A generating polynomial for the pretzel knot

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Abstract

We collect statistics which consist of the coefficients in the expansion of the generating polynomials that count the Kauffman states associated with certain classes of pretzel knots having n tangles, of r half-twists respectively.

Keywords: generating polynomial, shadow diagram, Kauffman state.

1 Introduction

The *generating polynomial* for the shadow diagram of the knot K provides a refinement of counting the corresponding Kauffman states [1]. By state is meant the diagram obtained by *splitting* each vertex representing the initial diagram crossings, i.e., each $\times$ to $\genfrac{}{}{0pt}{}{\diagup\,\diagdown}{\diagdown\,\diagup}$, and repasting the edges as either $)($ or $\asymp$. The generating polynomial for the knot K is then defined as the summation which is taken over all its states, namely

$$K(x) = \sum_{S} x^{|S|}, \tag{1}$$

with $|S|$ denoting the number of Jordan curves in the state diagram S. For instance, the generating polynomial for the Hopf link is $L(x) = 2x^2 + 2x$ (see Figure 1).

If K and K' are two arbitrary diagrams, and $\bigcirc$ denotes the unknot diagram, then we have the following properties and notations [2]:

(i) $\bigcirc(x) = x$;

(ii) $(\underbrace{\bigcirc \sqcup \bigcirc \sqcup \cdots \sqcup \bigcirc}_{n \text{ copies}})(x) := (\bigcirc\bigcirc\cdots\bigcirc)(x) = x^n$;

(iii) $(K \sqcup \bigcirc)(x) = xK(x)$;

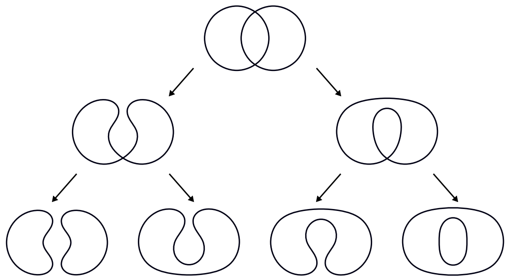


Figure 1: The states of the Hopf link.

(iv) $(K \sqcup K')(x) = K(x).K'(x)$;

(v) $(K \# \bigcirc)(x) = K(x)$;

(vi) $K_n(x) := (\underbrace{K \# K \# \cdots \# K}_{n \text{ copies}})(x) = x\left(x^{-1}K(x)\right)^n$, with $K_0 = \bigcirc$;

(vii) $(K \# K')(x) = x^{-1}K(x).K'(x)$,

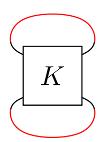
where $\#$ and $\sqcup$ are respectively the connected sum and the disjoint union. Moreover, if we let $\overline{K}$ denote the closure of K, i.e., the connected sum with itself, then there exist two polynomials $\alpha, \beta \in \mathbb{N}[x]$ such that

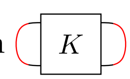
$$\overline{K}(x) = x^2\alpha(x) + x\beta(x), \text{ with } K(x) = x^2\beta(x) + x\alpha(x). \tag{2}$$

With the notation and property in (vi) we obtain $\overline{K_0}(x) = (\bigcirc\bigcirc)(x) = x^2$ and

$$\overline{K_n}(x) = \alpha(x)\overline{K_{n-1}}(x) + \beta(x)K_{n-1}(x) \tag{3}$$

$$= x^2\alpha(x)^n + \beta(x)\sum_{j=0}^{n-1}\alpha(x)^{n-j-1}K_j(x). \tag{4}$$

We can interpret (2) as follows: given the closure 

of a knot diagram 

, its state diagrams can be divided into two subsets that are respectively counted by $x^2\alpha(x)$ and $x\beta(x)$ as represented in Figure 2.

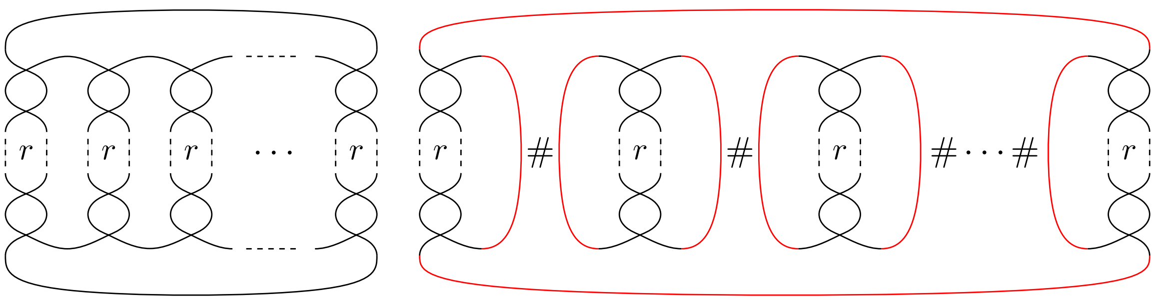
(a) States counted by $x^2\alpha(x)$. (b) States counted by $x\beta(x)$.

