ON CERTAIN COMBINATORIAL EXPANSIONS OF THE LEGENDRE-STIRLING NUMBERS

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ABSTRACT. The Legendre-Stirling numbers of the second kind were introduced by Everitt et al. in the spectral theory of powers of the Legendre differential expressions. In this paper, we provide a combinatorial code for Legendre-Stirling set partitions. As an application, we obtain combinatorial expansions of the Legendre-Stirling numbers of both kinds. Moreover, we present grammatical descriptions of the Jacobi-Stirling numbers of both kinds.

Keywords: Legendre-Stirling numbers; Jacobi-Stirling numbers; Context-free grammars

1. Introduction

Let $\ell[y](t) = -(1-t^2)y''(t)+2ty'(t)$ be the Legendre differential operator. Then the Legendre polynomial $y(t) = P_n(t)$ is an eigenvector for the differential operator ℓ corresponding to $n(n+1)$, i.e., $\ell[y](t) = n(n+1)y(t)$. Following Everitt et al. [7], for $n \in \mathbb{N}$, the *Legendre-Stirling numbers* $\mathrm{LS}\,(n,k)$ *of the second kind* appeared originally as the coefficients in the expansion of the n-th composite power of ℓ, i.e.,

$$\ell^n[y](t) = \sum_{k=0}^{n}(-1)^k\mathrm{LS}\,(n,k)((1-t^2)^k y^{(k)}(t))^{(k)}.$$

For each $k \in \mathbb{N}$, Everitt et al. [7, Theorem 4.1)] obtained that

$$\prod_{r=1}^{k}\frac{1}{1-r(r+1)x} = \sum_{n=0}^{\infty}\mathrm{LS}\,(n,k)x^{n-k},\ \left(|x| \le \frac{1}{k(k+1)}\right), \tag{1}$$

$$\mathrm{LS}\,(n,k) = \sum_{r=0}^{k}(-1)^{r+k}\frac{(2r+1)(r^2+r)^n}{(r+k+1)!(k-r)!}.$$

According to [2, Theorem 5.4], the numbers $\mathrm{LS}\,(n,k)$ have the following horizontal generating function

$$x^n = \sum_{k=0}^{n}\mathrm{LS}\,(n,k)\prod_{i=0}^{k-1}(x-i(1+i)). \tag{2}$$

It follows from (2) that the numbers $\mathrm{LS}\,(n,k)$ satisfy the recurrence relation

$$\mathrm{LS}\,(n,k) = \mathrm{LS}\,(n-1,k-1) + k(k+1)\mathrm{LS}\,(n-1,k).$$

with the initial conditions $\mathrm{LS}\,(n,0) = \delta_{n,0}$ and $\mathrm{LS}\,(0,k) = \delta_{0,k}$, where $\delta_{i,j}$ is the Kronecker's symbol.

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By using (1), Andrews et al. [2, Theorem 5.2] derived that the numbers $\mathrm{LS}\,(n,k)$ satisfy the vertical recurrence relation

$$\mathrm{LS}\,(n,j)=\sum_{k=j}^{n}\mathrm{LS}\,(k-1,j-1)(j(j+1))^{n-k}.$$

A particular values of $\mathrm{LS}\,(n,k)$ is provided at the end of [3]:

$$\mathrm{LS}\,(n+1,n)=2\binom{n+2}{3}. \tag{3}$$

In [6, Eq. (19)], Egge found that

$$\mathrm{LS}\,(n+2,n)=40\binom{n+2}{6}+72\binom{n+2}{5}+36\binom{n+2}{4}+4\binom{n+2}{3}.$$

Using the triangular recurrence relation $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$, we get

$$\mathrm{LS}\,(n+2,n)=40\binom{n+3}{6}+32\binom{n+3}{5}+4\binom{n+3}{4}. \tag{4}$$

Egge [6, Theorem 3.1] showed that for $k\geq 0$, the quantity $\mathrm{LS}\,(n+k,n)$ is a polynomial of degree $3k$ in n with leading coefficient $\frac{1}{3^k k!}$.

This paper is a continuation of [6], and it is motivated by the following problem.

Problem 1. *Let k be a given nonnegative integer. Could the numbers $\mathrm{LS}\,(n+k,n)$ be expanded in the binomial basis?*

The paper is organized as follows. In Section 2, by introducing a combinatorial code for Legendre-Stirling set partitions, we give a solution of Problem 1. Moreover, we get a combinatorial expansion of the Legendre-Stirling numbers of the first kind. In Section 3, we present grammatical interpretations of Jacobi-Stirling numbers of both kinds.

