now if $\pi(i)$ is a peak, then $(\pi(i), \pi(i+1), \pi(j))$ will produce a (213) pattern, so $\pi(i)$ must be a valley, then $\pi(i) < \pi(i-1)$ and consequently $(\pi(j), \pi(i))$ forms an admissible inversion pair in π^r. This completes the proof. □

With the above two lemmas we obtain another combinatorial interpretation of $C_n(q)$.

Proposition 3.12. *For any* $n \ge 1$,

$$C_n(q) = \sum_{\pi \in \widetilde{\mathfrak{S}}_{2n+1,n}(213)} q^{n^2 - (3\text{-}12)\,\pi} \tag{3.4}$$

Proof. By Lemma 3.6 and Lemma 3.7 concerning the MFS-action, we have

$$\sum_{\pi \in \mathfrak{S}_{2n+1}(213)} t^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = \sum_{k=0}^{n} \Bigg(\sum_{\pi \in \widetilde{\mathfrak{S}}_{2n+1,k}(213)} q^{(31\text{-}2)\,\pi} \Bigg) t^k (1+t)^{2n-2k}. \tag{3.5}$$

With Theorem 1.1 in mind, we take $t = -1$ in equations (3.5) and (1.10) to get

$$\sum_{\pi \in \widetilde{\mathfrak{S}}_{2n+1,n}(213)} q^{(31\text{-}2)\,\pi} = \sum_{\pi \in \widetilde{\mathfrak{S}}_{2n+1,n}(321)} q^{\mathsf{inv}\,\pi - \mathsf{exc}\,\pi} = q^n C_n(q^2).$$

Then we apply Lemmas 3.10 and 3.11 to get (3.4) after simplification.

□

4. A variant of q-Narayana polynomials

Definition 4.1. For $\pi \in \mathfrak{S}_n$, a value $x = \pi(i)$ $(i \in [n])$ is called

- a cyclic valley if $i = \pi^{-1}(x) > x$ and $x < \pi(x)$;
- a double excedance if $i = \pi^{-1}(x) < x$ and $x < \pi(x)$;
- a drop if $x = \pi(i) < i$.

Let $\mathsf{cvalley}$ (resp. cda, drop) denote the number of cyclic valleys (resp. double excedances, drops) in π. The following result is due to Shin-Zeng [38, Theorem 5].

Lemma 4.2 (Shin-Zeng)**.** *There is a bijection Υ on $\mathfrak{S}_n$ such that for all $\pi \in \mathfrak{S}_n$,*

$$(\mathsf{nest}, \mathsf{cros}, \mathsf{drop}, \mathsf{cda}, \mathsf{cdd}, \mathsf{cvalley}, \mathsf{fix})\, \pi = (2\text{-}31, 31\text{-}2, \mathsf{des}, \mathsf{da} - \mathsf{fmax}, \mathsf{dd}, \mathsf{valley}, \mathsf{fmax})\, \Upsilon(\pi),$$

where the linear statistics on the right-hand side are defined with the convention $\pi(0) = 0$ and $\pi(n+1) = n+1$ for $\pi \in \mathfrak{S}_n$.

Theorem 4.3. *we have*

$$\begin{aligned} W_n(t,q) &:= \sum_{\pi\in\mathfrak{S}_n(321)} t^{\mathsf{wex}\,\pi} q^{\mathsf{inv}\,\pi} \\ &= t^n \sum_{\pi\in\mathfrak{S}_n(231)} (q/t)^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = t^n \sum_{\pi\in\mathfrak{S}_n(231)} (q/t)^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi} = t^n \sum_{\pi\in\mathfrak{S}_n(231)} (q/t)^{\mathsf{des}\,\pi} q^{\mathsf{adi}^*\,\pi} \\ &= t^n \sum_{\pi\in\mathfrak{S}_n(312)} (q/t)^{\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi} = t^n \sum_{\pi\in\mathfrak{S}_n(312)} (q/t)^{\mathsf{des}\,\pi} q^{(2\text{-}13)\,\pi} = t^n \sum_{\pi\in\mathfrak{S}_n(312)} (q/t)^{\mathsf{des}\,\pi} q^{\mathsf{adi}\,\pi} \\ &= t^n \sum_{\pi\in\mathfrak{S}_n(213)} (q/t)^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = t^n \sum_{\pi\in\mathfrak{S}_n(213)} (q/t)^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi} \\ &= t^n \sum_{\pi\in\mathfrak{S}_n(132)} (q/t)^{\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi} = t^n \sum_{\pi\in\mathfrak{S}_n(132)} (q/t)^{\mathsf{des}\,\pi} q^{(2\text{-}13)\,\pi}. \end{aligned}$$

Moreover we have

$$\sum_{n=0}^{\infty} W_n(t,q) z^n = \cfrac{1}{1 - tz - \cfrac{tqz^2}{1-(1+t)qz - \cfrac{tq^3z^2}{1-(1+t)q^2z - \cfrac{tq^5z^2}{\ddots}}}}. \tag{4.1}$$

Proof. Since $\mathsf{drop}\,\pi = n - \mathsf{wex}\,\pi$ and $\mathsf{inv}\,\pi = n - \mathsf{wex}\,\pi + \mathsf{cros}\,\pi + 2\mathsf{nest}\,\pi$ ([37, Eq. (40)]), we have

$$\sum_{\pi\in\mathfrak{S}_n(321)} t^{\mathsf{wex}\,\pi} q^{\mathsf{inv}\,\pi} = \sum_{\pi\in\mathfrak{S}_n(321)} t^{n-\mathsf{drop}\,\pi} q^{\mathsf{inv}\,\pi} = t^n \sum_{\pi\in\mathfrak{S}_n(321)} (q/t)^{\mathsf{drop}\,\pi} q^{\mathsf{cros}\,\pi}.$$

By Theorem 1.1 and Lemma 4.2, we have

$$t^n \sum_{\pi\in\mathfrak{S}_n(321)} (q/t)^{\mathsf{drop}\,\pi} q^{\mathsf{cros}\,\pi} = t^n \sum_{\pi\in\mathfrak{S}_n(231)} (q/t)^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = t^n \sum_{\pi\in\mathfrak{S}_n(231)} (q/t)^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi},$$

and the remaining equalities follow similarly.

Finally, Eq. (4.1) is the special $y = t$ case of (2.7). ☐

We can now derive another q-γ-expansion for the joint distribution of wex and inv over $\mathfrak{S}_n(321)$.

Theorem 4.4. *For any* $n \geq 1$,

$$\sum_{\pi\in\mathfrak{S}_n(321)} t^{\mathsf{wex}\,\pi} q^{\mathsf{inv}\,\pi} = \sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}\Bigg(q^{n-k}\sum_{\pi\in\widetilde{\mathfrak{S}}_{n,k-1}(231)} q^{(13\text{-}2)\,\pi}\Bigg)t^k(1+t/q)^{n+1-2k}. \tag{4.2}$$

Proof. By Theorems 4.3 and 1.2,

$$t^n\sum_{\pi\in\mathfrak{S}_n(231)}(q/t)^{\mathsf{des}\,\pi}q^{(13\text{-}2)\,\pi} = t^n\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\Bigg(\sum_{\pi\in\widetilde{\mathfrak{S}}_{n,k}(231)} q^{(13\text{-}2)\,\pi}\Bigg)(q/t)^k(1+q/t)^{n-1-2k}.$$

For the right-hand side of above equation, by shifting k to $k-1$, we get (4.2). $\square$

Comparing (4.2) with (2.8), by utilizing Theorem 4.3 and Theorem 1.2, we obtain the following equivalent q-analogues of γ-coefficients with the same arguments in the proof of Theorem 4.4.