Figure 2: The two subsets of states associated with the closure of a knot.

In this note, we shall take advantage of these properties and establish the generating polynomial for a particular class of the pretzel knots.

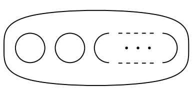
2 Pretzel knot

A *pretzel knot* $P_{n,r} := P(r, r, \ldots, r)$ [4] is a knot composed of n pairs of strands twisted r times and attached along the tops and bottoms as in Figure 3(a).



(a) $P(r, r, \ldots, r)$ (b) $P(r, r, \ldots, r) := \overline{F_r \# F_r \# \cdots \# F_r}$

Figure 3: The pretzel knot shadow and the corresponding connected sums for constructing it.

If F_r denotes the *r-foil* as pictured in Figure 3(b), then we have $P_{n,r} := \overline{(F_r)_n}$. For the convenience, we set $F_0 = \bigcirc\bigcirc$ and $(F_r)_0 = \bigcirc$ so that $\overline{(F_0)_0} = \bigcirc\bigcirc$ and $\overline{(F_0)_n} =$  for $n \geq 1$. Figure 4 displays some pretzel knots for small values of n and r.

3 Generating polynomial

We begin with the generating polynomial for the closure of the r-foil (see Figure 3(b)) which yields the r-twist loop (see Figure 4(e)).

Lemma 1 ([2]). *The generating polynomials for the r-twist loop and the r-foil knot are respectively given by*

$$T_r(x) = x(x+1)^r \tag{5}$$

and

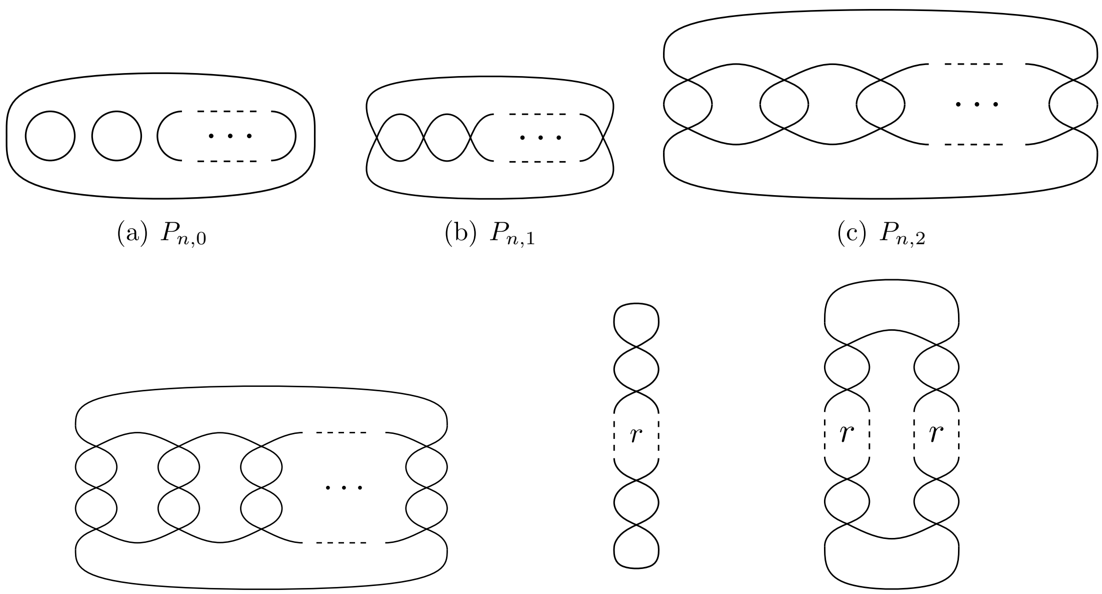
$$F_r(x) = \overline{T_r}(x) = (x+1)^r + x^2 - 1. \tag{6}$$

We shall deduce the two polynomials α_r, β_r associated with closure of the r-foil with the help of formula (5).

