

ENUMERATION OF BORDER-STRIP DECOMPOSITIONS & WEIL–PETERSSON VOLUMES

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ABSTRACT. We describe an injection from border-strip decompositions of certain shapes to permutations. This allows us to provide enumeration results, as well as q -analogues of enumeration formulas.

Finally, we use this injection to prove a connection between the number of border-strip decompositions of the $n \times 2n$ rectangle and the Weil–Petersson volume of the moduli space of an n -punctured Riemann sphere.

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1. INTRODUCTION

Border-strip *tableaux* have a rich history, originating with the celebrated Murnaghan–Nakayama rule, [Mur37, Nak40], which provides a combinatorial formula for computing character values of S_n . It is a signed sum over border-strip tableaux, but the sign only depends on the border-strip *decomposition*, *i.e.*, the “unlabeled version” of the tableaux. This gives a motivation to enumerate border-strip decompositions.

We note that there is a hook-formula for enumerating border-strip tableaux, see [FL97], but less study has been devoted to enumerating

Key words and phrases. Border-strip tableaux, border-strip decompositions, permutations, q -analogue, Weil–Petersson volume.

border-strip decompositions. Even determining if a region can be tiled by n -ribbons is non-trivial, see [Pak00].

We introduce a family of shapes (called *simple diagrams*) which have nice properties with respect to enumeration. These are parametrized by a binary word, and the size of the ribbons which are used to tile the region. In particular, we show that certain normalized enumerations grow as a polynomial in n (the size of the ribbons) thus reducing specific enumerations to a finite computation.

1.1. Overview of results. We show that border-strip tableaux and border-strip decompositions of simple diagrams are in bijection with certain classes of permutations, see Proposition 16 and Corollary 22. This allows us to study a certain q -analogue of border-strip decompositions, which generalize the classical inversion-statistic on permutations. For example, in Corollary 30, we give the formula

$$\sum_{\mathbf{w} \in \{r,c\}^k} \sum_{T \in \text{BSD}(\mathbf{w},n)} q^{\text{inv} T} = [n+1]_q^k [n]_q!$$

where the first sum is over all binary words of length k (defining a simple diagram), and $\text{BSD}(\mathbf{w}, n)$ is the set of border-strip decompositions with strips of size n , and shape determined by (\mathbf{w}, n) . In Proposition 24, we give an efficient way to compute the number of border-strip decompositions of simple diagrams, as a function of n — the strip size. This allows us to prove an inequality, showing that “straighter” simple shapes admit a larger number of border-strip decompositions, see Theorem 31. The maximum is attained for rectangles. In contrast, by Corollary 17 we know that these shapes admit the same number of *border-strip tableaux* whenever $n \geq k$.

Finally, we give a new interpretation of [Slo16, A115047] in the OEIS. We show that these numbers count the number of ways to tile a $2n \times n$ -rectangle with strips of size n , which gives a new simple combinatorial interpretation of certain Weil–Petersson volumes. We cannot give an intuitive explanation for this curious connection, and it invites for further research.

2. PRELIMINARIES

We first need to recall some general definitions — for a thorough background, see [Sta01].

A *tableau* of shape λ and type μ is a filling of the Young diagram λ , such that there are exactly μ_i boxes filled with i , for $i = 1, \dots, \ell(\mu)$. A *border-strip* (or simply *strip*) of a diagram is a subset of boxes that form a connected skew shape, and contains no 2×2 subdiagram. A

*border-strip tableau*¹ is a tableau such that rows and columns are weakly increasing, and for all i , the boxes filled with the number i , form a border-strip. We let $\text{BST}(\lambda, \mu)$ denote the set of border-strip tableaux of shape λ and type μ .

A *border-strip decomposition* of shape λ and type μ is a partition of λ into border-strips where the border-strip sizes are determined by the μ_i , and the set of such decompositions is denoted $\text{BSD}(\lambda, \mu)$. Hence, each border-strip tableau defines a border-strip decomposition. Finally, the definition of $\text{BSD}(\lambda, \mu)$ extends in the natural manner the case when λ is a skew shape.

Example 1. The following tableau T is an element in $\text{BST}(\lambda, \mu)$ with $\lambda = (5, 5, 4, 3, 3, 3)$ and $\mu = (5, 4, 3, 4, 3, 2, 2)$. To the right, we show the corresponding border-strip decomposition with the strips indicated by the colors.

1	1	1	1	4
1	2	4	4	4
2	2	5	5	
2	3	5		
3	3	7		
6	6	7		

(1)

It is clear that the number of elements in $\text{BST}(\lambda, \mu)$ depend on the order of the entries in μ , but this is not the case for $\text{BSD}(\lambda, \mu)$. In particular, $\text{BST}(\lambda, \mu)$ might be empty, while $\text{BSD}(\lambda, \mu)$ is not.

Recall that the *content*, $c(\square)$, of a box is defined as the difference $j - i$ of column-index minus row-index of the box. From the definition of border-strips, it is straightforward to show that the boxes in a border-strip B all have different content, and these numbers form the *content-interval* $a, a + 1, \dots, b$ with no gaps. We can thus define the *head*, $\mathcal{H}(B)$ of a border-strip is the box with maximal content, and its *tail*, $\mathcal{T}(B)$, which is the box with minimal content. In (2), the head and tail boxes have been marked.

(2)

3. ENUMERATION OF BORDER-STRIP DECOMPOSITIONS

In this section, we introduce a natural family of diagram shapes which have particularly nice properties.

We first describe a bijection from border-strip decompositions of such shapes to certain permutations. Using this bijection, we are able to give

¹Also known as rim-hook tableau

several q -refinements of enumerations of border-strip decompositions. In particular, this includes the classical q -analogue of permutations in S_n given by Mahonian statistics.

