On Calculating the Coefficients of a Polynomial Generated Sequence Using the Worpitzky Number Triangles

John K. Sikora

sikorajk@corning.com

June 5, 2018

Abstract

In this paper we show that two of the Worpitzky Number Triangles, OEIS A028246 and A019538, may each be used in look-up table fashion, along with specific diagonals of a polynomial sequence's difference triangle to easily solve for the unknown coefficients of the sequence. This is accomplished by using a method that isolates each of the coefficients as a single unknown in a series of simple linear equations. The method is first applied to a sequence generated using integer indexes with a starting index of 0, using the Alternate Worpitzky Number Triangle, A019538. Although the numbers in A019538 are less commonly referred to as a Worpitzky Number Triangle, a justification for such a reference is briefly presented. Next, the method is applied to a sequence generated using integer indexes with a starting index of 1, using the Mirrored Worpitzky Number Triangle, A028246. The method is then extended to solve for the coefficients of a sequence generated with input data consisting of an arbitrary starting number and an arbitrary constant differential.

1 Introduction

It is well known that the first step in using what is perhaps the most popular method to find the unknown generating polynomial of a sequence is to calculate the difference triangle of the sequence by subtracting successive terms of the sequence. The next row in the difference triangle is formed in a like manner, by subtracting successive terms. If the dth row of the difference triangle, as defined below in Definition 4, stays constant for a sufficient number of terms, then it is known that that the generating sequence is a polynomial of degree d.

Once it is established that the sequence is generated via a polynomial of degree d, there are various methods that may be used to obtain a formula for the generating polynomial. Such methods include generating and solving linear equations for the coefficients of each power in the polynomial, or using Newton's Divided Difference Formula.

We present what we believe is an easy and newly described approach of formulating the unknown polynomial using a Worpitzky Number Triangle and the main diagonal of the difference triangle of the sequence, given, or assuming that the sequence is generated with input data consisting of integers starting at either 0 or 1. We then show how the method may be generalized to allow it to be used on a sequence with input data consisting of an arbitrary starting number and an arbitrary constant differential.

2 Outline

The remainder of this paper consists of a number of sections. Following this outline, in the next section the definitions, notations, and the specific formulas that are used in the paper are given. In

1

addition, references to alternate versions of the Worpitzky Number Triangles are presented.

Next, three examples are given, with only the practical calculations shown. We feel that this is desirable in order to show the ease of using the method. The three examples show the calculations for a polynomial sequence generated with input data consisting of integers starting at 0, then for a sequence generated with input data consisting of integers starting at 1, and finally for a sequence generated with input data starting at 3.3 with an increment of 0.1.

After that, in the next section we provide the mathematical basis for the method, and in the following two sections we first present the the full rendition of one of the examples, and then a partial rendition of another example. We feel that this will make the mathematics behind the method more apparent. In the final section, we present our closing remarks.

3 Definitions, Notation, and Existing Terminology

Definition 1: Mirrored Worpitzky Number Triangle or $MWNT(n, k)$ – The triangle formed from the numbers, n and k, in OEIS [1] A028246 [2] as shown in [2, Example] and in Table 1. One formula for the numbers in the Mirrored Worpitzky Number Triangle is $(k-1)! \cdot S(n, k)$, where $S(n, k)$ is the Stirling Number of the Second Kind [18] [10]. A triangle of these numbers is given in OEIS A008277 [6].

The specific formula for $MWNT(n, k)$ used in this paper, equivalent to the one given above, is:

$$MWNT(n, k) = \frac{1}{k} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n \ ; \ n \geq 1, \ k \geq 1 \tag{3.1}$$

The mirror image of A028246 was recently referred to as the Worpitzky Number Triangle by Vandervelde [8], and we yield to that reference by using the term "Mirrored" in our definition. The referenced triangle may be found on the OEIS as A130850 [5]. However, it should be noted that in OEIS A028246, A130850 is referred to as "The mirror image of the Worpitzky triangle" [2, Comments].

In addition, what we refer to as the Mirrored Worpitzky Number Triangle (A028246) appears elsewhere in the OEIS, such as in OEIS sequence A005460 [4, Links] ([7] provides a direct link). A005460 is described [4, Comments] as: "third external diagonal of Worpitzky triangle A028246".[1] Obviously, the use of the term Worpitzky Number Triangle (or similar) varies.