2. Legendre-Stirling set partitions

The combinatorial interpretation of the Legendre-Stirling numbers $\mathrm{LS}\,(n,k)$ of the second kind was first given by Andrews and Littlejohn [3]. For $n\geq 1$, let M_n denote the multiset $\{1,\overline{1},2,\overline{2},\ldots,n,\overline{n}\}$, in which we have one unbarred copy and one barred copy of each integer i, where $1\leq i\leq n$. Throughout this paper, we always assume that the elements of M_n are ordered by

$$\overline{1}=1<\overline{2}=2<\cdots<\overline{n}=n.$$

A *Legendre-Stirling set partition* of M_n is a set partition of M_n with $k+1$ blocks $B_0,B_1,\ldots,B_k$ and with the following rules:

- (r_1) The 'zero box' B_0 is the only box that may be empty and it may not contain both copies of any number;
- (r_2) The 'nonzero boxes' $B_1,B_2,\ldots,B_k$ are indistinguishable and each is non-empty. For any $i\in[k]$, the box B_i contains both copies of its smallest element and does not contain both copies of any other number.

Let $\mathcal{LS}(n,k)$ denote the set of Legendre-Stirling set partitions of M_n with one zero box and k nonzero boxes. The *standard form* of an element of $\mathcal{LS}(n,k)$ is written as

$$\sigma = B_1 B_2 \cdots B_k B_0,$$

where B_0 is the zero box and the minima of B_i is less than that of B_j when $1 \le i < j \le k$. Clearly, the minima of B_1 are 1 and $\overline{1}$. Throughout this paper we always write $\sigma \in \mathcal{LS}(n,k)$ in the standard form. As usual, we let angle bracket symbol $< i, j, \ldots >$ and curly bracket symbol $\{k, \overline{k}, \ldots\}$ denote the zero box and nonzero box, respectively. In particular, let $<>$ denote the empty zero box. For example, $\{1, \overline{1}, 3\}\{2, \overline{2}\} < \overline{3} > \in \mathcal{LS}(3,2)$. A classical result of Andrews and Littlejohn [3, Theorem 2] says that

$$\mathrm{LS}\,(n,k) = \#\mathcal{LS}(n,k).$$

We now provide a combinatorial code for Legendre-Stirling partitions (CLS-sequence for short).

Definition 2. *We call $Y_n = (y_1, y_2, \ldots, y_n)$ a* CLS*-sequence of length n if $y_1 = X$ and*

$$y_{k+1} \in \{X, A_{i,j}, B_s, \overline{B}_s, 1 \le i, j, s \le n_x(Y_k), i \ne j\} \quad \text{for } k = 1, 2, \ldots, n-1,$$

where $n_x(Y_k)$ is the number of the symbol X in $Y_k = (y_1, y_2, \ldots, y_k)$.

For example, $(X, X, A_{1,2})$ is a CLS-sequence, while $(X, X, A_{1,2}, B_3)$ is not since $y_4 = B_3$ and $3 > n_x(Y_3) = 2$. Let $\mathcal{CLS}_n$ denote the set of CLS-sequences of length n.

The following lemma is a fundamental result.

Lemma 3. *For $n \ge 1$, we have* $\mathrm{LS}\,(n,k) = \#\{Y_n \in \mathcal{CLS}_n \mid n_x(Y_n) = k\}$.

Proof. Let

$$\mathcal{CLS}(n,k) = \{Y_n \in \mathcal{CLS}_n \mid n_x(Y_n) = k\}.$$

Now we start to construct a bijection, denoted by Φ, between $\mathcal{LS}(n,k)$ and $\mathcal{CLS}(n,k)$. When $n = 1$, we have $y_1 = X$. Set $\Phi(Y_1) = \{1, \overline{1}\} <>$. This gives a bijection from $\mathcal{CLS}(1,1)$ to $\mathcal{LS}(1,1)$. Let $n = m$. Suppose Φ is a bijection from $\mathcal{CLS}(n,k)$ to $\mathcal{CLS}(n,k)$ for all k. Consider the case $n = m+1$. Let

$$Y_{m+1} = (y_1, y_2, \ldots, y_m, y_{m+1}) \in \mathcal{CLS}_{m+1}.$$

Then $Y_m = (y_1, y_2, \ldots, y_m) \in \mathcal{CLS}(m,k)$ for some k. Assume $\Phi(Y_m) \in \mathcal{LS}(m,k)$. Consider the following three cases:

