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About Some Relatives of the Taxicab Number

Viorel Niţică
Department of Mathematics
West Chester University of Pennsylvania
West Chester, PA 19383
USA
vnitica@wcupa.edu

Abstract

The taxicab number, 1729, has the following property. If we add its digits we obtain 19. The number obtained from 19 by reversing the order of its digits is 91. If we multiply 19 by 91 we obtain again 1729. In the paper we study various generalizations of this property.

1 Introduction

The *taxicab number,* 1729, became well known due to a discussion between Hardy and Ramanujan. It is the smallest positive integer that can be written in two ways as a sum of two cubes: $1^3 + 12^3$ and $9^3 + 10^3$. The number 1729 has other interesting property: if we add its digits we obtain 19; multiplying 19 by 91, the number obtained from 19 by reversing the order of its digits, we obtain again 1729. It is not hard to show that the set of integers with this property is finite and equal to $\{1, 81, 1458, 1729\}$.

In a conversation that the author had with his colleague, Professor Shiv Gupta, Shiv asked if the second property can be generalized. One replaces the sum of the digits of an integer by the sum of the digits times an integer multiplier and then multiplies the product by the number obtained by reversing the order of the digits in the product. The taxicab number becomes a particular example with multiplier 1. A computer search produced a large number of examples with higher multiplier. For example, 2268 has multiplier 2. Indeed, the sum of the digits is $18, 18 \times 2 = 36$, and $36 \times 63 = 2268$.

One may replace the last product in the above procedure by a sum. A computer search showed that there are numbers that have this property for sums. For example, 121212 has multiplier 6734. The sum of the digits is $9, 9 \times 6732 = 60606$, and $60606 + 60606 = 121212$.

The paper is dedicated to the study of these properties. After the paper was submitted for publication we learned from the editor that our work may be related to the study of Niven (or Harshad) numbers. These are numbers divisible by the sum of their decimal digits. Niven numbers have been extensively studied as one can see for instance from Cai

[3], Cooper and Kennedy [4], De Koninck and Doyon [6], Grundman [7]. It follows from the formal definitions given in the paper that one of the classes of integers we study, that of multiplicative Ramanujan-Hardy numbers, is a subclass of the class of Niven numbers. Of interest are also q-Niven numbers, which are numbers divisible by the sum of their base q digits. See, for example, Fredricksen, Ionaşcu, Luca, and Stănică [8]. Some other variants of Niven numbers can be found in Boscaro [1] and Bloem [2].

2 Statements of the main results

In what follows let $b \geq 2$ be an arbitrary numeration base.

Definition 1. If N is a positive integer, written in base b, we call *reversal* of N and denote by N^R the integer obtained from N by writing its digits in reverse order.

We denote by $s_b(N)$ the sum of the base b digits of an integer N.

Definition 2. A positive integer N written in base b is called *b-additive Ramanujan-Hardy number,* or simply b-ARH number, if there exists a positive integer M, called *additive multiplier*, such that

$$N = Ms_b(N) + (Ms_b(N))^R, \tag{1}$$

where all arithmetic operations are done in base b.

Definition 3. A positive integer N written in base b is called *b-multiplicative Ramanujan-Hardy number,* or simply b-MRH number, if there exists a positive integer M, called *multiplicative multiplier*, such that

$$N = Ms_b(N) \cdot (Ms_b(n))^R. \tag{2}$$

where all arithmetic operations are done in base b.

To simplify the notation, we also denote $s_{10}(N), 10 - ARH, 10 - MRH$ respectively by $s(N), ARH, MRH$.

While b-MRH numbers are b-Niven numbers, b-Niven numbers are not necessarily b-MRH numbers.

Example 4. The number 144_7 is a 7-Niven number but not a 7-MRH number.

Once these notions are introduced and examples of such numbers found, several natural questions arise.

Question 5. Does there exist an infinite set of b-ARH numbers?

Question 6. Does there exist an infinite set of b-MRH numbers?

Question 7. Does there exist an infinite set of additive multipliers?

Question 8. Does there exist an infinite set of multiplicative multipliers?

In what follows, if x is a string of digits, we denote by $(x)^{\wedge k}$ the base 10 integer obtained by repeating x k-times. We denote by $[x]_b$ is the value of the string in base b.

The following theorem gives an explicit positive answer to Question 5 if $b = 10$.

Theorem 9. *Consider the numbers*

$$N_k = (12)^{\wedge k}, \tag{3}$$

where k is a positive integer. Then N_k is divisible by $s(N_k)$. Moreover all numbers N_k are ARH numbers and Niven numbers.

We state the following corollary due to the explicit nature of the example in Theorem 9.

Corollary 10. *There exists an infinite set of Niven numbers with no digit equal to zero.*

Remark 11. If we allow zero digits an example of infinite set of b-MRH numbers is given by $\{[1(0)^{\wedge k}]_b | k \in \mathbb{N}\}$. The example has the unpleasant feature that the apparent multiplicative multiplier of each b-MRH numbers is the number itself and the search for other multipliers is dependent on the base. In order to avoid trivial considerations, we consider from now on only examples of b-ARH and b-MRH numbers that have many digits different from zero.

It follows from the proof of Theorem 9 that $Ms(N_k) = (Ms(N_k))^R$. The following theorem gives an example in which it is clear from the proof that $Ms_b(N_k) \neq (Ms_b(N_k))^R$ for an arbitrary even base b. Other advantage of the example is that one can read from the proof the explicit base b expansion of the multipliers. It also shows that the set of multipliers corresponding to a b-ARH number can grow exponentially in terms of the number of digits of the b-ARH number.

Theorem 12. *Consider the numbers*

$$N_k = [(1)^{\wedge k}]_b, \tag{4}$$

where b is even and $k = [1(0)^{\wedge p}]_b, p \geq 1$. Then all numbers N_k are b-ARH numbers and not b-Niven numbers.

