A DUAL-RADIX APPROACH TO STEINER'S 1-CYCLE THEOREM

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Abstract. This article presents a variety of algebraic proofs of Steiner's *1-Cycle Theorem* [12]. It also demonstrates that, under an exponential upper-bound on the iterates, the only 1-cycles in the (accelerated) $3x-1$ dynamical system are (1) and $(5,7)$.

1. Introduction

Within the context of the $3x+1$ Problem, Steiner's *1-cycle Theorem* [12] is a result pertaining to the non-existence of 1-*cycles* (or *circuits*): for all $a, b \in \mathbb{N}$, Steiner shows that a rational expression of the form

$$\frac{2^a - 1}{2^{a+b} - 3^b} \tag{1}$$

does not assume a positive integer value except in the case where $a = b = 1$. In the proof, the author appeals to the continued fraction expansion of $\log_2 3$, transcendental number theory, and extensive numerical computation (see [11]). This argument serves as the basis for demonstrating the non-existence of 2-cycles in [10], and the non-existence of m-cycles in [11] where $m \leq 68$.

However, the author in [7] declares that the "most remarkable thing about [the theorem] is the weakness of its conclusion compared to the strength of the methods used in its proof." This article offers alternative proofs of this theorem using a variety of algebraic approaches; assuming the upper bound on periodic iterates established in [1], these proofs exploit that fact that the denominator in the above expression is coprime to both 2 and 3: this work simultaneously analyzes the residues of the circuit elements in a 2-adic and 3-adic setting. Based on the results in [9], the first proof employs elementary modular arithmetic, the second exploits identities on weighted binomial coefficients, and the third proof analyzes the 2-adic and 3-adic digits of such rational expressions.

2. Overview

2.1. **Notation.** This manuscript inherits all of the notation and definitions established in [9], which we summarize here. Let $\tau \in \mathbb{N}$, and let m and l be coprime integers exceeding 1. Let $\mathbf{e}, \mathbf{f} \in \mathbb{N}^\tau$ where $\mathbf{e} = (e_0, \ldots, e_{\tau-1})$ and $\mathbf{f} = (f_0, \ldots, f_{\tau-1})$. For each $u \in \mathbb{Z}$, define $E_u = \sum_{0 \leq w < u} e_{w \bmod \tau}$ and $\overline{E}_u = \sum_{0 \leq w < u} e_{(\tau-1-w) \bmod \tau}$; we will define F_u and $\overline{F}_u$ in an analogous manner with the elements of $\mathbf{f}$.

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For any integer a and positive base b ($b \geq 1$), let $[a]_b$ denote the element[1] of $[b)_0$ that satisfies the equivalence $[a]_b \equiv a \bmod b$. We may also express this element as $a \bmod b$. We will also write $[a]_b^{-1}$ to denote the element in $[b)_0$ that satisfies the equivalence $[a]_b\,[a]_b^{-1} \underset{b}{\equiv} 1$.

We will write $(-)^n = (-1)^n$ for each $n \in \mathbb{N}$.

2.2. **Argument Overview.** This dual-radix approach to the non-existence of circuits is based upon the following premises:

i. Let $\tau \in \mathbb{N}$, let m and l be coprime integers exceeding 1, and let n be a periodic orbit element from a given (m, l)-system of order τ satisfying the equivalences $\mu_\tau \equiv n \bmod m^{F_\tau}$ and $\lambda_\tau \equiv n \bmod l^{\overline{E}_\tau}$, where μ_τ and λ_τ are the canonical representatives of their corresponding equivalence classes.

   In [9], the equalities

$$n = \frac{\sum_{0 \leq w < \tau} m^{F_w} l^{\overline{E}_{\tau-1-w}} a_w}{l^{\overline{E}_\tau} - m^{F_\tau}} = \mu_\tau + m^{F_\tau}\left(\frac{\mu_\tau - \lambda_\tau}{l^{\overline{E}_\tau} - m^{F_\tau}}\right) = \lambda_\tau + l^{\overline{E}_\tau}\left(\frac{\mu_\tau - \lambda_\tau}{l^{\overline{E}_\tau} - m^{F_\tau}}\right)$$

   have been demonstrated for an admissible sequence of translation values $\mathbf{a} = (a_0, \ldots, a_{\tau-1})$; consequently, the denominator $l^{\overline{E}_\tau} - m^{F_\tau}$ divides the sum $\sum_{0 \leq w < \tau} m^{F_w} l^{\overline{E}_{\tau-1-w}} a_w$ if and only if it divides the arithmetic difference of canonical representatives $\mu_\tau - \lambda_\tau$. Furthermore, as $\mu_\tau \in \left[m^{F_\tau}\right)_0$ and $\lambda_\tau \in \left[l^{\overline{E}_\tau}\right)_0$, the iterate $n \in \mathbb{N}$ if and only if $\mu_\tau - \lambda_\tau \in D\mathbb{N}_0$.

ii. In the cases where $m = 3$, $l = 2$, $\mathbf{f} = \mathbf{1}^\tau = \mathbf{a}$, we apply the argument outlined in [9]: we will establish an upper bound of 3^τ for a potential, periodic iterate value over $\mathbb{N}$ for the $3x + 1$ Problem. In this context, the authors in [1] have demonstrated that the maximal iterate $n_{\max}$ within a periodic orbit admits the upper bound

$$n_{\max} < \frac{\left(\frac{3}{2}\right)^{\tau-1}}{1 - \frac{3^\tau}{2^{\overline{E}_\tau}}} \leq \tau^C \left(\frac{3}{2}\right)^{\tau-1} = o\left(3^{\tau-1}\right) \tag{2}$$

   for some effectively computable constant C (by applying the result in [13]). A recent upper bound on C is available in [8], in which the author establishes the inequality

$$\left|-\overline{E}_\tau \log 2 + \tau \log 3\right| \geq \overline{E}_\tau^{-13.3} \tag{3}$$

   (in their notation, we set $u_0 = 0$, $u_1 = -\overline{E}_\tau$, and $u_2 = \tau$); consequently, assuming $2^{\overline{E}_\tau} > 3^\tau$, we can bound[2] the denominator in (2) from below

$$1 - \frac{3^\tau}{2^{\overline{E}_\tau}} \geq \frac{\overline{E}_\tau^{-13.3}}{2}.$$

[1]This element is sometimes referred to as the *standard* (or *canonical*) *representative* of the equivalence class $\overline{a} \bmod b$.

[2]We can shed the logarithms: when $|w| < 1$, the power series expansion of $\log(1+w) = \sum_{u \geq 1}(-1)^{u-1}\frac{w^u}{u}$ yields $|\log(1 + w)| \leq 2|w|$ when $|w| \leq \frac{1}{2}$. See [4] (Corollary 1.6).

According to [3], in a periodic orbit over $\mathbb{N}$ of length $\overline{E}_\tau$, the ratio $\frac{\overline{E}_\tau}{\tau}$ satisfies the inequality

$$\frac{\overline{E}_\tau}{\tau} \leq \lg\left(3 + \frac{1}{n_{\min}}\right) \leq 2;$$

numerical computation yields

$$n_{\max} < \left(\frac{3}{2}\right)^{\tau-1} 2 \cdot (2\tau)^{13.3} < 3^\tau$$

when $\tau \geq 103$.