Definition 2. A *simple diagram* is parametrized by two parameters, a word \mathbf{w} with entries in $\{r, c\}$, and a natural number n .

The family of simple diagrams are constructed recursively as follows:

- If $\mathbf{w} = \emptyset$, then (\mathbf{w}, n) is the $n \times n$ -square.
- The diagram $(c\mathbf{w}, n)$ is obtained from (\mathbf{w}, n) by adding an additional column of size n on the left, such that the bottom-most square of the new column is in the bottommost row of (\mathbf{w}, n) .
- The diagram $(r\mathbf{w}, n)$ is obtained from (\mathbf{w}, n) by adding an additional row of size n on the bottom, such that the left-most square of the new row is in the leftmost column of (\mathbf{w}, n) .

We let $\text{BSD}(\mathbf{w}, n)$ denote the set of border-strip decompositions of (\mathbf{w}, n) , and $\text{BST}(\mathbf{w}, n)$ denotes the set of border-strip tableaux of (\mathbf{w}, n) , in both cases with strips of size n .

For a word \mathbf{w} , we define $C_{\mathbf{w}}$ the total number of c 's in \mathbf{w} , $R_{\mathbf{w}}$ the total number of r 's in \mathbf{w} . Furthermore, let $\text{hor}(\mathbf{w}) := C_{\mathbf{w}} - R_{\mathbf{w}}$. Intuitively, $\text{hor}(\mathbf{w})$ measures how “horizontal” the diagram is.

Example 3. The simple diagram determined by $(rcrcc, 2)$ is the following shape:


(3)

Below we can see how $(rcrcc, 2)$ is constructed from the 2×2 square by adding successively the blue, red, green, yellow and gray boxes to a 2×2 square.


(4)

We have $C_{rcrcc} = 3$, $R_{rcrcc} = 1$ and $\text{hor}(rcrcc) = 3 - 1 = 2$.

Definition 4. In a fixed border-strip decomposition, a border-strip B_a is *above* a border-strip B_b if there is a path from B_a to B_b going only down or right. In this case B_b is *below* B_a .

Definition 5. A border-strip B_a is *inner* to a border-strip B_b if there exists a sequence $B_a = B_1, B_2, \dots, B_k = B_b$ such that for all i B_i is above B_{i+1} . This means the relation *inner* is the transitive closure of the relation *above*.

If B_a is inner to B_b , B_b is *outer* to B_a .

Two border strips B_a and B_b are *comparable*, if B_a is inner or outer to B_b .

Remark 6. If B_1 is above B_2 , it implies B_1 must contain a smaller number than B_2 in any border-strip tableau, thus the existence of a BST for any BSD implies the transitive closure is well-defined.

Also, B_1 is inner to B_2 if and only if it contains a smaller number in every BST with the border-strip decomposition. We do not use this property, but it follows from the proof of Proposition 21 below.

Example 7. Here is an example in $\text{BSD}(ccrcc, 3)$:



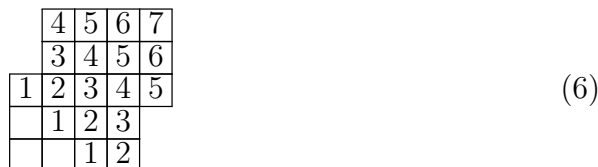
In this case the blue strip is above the red strip, and the red strip is above the yellow strip, which means the blue strip is inner to the yellow strip, and the blue and yellow strip are comparable. But the blue strip is neither above nor below the yellow strip.

Definition 8. Two border strips B_1 and B_2 in a decomposition form an *inversion* if the following three conditions are fulfilled:

- The content-sequences of B_1 and B_2 have a non-empty intersection,
- B_1 is inner to B_2 , and
- $\mathcal{H}(B_1) > \mathcal{H}(B_2)$.

We prove in Corollary 27 that this definition generalizes the notion of inversions in S_n in a natural manner.

Definition 9. For a word \mathbf{w} of length k , we number the diagonals of the simple diagram (\mathbf{w}, n) from $n + k$ to 1, starting in the top right corner, as shown in the example below for $(crrc, 3)$:

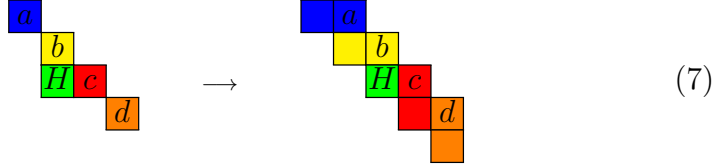


Lemma 10. Let \mathbf{w} be a word of length k . Then for any decomposition in $\text{BSD}(\mathbf{w}, n)$, there is a unique head in each diagonal from 1 to $n + k$, and the position of the heads uniquely determines the border-strip decomposition.

Proof. We will show that the position of the heads uniquely determines the decomposition, by processing the diagonals one by one and iteratively prolonging the strips, starting from diagonal $n + k$.

The only way to cover the single box in diagonal $n + k$ is for it to be a head.

For diagonal i with $k < i < n + k$ we have one box more in diagonal i than in diagonal $i + 1$, and all strips we already started have less than n squares, and must continue, therefore there is exactly one head in diagonal i . Furthermore, the position of the head H in diagonal i determines the continuation of the strips started, as shown in this figure:



For $i \leq k$, there is exactly one strip ending in diagonal $i + 1$, and diagonals i and $i + 1$ have the same size, therefore there must be exactly one head in diagonal i . Once we placed the head, there are $n - 1$ boxes left in diagonal i , and $n - 1$ strips must have a box in diagonal i . As strips cannot cross each other, this gives at most one solution.