Each N_k has a subset of additive multipliers of cardinality $2^{\frac{k-2p}{2}}$ consisting of all integers $[(1)^{\wedge p} I]_b$, where I is a sequence of 0 and 1 of length $k - 2p$ in which no two digits symmetric about the center of the sequence are identical.

Example 13. We show an example that illustrates the results in Theorem 12. Assume that $b = 2$, $k = 16 = 10000_2$, and $p = 4$. Then $N_{16} = [(1)^{\wedge 16}]_2$ and $s_2(N_{16}) = 2^4 = 10000_2$. The following $16 = 2^{\frac{16-2\cdot 4}{2}}$ numbers are additive multipliers of N_{16}:

- 111100001111_2
- 111100010111_2

- 111100101011_2
- 111100111100_2
- 111101001101_2
- 111101010101_2
- 111101110001_2
- 111110001110_2
- 111110010110_2
- 111110101010_2
- 111110110010_2
- 111111001100_2
- 111111010100_2
- 111111101000_2
- 111111110000_2.

Remark 14. It is possible that the numbers N_k have other multipliers, besides those listed in Theorem 12. The growth of the set of multipliers can be larger than that shown in Theorem 12 and depends on the numeration base. See Theorem 15. Nevertheless, for $b = 2$ there are no other multipliers of N_k besides those listed in Theorem 12. We also observe that the numbers N_k from Theorem 12 have an even number of digits and the numbers N_k from Theorem 15 have an odd number of digits.

Theorem 15. *Consider the numbers*

$$N_k = [(1)^{\wedge p}(10)^{\wedge k-2p}0(1)^{\wedge p}]_b, \tag{5}$$

where b is even and $k = [1(0)^{\wedge p}]_b, p \geq 1$. Then all numbers N_k are b-ARH numbers and not b-Niven numbers.

For each N_k the set of additive multipliers has cardinality $(b-1)^{\frac{k-2p}{2}}$ and consists of all integers $[(1)^{\wedge p}I0]_b$, where I is a concatenation of $k-2p$ two digits strings of type $0\alpha, \alpha \neq 0$, in which any pair of nonzero digits symmetric about the center of $I0$ have their sum equal to b.

Example 16. We show an example that illustrates the results in Theorem 15. Assume that $b = 4$, $k = 4^1 = 10_4$, and $p = 1$. Then $N_4 = 1101001_4$ and $s_4(N_4) = 4 = 10_4$. The following $3 = 3^{\frac{4-2\cdot 1}{2}}$ numbers are additive multipliers of N_4:

- 102020_4
- 101030_4
- 103010_4.

The following corollary of Theorem 37 gives a positive answer to Question 7.

Corollary 17. *If b is even there exists an infinite set of additive multipliers. Moreover, there exists an infinite set of b-ARH numbers that have at least two additive multipliers.*

It is easy to see that the numbers N_k from Theorems 12 and 15 are not b-MRH numbers.

Question 18. Does there exist an infinite set of b-MRH numbers that have at least two multiplicative multipliers?

Corollary 19. *For any base b there exists an infinite set of b-ARH numbers that are not b-MRH numbers.*

Motivated by the results in Theorems 12 and 15 and by the examples of ARH and MRH numbers shown later in Sections 13 and 14 we introduce the following definitions.

Definition 20. If N is a b-ARH number, we call the *multiplicity* of N the cardinality of the corresponding set of additive multipliers.

Definition 21. If N is a b-MRH number, we call the *multiplicity* of N the cardinality of the corresponding set of multiplicative multipliers.

Theorem 12 has the following corollary.

Corollary 22. *The multiplicity of b-ARH numbers is unbounded for any even base.*

We do not know how to answer the following questions for any numeration base.

Question 23. For a given b-ARH number N can one find bounds for the number of distinct additive multipliers in terms of the value of N or in terms of the number of digits of N?

Question 24. Is the multiplicity of b-MRH numbers bounded?

Question 25. For a given MRH number N can one find bounds for the number of distinct multiplicative multipliers in terms of the value of N or in terms of the number of digits of N?

Remark 26. For Questions 6 and 8 we do not have an answer with b-MRH numbers having all digits different from zero. But see Theorem 29 below for an infinite set of b-MRH numbers with half of the digits different from zero. Besides the set listed in Theorem 12 it is easy to find other infinite sets of integers that contain no MRH number. For example, no prime number can be MRH number. Also note that no other integer with two prime factors in the prime factorization can be MRH number. Indeed, such a MRH number has the additive multiplier equal to 1 and among the MRH numbers with additive multiplier 1 none has two factors in the prime factorization.

One observes that many numbers in Remark 26 are not Niven numbers. The following theorem shows an infinite set of b-Niven numbers that are not b-MRH numbers.

Theorem 27. *For $b \geq 2$ a numeration base define*

$$R_n = \frac{b^n - 1}{b} = [(1)^{\wedge n}]_b, n \geq 1.$$

Assume that $b-1$ does not divide n. Then $[(b-1)nR_n]_b$ are b-Niven numbers but not b-MRH numbers.

For b-ARH numbers one has the following result.

Theorem 28. *There exists an infinite set of integers that are not b-ARH numbers.*

The following Theorem gives a partial answer to Question 6.

Theorem 29. *Let b odd and $k \geq 2$. Then the numbers*

$$N_k = [(b-1)^{\wedge 2^{k-1}-1}(b-2)(0)^{\wedge 2^{k-1}-1}1]_b \tag{6}$$

are b-MRH numbers and $s_b(\sqrt{N_k}) = s_b(N_k)$.

Moreover, if $b \equiv 3 \pmod 4$ then $\sqrt{N_k}$ is itself a b-Niven number.

Example 30. We illustrate the result in Theorem 29.