Thus, if $n_{\max} > 3^\tau$ and $n_{\max} \in \mathbb{N}$, then $\tau < 103$. However, the author in [5] demonstrates that the length of a non-trivial periodic orbit (excluding 1) over $\mathbb{N}$ must satisfy the inequality $2\tau \geq \overline{E}_\tau \geq 35,400$.

Thus, if $n \in \mathbb{N}$, then $n < 3^\tau$, and the equalities

$$n = \mu_\tau = \lambda_\tau$$

must hold.

iii. Within a circuit of order τ in the (accelerated) $3x+1$ dynamical system, the maximal element equals

$$\frac{(2^e+1)3^{\tau-1} - 2^{e+\tau-1}}{2^{e+\tau-1} - 3^\tau} = 2 \cdot 3^{\tau-1}\left(\frac{2^{e-1}-1}{2^{e+\tau-1} - 3^\tau}\right) - 1$$

for some $e \in \mathbb{N}$ (see [2]).

When $\tau = 1$, we note that $2^e - 3 = 2^{e-1} - 1 + 2^{e-1} - 2 \geq 2^{e-1} - 1$ for $e \geq 2$; thus the ratio in (1), evaluated at $a = e-1$ and $b = 1$, is at most one. When $e = 1$, the left-hand side of the equality above is negative, and the ratio in (1) vanishes.

When $\tau > 1$, we will analyze the difference of canonical residues

$$\mu_\tau = \left[(2^e+1)3^{\tau-1} - 2^{e+\tau-1}\right]\left[2^{e+\tau-1}\right]^{-1} \bmod 3^\tau$$

and

$$\lambda_\tau = \left[(2^e+1)3^{\tau-1} - 2^{e+\tau-1}\right]\left[-3^\tau\right]^{-1} \bmod 2^{\overline{E}_\tau};$$

we will show that the difference $\mu_\tau - \lambda_\tau$ is non-zero (contradicting the assumption that $n = \mu_\tau = \lambda_\tau < 3^\tau$ as per above).

We will also perform similar analyses on the maximal element of a circuit within the (accelerated) $3x-1$ dynamical system; we will show that, assuming[3] the inequality $n < 2^{\overline{E}_\tau}$, a circuit over $\mathbb{N}$ exists if and only if either $e = 1$, or $\tau = e = 2$.

[3]Appealing to a similar argument outlined abve, this condition holds for finitely many τ for each fixed $e \in \mathbb{N}$.

3. Circuits in $(3,2)$-Systems

Throughout the remainder of the manuscript, unless otherwise stated, we assume that

i. $\tau \in \mathbb{N}$ with $\tau \geq 2$;
ii. $(m, l) = (3, 2)$;
iii. $\mathbf{f} = (1, \dots, 1) \in \mathbb{N}^\tau$;
iv. $\mathbf{e} = (\underbrace{1, \dots, 1}_{\tau-1}, e)$ for some $e \in \mathbb{N}$; and
v. $\mathbf{a} = (a_0, \dots, a_{\tau-1}) \in \{-1, +1\}^\tau$.

We begin with the following assumptions.

Assumptions 3.1. *Assume 3.1 and 3.3 from* [9], *and let* $\mathbf{a} = \mathbf{1}^\tau$. *Let* $N = \sum_{0 \leq w < \tau} 3^w 2^{e+\tau-2-w} = (2^e + 1)3^{\tau-1} - 2^{e+\tau-1}$, *and let* $D = 2^{e+\tau-1} - 3^\tau$ *where* $D > 0$.

Assume that

$$n = \frac{N}{D} < \min\left(3^\tau, 2^{\overline{E}_\tau}\right),$$

let $\mu_\tau = n \bmod 3^\tau$ *denote the* 3*-residue of* n, *and let* $\lambda_\tau = n \bmod 2^{e+\tau-1}$ *denote the* **2***-residue of* n.

Under these assumptions, if $n \in \mathbb{N}$, then the chain of equalities

$$n = \mu_\tau = \lambda_\tau$$

holds.

Our goal for the remainder of this subsection is to prove the following theorem:

Theorem 3.1. *Assume 3.1.*

We have the equalities

$$\mu_\tau = \begin{cases} 3^{\tau-1} - 1 & e \underset{2}{\equiv} 0 \\ 3^\tau - 1 & e \underset{2}{\equiv} 1 \end{cases}$$

when $\tau \underset{2}{\equiv} 0$, *and*

$$\mu_\tau = \begin{cases} 2 \cdot 3^{\tau-1} - 1 & e \underset{2}{\equiv} 0 \\ 3^\tau - 1 & e \underset{2}{\equiv} 1 \end{cases}$$

when $\tau \underset{2}{\equiv} 1$.

Furthermore, when $\tau \underset{2}{\equiv} 1 \underset{2}{\equiv} e - 1$, *then*

$$\lambda_\tau = 2^e \left(\frac{2^{\tau-1} - 1}{3}\right) + \frac{2^{e+\tau-1} - 1}{3} = \frac{(2^\tau - 1)2^e - 1}{3}.$$

For completeness, we have

$$\lambda_\tau = \begin{cases} \frac{(2^{\tau-1}-1)2^e-1}{3} & e \underset{2}{\equiv} 0 \\ 2^{e+\tau-1} - \frac{2^e+1}{3} & e \underset{2}{\equiv} 1 \end{cases}$$

when $\tau \underset{2}{\equiv} 0$, and

$$\lambda_\tau = \begin{cases} \frac{(2^\tau-1)2^e-1}{3} & e \underset{2}{\equiv} 0 \\ 2^{e+\tau-1} - \frac{2^e+1}{3} & e \underset{2}{\equiv} 1 \end{cases}$$

when $\tau \underset{2}{\equiv} 1$. However, in order to expedite the proofs, we exclude three out of four cases when the corresponding canonincal 3-residue μ_τ is even (assuring the inequality $\mu_\tau \neq \lambda_\tau$). We exclude the remaining case with the following lemma.

Lemma 3.2. *Assume that* $\tau \underset{2}{\equiv} 1 \underset{2}{\equiv} e - 1$*; furthermore, assume that*

$$\mu_\tau = 2 \cdot 3^{\tau-1} - 1,$$

and

$$\lambda_\tau = \frac{(2^\tau - 1)2^e - 1}{3}.$$

The inequality $\mu_\tau \neq \lambda_\tau$ *holds.*

Proof. By way of contradiction, assume e satisfies the equality

$$2 \cdot 3^{\tau-1} - 1 = \frac{(2^\tau - 1)2^e - 1}{3};$$

equivalently, we require that the equality

$$2\left(3^\tau - 1\right) = (2^\tau - 1)2^e$$

holds. However, we have

$$\frac{3^\tau - 1}{2} \underset{2}{\equiv} \sum_{0 \leq w < \tau} 3^w \underset{2}{\equiv} 1$$

for all odd, positive τ. When $e = 2$, the value of τ must satisfy the equality

$$2^{\tau+1} = 3^\tau + 1;$$

equivalently, we require that

$$2 - \frac{1}{2^\tau} = \left(\frac{3}{2}\right)^\tau;$$

however, this equality fails to hold for $\tau > 1$.