Table 1: The Mirrored Worpitzky Number Triangle, OEIS A028246, with zeros for $n < k$

n \ k	1	2	3	4	5	6	7	8	9
1	1	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0
3	1	3	2	0	0	0	0	0	0
4	1	7	12	6	0	0	0	0	0
5	1	15	50	60	24	0	0	0	0
6	1	31	180	390	360	120	0	0	0
7	1	63	602	2100	3360	2520	720	0	0
8	1	127	1932	10206	25200	31920	20160	5040	0
9	1	255	6050	46620	166824	317520	332640	181440	40320

---

[1]Although we had figured out that the first diagonal in what we would eventually call the Mirrored Worpitzky Number Triangle, $MWNT(n, k)$, is $(n-1)!$, and that the second diagonal is $n!/2$, we were perplexed about the third diagonal which is 1, 7, 50, 390, 3360, etc. A search on the OEIS turned up A005460, which referenced A028246, and all of the succeeding diagonals that we checked matched. Since these numbers were readily available on the OEIS in look up table form, we decided to write this paper.

Definition 2: Alternate Worpitzky Number Triangle or $\boldsymbol{AWNT(n,k)}$ – The triangle formed from the numbers, n and k, in OEIS A019538 [3] as shown in [3, Example] and in Table 2. Perhaps providing justification for referring to this triangle as a Worpitzky Number Triangle comes from Gould and Quaintance, (2016) [13, Equation 11.3], who provide an equation for Worpitzky Numbers in general. A specific case is mentioned [17] which results in the numbers in the Alternate Worpitzky Number Triangle, with a formula given as as $k! \cdot S(n,k)$.
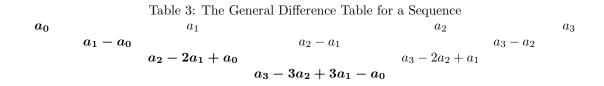
The specific formula for $AWNT(n,k)$ used in this paper, equivalent to the one given above, is:

$$AWNT(n,k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n \; ; \; n \geq 1, \; k \geq 1 \tag{3.2}$$

Table 2: The Alternate Worpitzky Number Triangle, OEIS A019538, with zeros for $n < k$

n \ k	1	2	3	4	5	6	7	8	9
1	1	0	0	0	0	0	0	0	0
2	1	2	0	0	0	0	0	0	0
3	1	6	6	0	0	0	0	0	0
4	1	14	36	24	0	0	0	0	0
5	1	30	150	240	120	0	0	0	0
6	1	62	540	1560	1800	720	0	0	0
7	1	126	1806	8400	16800	15120	5040	0	0
8	1	254	5796	40824	126000	191520	141120	40320	0
9	1	510	18150	186480	834120	1905120	2328480	1451520	362880

Definition 3: Polynomial Sequence – A sequence, a_i, a_{i+1}, a_{i+2}, etc., generated by the polynomial of finite degree d and written in long form as $c_d x^d + c_{d-1} x^{d-1} + c_{d-2} x^{d-2} + \cdots + c_1 x + c_0$. In this paper the more compact form, $\sum_{j=0}^{d} c_j x^j$ will primarily be used. The x values may be integers or real numbers with an arbitrary starting value and an arbitrary constant differential.

Definition 4: The Difference Triangle and the Main Diagonal – The difference triangle of a sequence is the triangle formed by subtracting the preceding element of a sequence from the current element, and continuing this process for successive rows. An example is shown in Table 3. Note that the row containing the sequence values, a_0, a_1, a_2, etc. is row number 0, and the succeeding rows are numbered $1, 2, 3$, etc. The left-most diagonal is shown in bold and is known as the main diagonal.