- (i) If $y_{m+1} = X$, then let $\Phi(Y_{m+1})$ be obtained from $\Phi(Y_m)$ by putting the box $\{m+1, \overline{m+1}\}$ just before the zero box. In this case, $\Phi(Y_{m+1}) \in \mathcal{LS}(m+1, k+1)$.
- (ii) If $y_{m+1} = A_{i,j}$, then let $\Phi(Y_{m+1})$ be obtained from $\Phi(Y_m)$ by inserting the entry $m+1$ to the ith nonzero box and inserting the entry $\overline{m+1}$ to the jth nonzero box. In this case, $\Phi(Y_{m+1}) \in \mathcal{LS}(m+1, k)$.
- (iii) If $y_{m+1} = B_s$ (resp. $y_{m+1} = \overline{B}_s$), then let $\Phi(Y_{m+1})$ be obtained from $\Phi(Y_m)$ by inserting the entry $m+1$ (resp. $\overline{m+1}$) to the sth nonzero box and inserting the entry $\overline{m+1}$ (resp. $m+1$) to the zero box. In this case, $\Phi(Y_{m+1}) \in \mathcal{LS}(m+1, k)$.

After the above step, it is clear that the obtained $\Phi(Y_{m+1})$ is in standard form. By induction, we see that Φ is the desired bijection from $\mathcal{CLS}(n,k)$ to $\mathcal{CLS}(n,k)$, which also gives a constructive proof of Lemma 3. $\square$

Example 4. *Let $Y_5 = (X, X, A_{2,1}, B_2, \overline{B}_1)$. The correspondence between Y_5 and $\Phi(Y_5)$ is built up as follows:*

$$\begin{aligned} X &\Leftrightarrow \{1,\overline{1}\} <>; \\ X &\Leftrightarrow \{1,\overline{1}\}\{2,\overline{2}\} <>; \\ A_{2,1} &\Leftrightarrow \{1,\overline{1},\overline{3}\}\{2,\overline{2},3\} <>; \\ B_2 &\Leftrightarrow \{1,\overline{1},\overline{3}\}\{2,\overline{2},3,4\} < \overline{4} >; \\ \overline{B}_1 &\Leftrightarrow \{1,\overline{1},\overline{3},\overline{5}\}\{2,\overline{2},3,4\} < \overline{4},5 > . \end{aligned}$$

As an application of the CLS-sequences, we present the following result.

Lemma 5. *Let k be a given positive integer. Then for $n \geq 1$, we have*

$$\mathrm{LS}\,(n+k,n) = 2^k \sum_{t_k=1}^{n} \binom{t_k+1}{n} \sum_{t_{k-1}=1}^{t_k} \binom{t_{k-1}+1}{2} \cdots \sum_{t_2=1}^{t_3} \binom{t_2+1}{2} \sum_{t_1=1}^{t_2} \binom{t_1+1}{2}. \tag{5}$$

Proof. It follows from Lemma 3 that

$$\mathrm{LS}\,(n+k,n) = \#\{Y_{n+k} \in \mathcal{CLS}_{n+k} \mid n_x(Y_{n+k}) = n\}.$$

Let $Y_{n+k} = y_1 y_2 \cdots y_{n+k}$ be a given element in $\mathcal{CLS}_{n+k}$. Since $n_x(Y_{n+k}) = n$, it is natural to assume that $y_i = X$ except $i = t_1+1, t_2+2, \cdots, t_k+k$. Let σ be the corresponding Legendre-Stirling partition of Y_{n+k}. For $1 \leq \ell \leq k$, consider the value of $y_{t_\ell+\ell}$. Note that the number of the symbol X before $y_{t_\ell+\ell}$ is t_ℓ. Let $\widehat{\sigma}$ be the corresponding Legendre-Stirling partition of $y_1 y_2 \cdots y_{t_\ell+\ell-1}$. Now we insert $y_{t_\ell+\ell}$. We distinguish two cases:

- (i) If $y_{t_\ell+\ell} = A_{i,j}$, then we should insert the entry $t_\ell+\ell$ to the ith nonzero box of $\widehat{\sigma}$ and insert $\overline{t_\ell+\ell}$ to the jth nonzero box. This gives $2\binom{t_\ell}{2}$ possibilities, since $1 \leq i,j \leq t_\ell$ and $i \neq j$.
- (ii) If $y_{t_\ell+\ell} = B_s$ (resp. $y_{t_\ell+\ell} = \overline{B}_s$), then we should insert the entry $t_\ell+\ell$ (resp. $\overline{t_\ell+\ell}$) to the sth nonzero box of $\widehat{\sigma}$ and insert $\overline{t_\ell+\ell}$ (resp. $t_\ell+\ell$) to the zero box. This gives $2\binom{t_\ell}{1}$ possibilities, since $1 \leq s \leq t_\ell$.

Therefore, there are exactly $2\binom{t_\ell}{2} + 2\binom{t_\ell}{1} = 2\binom{t_\ell+1}{2}$ Legendre-Stirling partitions of $\mathrm{M}_{\,t_\ell+\ell}$ can be generated from $\widehat{\sigma}$ by inserting the entry $y_{t_\ell+\ell}$. Note that $1 \leq t_{j-1} \leq t_j \leq n$ for $2 \leq j \leq k$. Applying the product rule for counting, we immediately get (5). $\square$

The following simple result will be used in our discussion.