□

Lemma 3.2, Assumptions 3.1, and Theorem 3.1, along with the bounds provided in [11], [3], and [5], demonstrate the non-existence of circuits in the $3x+1$ dynamical system.

Before proceeding, we remind the reader of some elementary identities.

Identity 3.1. *Let* a *and* b *be coprime, positive integers.*

i. *If* $g, h \in \mathbb{N}$ *with* $h > g$*, then* $b^g a \underset{b^h}{\equiv} b^g \left[a\right]_{b^{h-g}}$*;*

ii. $\left[a\right]_b^{-1} = \frac{b[-b]_a^{-1}+1}{a}$*;*

iii. *if* $a > b$*, then* $\left[a-b\right]_b^{-1} = \left[a\right]_b^{-1} = \frac{b\gamma+1}{a-b}$ *for some* $\gamma \in \left[a-b\right)_0$*;*

iv. if $a > b$, then $[a-b]_a^{-1} = [-b]_a^{-1} = \frac{a\gamma+1}{a-b} = \gamma + [a-b]_b^{-1}$.

Proof. The elementary proofs of these identities are left to the reader. Note that

i: if $a = [a]_{b^h} + b^h u$ for some $u \in \mathbb{Z}$, then

$$b^g a = b^g [a]_{b^h} + b^{g+h} u = b^g \left([a]_{b^{h-g}} + b^{h-g} a' \right) + b^{g+h} u = b^g [a]_{b^{h-g}} + b^h u'$$

for some $a' \in \mathbb{N}$;

iv, v: as $a \underset{a-b}{\equiv} b$, we can write $\gamma \underset{a-b}{\equiv} [-a]^{-1} \underset{a-b}{\equiv} [-b]^{-1}$.

□

3.1. **Elementary Modular Arithmetic.** Our first proof of Theorem 3.1 appeals to elementary modular arithmetic.

Proof. We can write

$$\begin{aligned} \mu_\tau &\underset{3^\tau}{\equiv} ND^{-1} \\ &\underset{3^\tau}{\equiv} \left[(2^e+1)3^{\tau-1} - 2^{e+\tau-1}\right] \left[2^{e+\tau-1}\right]_{3^\tau}^{-1} \\ &\underset{3^\tau}{\equiv} \left[\left[2^{\tau-1}\right]_{3^1}^{-1} + \left[2^{e+\tau-1}\right]_{3^1}^{-1} \right] 3^{\tau-1} - 1. \end{aligned}$$

It follows that

$$\mu_\tau \underset{3^\tau}{\equiv} 3^{\tau-1}(-1)^{\tau-1}\left[1 + (-1)^e\right] - 1.$$

Thus, when $e \underset{2}{\equiv} 1$, we have $\mu_\tau = 3^\tau - 1 \underset{2}{\equiv} 0$. Similarly, when $e \underset{2}{\equiv} 0$ and $\tau \underset{2}{\equiv} 0$, we have $\mu_\tau = 3^{\tau-1} - 1 \underset{2}{\equiv} 0$.

When $\tau \underset{2}{\equiv} 1 \underset{2}{\equiv} e-1$, we arrive at the equality $\mu_\tau = 2 \cdot 3^{\tau-1} - 1$.

For the **2**-residue, we begin by writing

$$\begin{aligned} \lambda_\tau &\underset{2^{e+\tau-1}}{\equiv} ND^{-1} \\ &\underset{2^{e+\tau-1}}{\equiv} \left[(2^e+1)3^{\tau-1} - 2^{e+\tau-1}\right] [-3^\tau]_{2^{e+\tau-1}}^{-1} \\ &\underset{2^{e+\tau-1}}{\equiv} 2^e [-3]_{2^{\tau-1}}^{-1} + [-3]_{2^{e+\tau-1}}^{-1}. \end{aligned}$$

When $\tau \underset{2}{\equiv} 1 \underset{2}{\equiv} e-1$, we have $\left[-3^1\right]_{2^{\tau-1}}^{-1} = \frac{2^{\tau-1}-1}{3}$ and $\left[-3^1\right]_{2^{e+\tau-1}}^{-1} = \frac{2^{e+\tau-1}-1}{3}$.

As

$$2^e \left(\frac{2^{\tau-1}-1}{3} \right) + \frac{2^{e+\tau-1}-1}{3} = \frac{(2^\tau-1)2^e - 1}{3} < 2^{e+\tau-1},$$

we arrive at the equality

$$\lambda_\tau = \frac{(2^\tau-1)2^e-1}{3}.$$

□

3.2. **Weighted Binomial Coefficients.** The previous approach is apparently limited; it is unclear to the author how to extrapolate this approach to admissible sequences of order τ with an arbitrary $\mathbf{2}$-grading $(e_0, \ldots, e_{\tau-1})$. In this subsection, we introduce a more robust approach to identifying the 3-residues and $\mathbf{2}$-residues of the iterates of an admissible cycle in a $(3, 2)$-system. Moreover, we do so by connecting the residues of $(3, 2)$-systems to the well-known *Fibonacci sequence* by way of elementary equivalence identities, which we establish first.

Lemma 3.3. *For $a, b, z \in \mathbb{N}$, the equivalence*

$$\left(\sum_{0\leq w<b} z^w\right)^a \underset{z^b}{\equiv} \sum_{0\leq w<b} \binom{a-1+w}{w} z^w$$

holds.

Proof. Define $S_b(z) = \sum_{0\leq w<b} z^w$, and define $T_{a,b}(z) = \sum_{0\leq w<b} \binom{a-1+w}{w} z^w$. The proof is by induction on b.

When $b = 1$, we arrive at the equivalence $1^a \underset{z}{\equiv} \binom{a-1}{0}$ for all $a, z \in \mathbb{N}$.

Assume the claim holds for $b \in \mathbb{N}$. The identity $S_{b+1}(z) = zS_b(z) + 1$ allows the chain of equivalences

$$\begin{aligned}
[S_{b+1}(z)]^a &\underset{z^{b+1}}{\equiv} \sum_{0\leq y<b+1} \binom{a}{y} z^y \, [S_b(z)]^y \\
&\underset{z^{b+1}}{\equiv} \binom{a}{0} z^0 + \sum_{1\leq y<b+1} \binom{a}{y} z^y T_{y,b}(z).
\end{aligned}$$

We will recast the coefficient of z^0 as $\binom{a-1}{0}$, and we will write

$$\sum_{1\leq y<b+1} \binom{a}{y} z^y T_{y,b}(z) = \sum_{1\leq y<b+1} \sum_{0\leq u<b} z^{u+y} \binom{a}{y} \binom{y-1+u}{u}.$$

For each $w \in [b+1)$, the coefficient of z^w is $\sum_{1\leq y\leq w} \binom{a}{y}\binom{w-1}{w-y} = \sum_{0\leq y<w} \binom{a}{w-y}\binom{w-1}{y}$, which equals $\binom{a-1+w}{w}$ as per the *Vandermonde-Chu* identity.