Similarly, for the diagonals below diagonal 1, the size of the diagonals decreases by 1 each step, and the number of strips too, so there cannot be any heads below diagonal 1, and there is a unique way to extend the border-strip decomposition. \square

Definition 11. Given a border-strip decomposition of a simple diagram, the unique strip with head in diagonal i is referred to as *strip i* .

Proposition 12. Let (\mathbf{w}, n) be a simple diagram. Then for any decomposition in $\text{BSD}(\mathbf{w}, n)$, if $|i - j| \leq n$, then strip i and j are comparable.

Proof. Without loss of generality, $i > j$. Then the tail of i is at most one diagonal higher than the head of j . As we can cover two consecutive diagonals with a path going only right and down, two elements that are at most one diagonal apart are comparable. \square

We noticed that the positions of the heads of the strips uniquely determine the border-strip decomposition. The next definition and proposition encodes the placements of the heads as a permutation with certain restrictions, giving an alternative description of border-strip tableaux of simple shapes.

Further down, we add more restrictions, so that the resulting set of permutations are in bijection with border-strip decompositions.

Definition 13. We define $\psi : \text{BST}(\mathbf{w}, n) \rightarrow S_{n+k}$ by $\psi(T) = \sigma$ such that if the unique head in diagonal i is numbered j then $\sigma(j) = i$.

We let $\text{BSP}(\mathbf{w}, n) \subseteq S_{n+k}$ denote the image of $\text{BST}(\mathbf{w}, n)$ under ψ .

Proposition 14. *The map ψ is injective.*

Proof. A permutation defines the value of the heads in each diagonal, and thus the value of all the boxes in each diagonal. As they have to be in increasing order to form a border-strip tableau, there is a unique way to do this.

Note that not every permutation give rise to a valid border-strip tableau, see Proposition 16 below. \square

Example 15. Here is an example $T \in \text{BST}(ccc, n)$:

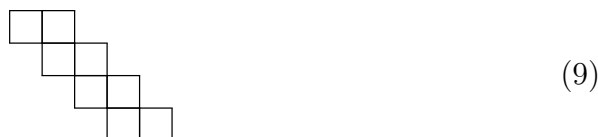
$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 3 & 4 \\ \hline 2 & 2 & 2 & 3 & 4 & 4 \\ \hline 5 & 5 & 5 & 6 & 6 & 6 \\ \hline \end{array} \quad \text{with} \quad \psi(T) = [3, 2, 5, 6, 1, 4]. \quad (8)$$

The strip labeled 1 has its head in diagonal 3, thus $\psi(T)(1) = 3$, the strip labeled 2 has its head in diagonal 2, thus $\psi(T)(2) = 2$ and so on.

Proposition 16. *Let $\mathbf{w} = (w_1, \dots, w_k)$ be a word of length k . A permutation $\sigma \in S_{n+k}$ is in $\text{BSP}(\mathbf{w}, n)$ if and only if for all i with $1 \leq i \leq k$ we have:*

- $\sigma^{-1}(i) < \sigma^{-1}(n + i)$ whenever $w_i = c$, and
- $\sigma^{-1}(i) > \sigma^{-1}(n + i)$ whenever $w_i = r$.

Proof. We construct the tableau from the last diagonal to the first one. For any i , the unique head in diagonal i must be filled with number $\sigma^{-1}(i)$. If $k < i \leq n + k$, diagonal i has one element more than diagonal $i + 1$, and it is always possible to extend a BSD. If $1 \leq i \leq k$, we have to look at w_i . If $w_i = c$, diagonals i and $i + 1$ are as follows:



We observe the new strip must be added above the strip starting in diagonal $n + i$ (ending in diagonal $i + 1$), which means it has to be a smaller number, *i.e.* $\sigma^{-1}(i) < \sigma^{-1}(n + i)$. If $w_i = r$, diagonals i and $i + 1$ must be as follows:



and the new strip must be below strip $n + i$, and it has to be filled with a larger number, *i.e.* $\sigma^{-1}(i) > \sigma^{-1}(n + i)$. \square

Corollary 17. *For a word \mathbf{w} of length k , with $k \leq n$, we have*

$$|\text{BST}(\mathbf{w}, n)| = (n + k)!/2^k$$

This means the number of border-strip tableaux only depends on the length of the word for $n \geq k$. In contrast, this count is word dependent for $n < k$.

Proof. From Proposition 14 we know ψ is injective, thus $|\text{BST}(\mathbf{w}, n)| = |\text{BSP}(\mathbf{w}, n)|$. From the conditions in Proposition 16 we know the relative order on all pairs $(i, n+i)$. As $n \geq k$, no such entry belongs to two such pairs, and thus $|\text{BSP}(\mathbf{w}, n)| = (n+k)!/2^k$. \square

Corollary 18. *For every permutation $\sigma \in S_{n+k}$ there is exactly one word \mathbf{w} of length k such that $\sigma \in \text{BSP}(\mathbf{w}, n)$. In particular, ψ is a bijection between $\{\text{BST}(\mathbf{w}, n) : \mathbf{w} \in \{r, c\}^k\}$ and S_{n+k} and*

$$\sum_{\mathbf{w} \in \{r, c\}^k} |\text{BST}(\mathbf{w}, n)| = (n+k)!.$$

Proof. From Proposition 14 we know ψ restricted to one word is injective. From Proposition 16 we deduce two different words cannot give the same permutation so ψ is injective over the set of all words of length k . On the other hand, for any permutation $\sigma \in S_{n+k}$ there is always one word $\mathbf{w} \in \{r, c\}^k$ such that $\sigma \in \text{BSP}(\mathbf{w}, n)$, as we can recover the word from the pairs $(i, n+i)$. Thus ψ is also surjective. \square

Definition 19. If $\sigma(i) - k > \sigma(i+1)$, i is called a k -descent of σ . Let $\text{DES}_k(\sigma)$ denote the set of k -descents of σ , and let $\text{des}_k(\sigma)$ be the number of such k -descents.