- For $b = 3, k = 2$ we get $N_2 = 2101_3$ which is a 3-MRH number. Then $\sqrt{2101_3} = 22_3$, $s_3(2101_3) = s_b(22_3) = 4$ and 22_3 is a 3-Niven number.
- For $b = 5, k = 2$ we get $N_2 = 4301_5$ which is a 5-MRH number. Then $\sqrt{4301_5} = 44_5$, $s_5(4301_5) = s(44_5) = 8$ and 44_5 is a 5-Niven number.
- For $b = 17, k = 5$, N_5 is a 17-MRH number, but $\sqrt{N_5}$ is not a 17-Niven number.
- For $b = 7, k = 2$ we get $N_2 = 6501_7$ which is a 7-MRH number. Then $\sqrt{6501_7} = 66_7$, $s_7(6501_7) = s_7(66_7) = 48$ and 66_7 is a 7-Niven number.

Third item shows that the congruence condition in Theorem 29 is necessary for a general result. Nevertheless, the second item shows that $\sqrt{N_k}$ may still be a b-Niven number even without this condition.

The following corollary of Theorem 29 gives a partial positive answer to Question 8.

Corollary 31. *If b is odd there exists an infinite set of multiplicative multipliers.*

We show two unexpected corollaries of the proof of Theorem 29.

Corollary 32. *If b is odd there exists an infinite set of b-MRH numbers that are perfect squares.*

Corollary 33. *If $b \equiv 3 \pmod 4$ then there exists an infinite set N_k of b-MRH numbers for which $\sqrt{N_k}$ are b-Niven numbers and for which $s_b(N_k) = s_b(\sqrt{N_k})$.*

The following notions of high degree b-Niven numbers blossom from Corollary 33, which provides plenty of examples.

Definition 34. An integer N is called *quadratic b-Niven number* if N and N^2 are b-Niven numbers. If in addition $s_b(N) = s_b(N^2)$ then N is called *strongly quadratic b-Niven number*.

The study of high degree b-Niven numbers is continued in Niţică [9] where it is shown that for each degrees there exists an infinite set of bases in which b-Niven numbers of that degree appear.

We show in Sections 13 that 6 is not an additive multiplier for base 10 and ARH numbers without zero digits and that 9 is not an additive multiplier for base 10. We show in Section 14 that 3 is not a multiplicative multiplier for base 10. Nevertheless, we do not know how to answer the following questions for any base:

Question 35. Does there exists an infinite set of integers that are not additive multipliers?

Question 36. Does there exists an infinite set of integers that are not multiplicative multipliers?

The following theorem gives bounds for the number of digits in a b-ARH number in terms of the multiplier, showing that if the multiplier is fixed, there exists only a finite set of b-ARH numbers corresponding to that multiplier.

In what follows we denote by $\lfloor x \rfloor$ the integer part, by $\ln x$ the natural logarithm and by $\log_b x$ the base b logarithm of the positive real number x.

Theorem 37. *Let N be a b-ARH number with k digits and with additive multiplier M. Then*

$$k \leq \begin{cases} M+2, & \text{if } b \geq 3; \\ M+4, & \text{if } b = 2. \end{cases}$$

Corollary 38. *For fixed additive multiplier M and base b, the set of b-ARH numbers with multiplier M is finite.*

Example 39. The result in Theorem 37 for $b = 2$ is optimal as shown by the following example: $N = [(1)^{\wedge 8}]_2, M = 111010_2$. Then $k = M + 4$.

If the additive multiplier is large, one can obtain better bounds for the number of digits of a b-ARH number in terms of the multiplier.

Theorem 40. *Let N be a b-ARH number with k digits and with additive multiplier M. Under any of the following assumptions:*

- $b \geq 10$ *and* $M \geq b^5$

- $3 \le b \le 9$ *and* $M \ge b^6$
- $b = 2$ *and* $M \ge b^7$

one has

$$k \le 2\lfloor \log_b M \rfloor. \tag{7}$$

The following theorem gives bounds for the number of digits in a b-MRH number in terms of the multiplier, showing that if the multiplier is fixed there exists only a finite set of b-MRH numbers corresponding to that multiplier.

Theorem 41. *Let N be a b-MRH number with k digits and with multiplicative multiplier M. Then*

$$k \le \begin{cases} M+4, & \text{if } b \ge 6; \\ M+5, & \text{if } 2 \le b \le 6. \end{cases}$$

Theorem 41 shows that a MRH number with multiplicity 1 can have at most 5 digits. A computer search shows that the set of all such numbers is indeed $\{1, 81, 1458, 1729\}$.

Corollary 42. *For fixed multiplicative multiplier M, the set of b-MRH numbers with multiplier M is finite.*

If the multiplicative multiplier is large, one can obtain better bounds for the number of digits of a b-MRH number in terms of the multiplier.

Theorem 43. *Let N be a b-MRH number with k digits and with multiplicative multiplier M. Under any of the following assumptions:*

- $b \ge 8$ *and* $M \ge b^9$
- $5 \le b \le 7$ *and* $M \ge b^{10}$
- $b = 4$ *and* $M \ge b^{11}$
- $b = 3$ *and* $M \ge b^{12}$
- $b = 2$ *and* $M \ge b^{15}$

one has

$$k \le 3\lfloor \log_b M \rfloor. \tag{8}$$

We summarize the rest of the paper. Theorem 9 is proved in Section 3, Theorem 12 is proved in Section 4, Theorem 5 is proved in Section 5, Theorem 28 is proved in Section 7, Theorem 27 is proved in Section 6, Theorem 29 is proved in Section 8, Theorem 37 is proved in Section 9, Theorem 40 is proved in Section 10, Theorem 41 is proved in Section 11, and Theorem 43 is proved in Section 12. In Section 13 we show examples of ARH numbers and ask additional questions and in Section 14 we show examples of MRH numbers and ask additional questions. In Section 15 we describe an approach to Question 6 if $b = 10$.