Corollary 4.5. *There holds*

$$\begin{aligned}\gamma_{n,k}(q) &:= \sum_{\pi\in\mathrm{NDW}_{n,k}(321)} q^{\mathsf{inv}\,\pi}\\ &= q^{n-k}\sum_{\pi\in\widetilde{\mathfrak{S}}_{n,k-1}(231)} q^{(13\text{-}2)\,\pi} = q^{n-k}\sum_{\pi\in\widetilde{\mathfrak{S}}_{n,k-1}(132)} q^{(2\text{-}31)\,\pi}\\ &= q^{n-k}\sum_{\pi\in\widetilde{\mathfrak{S}}_{n,k-1}(312)} q^{(2\text{-}13)\,\pi} = q^{n-k}\sum_{\pi\in\widetilde{\mathfrak{S}}_{n,k-1}(213)} q^{(31\text{-}2)\,\pi}.\end{aligned}$$

5. The Catalan case: a complete characterization

5.1. The 231-avoding des-case and its q-analogues.

The 231-avoiding alternating permutations were first enumerated by Mansour [30] (see also exercises 149 and 150 in [40]):

$$|\mathfrak{A}_{2n+1}(231)| = |\mathfrak{A}_{2n}(231)| = C_n, \text{ for } n\geq 0. \tag{5.1}$$

A bijective proof of this fact with further implications was given by Lewis [23]. Indeed, basing on (5.1) and utilizing the reverse map as well as the reverse complement map, one get the complete enumerations for all alternating permutations avoiding a single pattern of length three (see Table 1).

Recall the *standardization* of a word w with n distinct ordered letters, denoted as $\mathrm{st}(w)$, is the unique permutation in $\mathfrak{S}_n$ that is order isomorphic to w. We say a word w_1 is superior to another word w_2 and denote as $w_1 > w_2$, if for any two letters $l_1\in w_1, l_2\in w_2$, we always have $l_1 > l_2$. The following decomposition is crucial for deriving q-analogues of the (-1)-phenomenon on pattern-avoiding subsets of the coderangements.

Lemma 5.1. *Let* $P_0(t,q)=Q_0(t,q)=R_1(t,q)=1, P_1(t,q)=Q_1(t,q)=0,$ *and for* $n\geq 2$,

$$P_n(t,q) := \sum_{\pi\in\mathfrak{D}_n^*(231)} t^{\mathsf{des}\,\pi}q^{(13\text{-}2)\,\pi},$$

$$Q_n(t,q) := \sum_{\pi\in\mathfrak{D}_n^*(132)} t^{\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi},$$
$$R_n(t,q) := \sum_{\pi\in\mathfrak{D}_n^*(213)} t^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi}.$$

Then for $n \geq 2$,

$$P_n(t,q) = \sum_{m=0}^{n-2} tq^{n-m-1} P_m(t,q) N_{n-m-1}(t,q), \tag{5.2}$$

$$Q_n(t,q) = \sum_{m=0}^{n-2} tq^{m} Q_m(t,q) N_{n-m-1}(t,q), \tag{5.3}$$

$$R_n(t,q) = \sum_{m=1}^{n-1} tq^{n-m-1} R_m(t,q) N_{n-m-1}(t,q). \tag{5.4}$$

Proof. The key observation is that by definition, $\pi\in\mathfrak{D}_n^*(231)$ if and only if $\pi = \pi^{(1)} n \pi^{(2)}$, for some subwords $\pi^{(1)}$ and $\pi^{(2)} \neq \varnothing$, such that $\pi^{(1)} \in \mathfrak{D}_m^*(231)$ and $\mathrm{st}(\pi^{(2)}) \in \mathfrak{S}_{n-m-1}(231)$, for some $m, 0 \leq m \leq n-2$, with $\pi^{(2)} > \pi^{(1)}$. Then we use the appropriate interpretation for $N_{n-m-1}(t,q)$ taken from Theorem 1.1 and examine the change of des and (13-2) during this decomposition. This should give us (5.2), the proofs of (5.3) and (5.4) are similar and thus omitted. □

Now we can derive the following q-analogues for the strong (-1)-phenomenon on $\mathfrak{S}_n(231)$ concerning des, which parallels Theorem 1.5 nicely.

Theorem 5.2. *For any $n \geq 1$,*

$$\sum_{\pi\in\mathfrak{S}_n(231)} (-1)^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(231)} (-1)^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi} = \begin{cases} 0 & \text{if } n \text{ is even,}\\ (-q)^{\frac{n-1}{2}} C_{\frac{n-1}{2}}(q^2) & \text{if } n \text{ is odd,}\end{cases} \tag{5.5}$$

$$\sum_{\pi\in\mathfrak{D}_n^*(231)} (-q)^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = \begin{cases} (-q)^{\frac{n}{2}} C_{\frac{n}{2}}(q^2) & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd,}\end{cases} \tag{5.6}$$

$$\sum_{\pi\in\mathfrak{D}_n^*(231)} (-1)^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi} = \begin{cases} (-1)^{\frac{n}{2}} C^*_{\frac{n}{2}}(q) & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd,}\end{cases} \tag{5.7}$$

where $C_n^*(q) := \sum_{\pi\in\mathfrak{A}_{2n}(132)} q^{(2\text{-}31)\,\pi}$. *For example,*

$$C_0^*(q) = C_1^*(q) = 1,$$
$$C_2^*(q) = 2q,$$
$$C_3^*(q) = 3q^2 + 2q^4,$$

TABLE 1. The enumeration of $\mathfrak{A}_n(\tau)$, for $n \geq 3$ odd and even.

τ	123	132	213	231	312	321
$\mathfrak{A}_{2n+1}(\tau)$	C_{n+1}	C_n	C_{n+1}	C_n	C_{n+1}	C_{n+1}
$\mathfrak{A}_{2n}(\tau)$	C_n	C_n	C_n	C_n	C_n	C_{n+1}

TABLE 2. The (-1)-evaluation over $\mathfrak{S}_n(\tau)$ and $\mathfrak{D}^*_n(\tau)$ with respect to des. The signs $(-1)^n$ have all been removed.

des $\setminus \tau$	123	132	213	231	312	321
$\mathfrak{S}_{2n+1}$	$\star$	C_n	C_n	C_n	C_n	$\star$
$\mathfrak{D}^*_{2n}$	$\star$	C_n	C_{n-1}	C_n	$\star$	$\star$

$$C^*_4(q) = 4q^3 + 6q^5 + 2q^7 + 2q^9,$$
$$C^*_5(q) = 5q^4 + 12q^6 + 9q^8 + 8q^{10} + 4q^{12} + 2q^{14} + 2q^{16}.$$