Example 20. Let $\sigma = [2, 4, \mathbf{10}, 5, 6, 3, \mathbf{8}, 1, 7, 9]$, then the 3-descents of σ are 3 and 7, marked in bold.

Let s_1, \dots, s_{n-1} denote the simple transpositions in S_n .

Proposition 21. *Let \mathbf{w} be a word of length k , $\sigma \in \text{BSP}(\mathbf{w}, n)$ and $i \in \text{DES}_n(\sigma)$, then the border-strip tableaux $\psi^{-1}(s_i\sigma)$ and $\psi^{-1}(\sigma)$ give rise to the same border-strip decomposition. Moreover,*

$$|\text{BSD}(\mathbf{w}, n)| = |\{\sigma \in \text{BSP}(\mathbf{w}, n) : \text{des}_n(\sigma) = 0\}|$$

that is, the number of elements in $\text{BSD}(\mathbf{w}, n)$ is the number of permutations in $\text{BSP}(\mathbf{w}, n)$ without n -descent.

Proof. Let $\tau := s_i\sigma$ and T_σ, T_τ be the corresponding border-strip tableaux. First we show that $\tau \in \text{BSP}(\mathbf{w}, n)$. The only places where τ^{-1} differs from σ^{-1} are $\tau(i)$ and $\tau(i+1)$. As i is an n -descent, Proposition 16 does not give any condition on their order. Suppose $j \in [k]$. Then at most one from $\sigma^{-1}(j)$ and $\sigma^{-1}(n+j)$ is different for τ^{-1} and the quantities

$$\tau^{-1}(j) - \tau^{-1}(n+j) \text{ and } \sigma^{-1}(j) - \sigma^{-1}(n+j)$$

are either the same or differ by 1, so they can never have opposite signs. Since $\sigma \in \text{BSP}(\mathbf{w}, n)$, the conditions in Proposition 16 are still fulfilled for τ and we have that $\tau \in \text{BSP}(\mathbf{w}, n)$.

It remains to show that T_τ and T_σ have the same border-strip decomposition. The only strips which have a different number in T_σ and T_τ are strip $\tau(i)$ and strip $\tau(i+1)$ and the new numbers differ by ± 1 . Therefore, the only pair that has a different relative ordering under τ than under σ is the pair $(\tau(i), \tau(i+1))$. However, since i is an n -descent, it does not affect the construction in the proof of Proposition 16, and as ψ is injective, this implies that T_σ and T_τ have the same BSD.

For the second statement, we will prove that there is exactly one permutation without any n -descent in $\text{BSP}(\mathbf{w}, n)$ for a fixed border-strip decomposition.

We claim that if there are two strips, x and y , such that the three following conditions hold:

- (1) the strips x and y are not comparable in the sense of Definition 4
- (2) $x > y$
- (3) $\sigma^{-1}(x) < \sigma^{-1}(y)$

then σ has an n -descent.

We consider the sequence $\sigma^{-1}(x) = a_1, a_1 + 1 = a_2, \dots, a_m = \sigma^{-1}(y)$ and i such that $\sigma(a_i) - \sigma(a_{i+1})$ is maximal. If $\sigma(a_i) - \sigma(a_{i+1}) \leq n$, then we can find a subsequence $\sigma^{-1}(x) = a_{i_1}, a_{i_2}, \dots, a_{i_s} = \sigma^{-1}(y)$ such that for all j we have $|\sigma(a_{i_j}) - \sigma(a_{i_{j+1}})| \leq n$. But then Proposition 12 implies $\sigma(a_{i_j})$ and $\sigma(a_{i_{j+1}})$ are comparable, and by transitivity, x and y are comparable, which contradicts our assumption.

This implies to avoid an n -descent, we must fix the relative order of all non-comparable pairs, but the relative order of comparable pairs is always fixed, which means there is at most one permutation without n -descent for a given decomposition.

On the other hand, we can always find such a permutation, by starting from a permutation in $\text{BSP}(\mathbf{w}, n)$ and repeatedly remove n -descents until a permutation without n -descents is obtained. \square

Corollary 22. *Let $\mathbf{w} \in \{r, c\}^k$. The set of border-strip decompositions of the simple diagram (\mathbf{w}, n) is in bijection with the set of permutations in S_{n+k} such that for each $i \in [k]$,*

- $w_i = c \implies \sigma^{-1}(i) < \sigma^{-1}(n+i)$,
- $w_i = r \implies \sigma^{-1}(i) > \sigma^{-1}(n+i)$ and
- $\sigma(j) - \sigma(j+1) \leq n$ for all $j \in [n+k-1]$.

Definition 23. For a word $\mathbf{w} \in \{r, c\}^k$ let

$$\hat{f}_{\mathbf{w}}(n) := |\text{BSD}(\mathbf{w}, n)| \frac{(2k)!}{(n-k)!}. \quad (11)$$

Proposition 24. Whenever $n > 2k - 1$, the function $\hat{f}_{\mathbf{w}}(n)$ is equal to

$$f_{\mathbf{w}}(n) = \sum_{\tau \in \text{BSP}(\mathbf{w}, k)} (n + k - \text{des}_k(\tau))_{2k}. \quad (12)$$

As a consequence, $\hat{f}_{\mathbf{w}}(n)$ is a polynomial in n of degree $2k$ with integer coefficients when restricted to values $n > 2k - 1$. Moreover, $f_{\mathbf{w}}(n)$ is divisible by the falling factorial $(n + 1)_{k+1}$.