3 Proof of Theorem 9

One obtains a formula for N_k by adding two geometric series.

$$\begin{aligned}N_k &= 10^{2\cdot 3^k-1} + 10^{2\cdot 3^k-3} + \ldots + 10 \\ &+2(10^{2\cdot 3^k-2} + 10^{2\cdot 3^k-4} + \ldots + 1) \\ &= 12 \cdot \frac{10^{2\cdot 3^k}-1}{99} = 4 \cdot \frac{10^{2\cdot 3^k}-1}{33}.\end{aligned} \tag{9}$$

Note that $s(N_k) = 3^{k+1}$. We show by induction that $s(N_k)$ divides N_k. The case $k = 0$ gives $s(N_0) = 3$ which divides $N_0 = 12$. Assume that for fixed k $s(N_k)$ divides N_k.

$$\begin{aligned}N_{k+1} = 4 \cdot \frac{10^{2\cdot 3^{k+1}}-1}{33} = 4 \cdot \frac{(10^{2\cdot 3^k})^3 - 1^3}{33} \\ = 4 \cdot \frac{10^{2\cdot 3^k}-1}{33} \cdot (10^{4\cdot 3^k} + 10^{2\cdot 3^k} + 1) = N_k \cdot (10^{4\cdot 3^k} + 10^{2\cdot 3^k} + 1),\end{aligned} \tag{10}$$

which is clearly divisible by $s(N_{k+1}) = 3^{k+2}$ due to N_k divisible by $s(N_k) = 3^{k+1}$ and $10^{4\cdot 3^k} + 10^{2\cdot 3^k} + 1$ divisible by 3. This ends the proof of the first part of Theorem 9.

To prove the second part of Theorem 9 observe that the number $N_k/2 = (N_k/2)^R$. It follows from (3) and the fact that N_k is divisible by $s(N_k) = 3^{k+1}$ that $N_k/2$ is divisible by $s(N_k)$. We conclude that N_k is an ARH number with additive multiplier $M = N_k/(2s(N_k))$. It also is a Niven number. This ends the proof of Theorem 9.

4 Proof of Theorem 12

In this section all arithmetic computations are done in base b.

Let $N_k = [(1)^{\wedge k}]_b$ where k is even and $k = [1(0)^{\wedge p}]_b, p \geq 1$. Then $s_b(N_k) = [1(0)^{\wedge p}]_b$. Let $M = [(1)^{\wedge p} I]_b$, where I is a sequence of 0 and 1 of length $k - 2p$ in which no two digits symmetric about the center of the sequence are identical. Note that $M^R = [(I)^R (1)^{\wedge p}]_b$. The following calculation shows that N_k is a b-ARH number. Note that $I + (I)^R = [(1)^{\wedge k-2p}]_b$.

$$\begin{aligned}&s_b(N_k) \cdot M + (s_b(N_k) \cdot M)^R \\ &= [1(0)^{\wedge p}]_b \cdot [(1)^{\wedge p} I]_b + ([1(0)^{\wedge p}]_b \cdot [(1)^{\wedge p} I]_b)^R \\ &= [(1)^{\wedge p} I (0)^{\wedge p}]_b + ([(1)^{\wedge p} I (0)^{\wedge p}]_b)^R \\ &= [(1)^{\wedge p} I (0)^{\wedge p}]_b + [(0)^{\wedge p} (I)^R (1)^{\wedge p}]_b = [(1)^{\wedge k}]_b = N_k.\end{aligned}$$

In order to count the multipliers, observe that the length of the string I is $k - 2p$. If we know half of the digits we can find the other half using the condition that no two digits symmetric about the center of the sequence are identical. The number of strings of 0 and 1 of length $\frac{k-2p}{2}$ is $2^{\frac{k-2p}{2}}$.

Finally, to show that N_k is not a b-Niven number observe that N_k is not divisible by $s_b(N_k)$.

5 Proof of Theorem 15

In this section all arithmetic computations are done in base b.

Let $N_k = [(1)^{\wedge p}(10)^{\wedge k-2p}0(1)^{\wedge p}]_b$ where b is even and $k = [1(0)^{\wedge p}]_b, p \geq 1$. Then $s_b(N_k) = [1(0)^{\wedge p}]_b$. Let $M = [(1)^{\wedge p}I0]_b$. Note that $M^R = [0(I)^R(1)^{\wedge p}]_b$. The following calculation shows that N_k is a b-ARH number. Note that $I0 + 0(I)^R = [(10)^{\wedge k-2p}0]_b$.

$$\begin{gathered} s_b(N_k) \cdot M + (s_b(N_k) \cdot M)^R \\ = [1(0)^{\wedge p}]_b \cdot [(1)^{\wedge p}I0]_b + ([1(0)^{\wedge p}]_b \cdot [(1)^{\wedge p}I0]_b)^R \\ = [(1)^{\wedge p}I0(0)^{\wedge p}]_b + ([(1)^{\wedge p}0I(0)^{\wedge p}]_b)^R \\ = [(1)^{\wedge p}I0(0)^{\wedge p}]_b + [(0)^{\wedge p}0(I)^R(1)^{\wedge p}]_b = [(1)^{\wedge p}(10)^{\wedge k-2p}0(1)^{\wedge p}]_b = N_k. \end{gathered}$$

In order to count the multipliers, observe that the length of the string $I0$ is $k-2p+1$. If we know half of the nonzero digits we can find the other half using the condition that no two digits symmetric about the center of the string $I0$ are identical. There are $\frac{k-2p}{2}$ positions to be filled and each one can be filled in $b-1$ ways. To show that there are no other multiplier it is enough to observe, for example using induction on length, that the string $[(10)^{\wedge k-2p}0]_b$ cannot be written as a sum of a string J and its reversal except if $J = I0$, where I is as above.

Finally, to show that N_k is not a b-Niven number observe that N_k is not divisible by $s_b(N_k)$.