Lemma 2. *The generating polynomial for the closure of the r-foil verifies*

$$\overline{F_r}(x) = x^2\alpha_r(x) + x\beta_r(x), \tag{7}$$

where $\alpha_r(x) := \dfrac{(x+1)^r - 1}{x}$ *and* $\beta_r(x) = 1$.



(a) $P_{n,0}$ (b) $P_{n,1}$ (c) $P_{n,2}$

(d) $P_{n,3}$ (e) $P_{1,r}$ (f) $P_{2,r}$

Figure 4: For some values of n and r we have: (a) a disjoint union of n unknots ($r = 0$, $n \geq 1$); (b) an n-*foil* ($r = 1$); (b) an n-*chain link* ($r = 2$); (c) an n-*sinnet of square knotting* ($r = 3$); (e) an r-*twist loop* ($n = 1$) and (f) a $2r$-foil ($n = 2$).

Proof. First, note that $\overline{F_r} = \overline{(F_r)_1} = P_{1,r} = T_r$. Among the states of the r-twist loop, there is exactly one which is made up of one component as shown in Figure 5. Hence $\beta_r(x) = 1$. Then by (5), we get

$$T_r(x) = x^2 \left(\frac{(x+1)^r - 1}{x} \right) + x.$$

In fact if we let $\alpha_r(x) = \dfrac{(x+1)^r - 1}{x}$, then the expansion of $x^2\alpha_r(x)$, namely

$$\begin{aligned} x^2\alpha_r(x) &= x\Big(x(x+1)^0 + x(x+1)^1 + x(x+1)^2 + \cdots + x(x+1)^{r-1}\Big) \\ &= (\bigcirc \sqcup \bigcirc)(x) + (\bigcirc \sqcup \bigcirc\!\!\bigcirc)(x) + (\bigcirc \sqcup \bigcirc\!\!\bigcirc\!\!\bigcirc)(x) + \cdots + (\bigcirc \sqcup \bigcirc\!\!\bigcirc\!\!\cdots\!\!\bigcirc)(x), \end{aligned}$$

counts as expected the states which might result to that in Figure 55(a). ☐

Proposition 3. *The generating polynomial for the Pretzel knot* $P_{n,r}$ *is given by*

$$\begin{aligned} P_{n,r}(x) &= \alpha_r(x)P_{n-1,r}(x) + x\left(\alpha_r(x) + x\right)^{n-1} \\ &= x^2\alpha_r(x)^n + x\sum_{j=0}^{n-1} \alpha_r(x)^{n-j-1}\left(\alpha_r(x) + x\right)^j. \end{aligned}$$

Proof. By (3) and (7) we write

$$P_{n,r}(x) = \overline{(F_r)_n}(x) = \alpha_r(x)P_{n-1,r}(x) + \beta_r(x)\left(F_r\right)_{n-1}(x).$$

(a) $2^r - 1$ states of the kind. (b) One state of the kind.

Figure 5: Representatives of the r-twist loop states.

Formula (6) and property (vi) yield

$$(F_r)_{n-1}(x) = x\left(x^{-1}F_r(x)\right)^{n-1} = x\left(\frac{(x+1)^r - 1}{x} + x\right)^{n-1},$$

and we conclude by (4). $\square$

Remark 4. Formulas in Proposition 3 have the particularities below.

1. With $(F_r)_n(x) = x\left(\alpha_r(x) + x\right)^n$, $r = 1, 2, 3$, we have the following expressions:
   - $r = 1$, generating polynomial for the n-twist loop: $x(x+1)^n$ [2, 3, A097805, A007318];
   - $r = 2$, generating polynomial for the n-*link*: $x(2x+2)^n$ [2, 3, A038208];
   - $r = 3$, generating polynomial for the n-*overhand knot*: $x(x^2 + 4x + 3)$ [2, 3, A299989].

2. Let $\sum_{k=0}^{nr-n} a_r(n,k)x^k := \alpha_r(x)^n$ and $\sum_{k=0}^{\max(nr-n+1,n+1)} \overline{a}_r(n,k)x^k := x\big(\alpha_r(x) + x\big)^n$.