Proof. Interpreting permutations in S_{n+k} as sequences of $n+k$ numbers, we note that the first two conditions in Corollary 22 only apply to the relative order of the first and last k elements.

Thus, in order to construct a permutation σ in S_{n+k} fulfilling the three conditions in Corollary 22, we proceed in three initial steps:

- (1) Choose an ordering of the entries $1, 2, \dots, k, n+1, n+2, \dots, n+k$.
- (2) Choose the positions of the entries $1, 2, \dots, k, n+1, n+2, \dots, n+k$.
- (3) Choose an ordering of the entries $k+1, \dots, n$.

Not all choices here will fulfill the conditions in Corollary 22, we shall see below which ones are valid. For a choice in the first step, two things might happen:

- a) There is some pair $(i, i+k)$ in the wrong order — violating one of the first two conditions. In this case we do not have a BST, and thus no BSD corresponding to this choice.
- b) All pairs $(i, i+k)$ have the correct order. In this case, the ordering of the entries

$$1, 2, \dots, k, n+1, n+2, \dots, n+k$$

fulfill the conditions (after standardization) of being a permutation τ in $\text{BSP}(\mathbf{w}, k)$.

Now we need to ensure that there are no n -descents in the final permutation. If there are no k -descents in τ (from step *b* above), this is always the case. Otherwise, we need to insert another number after every k -descent of τ . This means we only have $\binom{n+k-\text{des}_k(\tau)}{2k}$ valid choices in step (2). The last step always has $(n-k)!$ valid choices as the order on

$k + 1, \dots, n$ does not matter. It follows that $f_{\mathbf{w}}(n)$ is given by

$$\begin{aligned} f_{\mathbf{w}}(n) &= \frac{(2k)!}{(n-k)!} \sum_{\tau \in \text{BSP}(\mathbf{w}, k)} \binom{n+k - \text{des}_k(\tau)}{2k} (n-k)! \\ &= \sum_{\tau \in \text{BSP}(\mathbf{w}, k)} (n+k - \text{des}_k(\tau))_{2k}. \end{aligned}$$

This function is obviously a polynomial of degree $2k$. Furthermore, since $\text{des}_k(\tau)$ is between 0 and $k-1$ it follows that $(n+k - \text{des}_k(\tau))_{2k}$ is divisible by $(n+1)_{k+1}$. \square

Corollary 25. *We have the enumeration*

$$|\text{BSD}(rc, n)| = (n+1)!(3n+2)/12 \text{ whenever } n \geq 2.$$

Proof. Using Proposition 24, we know that $|\text{BSD}(rc, n)|$ can be expressed as $(n-2)!\hat{f}_{rc}(n)/4!$. Since we know that $\hat{f}_{rc}(n)$ is a polynomial in n for $n \geq 4$, it suffices to verify the formula for the first few values of n . \square

The sequence $a_{n+1} = (n+1)!(3n+2)/12$ appear as [Slo16, A227404], where a_n count the total number of inversions in all permutations in S_n consisting of a single cycle. For example, the permutations (123) and (132) have four inversions in total, giving $a_3 = 4$.

Lemma 26. *Let $\sigma \in \text{BSP}(\mathbf{w}, n)$, with T_σ being the corresponding border-strip decomposition. Then the strips i and j in T_σ with $i < j$ form an inversion if and only if $j - i < n$ and $\sigma^{-1}(i) > \sigma^{-1}(j)$.*

Proof. If $j - i \geq n$ they do not have an element on the same diagonal, and by definition do not form an inversion. If $j - i < n$ they share an element on the same diagonal, and if $\sigma^{-1}(i) > \sigma^{-1}(j)$ strip j is above strip i , and we have an inversion. \square

Given a border-strip decomposition T , let $\text{inv}(T)$ denote the total number of inversions in T . Furthermore, for $\sigma \in S_{n+k}$ let

$$\text{inv}_n(\sigma) := \{(i, j) : 0 < j - i < n \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j)\}.$$

The q -analogue of $\text{BSD}(\mathbf{w}, n)$ is defined as

$$\sum_{T \in \text{BSD}(\mathbf{w}, n)} q^{\text{inv}(T)} \tag{13}$$

and by previous lemma we have that

$$\sum_{T \in \text{BSD}(\mathbf{w}, n)} q^{\text{inv}(T)} = \sum_{\sigma \in \text{BSP}(\mathbf{w}, n)} q^{\text{inv}_n(\sigma)}. \tag{14}$$

Corollary 27. *The q -analogue of the $n \times n$ -square, $\text{BSD}(\emptyset, n)$, satisfies the identity*

$$\sum_{T \in \text{BSD}(\emptyset, n)} q^{\text{inv}(T)} = [n]_q!.$$

Proof. From Proposition 16 we know all permutations in S_n are in $\text{BSP}(\emptyset, n)$, from Proposition 21, we know all BST correspond to BSD, and from the previous result we deduce the q -analogue is given by $[n]_q!$ \square

Corollary 28. *We have the following q -analogue for $\text{BSD}(c, n)$:*

$$\sum_{T \in \text{BSD}(c, n)} q^{\text{inv} T} = [n-1]_q! \sum_{i=1}^n i q^{i-1}.$$

Proof. We get a permutation corresponding to a decomposition by placing 1 and $n+1$ (i.e. choose $\sigma^{-1}(1)$ and $\sigma^{-1}(n+1)$), and then choose the order of $2, \dots, n$. This choice gives $[n-1]_q!$, and the possible positions of 1 and $n+1$ gives $\sum_{i=1}^n i q^{i-1}$, as 1 has to be before $n+1$ for it to be a BST. Note that there cannot be any n -descents and therefore the number of border-strip tableaux is equal to the number of decompositions. \square