6 Proof of Theorem 27

McDaniels proved in [5, Theorem 2] that if $m \leq 9R_n$ then $s(9mR_n) = 9n$. The proof is valid in any base b and follows readily upon writing m as:

$$m = \sum_{i=1}^{k} a_i b^i, k < n. \tag{11}$$

It gives that if $m \leq (b-1)[R_n]_b$ then $s_b((b-1)mR_n) = (b-1)n$. If $m = n$ one has $s_b((b-1)nR_n) = (b-1)n$, which shows that $[(b-1)nR_n]_b$ is a b-Niven number. By contradiction, assume that $[(b-1)nR_n]_b$ is a b-MRH number with multiplier M. It follows that:

$$(b-1)nM((b-1)nM)^R = (b-1)nR_n. \tag{12}$$

We recall that a base b number is divisible by $b-1$ if the sum of its base b digits is divisible by $b-1$. Base b divisibility test by $b-1$ and $b-1 \not| n$ implies that $b-1 \not| R_n$, but $b-1 | ((b-1)nM)^R$. As $b-1 | n$, due to our assumptions, there are at least two factors of $b-1$ in the factorization of the left hand side of (12) and only one factor of $b-1$ in the right hand side of (12). This gives a contradiction.

7 Proof of Theorem 28

Any b-ARH number is a sum of an integer and its reversal. In order to prove the theorem it is enough to show that there exists an infinite set of integers that are not a sum of an integer and its reversal. There are $b^k - b^{k-1} = b^{k-1}(b-1)$ base b k-digit numbers. Those of type $N + N^R$, either have $N = [a_k a_{k-1} \dots a_2 a_1]_b$ with $a_k + a_1 \leq b-1$, or have N with at most $k-1$ digits. There are $\frac{b(b-1)}{2} \cdot b^{k-2}$ k-digit numbers with $a_k + a_1 \leq b-1$ and there are at most $b^{k-1} - b^{k-2}$ $(k-1)$-digits number. Overall, we have at most

$$\frac{b(b-1)}{2} \cdot b^{k-2} + (b^{k-1} - b^{k-2}) = b^{k-1}\left(\frac{b+1}{2}\right) - b^{k-2}$$

k-digit numbers of type $N + N^R$. Hence there are at least

$$b^k - b^{k-1} - \left(b^{k-1}\left(\frac{b+1}{2}\right) - b^{k-2}\right) = b^{k-1}\left(\frac{b-1}{2}\right) + b^{k-2}$$

k-digit numbers that are not of type $N + N^R$. These numbers are not b-ARH either. One concludes that there exists an infinite set of integers that are not b-ARH numbers.

8 Proof of Theorem 29

As $\gcd(b,2) = 1$ Euler's Theorem implies that 2^k divides $b^{\phi(2^k)} - 1$. Clearly $b-1$ also divides $b^{\phi(2^k)} - 1$. Assume that $\gcd(2^k, b-1) = 2^\ell$. Then $2^{k-\ell}(b-1)$ divides $b^{\phi(2^k)} - 1 = b^{2^{k-1}} - 1$. Consider now the product

$$(b^{2^{k-1}} - 1)^2 = b^{2 \cdot 2^{k-1}} - 2b^{2^{k-1}} + 1,$$

which is divisible by $2^{k-1}(b-1)$, written in base b coincides with N_k and $s_b(N_k) = 2^{k-1}(b-1)$. We conclude that N_k is a b-MRH number.

To finish the proof of the theorem observe that if $b \equiv 3 \pmod 4$ then $\gcd(2^k, b-1) = 2$. Therefore $2^{k-1}(b-1)$ divides $b^{2^k} - 1 = [(b-1)^{2^k-1}]_b$. Finally $s_b[[(b-1)^{2^k-1}]_b = 2^{k-1}(b-1) = s_b(N_k)$.

9 Proof of Theorem 37

In this section all computations are done in base 10. The proof is valid for any numeration base.

As N has k digits one has that:

$$N \geq b^{k-1}. \tag{13}$$

The largest possible value for $s_b(N)$ is $(b-1)k$. We observe that reversing to order of the digits in an integer may increase its value by at most $b-1$ times. Hence one has that:

$$Ms_b(N) + (Ms_b(N))^R \leq b(b-1)kM. \tag{14}$$

Combining equations (1), (13), (14) one has that:

$$b^{k-1} \leq b(b-1)kM. \tag{15}$$

Now we prove by induction on the variable k that:

$$b^{k-1} > b(b-1)kM, \text{ for } k \geq M+3, M \geq 1, b \geq 3, \tag{16}$$

which combined with (15) finishes the proof of Theorem 37 for $b \geq 3$.

In the initial induction step $k = M+3$. The statement in (16) becomes:

$$b^{M+2} > b(b-1)(M^2+3M), \text{ for } M \geq 1, b \geq 3. \tag{17}$$

We prove (17) by induction on the variable M. In the initial step $M = 1$.

$$b^3 > 4b(b-1) \Leftrightarrow b(b-2)^2 > 0,$$

which is clearly true for $b \geq 3$.

We also need the case $M = 2$. The statement in (17) becomes:

$$b^4 > 10b(b-1) \Leftrightarrow b(b^3 - 10b + 10) > 0,$$

which is true for $b \geq 3$.

Now we assume that the inequality in (17) is true for M and prove it for $M+1$. Using the induction hypothesis one has that:

$$b^{M+3} = b \cdot b^{M+2} > b \cdot b(b-1)(M^2+3M). \tag{18}$$

In order to finish the proof by induction, we still need to check that:

$$b \cdot b(b-1)(M^2+3M) \geq b(b-1)\left((M+1)^2 + 3(M+1)\right). \tag{19}$$

After simplifications, (19) becomes:

$$(b-1)M^2 + (3b-5) - 4 \geq 0. \tag{20}$$

As the left hand side of (20) is larger than M^2+M-4, which is clearly positive if $M \geq 2$, we conclude that (20) is true for all $M \geq 1$. This finishes the proof of (17).