Proof. All we need to do for proving (5.5) (resp. (5.6)) is take $t = -1$ (resp. $(x, y, q, p, s) = (-1, 0, 1, 0, q)$) in Theorem 1.1 (resp. (2.1)), then apply Theorem 1.5. Next for (5.7), with the decomposition (5.2) in mind, we note that $P_{2n+1}(-1, q) = 0$ follows from induction on $P_m(-1, q)$ and using (1.20) for $N_{n-m-1}(-1, q)$. In the same vein, the even $2n$ case reduces to proving the following identity:

$$C^*_n(q) = \sum_{m=0}^{n-1} q^{3n-3m-2} C^*_m(q) C_{n-m-1}(q^2). \tag{5.8}$$

Combining Proposition 2.10 and Corollary 4.5, we get the desired interpretation that meshes well with that of $C^*_m(q)$:

$$q^{n-m-1} C_{n-m-1}(q^2) = \sum_{\pi \in \mathfrak{A}_{2n-2m-1}(132)} q^{(2\text{-}31)\,\pi}.$$

Next we plug this back to (5.8) and decompose permutations in $\mathfrak{A}_{2n}(132)$ similarly as in the proof of (5.2) to complete the proof. □

5.2. **Other des-cases avoiding one pattern of length three and their q-analogues.** In a search for results analogous to Theorems 5.2 and 1.5, we consider all the remaining Catalan subsets that avoid one pattern of length three, and summarize the results in Tables 2 and 3, where a "$\star$" means there is no such phenomenon in this case. Take the top-left $\star$ for example, we put it there to indicate that neither do $\sum_{\pi \in \mathfrak{S}_{2n}(123)} (-1)^{\mathsf{des}\,\pi}$ always vanish, nor do we recognize $\sum_{\pi \in \mathfrak{S}_{2n+1}(123)} (-1)^{\mathsf{des}\,\pi}$ as a familiar sequence. We have suppressed all the $(-1)^n$ for the non-star entries, so the second entry in the first row should read as $\sum_{\pi \in \mathfrak{S}_{2n+1}(132)} (-1)^{\mathsf{des}\,\pi} = (-1)^n C_n$. For all the des-cases, we actually obtain the stronger q-versions. We begin by proving three useful lemmas.

Lemma 5.3. *For any $n \geq 1$,*

$$\sum_{\pi \in \mathfrak{D}_n^*(213)} t^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi} = t \sum_{\pi \in \mathfrak{S}_{n-1}(213)} t^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi}.$$

Proof. It is easy to see from the definition of $\mathfrak{D}_n^*$ that $\pi \in \mathfrak{D}_n^*(213)$ if and only if $\pi = n\pi'$ with $\pi' \in \mathfrak{S}_{n-1}(213)$. Moreover, we note that $\mathsf{des}\,\pi = 1 + \mathsf{des}\,\pi'$ and $(13\text{-}2)\,\pi = (13\text{-}2)\,\pi'$. Summing over all the $\pi \in \mathfrak{D}_n^*(213)$ completes the proof. □

Lemma 5.4. *For $n \geq 1$ and any $\pi \in \mathfrak{S}_n$,*

$$\mathsf{des}\,\pi + (31\text{-}2)\,\pi + 1 = \mathsf{fl}\,\pi + (13\text{-}2)\,\pi, \tag{5.9}$$

where $\mathsf{fl}\,\pi = \pi(1)$ is the **f***irst* **l***etter of π.*

Proof. We use induction on n. The $n = 1$ case holds trivially. Assume (5.9) is true for any permutaion with length less than n. Now take any $\pi \in \mathfrak{S}_n$, we discuss the position of n by the following two case. Suppose $\pi(i) = n$ for some $1 \leq i \leq n$. The two extreme cases $i = 1$ and $i = n$ can be quickly checked so we assume $2 \leq i \leq n-1$ and let $\pi' = \pi(1)\cdots\pi(i-1)\pi(i+1)\cdots\pi(n)$.

- If $\pi(i-1) < \pi(i+1)$, then $\mathsf{des}\,\pi = \mathsf{des}\,\pi' + 1$, $\mathsf{fl}\,\pi = \mathsf{fl}\,\pi'$, and
$$(13\text{-}2)\,\pi - (13\text{-}2)\,\pi' = (31\text{-}2)\,\pi - (31\text{-}2)\,\pi' + 1,$$
where we only need to check the contributions for 13-2 and 31-2 coming from the triple with n playing the role of 3.
- $\pi(i-1) > \pi(i+1)$, then $\mathsf{des}\,\pi = \mathsf{des}\,\pi'$, $\mathsf{fl}\,\pi = \mathsf{fl}\,\pi'$, and
$$(13\text{-}2)\,\pi - (13\text{-}2)\,\pi' = (31\text{-}2)\,\pi - (31\text{-}2)\,\pi'.$$

In both cases, we see that (5.9) holds for n as well. □

Lemma 5.5. *For any $n \geq 2$,*

$$\sum_{\pi \in \mathfrak{D}_n^*(132)} t^{\mathsf{des}\,\pi} q^{\mathsf{fl}\,\pi} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \Bigg(\sum_{\pi \in \overline{\mathfrak{D}}_{n,k}^*(132)} q^{\mathsf{fl}\,\pi} \Bigg) t^k (1+t)^{n-2k}, \tag{5.10}$$

where $\overline{\mathfrak{D}}_{n,k}^*(132) := \{\pi \in \mathfrak{D}_n^*(132) : \mathsf{dd}^*\,\pi = 1, \mathsf{des}\,\pi = k\}$.

Proof. Since pattern 132 is concerned here, per Remark 3.4, we shall use Lin-Zeng's dual version of the MFS-action φ_x. In addition, we modify it differently in the following way. This new variant of MFS-action is denoted as $\overline{\varphi}_x$.

$$\overline{\varphi}_x(\pi) := \begin{cases} \pi, & \text{if } x \text{ is a valley, a peak, or a left-to-right maximum of } \pi; \\ \varphi_x(\pi), & \text{otherwise.} \end{cases}$$

We state without proving the following facts about $\overline{\varphi}_x$, all of which can be verified similarly as for φ'_x.

- $\overline{\varphi}_x$'s are involutions and commute;
- the map $\overline{\varphi}_S$ is closed on $\mathfrak{D}_n^*(132)$;

- for any $\pi \in \mathfrak{D}_n^*(132)$ and each $x \in [n]$, $\mathsf{fl}\,\pi = \mathsf{fl}\,\overline{\varphi}_x(\pi)$.

Let $\pi \in \mathfrak{D}_n^*(132)$. The above facts, together with a similar argument about the orbits under this new MFS-action, tell us that there is a unique permutation in $\mathsf{Orb}(\pi)$ which has exactly one double descent at the first letter (this is due to the definition of coderangements $\mathfrak{D}^*$ and the convention that $\pi(0) = \pi(n) = n+1$). Now, let $\bar{\pi}$ be this unique element in $\mathsf{Orb}(\pi)$, then $\mathsf{da}^*\,\bar{\pi} = n-1-\mathsf{peak}^*\,\bar{\pi} - \mathsf{valley}^*\,\bar{\pi}$ and $\mathsf{des}\,\bar{\pi} = \mathsf{peak}^*\,\bar{\pi} + 1 = \mathsf{valley}^*\,\bar{\pi}$. Thus

$$\sum_{\sigma \in \mathsf{Orb}\,\pi} t^{\mathsf{des}\,\sigma} q^{\mathsf{fl}\,\sigma} = q^{\mathsf{fl}\,\bar{\pi}} t^{\mathsf{des}\,\bar{\pi}} (1+t)^{\mathsf{da}^*\,\bar{\pi}} = q^{\mathsf{fl}\,\bar{\pi}} t^{\mathsf{des}\,\bar{\pi}} (1+t)^{n-2\mathsf{des}\,\bar{\pi}}.$$

Summing over all the orbits establishes (5.10). □

Now we are ready to present the q-analogues for all the remaining entries shown in Table 2.