Lemma 6. *Let a and b be given integers. Then*

$$\binom{x-b}{2}\binom{x}{a} = \binom{a+2}{2}\binom{x}{a+2} + (a+1)(a-b)\binom{x}{a+1} + \binom{a-b}{2}\binom{x}{a}.$$

In particular,

$$\binom{x-1}{2}\binom{x}{a} = \binom{a+2}{2}\binom{x}{a+2} + (a^2-1)\binom{x}{a+1} + \binom{a-1}{2}\binom{x}{a}.$$

Proof. Note that

$$\binom{a+2}{2}\frac{(x-a)(x-a-1)}{(a+2)(a+1)} + (a+1)(a-b)\frac{x-a}{a+1} + \binom{a-b}{2} = \binom{x-b}{2}.$$

This yields the desired result. □

We can now conclude the main result of this paper from the discussion above.

Theorem 7. *Let k be a given nonnegative integer. Then for $n \geq 1$, the numbers* $\mathrm{LS}\,(n+k,n)$ *can be expanded in the binomial basis as*

$$\mathrm{LS}\,(n+k,n) = 2^k \sum_{i=k+2}^{3k} \gamma(k,i)\binom{n+k+1}{i}, \tag{6}$$

where the coefficients $\gamma(k,i)$ are all positive integers for $k+2 \leq i \leq 3k$ and satisfy the recurrence relation

$$\gamma(k+1,i) = \binom{i-k-1}{2}\gamma(k,i-1) + (i-1)(i-k-2)\gamma(k,i-2) + \binom{i-1}{2}\gamma(k,i-3), \tag{7}$$

with the initial conditions $\gamma(0,0)=1$, $\gamma(0,i)=\gamma(i,0)=0$ for $i \neq 0$. Let $\gamma_k(x) = \sum_{i=k+2}^{3k} \gamma(k,i)x^i$. Then the polynomials $\gamma_k(x)$ satisfy the recurrence relation

$$\gamma_{k+1}(x) = \left(\frac{k(k+1)}{2} - kx + x^2\right)x\gamma_k(x) - (k+(k-2)x-2x^2)x^2\gamma_k'(x) + \frac{(1+x)^2x^3}{2}\gamma_k''(x), \tag{8}$$

with the initial conditions $\gamma_0(x)=1, \gamma_1(x)=x^3$ and $\gamma_2(x)=x^4+8x^5+10x^6$.

Proof. We prove (6) by induction on k. It is clear that

$$\mathrm{LS}\,(n,n) = 1 = \binom{n+1}{0}.$$

When $k=1$, by using the *Chu Shih-Chieh's identity*

$$\binom{n+1}{k+1} = \sum_{i=k}^{n}\binom{i}{k},$$

we obtain

$$\sum_{t_1=1}^{n}\binom{t_1+1}{2} = \binom{n+2}{3},$$

and so (3) is established. When $k=2$, it follows from Lemma 5 that

$$\begin{aligned}\mathrm{LS}\,(n+2,n) &= 4\sum_{t_2=1}^{n}\binom{t_2+1}{2}\sum_{t_1=1}^{t_2}\binom{t_1+1}{2}\\ &= 4\sum_{t_2=1}^{n}\binom{t_2+1}{2}\binom{t_2+2}{3}.\end{aligned}$$

Setting $x = t_2 + 2$ and $a = 3$ in Lemma 6, we get

$$\begin{aligned}\mathrm{LS}\,(n+2,n) &= 4\sum_{t_2=1}^{n}\left(10\binom{t_2+2}{5}+8\binom{t_2+2}{4}+\binom{t_2+2}{3}\right)\\ &= 4\left(10\binom{n+3}{6}+8\binom{n+3}{5}+\binom{n+3}{4}\right),\end{aligned}$$

which yields (4). Along the same lines, it is not hard to verify that

$$\begin{aligned}\mathrm{LS}\,(n+3,n) &= 8\sum_{t_3=1}^{n}\binom{t_3+1}{2}\left(10\binom{t_3+3}{6}+8\binom{t_3+3}{5}+\binom{t_3+3}{4}\right)\\ &= 8\left(280\binom{n+4}{9}+448\binom{n+4}{8}+219\binom{n+4}{7}+34\binom{n+4}{6}+\binom{n+4}{5}\right).\end{aligned}$$