□

Identity 3.2 (Fibonacci Identity)**.** *Let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The equality*

$$F_n = \sum_{0\leq k<n} \binom{n-1-k}{k}$$

holds.

We will use this identity to establish the *residue approximation functions* for $(3, 2)$-systems.

Lemma 3.4. *Define the map* $M_\tau : \mathbb{N}^\tau \times \mathbb{N}^\tau \to \mathbb{Z}$ *to be*

$$M_\tau = M_\tau(\mathbf{e}, \mathbf{a}) = \sum_{0 \le w < u} (-)^{E_{w+1}} 3^w a_w \sum_{0 \le y < \tau - w} \binom{E_{w+1} - 1 + y}{y} 3^y,$$

and define the map $\Lambda_\tau : \mathbb{N}^\tau \times \mathbb{N}^\tau \to \mathbb{Z}$ *to be*

$$\Lambda_\tau = \Lambda_\tau(\mathbf{e}, \mathbf{a}) = \sum_{0 \le w < \tau} (-)^w 2^{\overline{E}_w} a_{\tau-1-w} \sum_{0 \le y < \eta_w} \binom{w+y}{y} 4^y,$$

where $\eta_w = \left\lceil \frac{E_{\tau-w}}{2} \right\rceil$.

Then, the equivalences $M_\tau \underset{3^\tau}{\equiv} \mu_\tau$ *and* $\Lambda_\tau \underset{2^{\overline{E}_\tau}}{\equiv} \lambda_\tau$ *hold.*

Proof. We will make use of the following elementary identities involving *Euler's totient function* ϕ: we have $3^{\phi(2)} - 1 = 2$ and $2^{\phi(3)} - 1 = 3$. In light of these identities, we will appeal to Lemma 3.3: for $a, b \in \mathbb{N}$, we will write

$$[2^a]^{-1} \underset{3^b}{\equiv} \left(\frac{1 - 3^{\phi(2)\left\lceil \frac{b}{\phi(2)} \right\rceil}}{2} \right)^a \underset{3^b}{\equiv} (-)^a \left(\sum_{0 \le y < b} 3^y \right)^a \underset{3^b}{\equiv} (-)^a \sum_{0 \le y < b} \binom{a-1+y}{y} 3^y,$$

and

$$\left[3^b\right]^{-1} \underset{2^a}{\equiv} \left(\frac{1 - 2^{\phi(3)\left\lceil \frac{a}{\phi(3)} \right\rceil}}{3} \right)^b \underset{2^a}{\equiv} (-)^b \left(\sum_{0 \le y < \left\lceil \frac{a}{2} \right\rceil} 4^y \right)^b \underset{2^a}{\equiv} (-)^b \sum_{0 \le y < \left\lceil \frac{a}{2} \right\rceil} \binom{b-1+y}{y} 4^y.$$

We derive the 3-residue approximation function as follows:

$$\begin{aligned}
\mu_\tau &\underset{3^\tau}{\equiv} \left[ND^{-1}\right]_{3^\tau} \\
&\underset{3^\tau}{\equiv} \sum_{0 \le w < \tau} 3^w 2^{\overline{E}_{\tau-1-w}} a_w \left[2^{\overline{E}_\tau}\right]^{-1} \\
&\underset{3^\tau}{\equiv} \sum_{0 \le w < \tau} 3^w a_w \left[2^{E_{w+1}}\right]^{-1}_{3^{\tau-w}} \\
&\underset{3^\tau}{\equiv} \sum_{0 \le w < \tau} (-)^{E_{w+1}} 3^w a_w \sum_{0 \le y < \tau - w} \binom{E_{w+1} - 1 + y}{y} 3^y.
\end{aligned}$$

We derive the **2**-residue approximation function as follows:

$$\begin{aligned}
\lambda_\tau &\underset{2^{\overline{E}_\tau}}{\equiv} \left[ND^{-1}\right]_{2^{\overline{E}_\tau}} \\
&\underset{2^{\overline{E}_\tau}}{\equiv} \sum_{0\le w<\tau} 3^w 2^{\overline{E}_{\tau-1-w}} a_w \left[-3^\tau\right]^{-1} \\
&\underset{2^{\overline{E}_\tau}}{\equiv} \sum_{0\le w<\tau} -2^{\overline{E}_{\tau-1-w}} a_w \left[3^{\tau-w}\right]^{-1}_{2^{E_{w+1}}} \\
&\underset{2^{\overline{E}_\tau}}{\equiv} \sum_{0\le w<\tau} (-)^{\tau-1-w} 2^{\overline{E}_{\tau-1-w}} a_w \sum_{0\le y<\left\lceil \frac{E_{w+1}}{2}\right\rceil} \binom{\tau-1-w+y}{y} 4^y \\
&\underset{2^{\overline{E}_\tau}}{\equiv} \sum_{0\le w<\tau} (-)^{w} 2^{\overline{E}_{w}} a_{\tau-1-w} \sum_{0\le y<\eta_w} \binom{w+y}{y} 4^y.
\end{aligned}$$

□

It will prove useful to re-index these double-sums: for example, in the 3-residue approximation, for each fixed $w \in [\tau)_0$ the coefficient of 3^w is

$$S_w = \sum_{0\le y\le w} (-)^{E_{y+1}} \binom{E_{y+1}-1+w-y}{w-y} a_y;$$

thus, we can write $M_\tau = \sum_{0\le w<\tau} 3^w S_w$.

The following example will illustrate the connection between an orbit over $\mathbb{N}$ within the $3x+1$ dynamical system and the Fibonacci Sequence.

3.2.1. *Example: The* $(1,4,2)$*-Orbit in the* $3x+1$ *Dynamical System.* For this example, define $e_w = 2$ and $a_w = 1$ for each $w \in [\tau)_0$. The sum $E_{w+1} = 2(w+1) \underset{2}{\equiv} 0$ for all $w \in [\tau]_0$; therefore, we can express the 3-residue approximation as $M_\tau = \sum_{0\le w<\tau} 3^w S_w$, where

$$S_w := \sum_{0\le y\le w} \binom{2(y+1)-1+w-y}{w-y} = \sum_{0\le y\le w} \binom{2w+1-y}{y}.$$

The sequence $(S_w)_{w\ge 0}$ is the even-indexed bisection of the Fibonacci sequence $(F_w)_{w\ge 0}$ as per Identity 3.2; we have $S_w = F_{2(w+1)}$ for $w \ge 0$. It is known[4] that this bisection[5] satisfies the recurrence[6] $F_{2w} = 3F_{2(w-1)} - F_{2(w-2)}$ for $w \ge 0$; thus, induction yields the identity $M_\tau = 3^\tau F_{2(\tau-1)} + 1$ for $\tau \in \mathbb{N}$.

For the **2**-residue approximation, we have the equalities

$$\Lambda_\tau = \sum_{0\le w<\tau} 4^w \sum_{0\le y\le w} \binom{w}{y} (-1)^y = \sum_{0\le w<\tau} 4^w (1-1)^w = 1$$

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[4]OEIS:A001906

[5]The interested reader will find the elements of the odd-indexed bisection of the Fibonacci sequence in the 3-residue approximation of the same $(3,2)$ system (i.e., "$3x+1$") where $e_0 = 1$ and $e_w = 2$ for $w \in [\tau)$.