Theorem 5.6. *For any $n \geq 1$,*

$$\sum_{\pi\in\mathfrak{S}_n(132)} (-1)^{\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(132)} (-1)^{\mathsf{des}\,\pi} q^{(2\text{-}13)\,\pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-q)^{\frac{n-1}{2}} C_{\frac{n-1}{2}}(q^2) & \text{if } n \text{ is odd,} \end{cases} \tag{5.11}$$

$$\sum_{\pi\in\mathfrak{S}_n(213)} (-1)^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(213)} (-1)^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-q)^{\frac{n-1}{2}} C_{\frac{n-1}{2}}(q^2) & \text{if } n \text{ is odd,} \end{cases} \tag{5.12}$$

$$\sum_{\pi\in\mathfrak{S}_n(312)} (-1)^{\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi} = \sum_{\pi\in\mathfrak{S}_n(312)} (-1)^{\mathsf{des}\,\pi} q^{(2\text{-}13)\,\pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-q)^{\frac{n-1}{2}} C_{\frac{n-1}{2}}(q^2) & \text{if } n \text{ is odd,} \end{cases} \tag{5.13}$$

$$\sum_{\pi\in\mathfrak{D}_n^*(132)} (-1)^{\mathsf{des}\,\pi} q^{(2\text{-}31)\,\pi} = \begin{cases} (-1)^{\frac{n}{2}} \widehat{C}_{\frac{n}{2}}(q) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{5.14}$$

$$\sum_{\pi\in\mathfrak{D}_n^*(132)} (-q)^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = \begin{cases} (-q)^{\frac{n}{2}} \overline{C}_{\frac{n}{2}}(q) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{5.15}$$

$$\sum_{\pi\in\mathfrak{D}_n^*(213)} (-1)^{\mathsf{des}\,\pi} q^{(13\text{-}2)\,\pi} = \begin{cases} (-1)^{\frac{n}{2}} q^{\frac{n-2}{2}} C_{\frac{n-2}{2}}(q^2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{5.16}$$

$$\sum_{\pi\in\mathfrak{D}_n^*(213)} (-q)^{\mathsf{des}\,\pi} q^{(31\text{-}2)\,\pi} = \begin{cases} (-1)^{\frac{n}{2}} q^{\frac{3n-4}{2}} C_{\frac{n-2}{2}}(q^2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{5.17}$$

where

$$\widehat{C}_n(q) := \sum_{\pi\in\mathfrak{A}_{2n}(231)} q^{(13\text{-}2)\,\pi} \quad \text{and} \quad \overline{C}_n(q) := \sum_{\pi\in\mathfrak{A}_{2n}(231)} q^{(2\text{-}13)\,\pi}.$$

Proof. (5.11)–(5.13) follow directly by taking $t=-1$ in Theorem 1.1 and applying (1.20). The proof of (5.14) parallels that of (5.7), only that we use the decomposition in (5.3) this time. To prove (5.15), we first note that

$$\sum_{\pi\in\mathfrak{D}_n^*(132)}(-q)^{\mathsf{des}\,\pi}q^{(31\text{-}2)\,\pi}=\sum_{\pi\in\mathfrak{D}_n^*(132)}(-1)^{\mathsf{des}\,\pi}q^{\mathsf{des}\,\pi+(31\text{-}2)\,\pi}\overset{(5.9)}{=}\sum_{\pi\in\mathfrak{D}_n^*(132)}(-1)^{\mathsf{des}\,\pi}q^{\mathsf{fl}\,\pi-1},$$

which gives directly the odd $2n+1$ case in view of the expansion (5.10). For the even $2n$ case, we compute using (5.10) again that

$$\sum_{\pi\in\mathfrak{D}_{2n}^*(132)}(-1)^{\mathsf{des}\,\pi}q^{\mathsf{fl}\,\pi-1}=(-1)^n\sum_{\pi\in\overline{\mathfrak{D}}_{2n,n}^*(132)}q^{\mathsf{fl}\,\pi-1}\overset{(5.9)}{=}(-q)^n\sum_{\pi\in\overline{\mathfrak{D}}_{2n,n}^*(132)}q^{(31\text{-}2)\,\pi}.$$

Moreover, we note that $\pi\in\overline{\mathfrak{D}}_{2n,n}^*(132)$ if and only if $\pi^r\in\mathfrak{A}_{2n}(231)$, which implies (5.15).

Finally, (5.16) follows from Theorem 1.5 and Lemma 5.3. In view of the similarity between (5.16) and (5.17), it is a straightforward calculation basing on identity (5.9) and the first letter consideration in Lemma 5.3. □

The first few values for $\widehat{C}_n(q)$ and $\overline{C}_n(q)$ are:

$$\begin{aligned}
\widehat{C}_0(q)&=\widehat{C}_1(q)=1,\\
\widehat{C}_2(q)&=q+q^2,\\
\widehat{C}_3(q)&=q^2+q^3+q^4+q^5+q^6,\\
\widehat{C}_4(q)&=q^3+q^4+2q^5+2q^6+2q^7+q^8+2q^9+q^{10}+q^{11}+q^{12},\\
\widehat{C}_5(q)&=q^4+q^5+3q^6+3q^7+4q^8+3q^9+5q^{10}+3q^{11}+4q^{12}+3q^{13}+3q^{14}\\
&\quad+2q^{15}+2q^{16}+2q^{17}+q^{18}+q^{19}+q^{20},\\
\overline{C}_0(q)&=\overline{C}_1(q)=1,\\
\overline{C}_2(q)&=1+q,\\
\overline{C}_3(q)&=1+2q+2q^2,\\
\overline{C}_4(q)&=1+3q+5q^2+5q^3,\\
\overline{C}_5(q)&=1+4q+9q^2+14q^3+14q^4,\\
\overline{C}_6(q)&=1+5q+14q^2+28q^3+42q^4+42q^5.
\end{aligned}$$

The q-Catalan numbers $\overline{C}_n(q)$ merit some further investigation for their own sake. First we utilize (5.9) again to get another interpretation for $\overline{C}_n(q)$:

$$\sum_{\pi\in\mathfrak{A}_{2n}(231)}q^{(2\text{-}13)\,\pi}=\sum_{\pi\in\mathfrak{A}_{2n}(231)}q^{(31\text{-}2)\,\pi^r}\overset{(5.9)}{=}q^{-n-1}\sum_{\pi\in\mathfrak{A}_{2n}(231)}q^{\mathsf{fl}\,\pi^r}.$$