Hence the formula (6) holds for $k = 0, 1, 2, 3$, so we proceed to the inductive step. For $k \geq 3$, assume that

$$\mathrm{LS}\,(n+k,n) = 2^k\sum_{i=k+2}^{3k}\gamma(k,i)\binom{n+k+1}{i}.$$

It follows from Lemma 5 that

$$\mathrm{LS}\,(n+k+1,n) = 2^{k+1}\sum_{t_{k+1}=1}^{n}\binom{t_{k+1}+1}{2}\sum_{i=k+2}^{3k}\gamma(k,i)\binom{t_{k+1}+k+1}{i}$$

By using Lemma 6, it is routine to verify that the coefficients $\gamma(k,i)$ satisfy the recurrence relation (7), and so (6) is established for general k. Multiplying both sides of (7) by x^i and summing for all i, we immediately get (8). □

In [2], Andrews et al. introduced the *(unsigned) Legendre-Stirling numbers* $\mathrm{Lc}\,(n,k)$ *of the first kind*, which may be defined by the recurrence relation

$$\mathrm{Lc}\,(n,k) = \mathrm{Lc}\,(n-1,k-1) + n(n-1)\mathrm{Lc}\,(n-1,k),$$

with the initial conditions $\mathrm{Lc}\,(n,0) = \delta_{n,0}$ and $\mathrm{Lc}\,(0,n) = \delta_{0,n}$. Let $f_k(n) = \mathrm{LS}\,(n+k,n)$. According to Egge [6, Eq. (23)], we have

$$\mathrm{Lc}\,(n-1,n-k-1) = (-1)^k f_k(-n) \tag{9}$$

for $k \geq 0$. For $m, k \in \mathbb{N}$, we define

$$\binom{-m}{k} = \frac{(-m)(-m-1)\cdots(-m-k+1)}{k!}.$$

Combining (6) and (9), we immediately get the following result.

Corollary 8. *Let k be a given nonnegative integer. For $n \geq 1$, the numbers* $\mathrm{Lc}\,(n-1,n-k-1)$ *can be expanded in the binomial basis as*

$$\mathrm{Lc}\,(n-1,n-k-1) = (-1)^k 2^k\sum_{i=k+2}^{3k}\gamma(k,i)\binom{-n+k+1}{i}, \tag{10}$$

where the coefficients $\gamma(k,i)$ are defined by (7).

It follows from (8) that

$$\gamma(k+1,k+3)=\left(\frac{k(k+1)}{2}-k(k+2)+\frac{(k+2)(k+1)}{2}\right)\gamma(k,k+2),$$
$$\gamma(k+1,3k+3)=\left(1+6k+\frac{3k(3k-1)}{2}\right)\gamma(k,3k),$$
$$\gamma_{k+1}(-1)=-\left(\frac{k(k+1)}{2}+k+1\right)\gamma_k(-1).$$

Since $\gamma(1,3)=1$ and $\gamma_1(-1)=-1$, it is easy to verify that for $k\geq 1$, we have

$$\gamma(k,k+2)=1,\ \gamma(k,3k)=\frac{(3k)!}{k!(3!)^k},\ \gamma_k(-1)=(-1)^k\frac{(k+1)!k!}{2^k}.$$

It should be noted that the number $\gamma(k,3k)$ is the number of partitions of $\{1,2,\ldots,3k\}$ into blocks of size 3 (see [18, A025035]), and the number $\frac{(k+1)!k!}{2^k}$ is the product of first k positive triangular numbers (see [18, A006472]). Moreover, if the number $\mathrm{LS}\,(n+k,n)$ is viewed as a polynomial in n, then its degree is $3k$, which is implied by the quantity $\binom{n+k+1}{3k}$. Furthermore, the leading coefficient of $\mathrm{LS}\,(n+k,n)$ is given by

$$2^k\gamma(k,3k)\frac{1}{(3k)!}=2^k\frac{(3k)!}{k!(3!)^k}\frac{1}{(3k)!}=\frac{1}{k!3^k},$$

which yields [6, Theorem 3.1].

3. Grammatical interpretations of Jacobi-Stirling numbers of both kinds

In this section, a context-free grammar is in the sense of Chen [4]: for an alphabet A, let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in A. A context-free grammar over A is a function $G: A\rightarrow\mathbb{Q}[[A]]$ that replace a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G. More precisely, the derivative $D=D_G$: $\mathbb{Q}[[A]]\rightarrow\mathbb{Q}[[A]]$ is defined as follows: for $x\in A$, we have $D(x)=G(x)$; for a monomial u in $\mathbb{Q}[[A]]$, $D(u)$ is defined so that D is a derivation, and for a general element $q\in\mathbb{Q}[[A]]$, $D(q)$ is defined by linearity. The reader is referred to [5, 14] for recent progress on this subject.