We continue with the general step in the proof of (16). By induction:

$$b^k = b \cdot b^{k-1} > b^2(b-1)kM.$$

To finish the proof we still need to check that

$$b^2(b-1)kM \geq b(b-1)k(M+1),$$

which is obvious. This finishes the proof of (17) and that of Theorem 37 for base $b \geq 3$.

Now assume $b = 2$. Equation (15) becomes:

$$2^{k-1} \leq 2kM. \tag{21}$$

We prove by induction on the variable k that:

$$2^{k-1} > 2kM, \text{ for } k \geq M + 5, M \geq 1, \tag{22}$$

which combined with (21) finishes the proof of case $b = 2$.

If $k = M + 5$ one has that:

$$2^{M+4} > 2(M^2 + 4M), \text{ for } M \geq 1, \tag{23}$$

which we prove by induction on M.

The case $M = 1$ is true. We assume (23) true for M and prove it for $M+1$. By induction one has that:

$$2^{M+5} = 2 \cdot 2^{M+4} > 4(M^2 + 4M).$$

To finish the proof of (23) we still need to check:

$$4(M^2 + 4M) \geq 2\left((M+1)^2 + 4(M+1)\right),$$

which simplifies to $M^2 + 4M - 3 \geq 0$, which is true for $M \geq 1$.

10 Proof of Theorem 40

In this section all computations are done in base 10. The proof is valid for any numeration base.

It follows from formula (15) in the proof of Theorem 37 that:

$$b^{k-1} \leq b(b-1)kM. \tag{24}$$

We show by induction on the variable k that:

$$b^{k-1} > b(b-1)KM \text{ if } M \geq b^5, k \geq 2\lfloor \log_b M \rfloor + 1, b \geq 10 \tag{25}$$

which together with (24) finishes the proof of Theorem 40 for base $b \geq 10$.

First we show by induction on the variable M that:

$$M > b^3(b-1)\log_b M + b^3(b-1) \text{ if } M \geq b^5, b \geq 10. \tag{26}$$

If $M = b^5$ (26) becomes:

$$b^5 > 11b^3(b-1) \Leftrightarrow b^2 - 11b + 11 > 0, \tag{27}$$

which is true if $b \geq 10$.

Now assume that (26) is true for a fixed M. Due to this hypothesis one has

$$M + 1 > 2b^3(b-1)\log_b M + b^3(b-1) + 1.$$

To finish the proof of (26) we still need to check that:

$$2b^3(b-1)\log_b M + b^3(b-1) + 1 \geq 2b^3(b-1)\log_b M + b^3(b-1) + 1,$$

which after simplifications becomes:

$$1 \geq 2b^3(b-1)\left(\log_b(M+1) - \log_b M\right),$$

which is true due to $M \geq b^5$ and the Mean Value Theorem.

We start the proof of (25). In the first step $k = 2\lfloor \log_b M \rfloor + 1$ and (25) becomes

$$b^{2\lfloor \log_b M \rfloor} > b(b-1)M(2\lfloor \log_b M \rfloor + 1). \tag{28}$$

Due to $\log_b M - 1 \leq \lfloor \log_b M \rfloor \leq \log_b M$ one has

$$\begin{gathered} b^{2\lfloor \log_b M \rfloor} \geq b^{2(\log_b M - 1)} \\ b(b-1)M(2\lfloor \log_b M \rfloor + 1) \leq b(b-1)M(2\log_b M + 1). \end{gathered} \tag{29}$$

In order to prove (28) it is enough to show that

$$b^{2(\log_b M - 1)} > b(b-1)M(2\log_b M + 1),$$

which after some algebraic manipulations becomes (26). This finishes the proof of the first induction step.

Now assume that (25) is true for fixed k and show that it is true for $k+1$. Due to the induction hypothesis one has that:

$$b^k \geq b \cdot b(b-1)kM.$$

To finish the proof of (25) we still need to check that

$$b \cdot b(b-1)kM > b(b-1)(k+1)M,$$

which is obviously true.

The proofs of the other cases are similar. The only significant difference appears in (27). If $3 \leq b \leq 9$, (27) becomes $b^3 - 11b + 11 \geq 0$, which is true if $b \geq 3$ and if $b = 2$ (27) becomes $2^4 - 11 \cdot 2 + 11 \geq 0$, which is true.

11 Proof of Theorem 41

In this section all computations are done in base 10. The proof is valid for any numeration base.

As N has k digits one has that:

$$N \geq b^{k-1}. \tag{30}$$

The largest possible value for $s_b(N)$ is $(b-1)k$. We observe that reversing to order of the digits in an integer may increase its value by at most $b-1$ times. Hence one has that:

$$Ms_b(N) \cdot (Ms_b(N))^R \leq (b-1)^3k^2M^2. \tag{31}$$

Combining equations (2), (30), (31) one has that:

$$b^{k-1} \leq (b-1)^3k^2M^2. \tag{32}$$

Now we prove by induction on the variable k that:

$$b^{k-1} > (b-1)^3k^2M^2, \text{ for } k \geq M+5, M \geq 1, b \geq 6 \tag{33}$$

which combined with (32) finishes the proof of Theorem 41 for $b \geq 6$.

In the initial induction step $k = M + 5$. The statement in (33) becomes:

$$b^{M+4} > (b-1)^3(M+5)^2M^2, \text{ for } M \geq 1, b \geq 6. \tag{34}$$

We prove (34) by induction on the variable M. If $M = 1$ (34) becomes $b^5 > 36(b-1)^3$, which is true if $b \geq 4$.

Now we assume that the inequality in (34) is true for M and prove it for $M+1$. From the induction hypothesis one has that:

$$b^{M+5} = b \cdot b^{M+4} > b \cdot (b-1)^3(M+5)^2M^2. \tag{35}$$

In order to finish the proof, we still need to check that:

$$b \cdot (b-1)^3(M+5)^2M^2 \geq (b-1)^3(M+6)^2(M+1)^2 \tag{36}$$

for $M \geq 1$.