Definition 5.7. Let $\overline{C}_n(q) = q^{-n-1} \sum_{\pi \in \mathfrak{A}_{2n}(231)} q^{\mathsf{fl}\,\pi^r} := \sum_{k=0}^{n-1} a_{n,k} q^k$, where

$$\mathfrak{a}_{n,k} = \{\pi \in \mathfrak{A}_{2n}(231) : \mathsf{fl}\,\pi^r = n + k + 1\} \text{ and } a_{n,k} = |\mathfrak{a}_{n,k}|.$$

The first few examples are :

$$\begin{aligned}
&\mathfrak{a}_{1,0} = \{12\};\\
&\mathfrak{a}_{2,0} = \{1423\} \quad \text{and} \quad \mathfrak{a}_{2,1} = \{1324\};\\
&\mathfrak{a}_{3,0} = \{162534\},\ \mathfrak{a}_{3,1} = \{162435, 132645\} \quad \text{and} \quad \mathfrak{a}_{3,2} = \{132546, 152436\}.
\end{aligned}$$

Recall that the ballot numbers $f(n,k)$ satisfy (see [1, 7]) the recurrence relation

$$f(n,k) = f(n,k-1) + f(n-1,k), \quad (n,k \geq 0), \tag{5.18}$$

where $f(n,k) = 0$ if $n < k$ and $f(0,0) = 1$, and have the explicit formula

$$f(n,k) = \frac{n-k+1}{n+1}\binom{n+k}{k}, \quad (n \geq k \geq 0).$$

With the initial values $a_{1,0} = a_{2,0} = a_{2,1} = 1$, and compairing (5.18) and (5.19), we establish the following connection.

Proposition 5.8. *For* $0 \leq k \leq n-1$,

$$a_{n,k} = f(n-1,k) = \frac{n-k}{n}\binom{n-1+k}{k}.$$

Proof. For $n, k \geq 0$ let $a_{0,0} = 1$ and $a_{n,k} = 0$ if $k \geq n$ or $k < 0$. It suffices to prove the following recurrence relation for $a_{n,k}$:

$$a_{n+1,k} = a_{n+1,k-1} + a_{n,k}. \tag{5.19}$$

First note two useful facts for any $\pi \in \mathfrak{A}_{2n}(231)$.

a) $\mathsf{fl}\,\pi = 1$, since otherwise $(\pi(1), \pi(2), 1)$ will form a 231 pattern.
b) $\pi(1) < \pi(3) < \cdots < \pi(2n-1)$, i.e., the valleys of π form an increasing subsequence.

Due to fact a), we can assume $\mathsf{fl}\,\pi^r = \pi(2n) > 1$. Now we decompose $\mathfrak{a}_{n,k}$ as the union of two disjoint subsets:

$$\begin{aligned}
\mathfrak{a}^p_{n,k} &:= \{\pi \in \mathfrak{a}_{n,k} : \pi(2n) - 1 \text{ is a peak}\},\\
\mathfrak{a}^v_{n,k} &:= \{\pi \in \mathfrak{a}_{n,k} : \pi(2n) - 1 \text{ is a valley}\}.
\end{aligned}$$

We proceed to show that $|\mathfrak{a}^p_{n+1,k}| = |\mathfrak{a}_{n+1,k-1}|$ and $|\mathfrak{a}^v_{n+1,k}| = |\mathfrak{a}_{n,k}|$ via two bijections between the concerned sets, and thus proving (5.19).

The first map α is relatively easier. For any $\pi \in \mathfrak{a}^p_{n+1,k}$, we get its image $\alpha(\pi)$ by switching the position of two peaks $\pi(2n)$ and $\pi(2n) - 1$. A moment of reflection should reveal that $\alpha : \mathfrak{a}^p_{n+1,k} \to \mathfrak{a}_{n+1,k-1}$ is indeed well-defined and bijective.

We have to lay some ground work for the second map $\beta : \mathfrak{a}^v_{n+1,k} \to \mathfrak{a}_{n,k}$. The key observation is on the last three letters. We claim that for any $\pi \in \mathfrak{a}^v_{n+1,k}$,

$$\pi(2n) = \pi(2n+2)+1,\ \pi(2n+1) = \pi(2n+2)-1. \tag{5.20}$$

First we see $2n+2 \neq \pi(2n+2)$, since otherwise $2n+1 = \pi(2n+2)-1$ cannot be a valley. So $2n+2$ must be a non-terminal peak. Now consider $2n+1$, it cannot appear to the left of $2n+2$, otherwise it will cause a 231 pattern. It must also be a peak, since there are no other letters larger than it except for $2n+2$. If $2n+1 = \pi(2n+2)$ is the last peak, then $2n$ being a valley forces $(\pi(2n), \pi(2n+1), \pi(2n+2)) = (2n+2, 2n, 2n+1)$, which means (5.20) holds true. Otherwise $2n+1$ is a non-terminal peak and we consider $2n$ next. This deduction must end in finitely many steps since the total number of peaks is n (and finite). At this ending moment we find some m as the last peak, and $2n+2, 2n+1, \ldots, m+1$ are all peaks decreasingly ordered to its left, then $m-1$ being a valley, together with fact b) force us to have (5.20) again. So the claim is proved.

The definitions and validity of β and its inverse become transparent, in view of (5.20).

- β: For $\pi \in \mathfrak{a}^v_{n+1,k}$, delete $\pi(2n+1)$ and $\pi(2n+2)$, then decrease the remaining letters larger than $\pi(2n+2)$ by 2.
- β^{-1}: For $\sigma \in \mathfrak{a}_{n,k}$, increase the letters no less than $\sigma(2n)$ by 2, and append two letters $\sigma(2n)$ and $\sigma(2n)+1$ to the right of σ, in that order.

The proof ends here and we give the following example for illustration. □

Example 5.9. The two bijections $\alpha : \mathfrak{a}^p_{n+1,k} \to \mathfrak{a}_{n+1,k-1}$ and $\beta : \mathfrak{a}^v_{n+1,k} \to \mathfrak{a}_{n,k}$ for the case of $n=3$ are shown below.

$$\mathfrak{a}^p_{4,3}\left\{\begin{array}{c}13254768\\13274658\\15243768\\17243658\\17263548\end{array}\right. \xrightarrow{\alpha} \left.\begin{array}{c}13254867\\13284657\\15243867\\18243657\\18263547\end{array}\right\}\mathfrak{a}_{4,2}$$

$$\mathfrak{a}^p_{4,2}\left\{\begin{array}{c}13284657\\18243657\\18263547\end{array}\right. \xrightarrow{\alpha} \left.\begin{array}{c}13284756\\18243756\\18273546\end{array}\right\}\mathfrak{a}_{4,1}$$

$$\mathfrak{a}^p_{4,1}\left\{\ 18273546\right. \xrightarrow{\alpha} \left.18273645\ \right\}\mathfrak{a}_{4,0}$$

$$\mathfrak{a}^v_{4,2}\left\{\begin{array}{c}13254867\\15243867\end{array}\right. \xrightarrow{\beta} \left.\begin{array}{c}132546\\152436\end{array}\right\}\mathfrak{a}_{3,2}$$

$$\mathfrak{a}^v_{4,1}\left\{\begin{array}{c}13284756\\18243756\end{array}\right. \xrightarrow{\beta} \left.\begin{array}{c}132645\\162435\end{array}\right\}\mathfrak{a}_{3,1}$$

$$\mathfrak{a}^v_{4,0}\left\{\ 18273645\right. \xrightarrow{\beta} \left.162534\ \right\}\mathfrak{a}_{3,0}$$

5.3. **Other exc-cases avoiding one pattern of length three.** In this subsection we present the parallel (−1)-phenomena with respect to exc, note the differences when one

TABLE 3. The (-1)-evaluation over $\mathfrak{S}_n(\tau)$ and $\mathfrak{D}_n(\tau)$ with respect to exc. The signs $(-1)^n$ have all been removed.