[6]We assume the definition of the sequence to be $F_{-n} = (-)^{n-1} F_n$.

for $\tau \in \mathbb{N}$.

The Fibonacci sequence appears within the **2**-residue approximation for the following proof of Theorem 3.1. In order to expedite the derivation of this **2**-residue, we will first prove the following lemma.

Lemma 3.5. *For $a \in \mathbb{N}$, let F_a denote the a-th Fibonacci number; furthermore, for $k \in \mathbb{N}_0$, define $\sigma(a,k) = 2\binom{a+1}{k} - \binom{a}{k}$, and define $\mathcal{S}(k) = \sum_{0 \leq i < k} \sigma(2k-i, i+1)$.*

We have the equality $\mathcal{S}(0) = 0$, and, for $k > 0$, the equality

$$\mathcal{S}(k) = F_{2k+2} + 2F_{2k+1} - 3$$

holds.

Proof. Assume the conditions within the statement of the lemma. Clearly, $\mathcal{S}(0) = 0$. As per Identity 3.2, when $k > 0$, we will write

$$\begin{aligned}\mathcal{S}(k) &= \sum_{0 \leq i < k} \left[2\binom{2k-i+1}{i+1} - \binom{2k-i}{i+1}\right] \\ &= \sum_{1 \leq i < k+1} \left[2\binom{2k+2-i}{i} - \binom{2k+1-i}{i}\right] \\ &= 2\left[F_{2k+3} - \binom{2k+2}{0} - \binom{k+1}{k+1}\right] - \left[F_{2k+2} - \binom{2k+1}{0}\right] \\ &= F_{2k+2} + 2F_{2k+1} - 3.\end{aligned}$$

□

We proceed with the proof of the theorem.

Proof. First, we will demonstrate the equality

$$M_\tau = -1 + 3^{\tau-1}(-1)^{\tau-1}\left[1 + (-1)^e\right];$$

afterwards, by assuming $\tau \underset{2}{\equiv} 1 \underset{2}{\equiv} e - 1$, we will show that

$$\Lambda_\tau = 2^e\left(\frac{2^{\tau-1}-1}{3}\right) + \frac{2^{e+\tau-1}-1}{3} + 2^{e+\tau-1}\left(F_{\tau-2} - 1\right).$$

In circuits, we have

$$E_w = \begin{cases} w & w < \tau \\ e + \tau - 1 & w = \tau, \end{cases}$$

and $\overline{E}_w = e + w - 1$ for $w \in [\tau)$. Thus, when $w < \tau - 1$, we have

$$\begin{aligned}
S_w &= \sum_{0 \le y \le w} (-)^{E_{y+1}} \binom{E_{y+1} - 1 + w - y}{w - y} \\
&= \sum_{0 \le y \le w} (-)^{y+1} \binom{w}{w-y} \\
&= -\sum_{0 \le y \le w} (-)^{w-y} \binom{w}{y} \\
&= -(1-1)^w \\
&= \begin{cases} 0 & w > 0 \\ -1 & w = 0. \end{cases};
\end{aligned}$$

when $w = \tau - 1 \ge 1$, we have

$$\begin{aligned}
S_{\tau-1} &= \sum_{0 \le y \le \tau-1} (-)^{E_{y+1}} \binom{E_{y+1} - 1 + \tau - 1 - y}{\tau - 1 - y} \\
&= \sum_{0 \le y \le \tau-2} (-)^{y+1} \binom{\tau - 1}{\tau - 1 - y} + (-)^{e+\tau-1} \binom{e + \tau - 2}{0} \\
&= -(1-1)^{\tau-1} + (-)^{\tau-1} \binom{\tau-1}{\tau-1} + (-)^{e+\tau-1} \binom{e+\tau-2}{0} \\
&= (-)^{\tau-1} \left[1 + (-1)^e\right].
\end{aligned}$$

It follows that

$$M_\tau = -1 + 3^{\tau-1}(-1)^{\tau-1}\left[1 + (-1)^e\right].$$

Thus, when $e \underset{2}{\equiv} 1$, we have $\mu_\tau = 3^\tau - 1$. Similarly, when $e \underset{2}{\equiv} 0$ and $\tau \underset{2}{\equiv} 0$, we have $\mu_\tau = 3^{\tau-1} - 1$.

When $\tau \underset{2}{\equiv} 1 \underset{2}{\equiv} e - 1$, we arrive at the equality $\mu_\tau = 2 \cdot 3^{\tau-1} - 1$. Continuing with these parity conditions, we let T_w denote the sum $\sum_{0 \le y < \left\lceil \frac{E_{\tau-w}}{2} \right\rceil} \binom{w+y}{y} 4^y$. We write

$$\begin{aligned}
\Lambda_\tau &= \sum_{0 \le w < \tau} (-)^w 2^{\overline{E}_w} T_w \\
&= T_0 + \sum_{1 \le w < \tau} (-)^w 2^{\overline{E}_w} T_w \\
&= \sum_{0 \le y < \frac{e+\tau-1}{2}} \binom{y}{y} 4^y + \sum_{1 \le w < \tau} (-)^w 2^{\overline{E}_w} \binom{w}{0} + \sum_{1 \le w < \tau} (-)^w 2^{\overline{E}_w} \left[T_w - \binom{w}{0}\right].
\end{aligned}$$

We proceed with the first two sums in this expression. When $e+\tau-1 \underset{2}{\equiv} 0$, we can write

$$T_0 = \sum_{0 \le y < \frac{e+\tau-1}{2}} \binom{y}{y} 4^y = \frac{2^{e+\tau-1}-1}{3};$$

furthermore, as $\tau - 1 \underset{2}{\equiv} 0$, we can also write

$$\begin{aligned}\sum_{1 \le w < \tau} (-)^w 2^{\overline{E}_w} &\underset{2^{e+\tau-1}}{\equiv} 2^e \sum_{0 \le w < \tau-1} (-)^{w+1} 2^w \\ &\underset{2^{e+\tau-1}}{\equiv} 2^e \sum_{0 \le w < \frac{\tau-1}{2}} \left[2^{2w+1} - 2^{2w}\right] \\ &\underset{2^{e+\tau-1}}{\equiv} 2^e \sum_{0 \le w < \frac{\tau-1}{2}} 4^w \\ &\underset{2^{e+\tau-1}}{\equiv} 2^e \left(\frac{2^{\tau-1}-1}{3}\right).\end{aligned}$$

What remains to be shown is that

$$\sum_{1 \le w < \tau} (-)^w 2^{\overline{E}_w} \left[T_w - \binom{w}{0}\right] \underset{2^{e+\tau-1}}{\equiv} 0.$$

To this end, for each $k \in \mathbb{N}$, we will define

$$\widehat{\Lambda}_{2k+1} = \sum_{1 \le w < 2k-1} (-)^w 2^{w-1} \sum_{1 \le y < \lceil \frac{2k+1-w}{2} \rceil} \binom{w+y}{y} 4^y;$$

we will show that

$$\sum_{1 \le w < \tau} (-)^w 2^{\overline{E}_w} \left[T_w - \binom{w}{0}\right] = 2^e \widehat{\Lambda}_\tau = 2^{e+\tau-1} \left(F_{\tau-2} - 1\right).$$