Let $[n]=\{1,2,\ldots,n\}$. The Stirling number $\left\{{n\atop k}\right\}$ of the second kind is the number of ways to partition $[n]$ into k blocks. Chen [4, Eq. 4.8] showed that if $G=\{x\rightarrow xy,y\rightarrow y\}$, then

$$D^n(x)=x\sum_{k=0}^{n}\left\{{n\atop k}\right\}y^k.$$

Let $\mathfrak{S}_n$ be the symmetric group of all permutations of $[n]$. Let $\mathrm{cyc}\,(\pi)$ be the number of cycles of π. The (unsigned) *Stirling number of the first kind* is defined by

$$\left[{n\atop k}\right]=\#\{\pi\in\mathfrak{S}_n\mid \mathrm{cyc}\,(\pi)=k\}.$$

From [13, Eq. 4.8], we see that if $G=\{x\rightarrow xy,y\rightarrow yz,z\rightarrow z^2\}$, then

$$D^n(x)=x\sum_{k=0}^{n}\left[{n\atop k}\right]y^kz^{n-k}.$$

According to [8, Theorem 4.1], the *Jacobi-Stirling number* $\mathrm{JS}_{n}^{k}(z)$ *of the second kind* is defined by

$$x^n = \sum_{k=0}^{n} \mathrm{JS}_{n}^{k}(z) \prod_{i=0}^{k-1} (x - i(z+i)). \tag{11}$$

It follows from (11) that the numbers $\mathrm{JS}_{n}^{k}(z)$ satisfy the recurrence relation

$$\mathrm{JS}_{n}^{k}(z) = \mathrm{JS}_{n-1}^{k-1}(z) + k(k+z)\mathrm{JS}_{n-1}^{k}(z),$$

with the initial conditions $\mathrm{JS}_{n}^{0}(z) = \delta_{n,0}$ and $\mathrm{JS}_{0}^{k}(z) = \delta_{0,k}$. It is clear that $\mathrm{JS}_{n}^{k}(1) = \mathrm{LS}\,(n,k)$. Following [10, Eq. (1.3), Eq. (1.5)], *the (unsigned) Jacobi-Stirling number* $\mathrm{Jc}_{n}^{k}(z)$ *of the first kind* is defined by

$$\prod_{i=0}^{n-1} (x + i(z+i)) = \sum_{k=0}^{n} \mathrm{Jc}_{n}^{k}(z) x^k,$$

and the numbers $\mathrm{Jc}_{n}^{k}(z)$ satisfy the following recurrence relation

$$\mathrm{Jc}_{n}^{k}(z) = \mathrm{Jc}_{n-1}^{k-1}(z) + (n-1)(n-1+z)\mathrm{Jc}_{n-1}^{k}(z),$$

with the initial conditions $\mathrm{Jc}_{n}^{0}(z) = \delta_{n,0}$ and $\mathrm{Jc}_{0}^{k}(z) = \delta_{k,0}$. In particular, $\mathrm{Jc}_{n}^{k}(1) = \mathrm{Lc}\,(n,k)$.

Properties and combinatorial interpretations of the Jacobi-Stirling numbers of both kinds were extensively studied in [1, 10, 11, 12, 15, 16, 17]. The Jacobi-Stirling numbers share many similar properties to those of the Stirling numbers. A question arises immediately: are there grammatical descriptions of the Jacobi-Stirling numbers of both kinds? In this section, we give the answer.

As a variant of the CLS-sequence, we now introduce a marked scheme for Legendre-Stirling partitions. Given a Legendre-Stirling partition $\sigma = B_1B_2\cdots B_kB_0 \in \mathcal{LS}(n,k)$, where B_0 is the zero box of σ. We mark the box vector $(B_1, B_2, \ldots, B_k)$ by the label a_k. We mark any box pair (B_i, B_j) by a label b and mark any box pair (B_s, B_0) by a label c, where $1 \leq i < j \leq k$ and $1 \leq s \leq k$. Let σ' denote the Legendre-Stirling partition that generated from σ by inserting $n+1$ and $\overline{n+1}$. If $n+1$ and $\overline{n+1}$ are in the same box, then

$$\sigma' = B_1B_2\cdots B_kB_{k+1}B_0,$$

where $B_{k+1} = \{n+1, \overline{n+1}\}$. This case corresponds to the operator $a_k \to a_{k+1}b^kc$. If $n+1$ and $\overline{n+1}$ are in different boxes, then we distinguish two cases:

- (i) Given a box pair (B_i, B_j), where $1 \leq i < j \leq k$. We can put $n+1$ (resp. $\overline{n+1}$) into the box B_i and put $\overline{n+1}$ (resp. $n+1$) into the box B_j. This case corresponds to the operator $b \to 2b$.
- (ii) Given a box pair (B_i, B_0), where $1 \leq i \leq k$. We can put $n+1$ (resp. $\overline{n+1}$) into the box B_i and put $\overline{n+1}$ (resp. $n+1$) into the zero box B_0. Moreover, we mark any barred entry in the zero box B_0 by a label z. This case corresponds to the operator $c \to (1+z)c$.