Table 3: The General Difference Table for a Sequence

$\boldsymbol{a_0}$ $\quad\quad\quad\quad a_1 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad a_2 \quad\quad\quad\quad\quad a_3$

$\boldsymbol{a_1 - a_0} \quad\quad\quad\quad\quad a_2 - a_1 \quad\quad\quad\quad a_3 - a_2$

$\boldsymbol{a_2 - 2a_1 + a_0} \quad\quad\quad\quad a_3 - 2a_2 + a_1$

$\boldsymbol{a_3 - 3a_2 + 3a_1 - a_0}$

Definition 5: $\boldsymbol{0^0 = 1}$ – More specifically, $x^0 = 1$ for all x, per Graham, Knuth, and Patashnik, (1994) [9]. This is common when using binomials, and it allows for the $c_0 x^0$ term to be c_0 when $x = 0$ (as is necessary) in the compact formula for the polynomial sequence given in Definition 3.
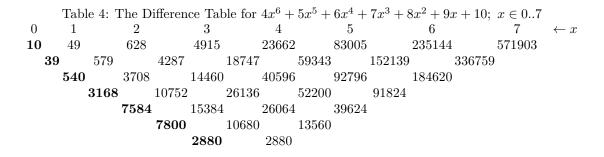
4 Examples of Using the Method

The following examples provide a brief description of how to use the method.

***Example* 1** – We first consider the sequence generated by the polynomial:

$$4x^6 + 5x^5 + 6x^4 + 7x^3 + 8x^2 + 9x + 10; \ x \in 0..7$$

The difference table for this sequence is given in Table 4, with the main diagonal in bold. The sequence values are in row 0 in accordance with Definition 4. In this example it is either given or assumed that the the integers starting with 0 were used to generate the sequence. Therefore, we will use the Alternate Worpitzky Number Triangle and the main diagonal values to easily solve for the "unknown" coefficients.

Table 4: The Difference Table for $4x^6 + 5x^5 + 6x^4 + 7x^3 + 8x^2 + 9x + 10; \ x \in 0..7$

0	1	2	3	4	5	6	7	$\leftarrow x$
10	49	628	4915	23662	83005	235144	571903	
	39	579	4287	18747	59343	152139	336759	
		540	3708	14460	40596	92796	184620	
			3168	10752	26136	52200	91824	
				7584	15384	26064	39624	
					7800	10680	13560	
						2880	2880	

Since the integers that generate the sequence start at 0, we know that $c_0 = 10$, as 10 is the initial value in row 0. Row 6 of the difference triangle is constant, which, of course, matches the degree, d, of the polynomial. Therefore, we start at $n, k = d = 6$ in Table 2, and use column k to calculate $c_{n=k}$ of the polynomial using the entries in the table as multipliers for each $c_n, n \in 1..6$. The c_n values are multiplied by the values in row n of Table 2 for that column. As can be seen in the calculations below, this process turns the rows of the table into columns, and the columns of the table into rows, when presented in the manner shown.

We will move backwards along the columns, k, in turn calculating the $c_{n=k}$ values as we go. The reasons for these steps and the mathematical relationships between the coefficients and the multiplier values in the table will be shown in Section 5.2. The multipliers (elements of the table) appear in parenthesis below, with multipliers of 0 not shown. Thus, we have:

$$
\begin{aligned}
2880 &= c_6(720) & &&&&&&&&& \Rightarrow \ c_6 &= 4 \\
7800 &= 4(1800) &+ \ c_5(120) & &&&&&&&& \Rightarrow \ c_5 &= 5 \\
7584 &= 4(1560) &+ \ 5(240) &+ \ c_4(24) &&&&&&&& \Rightarrow \ c_4 &= 6 \\
3168 &= 4(540) &+ \ 5(150) &+ \ 6(36) &+ \ c_3(6) &&&&&& \Rightarrow \ c_3 &= 7 \\
540 &= 4(62) &+ \ 5(30) &+ \ 6(14) &+ \ 7(6) &+ \ c_2(2) &&&& \Rightarrow \ c_2 &= 8 \\
39 &= 4(1) &+ \ 5(1) &+ \ 6(1) &+ \ 7(1) &+ \ 8(1) &+ \ c_1(1) &\Rightarrow \ c_1 &= 9
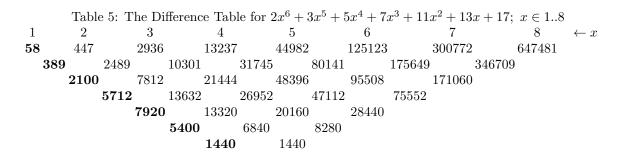\end{aligned}
$$

Since we already know that $c_0 = 10$, we have the complete solution for Example 1.

***Example* 2** – In this example, we show how to directly calculate the coefficients of a polynomial sequence given that the integers starting with 1 (instead of 0) are used to generate the sequence. We will use the values in the Mirrored Worpitzky Number Triangle (instead of the Alternate Worpitzky Number Triangle) and in the main diagonal of the difference triangle.