Assume the notation from the statement of Lemma 3.5. We will demonstrate the chain of equalities

$$\widehat{\Lambda}_{2k+1} = \widehat{\Lambda}_{2k-1} + 4^{k-1} \mathcal{S}(k-1) = 4^k \left(F_{2k-1} - 1\right)$$

inductively for $k \in \mathbb{N}$. Firstly, we have

$$\widehat{\Lambda}_3 = 0 = 4^0 \mathcal{S}(0) = 4^0 \left(F_1 - 1\right)$$

for $k = 1$. Assuming the inductive claim, we proceed with the chain of equalities for $k \ge 2$:

$$\begin{aligned}\widehat{\Lambda}_{2k+1} &= \sum_{1 \le w < 2k-1} (-)^w 2^{w-1} \sum_{1 \le y < \lceil \frac{2k+1-w}{2} \rceil} \binom{w+y}{y} 4^y \\ &= \widehat{\Lambda}_{2k-1} + A_k + B_k,\end{aligned}$$

where

$$A_k = \sum_{1\le w<2k-1} (-)^w 2^{w-1} \binom{w + \lceil \frac{2k-1-w}{2} \rceil}{\lceil \frac{2k-1-w}{2} \rceil} 4^{\lceil \frac{2k-1-w}{2} \rceil},$$

and

$$B_k = \sum_{2k-1\le w<2k+1} (-)^w 2^{w-1} \sum_{1\le y<\lceil \frac{2k-1-w}{2} \rceil} \binom{w+y}{y} 4^y.$$

Firstly, the sum $B_k = \sum_{2k-1\le w<2k+1} (-)^w 2^{w-1} \cdot \emptyset = 0$, and the sum

$$\begin{aligned}
A_k &= \sum_{1\le w<2k-1} (-)^w 2^{w-1} \binom{k + w + \lceil \frac{-1-w}{2} \rceil}{k + \lceil \frac{-1-w}{2} \rceil} 4^{k+\lceil \frac{-1-w}{2} \rceil} \\
&= \sum_{1\le w<\frac{2k-1}{2}} \left[2^{2w-1} \binom{k+w}{k-w} - 2^{2w-2} \binom{k-1+w}{k-w} \right] 4^{k-w} \\
&= 4^{k-1} \sum_{1\le w\le k-1} \left[2\binom{k+w}{k-w} - \binom{k-1+w}{k-w} \right] \\
&= 4^{k-1} \sum_{1\le w\le k-1} \left[2\binom{2k-w}{w} - \binom{2k-1-w}{w} \right] \\
&= 4^{k-1} \sum_{0\le w<k-1} \left[2\binom{2k-1-w}{w+1} - \binom{2k-2-w}{w+1} \right] \\
&= 4^{k-1} \mathcal{S}(k-1).
\end{aligned}$$

Thus, with Lemma 3.5 and the inductive hypothesis, we can write

$$\begin{aligned}
\widehat{\Lambda}_{2k+1} &= \widehat{\Lambda}_{2k-1} + 4^{k-1} \mathcal{S}(k-1) \\
&= 4^{k-1} \left[F_{2k-3} - 1 + F_{2k} + 2F_{2k-1} - 3 \right] \\
&= 4^{k-1} \left[F_{2k-3} + F_{2k-2} + 3F_{2k-1} - 4 \right] \\
&= 4^k \left[F_{2k-1} - 1 \right]
\end{aligned}$$

as required.

Consequently, when $\tau \underset{2}{\equiv} 1 \underset{2}{\equiv} e-1$, the **2**-approximation

$$\Lambda_\tau = 2^e \left(\frac{2^{\tau-1}-1}{3} \right) + \frac{2^{e+\tau-1}-1}{3} + 2^{e+\tau-1} \left(F_{\tau-2} - 1 \right),$$

and we conclude that

$$\lambda_\tau = 2^e \left(\frac{2^{\tau-1}-1}{3} \right) + \frac{2^{e+\tau-1}-1}{3} = \frac{(2^\tau-1)2^e-1}{3}.$$

□

Note that the approach within this subsection exploits the serendipitous pair of identities $3^{\phi(2)} - 1 = 2$ and $2^{\phi(3)} - 1 = 3$. In general, Euler's Theorem allows one to write

$$m^{\phi(l)} - 1 = [-l]^{-1}_{m^{\phi(l)}}\, l$$

and

$$l^{\phi(m)} - 1 = [-m]^{-1}_{l^{\phi(m)}}\, m;$$

however, for arbitrary, coprime m and l exceeding 1, the terms $[-l]^{-1}_{m^{\phi(l)}}$ and $[-m]^{-1}_{l^{\phi(m)}}$ may prevent one from executing the approach above in an analogous manner.

3.3. **Dual-Radix Modular Division.** The approach in this section, based on the work in [9], demonstrates a different method of proving Theorem 3.1 using *dual-radix modular division*.

Proof. Under the assumption that

$$e_w = \begin{cases} 1 & w \in [\tau - 1)_0 \\ e & w = \tau - 1, \end{cases}$$

we have the following initial conditions for the recurrence in Theorem 4.4 in [9]. For $w \in [\tau)_0$, the 3-adic digit $d_{w,0} \underset{3}{\equiv} [2^{e_w}]^{-1}$; thus, we have

$$d_{w,0} = \begin{cases} 2 & w \in [\tau - 1)_0 \\ 1 + e \bmod 2 & w = \tau - 1; \end{cases}$$

furthermore, the **2**-adic digit $b_{w,0} \underset{2^{e_w - 1}}{\equiv} [-3]^{-1}$; thus, we have

$$b_{w,0} = \begin{cases} \frac{2^{2\lceil \frac{e}{2} \rceil - 1}}{3} & w = 0 \\ 1 & w \in [\tau - 1]. \end{cases}$$

For $u > 0$, the equivalences

$$d_{v,u} \underset{3}{\equiv} [2^{e_v}]^{-1} [d_{v+1,u-1} - b_{v+u,u-1}]$$

and

$$b_{v,u} \underset{2^{e_v - 1 - u}}{\equiv} [-3]^{-1} [d_{v-u,u-1} - b_{v-1,u-1}]$$

yields, by induction on u, the equalities $d_{v,u} = 2[2 - 1] = 2$ for $v < \tau - 1 - u$, and $b_{v,u} = 1[2 - 1] = 1$ for $v > u$.