After simplifications, (36) becomes:

$$(b-1)M^4 + (10b-14)M^3 + (25b-61)M^2 - 84M - 36 \geq 0, \tag{37}$$

which is true for $M \geq 1$ and $b \geq 6$.

This finishes the proof of (34).

We continue with the general step in the proof of (33). One has from the induction hypothesis that:

$$b^k = b^{\cdot k-1} > 10 \cdot (b-1)^3k^2M^2.$$

To finish the proof of (33) we still need to check that

$$b \cdot (b-1)^3 k^2 M^2 \geq (b-1)^3 (k+1)^2 M^2,$$

which after simplifications becomes $(b-1)k^2 - 2k - 1 \geq 0$. This is obvious if $k \geq 3$ and in the cases $k = 1, k = 2$ the theorem is trivially true.

This finishes the proof of Theorem 41 for $b \geq 6$.

The proof of the case $2 \leq b \leq 5$ is similar. The only significant changes appear in (34), which becomes $b^6 > 49(b-1)^3$, true for $b \geq 2$, and in (37) which becomes

$$(b-1)M^4 + (12b-16)M^3 + (36b-78)M^2 - 112M - 49 \geq 0,$$

which is true if $b \geq 2$.

12 Proof of Theorem 43

It follows from formula (32) in the proof of Theorem 41 that:

$$b^{k-1} \leq (b-1)^3 k^2 M^2. \tag{38}$$

We prove by induction on the variable k that:

$$b^{k-1} > (b-1)^3 k^2 M^2 \text{ for } M \geq b^9, k \geq 3\lfloor \log_b M \rfloor + 1, b \geq 8, \tag{39}$$

which combined with (38) finishes the proof of Theorem 43.

We start showing by induction on M that:

$$M > (b-1)^3 b^3 (3 \log_b M + 1)^2 \text{for } M \geq b^9, b \geq 8. \tag{40}$$

If $M = b^9$ (40) becomes, after cancellations,

$$b^6 > 28^2 (b-1)^3 \tag{41}$$

which is true for $b \geq 8$.

We assume now that (40) is true for fixed M. We show that it is true for $M+1$. From the induction hypothesis one has that:

$$M + 1 > (b-1)^3 b^3 (3 \log_b M + 1)^2 + 1.$$

To finish the proof of (40) one still needs to check that:

$$(b-1)^3 b^3 (3 \log_b M + 1)^2 + 1 \geq (b-1)^3 b^3 \left(3 \log_b (M+1) + 1\right)^2,$$

which after algebraic manipulations becomes:

$$1 \geq b^3 (b-1)^3 (3 \log_b (M+1) - 3 \log_b M)(3 \log_b (M+1) M + 2). \tag{42}$$

Due to the Mean Value Theorem, (42) follows if we show that:

$$1 \geq 3b^3(b-1)^3 \cdot \frac{1}{M}\left(3\log_b(M^2+M)+2\right). \tag{43}$$

Consider the function $g(M) = \frac{1}{M}[3\log_b(M^2+M)+2]$, with derivative:

$$g'(M) = \frac{\frac{1}{\ln b} \cdot \frac{3}{M^2+M} \cdot (2M+1)M - (3\log_b(M^2+M)+2)}{M^2}.$$

For $M \geq b^9$ the first term in the denominator of $g'(M)$ is ≤ 6 and the second term is ≥ 30. We conclude that $g'(M)$ is negative and $g(M)$ is decreasing on the interval $[b^9, +\infty)$. The value of

$$3 \cdot b^3(b-1)^3 \cdot g(M) \tag{44}$$

for $M = b^9$ is larger than $\frac{114}{b^3}$, which shows that (43) is true if $b \geq 5$. Consequently (42) and (40) are true.

We start the proof of (39). In the first step $k = 3\lfloor \log_b M \rfloor + 1$. Equation (39) becomes

$$b^{3\lfloor \log_b M \rfloor} > (b-1)^3(3\lfloor \log_b M \rfloor + 1)^2 M^2. \tag{45}$$

Due to $\log_b M - 1 \leq \lfloor \log_b M \rfloor \leq \log_b M$, (45) follows if we prove that

$$b^{3(\log_b M - 1)} \geq (b-1)^3 M^2 (3\log_b M + 1)^2 \text{ for } M \geq b^9. \tag{46}$$

After algebraic manipulations (46) is exactly (40), so it is true. Now we show the general induction step for (39). Assume (39) valid for fixed M. Then one has that:

$$b^k \geq b \cdot b^{k-1} \geq b \cdot (b-1)^3 k^2 M^2.$$

To finish we still need to check that:

$$b \cdot (b-1)^3 k^2 M^2 \geq (b-1)^3 (k+1)^2 M^2,$$

which after simplifications becomes $bk^2 \geq k^2 + 2k + 1$ which is true for $k \geq 3$ and not needed for $k = 1$ or $k = 2$.

This ends the proof of the case $b \geq 8$. The proofs of the other cases are similar and the only significant changes appear in checking the equation (41) and checking that expression (44) is less than 1. Due to our assumptions the equation remains valid and the expression is less than 1.

13 Examples of ARH numbers

We list in Table 13 a sequence of small additive multipliers M and the corresponding ARH numbers N without zero digits, if any.

M	N
1	18, 99
2	12, 33, 66, 99
3	99
4	99
5	11,22,33,44,55,66,77,88,99
6	
7	747

Table 1: ARH numbers with multipliers 1, 2, 3, 4, 5, 7.

Theorem 37 shows that an ARH number with multiplier 6 has at most 8 digits. A computer search through all integers with at most 8 digits and all digits different from zero, shows that 6 is not an additive multiplier for numbers with all digits different from zero. If we allow for zero digits one finds that 909 is an ARH number with multiplier 6. A computer search through all integers with at most 11 digits shows that 9 is not an additive multiplier. These observations motivate Question 35.