$\mathsf{exc} \setminus \tau$	123	132	213	231	312	321
$\mathfrak{S}_{2n+1}$	$\star$	C_n	C_n	$\star$	$\star$	C_n
$\mathfrak{D}_{2n}$	F_n	C_n	C_n	$\star$	$\star$	C_n

compares Table 3 with Table 2. Unfortunately we have not found any q-analogues at this moment.

Theorem 5.10. *For any $n \geq 1$,*

$$\sum_{\pi\in\mathfrak{S}_n(213)} (-1)^{\mathsf{exc}\,\pi} = \sum_{\pi\in\mathfrak{S}_n(132)} (-1)^{\mathsf{exc}\,\pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} C_{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \end{cases} \tag{5.21}$$

$$\sum_{\pi\in\mathfrak{D}_n(213)} (-1)^{\mathsf{exc}\,\pi} = \sum_{\pi\in\mathfrak{D}_n(132)} (-1)^{\mathsf{exc}\,\pi} = \begin{cases} (-1)^{\frac{n}{2}} C_{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{5.22}$$

Proof. We first apply the $q = 1$ case of Theorem 1.5, and the following lemma due to Elizalde [12] to derive the second equalities in both (5.21) and (5.22). Next we observe the following facts, which can be easily checked.

$$\pi \in \mathfrak{S}_n(132) \Leftrightarrow \pi^{rc} \in \mathfrak{S}_n(213),$$
$$\mathsf{exc}(\pi) = n - \mathsf{exc}(\pi^{rc}) - \mathsf{fix}(\pi),\ \mathsf{fix}(\pi) = \mathsf{fix}(\pi^{rc}).$$

Consequently we have

$$\sum_{\pi\in\mathfrak{S}_n(132)} t^{\mathsf{exc}\,\pi} y^{\mathsf{fix}\,\pi} = t^n \sum_{\pi\in\mathfrak{S}_n(213)} t^{-\mathsf{exc}\,\pi - \mathsf{fix}\,\pi} y^{\mathsf{fix}\,\pi}. \tag{5.23}$$

Plugging in $t = -1, y = 0$ gives us directly the first equality in (5.22). Finally, taking $t = y = -1$ in (5.23), (5.24) and $t = -1, q = 1$ in (4.2) leads to:

$$(-1)^n \sum_{\pi\in\mathfrak{S}_n(213)} (-1)^{\mathsf{exc}\,\pi} = \sum_{\pi\in\mathfrak{S}_n(321)} (-1)^{\mathsf{wex}\,\pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n+1}{2}} C_{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \end{cases}$$

which is exactly the first equality in (5.21). □

Lemma 5.11 (Elizalde). *For any $n \geq 1$,*

$$\sum_{\pi\in\mathfrak{S}_n(321)} t^{\mathsf{exc}\,\pi} y^{\mathsf{fix}\,\pi} = \sum_{\pi\in\mathfrak{S}_n(132)} t^{\mathsf{exc}\,\pi} y^{\mathsf{fix}\,\pi}. \tag{5.24}$$

The only entry in Table 3 that is not covered by Theorems 1.5 and 5.10 is still a conjecture. Define the polynomials $G_n(t) := \sum_{\pi\in\mathfrak{D}_n(123)} t^{\mathsf{exc}\,\pi}$ for $n \geq 1$.

Conjecture 5.12. There is a sequence $\{F_n\}_{n\geq 1}$ of positive inetergs such that

$$\sum_{\pi\in\mathfrak{D}_n(123)} (-1)^{\mathsf{exc}\,\pi} = \begin{cases} (-1)^{\frac{n}{2}} F_{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{5.25}$$

and the polynomials $G_n(t)$ are γ-positive.

We note that neither of the sequences $G_n(1)$ and F_n $(n \geq 1)$ is registered in OEIS. The first values are given by $G_n(1) = 0, 1, 2, 7, 20, 66, 218, 725, \ldots$ and $F_n = 1, 7, 58, 545, 5570, \ldots$. For the first few $n \geq 1$, we have

$$\begin{aligned}
G_1(t) &= 0, G_2(t) = t, \\
G_3(t) &= t + t^2 = t(1+t), \quad G_4(t) = 7t^2, \\
G_5(t) &= 10t^2 + 10t^3 = 10t^2(1+t), \\
G_6(t) &= 2t^2 + 62t^3 + 2t^4 = 2t^2(1+t)^2 + 58t^3, \\
G_7(t) &= 109t^3 + 109t^4 = 109t^3(1+t), \\
G_8(t) &= 45t^3 + 635t^4 + 45t^5 = 45t^3(1+t)^2 + 545t^4, \\
G_9(t) &= 5t^3 + 1264t^4 + 1264t^5 + 5t^6 = 5t^3(1+t)^3 + 1249t^4(1+t), \\
G_{10}(t) &= 769t^4 + 7108t^5 + 769t^6 = 769t^4(1+t)^2 + 5570t^5.
\end{aligned}$$

The symmetry of $G_n(t)$ follows from the map $\pi \mapsto \pi^{rc}$, which is stable on $\mathfrak{S}_n(123)$ and $\mathfrak{D}_n(123)$, and satisfies $\mathsf{exc}(\pi) = n - \mathsf{exc}(\pi^{rc}) - \mathsf{fix}(\pi)$. Thus, if $\pi \in \mathfrak{D}_n(123)$, we obtain the symmetry.

6. Two cases avoiding two patterns of length four

Motivated by Lewis' work [23–26], many authors [5,8,31,41,42] have studied the pattern avoidance on alternating permutations, especially the Wilf-equivalence problem for patterns of length four. As for alternating permutations that avoid two patterns of length four simultaneously, our results in this section appear to be new. We first enumerate $\mathfrak{A}_n(2413, 3142)$ and $\mathfrak{A}_n(1342, 2431)$, then put these results in the context of (-1)-evaluations of the descent polynomials over $\mathfrak{S}_n(2413, 3142)$ and $\mathfrak{S}_n(1342, 2431)$. The following two γ-expansions (6.1) and (6.2), which were obtained recently by Fu-Lin-Zeng [20] and Lin [27], respectively, will be crucial in our (-1)-evaluations.

$$S_n(t) := \sum_{\pi\in\mathfrak{S}_n(2413,3142)} t^{\mathsf{des}\,\pi} = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \gamma^S_{n,k} t^k (1+t)^{n-1-2k}, \tag{6.1}$$

$$Y_n(t) := \sum_{\pi\in\mathfrak{S}_n(1342,2431)} t^{\mathsf{des}\,\pi} = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \gamma^Y_{n,k} t^k (1+t)^{n-1-2k}, \tag{6.2}$$

where

$$\gamma^S_{n,k} = \#\{\pi \in \mathfrak{S}_n(2413, 3142) : \mathsf{dd}^* \pi = 0, \mathsf{des}\, \pi = k\}, \tag{6.3}$$

$$\gamma^Y_{n,k} = \#\{\pi \in \mathfrak{S}_n(1342, 2431) : \mathsf{dd}^* \pi = 0, \mathsf{des}\, \pi = k\}. \tag{6.4}$$