We will first identify the 3-adic digits of the 3-residue of $n (= n_0)$. When $e \underset{2}{\equiv} 1$, we have the initial condition $d_{\tau-1,0} = 2$. Thus, for $u \in [\tau)$, we have

$$\begin{aligned} d_{\tau-1-u,u} &\underset{3}{\equiv} [2^{e_{\tau-1-u}}]^{-1} [d_{\tau-u,u-1} - b_{\tau-1,u-1}] \\ &\underset{3}{\equiv} 2\,[2 - 1] \\ &\underset{3}{\equiv} 2. \end{aligned}$$

Consequently, we have

$$\mu_\tau = \sum_{0\le w<\tau} 3^w d_{0,w} = 2\left(\frac{3^\tau-1}{2}\right) = 3^\tau - 1.$$

When $e \underset{2}{\equiv} 0$, we have the initial condition $d_{\tau-1,0} = 1$, and

$$d_{\tau-2,1} \underset{3}{\equiv} \left[2^1\right]^{-1}\left[d_{\tau-1,0} - b_{\tau-1,0}\right] \underset{3}{\equiv} \left[2^1\right]^{-1}[1-1] \underset{3}{\equiv} 0.$$

By induction, for $u \in [\tau)$ where $u \underset{2}{\equiv} 0$, we have

$$\begin{aligned} d_{\tau-1-u,u} &\underset{3}{\equiv} [2^{e_{\tau-1-u}}]^{-1}\left[d_{\tau-u,u-1} - b_{\tau-1,u-1}\right] \\ &\underset{3}{\equiv} 2\,[0-1] \\ &\underset{3}{\equiv} 1. \end{aligned}$$

For $u \underset{2}{\equiv} 1$, we have

$$\begin{aligned} d_{\tau-1-u,u} &\underset{3}{\equiv} [2^{e_{\tau-1-u}}]^{-1}\left[d_{\tau-u,u-1} - b_{\tau-1,u-1}\right] \\ &\underset{3}{\equiv} 2\,[1-1] \\ &\underset{3}{\equiv} 0. \end{aligned}$$

Thus,

$$d_{0,\tau-1} = \begin{cases} 0 & \tau \underset{2}{\equiv} 0 \\ 1 & \tau \underset{2}{\equiv} 1. \end{cases} \tag{4}$$

Thus, when $\tau \underset{2}{\equiv} 0$, the 3-adic residue

$$\mu_\tau = \sum_{0\le w<\tau-1} 3^w(2) = 3^{\tau-1} - 1 \underset{2}{\equiv} 0;$$

and, when $\tau \underset{2}{\equiv} 1$, the 3-adic residue

$$\mu_\tau = 2\left(\frac{3^{\tau-1}-1}{2}\right) + 3^{\tau-1} = 2\cdot 3^{\tau-1} - 1.$$

We will now determine the **2**-adic digits of n when $\tau \underset{2}{\equiv} 1 \underset{2}{\equiv} e-1$: when $e \underset{2}{\equiv} 0$, the **2**-adic digit

$$b_{0,0} = \frac{2^e-1}{3},$$

and the digit

$$b_{0,1} \underset{2^{e_{\tau-2}}}{\equiv} [-3]^{-1}\left[d_{\tau-1,0} - b_{\tau-1,0}\right] \underset{2^1}{\equiv} (1)\cdot[1-1] \underset{2^1}{\equiv} 0.$$

For $u \in [\tau)$ where $u \underset{2}{\equiv} 0$, we have

$$b_{0,u} \underset{2^{e_{\tau-1-u}}}{\equiv} [-3]^{-1} [d_{\tau-u,u-1} - b_{\tau-1,u-1}] \underset{2^1}{\equiv} (1) \cdot [0-1] \underset{2^1}{\equiv} 1,$$

and, when $u \underset{2}{\equiv} 1$, we have

$$b_{0,u} \underset{2^{e_{\tau-1-u}}}{\equiv} [-3]^{-1} [d_{\tau-u,u-1} - b_{\tau-1,u-1}] \underset{2^1}{\equiv} (1) \cdot [1-1] \underset{2^1}{\equiv} 0.$$

Thus, when $\tau \underset{2}{\equiv} 1 \underset{2}{\equiv} e-1$, the **2**-adic residue

$$\begin{aligned}
\lambda_\tau &= b_{0,0} + \sum_{1 \le u < \tau} 2^{\overline{E}_u} b_{0,u} \\
&= \frac{2^e - 1}{3} + 2^e \sum_{2 \le u < \tau} 2^{u-1} [u \underset{2}{\equiv} 0] \\
&= \frac{2^e - 1}{3} + 2^{e+1} \sum_{0 \le u < \tau - 2} 2^u [u \underset{2}{\equiv} 0] \\
&= \frac{2^e - 1}{3} + 2^{e+1} \sum_{0 \le u \le \frac{\tau-3}{2}} 4^u \\
&= \frac{2^e - 1}{3} + 2^{e+1} \left(\frac{4^{\frac{\tau-1}{2}} - 1}{3} \right) \\
&= \frac{2^{e+\tau} - 2^e - 1}{3} \\
&= 2^e \left(\frac{2^{\tau-1} - 1}{3} \right) + \frac{2^{e+\tau-1} - 1}{3}.
\end{aligned}$$

□

3.4. **Circuits in the $3x-1$ Dynamical System.** We conclude this article by applying the previous analyses to the $3x-1$ dynamical system; now, we will consider the case where $a_w = -1$ for all $w \in [\tau)_0$.

We will extend the argument in [1] to the case where $3^\tau > 2^{\overline{E}_\tau}$: the magnitude of the numerator of a maximal iterate in a periodic orbit can be bound from above as follows:

$$\left| (2^e + 1) \, 3^{\tau-1} - 2^{\overline{E}_\tau} \right| = 3^\tau \left[\frac{2^e + 1}{3} - \frac{2^{\overline{E}_\tau}}{3^\tau} \right] < 3^{\tau-1} \left(2^e + 1 \right).$$

We can bound the denominator $3^\tau - 2^{\overline{E}_\tau}$ from below by appealing to the inequality (3) once again[7] to conclude that the maximal iterate within a periodic orbit in the $3x-1$

[7]The changing of the signs of u_1 and u_2 does not alter the bound.

dynamical system satisfies the inequality

$$n_{\max} < \frac{\frac{2^e+1}{3}}{1-\frac{2^{e+\tau-1}}{3^\tau}} < \left(\frac{2^e+1}{3}\right) 2\,(e+\tau-1)^{13.3} = o(2^{e+\tau-1})$$

for any fixed $e \in \mathbb{N}$. Thus, we will reuse the notation of the previous section and begin with the following assumptions.

Assumptions 3.2. *Assume 3.1, except that now we assume that* $N = 2^{e+\tau-1} - (2^e + 1)3^{\tau-1} < 0$, *and* $D = 2^{e+\tau-1} - 3^\tau < 0$.

As before, under these assumptions, if $n \in \mathbb{N}$, then the chain of equalities

$$n = \mu_\tau = \lambda_\tau$$

holds.

Our goal for the remainder of this subsection is to prove the following theorem:

Theorem 3.6. *Assume 3.2.*

The 3*-residue*

$$\mu_\tau = \begin{cases} 2\cdot 3^{\tau-1}+1 & e \underset{2}{\equiv} 0 \\ 1 & e \underset{2}{\equiv} 1 \end{cases}$$

when $\tau \underset{2}{\equiv} 0$, *and*

$$\mu_\tau = \begin{cases} 3^{\tau-1}+1 & e \underset{2}{\equiv} 0 \\ 1 & e \underset{2}{\equiv} 1 \end{cases}$$

when $\tau \underset{2}{\equiv} 1$.