We observe that certain ARH numbers, for example 99, have several additive multipliers, respectively $1, 2, 3, 4, 5$. We also observe that certain multipliers, for example 5, have associated several ARH numbers, respectively $11, 22, 33, 44, 55, 66, 77, 88, 99$. The last observation motivates the following definition and questions.

Definition 44. If M is an additive multiplier in a base b, we call the *multiplicity* of M the cardinality of the corresponding set of b-ARH numbers.

Question 45. If we fix the multiplicity and the base, is the set of additive multipliers that have that multiplicity infinite?

Question 46. If we fix the base, is the multiplicity of additive multipliers bounded?

We observe that the notion of b-ARH number is dependent on the base. For example, 12 is a b-ARH number when written in base 10, but it is not a b-ARH number when written in base 9. We observe that if N is a b-ARH number then $N \geq b$. If $N < b$ then base b representation of N has a single digit and there are no such b-ARH numbers. we also observe that $[1(0)^{\wedge k}]_b, k \geq 0$ b-ARH number for any even b with multiplier $[(b/2)^{\wedge k}]_b$.

Question 47. Does there exist a numerical representation that is a b-ARH number for any b?

14 Examples of MRH numbers

We list in Table 2 a sequence of small multiplicative multipliers M and the corresponding MRH numbers N, if any.

M	N
1	1, 18, 1458, 1729
2	2268, 736
3	
4	1944, 7744
5	71685

Table 2: MRH numbers with multipliers $1, 2, 3, 4, 5$.

Theorem 41 shows that a MRH number with multiplier 3 has at most 7 digits. A computer search through all integers with at most 7 digits shows that 3 is not a multiplicative multiplier. This motivates Question 36.

One can also arrange the data as in Tables 3 and 4, where we list multiplicative multipliers M and the corresponding MRH numbers N with k digits, for small values of k.

We observe from Tables 3 and 4 that certain MRH numbers, for example, 332424, 132192, and 3252312, have several multipliers (respectively $\{27, 38\}$, $\{12, 34\}$, $\{72, 82\}$). We also observe from Table 2 that certain multipliers, for example 4, have associated several MRH numbers, respectively $1944, 7744$. The last observation motivates the following definition and questions.

Definition 48. If M is a multiplicative multiplier in base b, we call the *multiplicity* of M the cardinality of the corresponding set of b-MRH numbers.

Question 49. If we fix the multiplicity and the base, is the set of multiplicative multipliers that have that multiplicity infinite?

Question 50. If we fix the base, is the multiplicity of multiplicative multipliers bounded?

We observe that the notion of b-MRH number is dependent on the base. For example, 81 is a b-MRH number when written in base 10 or 9, but it is not a b-MRH number when written in base 8.

Question 51. Does there exist a numerical representation, besides $[1(0)^{\wedge k}]_b, k \geq 0$, that is a b-MRH number for any b?

15 Conclusion

In this paper for any numeration base b we introduce two new classes of integers, b-ARH numbers and b-MRH numbers. They have properties that generalize a property of the taxicab number 1729. The second class is a subclass of the well studied class of b-Niven numbers. We ask several natural questions about these classes and answer some of them, Questions 5, 6, 7 and 8, partially. In particular, we show that the class of b-ARH numbers is infinite if b is even and that the class of b-MRH numbers is infinite if b is odd.

k	M	N
1	1	1
2	1	81
3	2	736
4	1	1458, 1729
	2	2268
	4	1944, 7744
5	5	71685
	7	23632
	8	94528
	9	42282
	14	51142
	23	78246
6	12	132192
	14	188356, 247324
	19	161595
	21	433755, 496692
	22	234256
	23	685584
	26	258778
	27	332424
	29	679354
	31	122512
	33	176418
	34	132192, 751842
	36	271188
	37	215821
	38	332424
	39	145314
	44	235224

Table 3: MRH numbers with $1, 2, 3, 4, 5, 6$ digits.

k	M	N
7	22	9379678
	28	6527836
	29	9253987
	32	2892672
	33	8673885
	34	7526716
	38	3773932, 6362226
	39	5673564
	41	2187391
	49	4274613, 8239644
	63	1821771
	72	7651584
	73	2895472
	82	7651584
	84	3252312
	89	7331464
8	37	13184839
	46	11361448
	48	14292288
	53	15437628
	61	15178752
	66	15995232
	68	11715516
	71	16746912
	74	12419568, 15478432
	75	19348875
	76	17433792
	77	19552995
	78	12661272, 22694256
	79	11437225
	86	21371688
	89	12918439

Table 4: MRH numbers with $7, 8$ digits.

Among the questions left open, the most intriguing for us is if the set of MRH numbers with all digits different from zero is infinite. One way to attack it is to find an infinite sequence of integers N such that $N = N^R$, which are divisible by $s(N^2)$ and for which N^2 has no digit equal to zero. Then their squares form an infinite sequence of MRH numbers with all digits different from zero. Our numerical data shows some examples of such integers in base 10:

- $N^2 = 188356 = 434^2, s(N^2) = 31|434$,
- $N^2 = 234256 = 484^2, s(N^2) = 22|484$,
- $N^2 = 685584 = 828^2, s(N^2) = 36|828$.

A solution having the numbers N with all digits different from zero answers the following question.

Question 52. Find an infinite sequence of integers N with all digits different from zero that are divisible by the sum of the digits of their squares.

An example of such number is 424242. Indeed:

$$179981274564 = 424242^2,$$

which has the sum of the digits 63 and $424242 = 63 \times 6784$.

Motivated by this example one may consider the sequence $((42)^{\wedge k})_{k\geq 1}$. We conjecture that a sub-sequence of this sequence gives a positive answer to Question 52. Due to some numerical experiments we are very confident in this conjecture. Nevertheless, we do not believe that solving it will give a positive answer to the question at the beginning of this section.

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