Il follows that

$$|\mathfrak{A}_n(2413, 3142)| = \gamma^S_{n,\lfloor \frac{n-1}{2} \rfloor},\ |\mathfrak{A}_n(1342, 2431)| = \gamma^Y_{n,\lfloor \frac{n-1}{2} \rfloor}. \tag{6.5}$$

Recall the γ-coefficients in the expansion (1.2), (6.1)–(6.2). For $* = N, S, Y$, let

$$\Gamma_*(x, z) := \sum_{n=1}^{\infty} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma^*_{n,k} x^k z^n$$

be the generating functions for $\gamma^N_{n,k}$, $\gamma^S_{n,k}$ and $\gamma^Y_{n,k}$, respectively. We need the following two algebraic equations for $\Gamma_S(x, z)$ and $\Gamma_Y(x, z)$, which were first derived by Lin [27].

$$\Gamma_S = z + z\Gamma_S + xz\Gamma_S^2 + x\Gamma_S^3, \tag{6.6}$$

$$\Gamma_Y = z + z\Gamma_Y + 2xz\Gamma_N\Gamma_Y + x\Gamma_N^2(\Gamma_Y - z). \tag{6.7}$$

6.1. **The case of (2413,3142)–avoiding alternating permutations.**

Theorem 6.1. *Let* $r_n := |\mathfrak{A}_{2n+1}(2413, 3142)|$, $R(x) := \sum_{n=1}^{\infty} r_n x^n$, *then*

$$R(x) = x(R(x) + 1)^2 + x(R(x) + 1)^3. \tag{6.8}$$

Consequently, $r_0 = 1$ *and for* $n \geq 1$,

$$r_n = \frac{2}{n} \sum_{i=0}^{n-1} 2^i \binom{2n}{i} \binom{n}{i+1}. \tag{6.9}$$

Proof. First, (1.16) gives us $r_n = \gamma^S_{2n+1,n}$. Therefore, in order to get a recurrence relation for r_n, we should extract the coefficient of z^{2n+1} in (6.6) and then compare the coefficients of x^n from both sides. This gives us, for $n \geq 1$,

$$r_n = [x^{n-1}] \left([z^{2n}]\Gamma_S^2(x, z)\right) + [x^{n-1}] \left([z^{2n+1}]\Gamma_S^3(x, z)\right).$$

Now we take a closer look at $[z^{2n}]\Gamma_S^2(x, z)$.

$$[z^{2n}]\Gamma_S^2(x, z) = \sum_{m=1}^{2n-1} \left(\sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \gamma^S_{m,j} x^j \right) \cdot \left(\sum_{k=0}^{\lfloor \frac{2n-m-1}{2} \rfloor} \gamma^S_{2n-m,k} x^k \right),$$

So for each term in this summation, the power of x is

$$j + k \leq \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{2n-m-1}{2} \right\rfloor \leq n - 1.$$

Hence we get contributions for x^{n-1} only from odd m's. Similar analysis applies to the term involving Γ_S^3 and the details are omitted. All these amount to

$$r_n = \sum_{m=0}^{n-1} r_m r_{n-m-1} + \sum_{m,l=0}^{n-1} r_m r_l r_{n-m-l-1}.$$

In terms of the generating function $R(x)$, we obtain (6.8). Next we rewrite (6.8) as

$$x = \frac{R}{(R+1)^2(R+2)}, \tag{6.10}$$

which is ripe for applying the *Lagrange inversion.* A straightforward computation leads to (6.9) and completes the proof. □

Theorem 6.2. *For $n \geq 1$, let $t_n := |\mathfrak{A}_{2n}(2413, 3142)|$, $n \geq 1$, $T(x) := \sum_{n=1}^{\infty} t_n x^n$, then*

$$\frac{1}{2}R(x) = \frac{1}{2}R(x) \cdot T(x) + T(x). \tag{6.11}$$

Consequently, $t_1 = 1$ and for $n \geq 2$,

$$t_n = \frac{4}{n-1} \sum_{i=0}^{n-2} 2^i \binom{2n-1}{i} \binom{n-1}{i+1}. \tag{6.12}$$

Proof. It may still be possible to establish (6.11) using the algebraic equation (6.6), but this time we present a combinatorial argument, showing both sides generate the same set of permutations.

The first thing to notice is that for an alternating permutation $\pi \in \mathfrak{A}_{2n+1}(2413, 3142)$, $n \geq 1$, its reverse $\pi^r \neq \pi$ is also in $\mathfrak{A}_{2n+1}(2413, 3142)$. This implies that r_n is even for $n \geq 1$. Moreover, we call a permutation $\pi \in \mathfrak{S}_n, n \geq 2$ *normal* if 1 appears to the left of n. For example, there are three normal permutations in $\mathfrak{S}_3$: $213, 123, 132$. Now we see that exactly one permutation in the pair $\{\pi, \pi^r\}$ is normal, and consequently the number of normal permutations in $\mathfrak{A}_{2n+1}(2413, 3142)$ is $r_n/2$. Therefore the left-hand side of (6.11) generates all normal, alternating, and $(2413, 3142)$-avoiding permutations of odd length larger than 1. Next we show that the right-hand side does precisely the same. We state the following fact about any normal permutation $\pi \in \mathfrak{A}_{2n+1}(2413, 3142)$, $n \geq 1$, omitting the proof. See [20] for the definition of the operation $\oplus$.

Claim 6.3. *There exists a unique pair of permutations $(\pi^{(1)}, \pi^{(2)})$, such that*

- *(1) $\pi = \pi^{(1)} \oplus \pi^{(2)}$,*
- *(2) either $\pi^{(1)} = 1$ or $\pi^{(1)}$ is of odd length and not normal, alternating and $(2413, 3142)$-avoiding,*
- *(3) $\pi^{(2)}$ is of even length (≥ 2) and $(2413, 3142)$-avoiding, its reverse is alternating.*

In view of the claim above, $\frac{1}{2}R(x) \cdot T(x)$ accounts for the cases when $\pi^{(1)}$ is of length 3 or longer, while $T(x)$ corresponds to the case when $\pi^{(1)} = 1$. Now since the above decomposition using $\oplus$ is unique, we get (6.11).