The **2***-residue*

$$\lambda_\tau = \begin{cases} \frac{2^e(2^\tau+1)+1}{3} & e \underset{2}{\equiv} 0 \\ \frac{2^e+1}{3} & e \underset{2}{\equiv} 1 \end{cases}$$

when $\tau \underset{2}{\equiv} 0$, *and*

$$\lambda_\tau = \begin{cases} \frac{2^e\left(2^{\tau-1}+1\right)+1}{3} & e \underset{2}{\equiv} 0 \\ \frac{2^e+1}{3} & e \underset{2}{\equiv} 1 \end{cases}$$

when $\tau \underset{2}{\equiv} 1$.

Analogous to Lemma 3.2, the following lemma will aid in identifying circuits within the $3x-1$ Dynamical System.

Lemma 3.7. *Assume that the* 3*-residue is*

$$\mu_\tau = \begin{cases} 2\cdot 3^{\tau-1}+1 & e \underset{2}{\equiv} 0 \\ 1 & e \underset{2}{\equiv} 1 \end{cases}$$

when $\tau \underset{2}{\equiv} 0$, *and*

$$\mu_\tau = \begin{cases} 3^{\tau-1}+1 & e \underset{2}{\equiv} 0 \\ 1 & e \underset{2}{\equiv} 1 \end{cases}$$

when $\tau \underset{2}{\equiv} 1$. *Moreover, assume that the* **2***-residue is*

$$\lambda_\tau = \begin{cases} \frac{2^e(2^\tau+1)+1}{3} & e \underset{2}{\equiv} 0 \\ \frac{2^e+1}{3} & e \underset{2}{\equiv} 1 \end{cases}$$

when $\tau \underset{2}{\equiv} 0$, *and*

$$\lambda_\tau = \begin{cases} \frac{2^e\left(2^{\tau-1}+1\right)+1}{3} & e \underset{2}{\equiv} 0 \\ \frac{2^e+1}{3} & e \underset{2}{\equiv} 1 \end{cases}$$

when $\tau \underset{2}{\equiv} 1$.

The equality $\mu_\tau = \lambda_\tau$ *holds if and only if either i.)* $e = 1$ *or ii.)* $e = \tau = 2$.

Proof. When $e \underset{2}{\equiv} 1$, we require that the equality $\frac{2^e+1}{3} = 1$ holds; consequently, we require that $e = 1$ (irrespective of the parity of τ).

When $e \underset{2}{\equiv} 0$ and $\tau \underset{2}{\equiv} 0$, we require that the equality

$$2 \cdot 3^{\tau-1} + 1 = \frac{2^e\left(2^\tau+1\right)+1}{3}$$

holds. Equivalently, we require that $2 \cdot 3^\tau + 3 = 2^e\left(2^\tau+1\right)+1$; after simplifying, we require that $\frac{3^\tau+1}{2^{e-1}} = 2^\tau + 1$. When $\tau \underset{2}{\equiv} 0$, the numerator on the left-hand side $9^{\frac{\tau}{2}} + 1 \underset{4}{\equiv} 2$; thus, it follows that we require that $e = 2$. The equality $3^\tau = 2^{\tau+1} + 1$ holds only when $\tau = 2$ as per a result of Gersonides[8] on *harmonic numbers*.

When $e \underset{2}{\equiv} 0$ and $\tau \underset{2}{\equiv} 1$, we have $\mu_\tau \underset{2}{\equiv} 0$ and $\lambda_\tau \underset{2}{\equiv} 1$.

□

We offer one proof of Theorem 3.6.

Proof. We can write

$$\begin{aligned} \mu_\tau &\underset{3^\tau}{\equiv} -N\left[3^\tau - 2^{e+\tau-1}\right]^{-1} \\ &\underset{3^\tau}{\equiv} \left[(2^e+1)3^{\tau-1} - 2^{e+\tau-1}\right]\left[-2^{e+\tau-1}\right]_{3^\tau}^{-1} \\ &\underset{3^\tau}{\equiv} \left[\left[-2^{\tau-1}\right]_{3^1}^{-1} + \left[-2^{e+\tau-1}\right]_{3^1}^{-1}\right]3^{\tau-1} + 1. \end{aligned}$$

It follows that

$$\mu_\tau \underset{3^\tau}{\equiv} 3^{\tau-1}(-1)^\tau\left[1 + (-1)^e\right] + 1.$$

[8]Levi Ben Gerson, 1342 AD. See [6].

For the **2**-residue, we begin by writing

$$\begin{aligned}\lambda_\tau &\underset{2^{e+\tau-1}}{\equiv} -N\left[3^\tau - 2^{\overline{E}_\tau}\right]^{-1}\\ &\underset{2^{e+\tau-1}}{\equiv} \left[(2^e+1)3^{\tau-1} - 2^{e+\tau-1}\right][3^\tau]^{-1}_{2^{e+\tau-1}}\\ &\underset{2^{e+\tau-1}}{\equiv} 2^e\,[3]^{-1}_{2^{\tau-1}} + [3]^{-1}_{2^{e+\tau-1}}\,.\end{aligned}$$

We have the identities $[3]^{-1}_{2^{\tau-1}} = \frac{2^{\tau-(\tau-1) \bmod 2}+1}{3}$, and $[3]^{-1}_{2^{e+\tau-1}} = \frac{2^{e+\tau-(e+\tau-1) \bmod 2}+1}{3}$. We complete the proof by cases.

i. $(e \underset{2}{\equiv} 0, \tau \underset{2}{\equiv} 0)$ $\mu_\tau = 2\cdot 3^{\tau-1}+1$, and $\lambda_\tau = \left[2^e\left(\frac{2^{\tau-1}+1}{3}\right) + \frac{2^{e+\tau-1}+1}{3}\right] \bmod 2^{e+\tau-1} = \frac{2^{e+\tau}+2^e+1}{3}$

ii. $(e \underset{2}{\equiv} 0, \tau \underset{2}{\equiv} 1)$ $\mu_\tau = 3^{\tau-1}+1$, and $\lambda_\tau = \left[2^e\left(\frac{2^{\tau}+1}{3}\right) + \frac{2^{e+\tau}+1}{3}\right] \bmod 2^{e+\tau-1} = \frac{2^{e+\tau-1}+2^e+1}{3}$.

iii. $(e \underset{2}{\equiv} 1, \tau \underset{2}{\equiv} 0)$ $\mu_\tau = 1$, and $\lambda_\tau = \left[2^e\left(\frac{2^{\tau-1}+1}{3}\right) + \frac{2^{e+\tau}+1}{3}\right] \bmod 2^{e+\tau-1} = \frac{2^e+1}{3}$.

iv. $(e \underset{2}{\equiv} 1, \tau \underset{2}{\equiv} 1)$ $\mu_\tau = 1$, and $\lambda_\tau = \left[2^e\left(\frac{2^{\tau}+1}{3}\right) + \frac{2^{e+\tau-1}+1}{3}\right] \bmod 2^{e+\tau-1} = \frac{2^e+1}{3}$.

□

Thus, under the assumption that $n < 2^{e+\tau-1}$, the only circuits within the $3x-1$ dynamical system are (1) and $(5,7)$.

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