Proposition 29. *If \mathbf{w} is a word of a simple diagram, then*

$$|\text{BSD}(c\mathbf{w}, n)| + |\text{BSD}(r\mathbf{w}, n)| = (n+1)|\text{BSD}(\mathbf{w})|.$$

Furthermore, this relation extends to the following q -analogue:

$$\sum_{T \in \text{BSD}(c\mathbf{w}, n)} q^{\text{inv} T} + \sum_{T \in \text{BSD}(r\mathbf{w}, n)} q^{\text{inv} T} = [n+1]_q \sum_{T \in \text{BSD}(\mathbf{w}, n)} q^{\text{inv} T}$$

Proof. If we fix the positions of the heads in (\mathbf{w}, n) , the new head in $(c\mathbf{w}, n)$ must be above the strip it replaces, where as in $(r\mathbf{w}, n)$ it must be below. Together, this gives $n+1$ possibilities to complete a BSD of (\mathbf{w}, n) . If, in $(c\mathbf{w}, n)$ or in $(r\mathbf{w}, n)$, we place the new head in position i of the diagonal, the new strip forms an inversion with all $i-1$ strips above it, thus the q -analogue. \square

Corollary 30. *We can count the total number of border-strip decompositions for all words of length k , more precisely:*

$$\sum_{\mathbf{w} \in \{r, c\}^k} |\text{BSD}(\mathbf{w}, n)| = (n+1)^k n!$$

and this relation extends to the q -analogue:

$$\sum_{\mathbf{w} \in \{r, c\}^k} \sum_{T \in \text{BSD}(\mathbf{w}, n)} q^{\text{inv} T} = [n+1]_q^k [n]_q!$$

Proof. It suffices to show the q -analogue, by taking $q = 1$ we obtain the enumeration. We proceed by induction.

The base case, $k = 0$, is given by Corollary 27. The previous result gives the induction step:

$$\begin{aligned} & \sum_{\mathbf{w} \in \{r,c\}^k} \sum_{T \in \text{BSD}(\mathbf{w},n)} q^{\text{inv} T} = \\ & \sum_{\mathbf{w} \in \{r,c\}^{k-1}} \sum_{T \in \text{BSD}(r\mathbf{w},n)} q^{\text{inv} T} + \sum_{\mathbf{w} \in \{r,c\}^{k-1}} \sum_{T \in \text{BSD}(c\mathbf{w},n)} q^{\text{inv} T} = \\ & [n+1]_q \sum_{\mathbf{w} \in \{r,c\}^{k-1}} \sum_{T \in \text{BSD}(\mathbf{w},n)} q^{\text{inv} T} \end{aligned}$$

□

If we let $n = k - 1$, we note that the sequence $a(n) = (n+1)^{n-1}n!$ is A066319. This sequence also show up in [Wei12, Thm. 5.4]. Let $K_{n,n+1}$ be the complete bipartite graph with n sources and $n+1$ sinks. Then there are $a(n)$ spanning trees such that every source has exactly 2 incident edges. This is related to computing the Euler characteristic of certain moduli spaces, see [Wei12] for details. This connection is quite interesting, as it is perhaps related to what we discuss in Section 4 below.

Recall the definition of $\text{hor}(\mathbf{w})$ as the difference between the number of occurrences of c and r in \mathbf{w} . The following theorem shows that “straighter” shapes admits a larger number of decompositions, in a precise sense:

Theorem 31. *If \mathbf{v} and \mathbf{w} are words of length k and $|\text{hor}(\mathbf{v})| < |\text{hor}(\mathbf{w})|$, then*

$$|\text{BSD}(\mathbf{v}, n)| > |\text{BSD}(\mathbf{w}, n)| \text{ for } n \text{ sufficiently large.}$$

In fact,

$$\frac{|\text{BSD}(\mathbf{v}, n)| - |\text{BSD}(\mathbf{w}, n)|}{(n-k)!} = O(n^{2k-1}).$$

Proof. Recall from Proposition 24 that

$$f_{\mathbf{v}}(n) = \sum_{\sigma \in \text{BSP}(\mathbf{v},k)} (n+k - \text{des}_k(\sigma))_{2k}.$$

From Corollary 17, we know that $|\text{BSP}(\mathbf{v}, k)| = (2k)!/2^{2k}$. It then follows that

$$f_{\mathbf{v}}(n) = \frac{(2k)!}{2^k} n^{2k} + \alpha n^{2k-1} + \text{l.o.t} \text{ and } f_{\mathbf{w}}(n) = \frac{(2k)!}{2^k} n^{2k} + \beta n^{2k-1} + \text{l.o.t.}$$

Our goal is to prove that $\alpha < \beta$.

For a fixed permutation $\sigma \in \text{BSP}(\mathbf{v}, k)$, its contribution to α is given by

$$\sum_{i=0}^{2k-1} (k - \text{des}_k(\sigma) - i) = 2k^2 - k(2k-1) - 2k \text{des}_k(\sigma).$$

Hence,

$$\alpha = k|\text{BSP}(\mathbf{v}, k)| - 2k \sum_{\sigma \in \text{BSP}(\mathbf{v}, k)} \text{des}_k(\sigma).$$

As $|\text{BSP}(\mathbf{v}, k)|$ does not depend on \mathbf{v} , the only part depending on \mathbf{v} is

$$J_{\mathbf{v}} := \sum_{\sigma \in \text{BSP}(\mathbf{v}, k)} \text{des}_k(\sigma),$$

and it suffices to prove $J_{\mathbf{v}}$ is strictly smaller for a straighter word.