We could have repeated Example 1 using the next diagonal (adjacent to the main diagonal) of Table 4, but we elect to use a different sequence to add more variety to the examples. We consider

the sequence generated by the polynomial:

$$2x^6 + 3x^5 + 5x^4 + 7x^3 + 11x^2 + 13x + 17; \ x \in 1..8$$

The difference table for this sequence is given in Table 5, with the main diagonal in bold. Again, the sequence values are in row 0 in accordance with Definition 4.

Table 5: The Difference Table for $2x^6 + 3x^5 + 5x^4 + 7x^3 + 11x^2 + 13x + 17; \ x \in 1..8$

1	2	3	4	5	6	7	8	$\leftarrow x$
58	447	2936	13237	44982	125123	300772	647481	
	389	2489	10301	31745	80141	175649	346709	
		2100	7812	21444	48396	95508	171060	
			5712	13632	26952	47112	75552	
				7920	13320	20160	28440	
					5400	6840	8280	
						1440	1440	

Row 6 of the difference triangle is constant, again matching the degree, d, of the polynomial. We start at $n, k = d+1 = 7$ in Table 1, and use column k to calculate $c_{n-1=k-1}$ of the polynomial using the entries in the table as multipliers for each $c_{n-1}, n \in 1..7$. The c_{n-1} values are multiplied by the values in row n of Table 1 for that column. Again, we will move backwards along the columns, calculating the $c_{n-1=k-1}$ values as we go. The reasons for these steps and the mathematical relationships between the coefficients and the multiplier values in the table will be shown in Section 5.3. The multipliers (elements of the table) appear in parenthesis below, with multipliers of 0 not shown. Thus, we have:

$$
\begin{aligned}
1440 &= c_6(720) & \Rightarrow c_6 &= 2 \\
5400 &= 2(2520) + c_5(120) & \Rightarrow c_5 &= 3 \\
7920 &= 2(3360) + 3(360) + c_4(24) & \Rightarrow c_4 &= 5 \\
5712 &= 2(2100) + 3(390) + 5(60) + c_3(6) & \Rightarrow c_3 &= 7 \\
2100 &= 2(602) + 3(180) + 5(50) + 7(12) + c_2(2) & \Rightarrow c_2 &= 11 \\
389 &= 2(63) + 3(31) + 5(15) + 7(7) + 11(3) + c_1(1) & \Rightarrow c_1 &= 13 \\
58 &= 2(1) + 3(1) + 5(1) + 7(1) + 11(1) + 13(1) + c_0(1) & \Rightarrow c_0 &= 17
\end{aligned}
$$

Example 3 – In this example, we show how to extend the method to polynomial sequences generated with integers not starting at either 0 or 1, or with input data with non-unity differentials. Since it is probably easiest to start with an integer index of 0 rather than 1 for the extension of the method, we will assign a function relating the input data to the integers $0, 1, 2$, etc., and we will use the values in the Alternate Worpitzky Number Triangle as in Example 1. We consider the sequence generated by the polynomial:

$$3x^5 + 1x^4 + 4x^3 + 1x^2 + 5x + 9; \ x \in 3.3, 3.4..3.9$$

The difference table for this sequence and input data is given in Table 6, with the main diagonal in bold. Again, the sequence values are in row 0 in accordance with Definition 4. This row appears directly below the integers, starting at 0, that we have calculated and assigned to the input data entries using the equation:

$$g(x) = 10.0(x - 3.3) \tag{4.1}$$

This allows us to use the method using the Alternate Worpitzky Number Triangle.

Table 6: The Difference Table for $3x^5 + 1x^4 + 4x^3 + 1x^2 + 5x + 9$; $x \in 3.3, 3.4..3.9$

3.3	3.4	3.5	3.6	3.7	3.8	3.9	$\leftarrow x$
0	1	2	3	4	5	6	$\leftarrow g(x)$
1472.79189	1691.47232	1935.96875	2208.53088	2511.53681	2847.49664	3219.05607	
	218.68043	244.49643	272.56213	303.00593	335.95983	371.55943	
		25.816	28.0657	30.4438	32.9539	35.5996	
			2.2497	2.3781	2.5101	2.6457	
				0.1284	0.132	0.1356	
					0.0036	0.0036	