Let $A = \{a_0, a_1, a_2, a_3, \ldots, b, c\}$ be a set of alphabet. Using the above marked scheme, it is natural to consider the following grammars:

$$G_k = \{a_0 \to a_1c, a_1 \to a_2bc, \ldots, a_{k-1} \to a_kb^{k-1}c, b \to 2b, c \to (1+z)c\} \tag{12}$$

where $k \geq 1$.

Theorem 9. *Let G_k be the grammars defined by* (12)*. Then we have*

$$D_nD_{n-1}\cdots D_1(a_0)=\sum_{k=1}^{n}\mathrm{JS}\,{}_n^k(z)a_kb^{\binom{k}{2}}c^k.$$

Proof. Note that $D_1(a_0)=a_1c$ and $D_2D_1(a_0)=a_2bc^2+(1+z)a_1c$. Thus the result holds for $n=1,2$. For $m\geq 2$, we define $P_m^k(z)$ by

$$D_mD_{m-1}\cdots D_1(a_0)=\sum_{k=1}^{n}P_m^k(z)a_kb^{\binom{k}{2}}c^k.$$

We proceed by induction. Consider the case $n=m+1$. Since

$$D_{m+1}D_mD_{m-1}\cdots D_1(a_0)=D_{m+1}(D_mD_{m-1}\cdots D_1(a_0)),$$

it follows that

$$\begin{aligned}D_{m+1}D_m\cdots D_1(a_0)&=D_{m+1}\left(\sum_{k=1}^{n}P_m^k(z)a_kb^{\binom{k}{2}}c^k\right)\\&=\sum_{k=1}^{n}P_m^k(z)\left(a_{k+1}b^{\binom{k+2}{2}}c^{k+1}+k(k-1)a_kb^{\binom{k}{2}}c^k+(1+z)ka_kb^{\binom{k}{2}}c^k\right).\end{aligned}$$

Therefore, we obtain $P_{m+1}^k(z)=P_m^{k-1}(z)+k(k+z)P_m^k(z)$. Since the numbers $P_n^k(z)$ and $\mathrm{JS}\,{}_n^k(z)$ satisfy the same recurrence relation and initial conditions, so they agree. □

Combining the marked scheme for Legendre-Stirling partitions and Theorem 9, it is clear that for $n\geq k$, the number $\mathrm{JS}\,{}_n^k(z)$ is a polynomial of degree $n-k$ in z, and the coefficient z^i of $\mathrm{JS}\,{}_n^k(z)$ is the number of Legendre-Stirling partitions in $\mathcal{LS}(n,k)$ with exactly i barred entries in the zero box, which gives a proof of [10, Theorem 2].

We end this section by giving the following result.

Theorem 10. *Let $A=\{a,b_0,b_1,\ldots\}$ be a set of alphabet. Let G_k be the grammars defined by*

$$G_k=\{a\to(k-1)(k-1+z)a,b_0\to b_1,b_1\to b_2,\ldots,b_{k-1}\to b_k\},$$

where $k\geq 1$. Then we have

$$D_nD_{n-1}\cdots D_1(ab_0)=a\sum_{k=1}^{n}\mathrm{Jc}\,{}_n^k(z)b_k.$$

Proof. Note that $D_1(ab_0)=ab_1$ and $D_2D_1(ab_0)=(1+z)ab_1+ab_2$. Hence the result holds for $n=1,2$. For $m\geq 2$, we define $Q_m^k(z)$ by $D_mD_{m-1}\cdots D_1(ab_0)=a\sum_{k=1}^{m}Q_m^k(z)b_k$. We proceed by induction. Consider the case $n=m+1$. Since

$$D_{m+1}D_mD_{m-1}\cdots D_1(ab_0)=D_{m+1}(D_mD_{m-1}\cdots D_1(ab_0)),$$

it follows that

$$\begin{aligned}D_{m+1}D_m\cdots D_1(ab_0)&=D_{m+1}\left(a\sum_{k=1}^{m}Q_m^k(z)b_k\right)\\&=a\sum_{k=1}^{m}Q_m^k(z)m(m+z)b_k+a\sum_{k=1}^{m}Q_m^kb_{k+1}.\end{aligned}$$

Therefore, we obtain $Q_{m+1}^k(z) = Q_m^{k-1}(z) + m(m+z)Q_m^k(z)$. Since the numbers $Q_n^k(z)$ and $\mathrm{Jc}\,_n^k(z)$ satisfy the same recurrence relation and initial conditions, so they agree. □

4. Concluding remarks

Note that the Jacobi-Stirling numbers are polynomial refinements of the Legendre-Stirling numbers. It would be interesting to explore combinatorial expansions of Jacobi-Stirling numbers of both kinds.