To do this, we count the number of permutations where $b+k$ is a k -descent with a , for $1 \leq a < b \leq k$ fixed (*i.e.* we have $\dots, b+k, a, \dots$ in the permutation). To create such a permutation, we can choose the order of all elements different from $a, b, a+k, b+k$ in any way respecting the orders of pairs $\sigma(i), \sigma(i+k)$, which gives $(2k-4)!/2^{k-2}$ choices. Then we must choose the order of the three blocks $a+k, (b+k)a, b$. If $a+k$ and b are on the same side of $(b+k)a$, this gives two possibilities, otherwise there is only one way. We observe $a+k$ and b are on the same side if and only if $\mathbf{v}_a \neq \mathbf{v}_b$. Finally, we can choose the position of the three blocks $a+k, b, (b+k)a$, which gives $\binom{2k}{3}$ choices. So the number of permutations where $b+k$ is a k -descent with a is exactly

$$\begin{cases} 2\binom{2k}{3}(2k-4)!/2^{k-2} & \text{if } \mathbf{v}_a \neq \mathbf{v}_b \\ \binom{2k}{3}(2k-4)!/2^{k-2} & \text{otherwise.} \end{cases}$$

Recall $C_{\mathbf{v}}$ is the number of c 's in \mathbf{v} , and $R_{\mathbf{v}}$ is the number of r 's in \mathbf{v} . The previous result implies

$$J_{\mathbf{v}} = \binom{2k}{3} \frac{(2k-4)!}{2^{k-2}} \left[\binom{C_{\mathbf{v}}}{2} + 2C_{\mathbf{v}}R_{\mathbf{v}} + \binom{R_{\mathbf{v}}}{2} \right]$$

Since $R_{\mathbf{v}} = k - C_{\mathbf{v}}$, it follows that

$$\binom{C_{\mathbf{v}}}{2} + 2C_{\mathbf{v}}R_{\mathbf{v}} + \binom{R_{\mathbf{v}}}{2} = \binom{k}{2} + C_{\mathbf{v}}R_{\mathbf{v}}$$

which is increasing as $|\text{hor}(\mathbf{v})|$ decreases. \square

Conjecture 32. *The function $f_{\mathbf{w}}(n)$ uniquely define \mathbf{w} up to isometry of the shape \mathbf{w} , i.e. up to exchanging r and c and reversing the word.*

Note that for fixed k , the polynomials (in n)

$$(n+k-i)_{2k} \text{ with } i = 0, \dots, k-1$$

are linearly independent: these span the same space as

$$\left\{ \frac{(n+i)_{2k}}{(2k)!} \right\}_{i=1}^k = \left\{ \binom{n+i}{2k} \right\}_{i=1}^k,$$

and the latter collection of polynomials can be seen to be linearly independent.

As a consequence, given $f_{\mathbf{w}}(n)$, which is a sum over permutations in $\text{BSP}(\mathbf{w}, k)$, for any i we can extract the number of permutations $\sigma \in \text{BSP}(\mathbf{w}, k)$ with $\text{des}_k(\sigma) = i$. Hence, the conjecture is reduced to determining if the multi-set of des_k -values of the elements in $\text{BSP}(\mathbf{w}, k)$ uniquely determines \mathbf{w} up to isometry.

In particular, if the number of terms without k -descents is different, the polynomial is also different, so we can formulate the stronger conjecture that $|\text{BSD}(\mathbf{w}, k)|$ uniquely determines a word \mathbf{w} of length k up to isometry.

4. A CONNECTION WITH THE WEIL–PETERSSON VOLUME

It follows from Corollary 22 that the set $\text{BSD}(2n \times n, n)$ is in bijection with the set of permutations of $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ such that x_i appear before y_i for all i , and we do not have \dots, x_i, y_j, \dots (consecutive), such that $i > j$.

Lemma 33 (Adaptation of [Xia]). *The cardinality of $\text{BSD}(2n \times n, n)$ is given by the formula*

$$|\text{BSD}(2n \times n, n)| = \sum_{p \vdash n} (-1)^{|p|-1} \frac{1}{m!} \binom{|p|}{p} \binom{|p+1|}{p+1}, \quad (15)$$

where $m = (m_1, m_2, \dots, m_k)$ and m_i is the multiplicity of i in p , and we use the notation $p \pm 1 := (p_1 \pm 1, \dots, p_k \pm 1)$ and $|p| = p_1 + \dots + p_k$. Note that $\binom{|p|}{p}$ and $\binom{|p+1|}{p+1}$ denote a multinomial coefficients.

Proof. For a permutation $\sigma \in S_{2n}$ corresponding to a *border-strip tableau*, let Γ_σ be the graph on the vertex set $[n]$ with edge set

$$\{(\sigma(i) - n, \sigma(i+1)) : \sigma(i) - n > \sigma(i+1)\}.$$

Let G be the set of graphs obtained from such border-strip tableaux. Let E be $\binom{[n]}{2}$, that is, the set of all possible edges on the vertex set $[n]$ and let $G(e_1, \dots, e_r) \subseteq G$ be the set of graphs that include the edges $\{e_1, \dots, e_r\} \subseteq E$. By definition, elements in $\text{BST}(2n \times n, n)$ are in bijection with $G(\emptyset)$, and Proposition 21 tells us that

$$|\text{BSD}(2n \times n, n)| = G \setminus \left(\bigcup_{r=1}^n \bigcup_{e_1, \dots, e_r \in E} G(e_1, \dots, e_r) \right).$$

Using the inclusion-exclusion principle, it follows that

$$|\text{BSD}(2n \times n, n)| = |\text{BST}(2n \times n, n)| - \sum_{e_1 \in E} |G(e_1)| + \sum_{e_1, e_2 \in E} |G(e_1, e_2)| - \dots$$

We then observe that these graphs are characterized by the connected components induced by the forced edges e_1, \dots, e_r , determining a partition p of n . Furthermore, the sign in the above formula only depends on the number of forced edges, which is equal to $|p - 1|$, so we can transform this into a sum over all partitions of n . Given a partition $p \vdash n$, the number of graphs with component sizes p_1, p_2, \dots, p_k is given by $\frac{1}{m!} \binom{|p|}{p}$, with m given as above.