Applying (6.10), we can rewrite (6.11) as

$$T = \frac{R}{R+2} = x(R+1)^2.$$

This form is suitable for the more general *Lagrange-Bürmann formula*, and we get for $n \geq 2$,

$$\begin{aligned} t_n = [x^{n-1}](R+1)^2 &= \frac{1}{n-1}[R^{n-2}]\left(2(R+1)(R+1)^{2n-2}(R+2)^{n-1}\right) \\ &= \frac{2}{n-1}\sum_{i=0}^{n-2} 2^{n-1-i}\binom{n-1}{i}\binom{2n-1}{n-2-i} \\ &= \frac{4}{n-1}\sum_{i=0}^{n-2} 2^{i}\binom{n-1}{i+1}\binom{2n-1}{i}. \end{aligned}$$

The proof is now completed. □

Remark 6.4. In view of the similarity in the expressions for r_n and t_n, we can unify them as the following formula:

$$|\mathfrak{A}_n(2413,3142)| = \frac{2^{n-2m}}{m}\sum_{i=0}^{m-1} 2^i \binom{m}{i+1}\binom{n-1}{i}, \text{ where } m = \left\lfloor \frac{n-1}{2} \right\rfloor, \text{ and } n \geq 3.$$

Moreover, the two sequences $\{r_n\}_{n\geq 0}$ and $\{t_n\}_{n\geq 1}$ have been cataloged in the OEIS (see oeis:A027307 and oeis:A032349), and were considered, for instance, by Deutsch et al. [11] as enumerating certain type of lattice paths. Then a natural question would be to find a bijection between these two combinatorial models.

Now we turn to the (-1)-evaluation for the Schröder case, which is a direct result of (6.1) and (6.5).

Theorem 6.5. *For any $n \geq 1$, there holds*

$$S_n(-1) = \sum_{\pi \in \mathfrak{S}_n(2413,3142)} (-1)^{\mathsf{des}\,\pi} = \begin{cases} 0 & \text{if } n \text{ is even}, \\ (-1)^{\frac{n-1}{2}} r_{\frac{n-1}{2}} & \text{if } n \text{ is odd}. \end{cases} \tag{6.13}$$

6.2. **The case of (1342,2431)–avoiding alternating permutations.**

Theorem 6.6. *Let $u_n := |\mathfrak{A}_{2n+1}(1342,2431)|$, $U(x) := \sum_{n=0}^{\infty} u_n x^n$, then*

$$U(x) = \frac{\sqrt{1-4x}}{\sqrt{1-4x}-2x} = \cfrac{1}{1-\cfrac{2x}{1-\cfrac{2x}{1-\cfrac{x}{1-\cfrac{x}{\ddots}}}}}. \tag{6.14}$$

Proof. We use (6.7) in a similar way as we use (6.6) in the proof of (6.8), i.e., we extract the coefficients of z^{2n+1} from both sides and then compare the coefficients of x^n. This leads to the following recurrence relation that involves the Catalan number C_n, since we have already shown that $\gamma^N_{2n+1,n} = |\mathfrak{A}_{2n+1}(231)| = C_n$. For $n \geq 1$, we have:

$$\begin{aligned} u_n &= 2\sum_{m=0}^{n-1} u_m C_{n-1-m} + \sum_{m=1}^{n-1} u_m \sum_{l=0}^{n-m-1} C_l C_{n-m-l-1} \\ &= 2\sum_{m=0}^{n-1} u_m C_{n-1-m} + \sum_{m=1}^{n-1} u_m C_{n-m}. \end{aligned}$$

In terms of generating function, this means

$$2(U(x)-1) = 2xU(x)C(x) + (U(x)-1)C(x),$$

where

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

is the generating function for the Catalan numbers. We plug in $C(x)$ and solve for $U(x)$ to finish the proof. $\square$

Remark 6.7. Two remarks on Theorem 6.6 are in order. First, our result above seems to be the first combinatorial interpretation for u_n, and the sequence $\{u_n\}_{n\geq 0}$ is also on OEIS (see oeis:A084868), but there is no simple sum formula for u_n. A multiple sum formula can be derived as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} u_n x^n &= \frac{1}{1-\dfrac{2x}{\sqrt{1-4x}}} = \sum_{m=0}^{\infty}\left(\frac{2x}{\sqrt{1-4x}}\right)^m \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty} 2\binom{2k}{k} x^{k+1}\right)^m. \end{aligned}$$

Thus we obtain

$$u_n = \sum_{m=0}^{n} 2^m \sum_{k_1+\cdots+k_m=n-m} \prod_{i=1}^{m} \binom{2k_i}{k_i}. \tag{6.15}$$

On the other hand, by (6.15)

$$u_n = 4\binom{2n-3}{n-2} + \sum_{m=2}^{n} 2^m \sum_{k_1+\cdots+k_m=n-m} \prod_{i=1}^{m} \binom{2k_i}{k_i},$$

from which we obtain $u_n \equiv 0 \pmod 4$ if $n \geq 2$.

With the aid of (6.2) and (6.5), we obtain

Theorem 6.8. *For any $n \geq 1$, there holds*

$$Y_n(-1) = \sum_{\pi \in \mathfrak{S}_n(1342,2413)} (-1)^{\mathsf{des}\,\pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} u_{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases} \tag{6.16}$$

We end this section by noting that both $\mathfrak{S}_n(2413, 3142)$ and $\mathfrak{S}_n(1342, 2431)$ exhibit only (-1)-phenomenon (not strong). This should not come as a surprise in view of the reversal relations between the two patterns that we avoid, namely $(2413)^r = 3142, (1342)^r = 2431$, and the fact that the definition of coderangements is incompatible with the reverse map. Other subsets of $\mathfrak{S}_n$ instead of $\mathfrak{D}_n^*$ should be examined to hunt for the other half of the (-1)-phenomenon.

7. Final remarks

It would be interesting to give direct combinatorial proofs of the (-1)-phenomena of this paper. In view of Theorem 1.1 (resp. Theorem 4.3), each interpretation listed should have a q-γ-expansion in theory. Namely, once we have an expansion for one of them, the others all share this expansion. But expansions derived this way are not "natural" (for instance, (4.2) is unnatural). The expansions we have in Theorems 1.2 and 1.4, Lemma 5.5 are all natural, in the sense that the statistics (powers of q) appear in the γ-coefficients on the expansion side, are the same as those appear on the left-hand side, the avoiding patterns are also the same. And we prove them uniformly using the MFS-action and its variation. So now the question is, do the other ones that we are missing in Theorem 1.2 have such "natural" expansions? It seems the MFS-action cannot help anymore.

Considering the ubiquity of Catalan numbers (cf. [40]), the interpretations we have in Theorem 1.1 are by no means exhaustive. We mention here one more connection that was suggested by the online database of combinatorial statistics FindStat [34].

Theorem 7.1. *For $n \geq 1$, let $\mathfrak{M}_n$ be the set of $n \times n$ alternating sign matrices that are determined by their X-rays (cf. [33]). Then*

$$N_n(t, q) = \sum_{M \in \mathfrak{M}_n} t^{\mathsf{pa}\,M-1} q^{\mathsf{neg}\,M},$$

where $\mathsf{pa}\,M$ (resp. $\mathsf{neg}\,M$) is the number of antidiagonals with 1s (resp. the total number of -1s) in M.

Another direction to extend the results presented here is to place $\mathfrak{S}_n$ in the broader context of Coxeter groups, and consider the so-called types B and D Narayana polynomials (see [2, Theorems 2.32 and 2.33]). This approach was shown fruitful for permutations in a recent work of Eu, Fu, Hsu and Liao [13].

It would be appealing to establish a multivariate generating function (in the spirit of Shin-Zeng's Lemma 2.5) that specializes to the $(2413, 3142)$-avoiding permutations or $(1342, 2413)$-avoiding permutations, and consequently giving us q-analogues of (6.13) or (6.16).

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