   If $\sum_{k=0}^{\max(nr-n+2,n)} p_r(n,k)x^k := P_{n,r}(x)$, then we have

$$p_r(n,k) = \overline{a}_r(n-1,k) + \sum_{j=0}^{r-1}\binom{r}{j} \times p_r(n-1, k-r+j+1) \tag{8}$$

$$= a_r(n,k-2) + \sum_{j=0}^{n-1}\sum_{i=0}^{k} a_r(n-j-1,i) \times \overline{a}_r(j,k-i), \tag{9}$$

   where

$$a_r(n,k) = \sum_{j=0}^{n}\binom{n}{j}\binom{rj}{n+k}(-1)^{n-j} \text{ and } \overline{a}_r(n,k) = \sum_{j=0}^{n}\binom{n}{j}a_r(j,k-n+j-1).$$

4 Results

In this section, we retrieve some of our previous results (case $r = 1, 2, 3$) [2] which confirm that the generating polynomial agrees with the construction in section 2.

1. **Case $r = 0$.**

   (a) Generating polynomial:

$$P_{n,0}(x) = \begin{cases} x^2 & \text{if } n = 0; \\ x^n & \text{if } n \geq 1. \end{cases}$$

   (b) Coefficients table: [3, A010054, A023531, A073424 ($n \geq 1$, all read as triangle)]

$n \setminus k$	0	1	2	3	4	5	6	7
0	0	0	1					
1	0	1						
2	0	0	1					
3	0	0	0	1				
4	0	0	0	0	1			
5	0	0	0	0	0	1		
6	0	0	0	0	0	0	1	
7	0	0	0	0	0	0	0	1

Table 1: Values of $p_0(n, k)$ for $0 \leq n \leq 7$ and $0 \leq k \leq 7$.

2. **Case $r = 1$.**

   (a) Generating polynomial:

$$P_{n,1}(x) = (x + 1)^n + x^2 - 1.$$

   (b) Coefficients table: [2, 3, A007318 ($3 \leq k \leq n$)]

$n \setminus k$	0	1	2	3	4	5	6	7	8
0	0	0	1						
1	0	1	1						
2	0	2	2						
3	0	3	4	1					
4	0	4	7	4	1				
5	0	5	11	10	5	1			
6	0	6	16	20	15	6	1		
7	0	7	22	35	35	21	7	1	
8	0	8	29	56	70	56	28	8	1

Table 2: Values of $p_1(n, k)$ for $0 \leq n \leq 8$ and $0 \leq k \leq 8$.

3. **Case $r = 2$.**

(a) Generating polynomial:

$$P_{n,2}(x) = x^2(x+2)^n + x\sum_{j=0}^{n-1}(x+2)^{n-j-1}\,(2x+2)^j\,.$$

(b) Coefficients table: [3, A300184]

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	1									
1	0	1	2	1								
2	0	4	7	4	1							
3	0	12	26	19	6	1						
4	0	32	88	88	39	8	1					
5	0	80	272	360	1230	71	10	1				
6	0	192	784	1312	1140	532	123	12	1			
7	0	448	2144	4368	4872	3164	1162	211	14	1		
8	0	1024	5632	13568	18592	15680	8176	2480	367	16	1	
9	0	2304	14336	39936	65088	67872	46368	20304	5262	655	18	1

Table 3: Values of $p_2(n,k)$ for $0 \le n \le 9$ and $0 \le k \le 11$.

4. **Case $r = 3$.**

(a) Generating polynomial:

$$P_{n,3}(x) = x^2\left(x^2+3x+3\right) + x\sum_{j=0}^{n-1}\left(x^2+3x+3\right)^{n-j-1}\left(x^2+4x+3\right)^j\,.$$

(b) Coefficients table: [2, Table 14]

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	1										
1	0	1	3	3	1								
2	0	6	16	20	15	6	1						
3	0	27	90	136	129	84	36	9	1				
4	0	108	459	876	1021	832	501	220	66	12	1		
5	0	405	2133	5085	7350	7321	5420	3103	1375	455	105	15	1

Table 4: Values of $p_3(n,k)$ for $0 \le n \le 5$ and $0 \le k \le 12$.

5. **Case $r = n$.**

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	1												
1	0	1	1												
2	0	4	7	4	1										
3	0	27	90	136	129	84	36	9	1						
4	0	256	1504	4336	8273	11744	13036	11488	8014	4368	1820	560	120	16	1

Table 5: Values of $p_n(n,k)$ for $0 \le n \le 4$ and $0 \le k \le 14$.