Claim: Let e_1, e_2, \dots, e_r be fixed edges such that the component sizes are given by p . Then

$$G(e_1, \dots, e_r) = \binom{|p+1|}{p+1}.$$

Proof: Suppose $\Gamma_\sigma \in G(e_1, \dots, e_r)$ has a component (i_1, i_2, \dots, i_j) , in increasing order. From Proposition 16, we know $\sigma^{-1}(i_s) < \sigma^{-1}(i_s + n)$ for all $1 \leq s \leq j$, and for (i_1, i_2, \dots, i_j) to be connected, we need $i_s + n$ to form an n -descent with i_{s-1} for all $1 < s \leq j$ i.e. $\sigma^{-1}(i_s + n) = \sigma^{-1}(i_{s-1}) - 1$.

Together these two statements imply that σ , has the following structure:

$$\begin{aligned} & \sigma^{-1}(i_j) < \sigma^{-1}(i_j + n) \ll \sigma^{-1}(i_{j-1}) < \sigma^{-1}(i_{j-1} + n) \ll \dots \\ & \dots < \sigma^{-1}(i_3 + n) \ll \sigma^{-1}(i_2) < \sigma^{-1}(i_2 + n) \ll \sigma^{-1}(i_1) < \sigma^{-1}(i_1 + n) \end{aligned}$$

where $a \ll b$ means that $a + 1 = b$. Thus, we have $j + 1$ blocks,

$[\sigma^{-1}(i_j)]$, $[\sigma^{-1}(i_{s+1} + n) \ll \sigma^{-1}(i_s)]$ for $s = 1, \dots, j - 1$ and $[\sigma^{-1}(i_1 + n)]$ which need to appear in order, but there is no further restriction. The number of Γ_σ with component sizes determined by p is therefore $\binom{|p+1|}{p+1}$, which concludes the proof. \square

Let us now dive into a completely different part of mathematics. The *Weil–Petersson volume*, $\text{Vol}_{WP}(\cdot)$, is defined as

$$\text{Vol}_{WP}(M) := \frac{1}{(n-3)!} \int_M \wedge^{n-3}(\omega_M).$$

where ω_M is the Weil–Petersson symplectic form on M .

Let $M_{0,n}$ be the moduli space of an n -punctured Riemann sphere, that is

$$M_{0,n} := \{(z_1, \dots, z_n) \in \hat{\mathbb{C}}^n : z_i \neq z_j\} / S_n \times \text{PSL}(2, \mathbb{C})$$

and S_n acts by permuting variables, and $\text{PSL}(2, \mathbb{C})$ acts as a linear fractional transformation.

Theorem 34 ([Zog93]). *The Weil–Petersson volume of the moduli space of an n -punctured Riemann sphere $M_{0,n}$ is given by the formula*

$$\text{Vol}_{WP}(M_{0,n}) = \frac{\pi^{2(n-3)} v_n}{n!(n-3)!}, \text{ for } n \geq 4,$$

where the sequence v_n for $n \geq 3$, $v_3 = 1$ be defined via the recursion

$$v_n = \frac{1}{2} \sum_{i=1}^{n-3} \frac{i(n-i-2)}{n-1} \binom{n-4}{i-1} \binom{n}{i+1} v_{i+2} v_{n-i}, \quad n \geq 4. \quad (16)$$

This sequence shows up as A115047 in the OEIS, [Slo16], see [KMZ96, Mat01] for more background.

In [KMZ96, Equation (0.7)], the following relationship is shown

Proposition 35. *Let the sequence v_n be defined as in (16). Then*

$$v_n = \sum_{k=1}^{n-3} \frac{(-1)^{n-3-k}}{k!} \sum_{\substack{m_1, \dots, m_k > 0 \\ m_1 + \dots + m_k = n-3}} \binom{n-3}{m_1, \dots, m_k} \binom{n-3+k}{m_1+1, \dots, m_k+1}.$$

We are now ready to prove the following connection between the sequence $v(n)$ and border-strip decompositions:

Theorem 36. *Let*

$$a(n) = \frac{1}{2} \sum_{i=1}^n \frac{i(n-i+1)}{(n+2)} \binom{n-1}{i-1} \binom{n+3}{i+1} a(i-1) a(n-i). \quad (17)$$

and let v_n be given as in (16). Then $a(n) = v_{n+3} = |\text{BSD}(2n \times n, n)|$.

Proof. The first equality, $v_{n+3} = a(n)$ follows from comparing (16) and (17). It is a straightforward calculation to verify that these are equal.

To get the second identity, note that we can get the formula in Proposition 35 from Equation (15) by replacing partitions with compositions, and then refining the sum over the number of parts (denoted k in Proposition 35). \square

5. FURTHER DIRECTIONS

Given the connection with Euler characteristics of moduli spaces mentioned after Corollary 30, and the connection with moduli spaces in Theorem 36, is there a generalization of this mysterious connection? For example, there are formula for the volumes of surfaces of other genus, see [Mat01].

Another interesting direction is to consider the q -analogue of border-strip tableaux rather than decompositions.

Acknowledgements. The authors are thankful for the observation made by Ziqing Xiang in [Xia]. We also thank Richard Stanley for pointing out the reference [Pak00], and Justin Troyka for pointing out an error in an earlier draft.

The first author is funded by the *Knut and Alice Wallenberg Foundation* (2013.03.07).

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