Let $\gamma_k(x)$ be the polynomials defined by (8). We end our paper by proposing the following.

Conjecture 11. *For any $k \geq 1$, the polynomial $\gamma_k(x)$ has only real zeros. Set*

$$\gamma_k(x) = \gamma(k, 3k)x^{k+2}\prod_{i=1}^{2k-2}(x - r_i),\ \ \gamma_{k+1}(x) = \gamma(k+1, 3k+3)x^{k+3}\prod_{i=1}^{2k}(x - s_i),$$

where $r_{2k-2} < r_{2k-3} < \cdots < r_2 < r_1$ and $s_{2k} < s_{2k-1} < s_{2k-2} < \cdots < s_2 < s_1$. Then

$$s_{2k} < r_{2k-2} < s_{2k-1} < r_{2k-3} < s_{2k-2} < \cdots < r_k < s_{k+1} < s_k < r_{k-1} < \cdots < s_2 < r_1 < s_1,$$

in which the zeros s_{k+1} and s_k of $\gamma_{k+1}(x)$ are continuous appearance, and the other zeros of $\gamma_{k+1}(x)$ separate the zeros of $\gamma_k(x)$.

References

[1] G.E. Andrews, E.S Egge, W. Gawronski, L.L. Littlejohn, The Jacobi-Stirling numbers, *J. Combin. Theory Ser. A*, 120(1) (2013), 288–303.

[2] G.E. Andrews, W. Gawronski, L.L. Littlejohn, The Legendre-Stirling numbers, *Discrete Math.*, 311 (2011), 1255–1272.

[3] G.E. Andrews, L.L. Littlejohn, A combinatorial interpretation of the Legendre-Stirling numbers, *Proc. Amer. Math. Soc.*, 137 (2009), 2581–2590.

[4] W.Y.C. Chen, Context-free grammars, differential operators and formal power series, *Theoret. Comput. Sci.*, 117 (1993), 113–129.

[5] W.Y.C. Chen, A.M. Fu, Context-free grammars for permutations and increasing trees, *Adv. in Appl. Math.*, 82 (2017), 58–82.

[6] E.S. Egge, Legendre-Stirling permutations, *European J. Combin.*, 31(7) (2010), 1735–1750.

[7] W.N. Everitt, L.L. Littlejohn, R. Wellman, Legendre polynomials, Legendre-Stirling numbers, and the left-definite analysis of the Legendre differential expression, *J. Comput. Appl. Math.*, 148 (1) (2002), 213–238.

[8] W.N. Everitt, K.H. Kwon, L.L. Littlejohn, R. Wellman, G.J. Yoon, Jacobi-Stirling numbers, Jacobi polynomials, and the left-definite analysis of the classical Jacobi differential expression, *J. Comput. Appl. Math.*, 208 (2007), 29–56.

[9] W. Gawronski, L.L. Littlejohn, T. Neuschel, On the asymptotic normality of the Legendre-Stirling numbers of the second kind, *European J. Combin.*, 49 (2015), 218–231.

[10] Y. Gelineau, J. Zeng, Combinatorial interpretations of the Jacobi-Stirling numbers, *Electron. J. Combin.*, 17 (2010), #R70.

[11] I.M. Gessel, Z. Lin, J. Zeng, Jacobi-Stirling polynomials and *P*-partitions, *European J. Combin.*, 33 (2012), 1987–2000.

[12] Z. Lin, J. Zeng, Positivity properties of Jacobi-Stirling numbers and generalized Ramanujan polynomials, *Adv. in Appl. Math.*, 53 (2014), 12–27.

[13] S.-M. Ma, Some combinatorial arrays generated by context-free grammars, *European J. Combin.*, 34 (2013), 1081–1091.

[14] S.-M. Ma, J. Ma, Y.-N. Yeh, B.-X. Zhu, Context-free grammars for several polynomials associated with Eulerian polynomials, *Electron. J. Combin.*, 25(1) (2018), #P1.31.

[15] M. Merca, A connection between Jacobi-Stirling numbers and Bernoulli polynomials, J. Number Theory, 151(2015), 223–229.

[16] P. Mongelli, Combinatorial interpretations of particular evaluations of complete and elementary symmetric functions, *Electron. J. Combin.*, 19 (2012), #P60.

[17] P. Mongelli, Total positivity properties of Jacobi-Stirling numbers, *Adv. in Appl. Math.*, 48 (2012), 354–364.

[18] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2010.

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