The next tables are obtained from formula (9) and Table 1.

6. **Case $k = 1$**: $p_r(n,1) = nr^{n-1},\ r \ge 1$ [3, A104002 $(1 \le r \le n)$].

$n \setminus r$	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1
2	0	2	4	6	8	10	12	14	16	18
3	0	3	12	27	48	75	108	147	192	243
4	0	4	32	108	256	500	864	1372	2048	2916
5	0	5	80	405	1280	3125	6480	12005	20480	32805
6	0	6	192	1458	6144	18750	46656	100842	196608	354294
7	0	7	448	5103	28672	109375	326592	823543	1835008	3720087
8	0	8	1024	17496	131072	625000	2239488	6588344	16777216	38263752

Table 6: Values of $p_r(n,1)$ for $0 \le n \le 8$ and $0 \le r \le 9$.

7. **Case $k = 2$**: $p_r(n,2) = \binom{n}{2} r^{n-2}\left(2\binom{r}{2}+1\right) + r^n,\ r \ge 1.$

$n \setminus r$	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8
2	1	2	7	16	29	46	67	92	121
3	0	4	26	90	220	440	774	1246	1880
4	0	7	88	459	1504	3775	7992	15043	25984
5	0	11	272	2133	9344	29375	74736	164297	324608
6	0	16	784	9234	54016	212500	649296	1666294	3764224
7	0	22	2144	37908	295936	1456250	5342112	16000264	41320448
8	0	29	5632	149445	1556480	9578125	42177024	147414197	435159040
9	0	37	14336	570807	7929856	61015625	322486272	1315198171	4437573632

Table 7: Values of $p_r(n,2)$ for $0 \le n \le 9$ and $0 \le r \le 8$.

8. **Case $k = r$.**

$n \setminus r$	0	1	2	3	4	5	6	7	8
0	0	0	1	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	7	20	70	252	924	3432	12870
3	0	3	26	136	804	5020	31842	203511	1307528
4	0	4	88	876	8273	78340	738093	6915552	64541512
5	0	5	272	5085	76500	1063926	14127730	182047131	2299773720
6	0	6	784	26784	635376	12821530	235289467	4060838166	67254512148
7	0	7	2144	130410	4804800	139108025	3486149646	79472545860	1697604751528

Table 8: Values of $p_r(n, r)$ for $0 \le n \le 7$ and $0 \le r \le 8$.

9. **Case $k = n$.**

$n \setminus r$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
2	1	2	7	16	29	46	67	92	121
3	1	1	19	136	511	1381	3061	5944	10501
4	1	1	39	1021	8273	38741	131911	364449	868561
5	1	1	71	7321	130151	1063926	5587921	22023156	70930861
6	1	1	123	51517	2025073	28991811	235289467	1324261975	5768165145
7	1	1	211	359815	31368807	787565521	9883571713	79472545860	468290153045

Table 9: Values of $p_r(n, n)$ for $0 \le n \le 7$ and $0 \le r \le 8$.

We observe the following formulas:

- First column in Table 7 is $1, 0, 1$ followed by $0, 0, 0, \ldots$ [3, A154272];
- $p_1(n, 2) = \binom{n}{2} + 1$ [3, A152947];
- $p_0(2, 0) = 0$, $p_1(2, 1) = 2$, $p_2(2, 2) = 7$ and $p_n(2, n) = \binom{2n}{n}$ [3, A000984];
- $p_2(n, 2) = (3n^2 - 3n + 8)2^{n-3}$ [3, A300451];
- $p_2(n, n) = 2n(n-1) + 2^n - 1$ [3, A295077];
- $p_3(n, 2n-1) = \binom{3n}{3}$ [3, A006566];

- $p_3(n, 2n) = \binom{3n}{2}$ [3, A062741];
- $p_0(0, 1) = 0$ and $p_n(n, 1) = n^n$ [3, A000312];
- $p_n(2, 2) = 2n^2 - n + 1$ [3, A130883];
- $p_n(n, n^2 - n + 1) = n^2$ [3, A000290];
- $p_0(0, 0) = p_1(1, 0) = 0$, $p_2(2, 2) = 7$ and $p_n(n, n^2 - n) = \binom{n^2}{2}$ [3, A083374];
- $p_n(n, n^2 - n - 1) = \binom{n^2}{3}$ [3, A178208].

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