Row 5 of the difference triangle is constant, matching the degree, d, of the polynomial as expected. We start at $n, k = d = 5$ in Table 2, and use column k to calculate $c_{n=k}$ of the polynomial using the entries in the table as multipliers for each $c_n, n \in 1..5$. By inspecting the first element in row 0 of Table 6, we see that c_0 is 1472.79189. Furthermore, we have:

$$
\begin{array}{lllllllll}
0.0036 & = & c_5(120) & & & & & \Rightarrow c_5 & = 0.00003 \\
0.1284 & = & 0.00003(240) & + & c_4(24) & & & \Rightarrow c_4 & = 0.00505 \\
2.2497 & = & 0.00003(150) & + & 0.00505(36) & + & c_3(6) & \Rightarrow c_3 & = 0.3439 \\
25.816 & = & 0.00003(30) & + & 0.00505(14) & + & 0.3439(6) + c_2(2) & \Rightarrow c_2 & = 11.8405 \\
218.68043 & = & 0.00003(1) & + & 0.00505(1) & + & 0.3439(1) + 11.8405(1) + c_1(1) & \Rightarrow c_1 & = 206.49095
\end{array}
$$

The value of the sequence for other input values of x may be calculated using these coefficients with an input value of $g(x)$ as given in Equation (4.1). Alternatively, we could calculate the coefficients for use with x directly, as opposed to $g(x)$, by symbolic evaluation of:

$$0.00003(g(x))^5 + 0.00505(g(x))^4 + 0.3439(g(x))^3 + 11.8405(g(x))^2 + 206.49095(g(x)) + 1472.79189$$

which simplifies to:

$$3x^5 + 1x^4 + 4x^3 + 1x^2 + 5x + 9$$

Obviously, the same method may be used for integer generated sequences with a starting integer value other than 0 or 1 by making a substitution of $g(x) = x - y$, with y as the appropriate integer. The value of y will depend upon whether the Mirrored Worpitzky Number Triangle or the Alternate Worpitzky Number Triangle was used in the calculation, and upon the starting integer value of the sequence.

5 Mathematical Basis

5.1 A Formula for the Main Diagonal of the Difference Triangle of a Polynomial Sequence

A polynomial sequence, given by a_i, a_{i+1}, a_{i+2}, etc., has a difference triangle as defined in Definition 4. A general example showing a_0, a_1, a_2, and a_3 was given in Table 3. It is known [16] [11] that the kth term of the main diagonal of the sequence of the difference triangle is:

$$D_k = \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} a_i \qquad k \geq 0$$

This may be proved via induction, or by the method given by Graham, et al., (1994) [12]. If we multiply by $(-1)^{-2i} = 1$, for each i in turn, we get:

$$D_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} a_i \qquad k \geq 0 \tag{5.1}$$

5.2 Using the Main Diagonal with the Alternate Worpitzky Number Triangle

5.2.1 Mathematical Derivation:

In this section we derive the expressions linking the main diagonal of the difference triangle of a polynomial sequence and the polynomial's coefficient multipliers to the equation for the Alternate Worpitzky Number Triangle, $AWNT(n, k)$. This leads to the method of solving for the polynomial's unknown coefficients as shown in Example 1.

First, since the starting integer used to generate the polynomial is $x = 0$, it is obvious that $c_0 = a_0$. Furthermore, recalling Equation (5.1), the equation for the kth term of the main diagonal, and substituting $a_i = \sum_{n=0}^{d} c_n i^n$, yields:

$$D0_k = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\sum_{n=0}^{d}c_n i^n \qquad k \geq 0$$

where d is the degree of the polynomial.

Since both sums have a finite number of terms, and due to the commutative property of multiplication and addition, and the distributive property of multiplication over addition, we may rearrange the above equation into (c_0 is left out as it is already known from the first value in the main diagonal):

$$D0_k = \sum_{n=1}^{d} c_n \underbrace{\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n}_{AWNT(n,k)} \qquad n \geq 1, \ k \geq 1 \qquad (5.2)$$

This equation shows that that the kth main diagonal element value for $k \geq 1$, is composed of c_n, $n \in 1..d$, multiplied by $\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n$. This is column k, with corresponding row, n, in A019538, and is seen by comparison of Equation (5.2) with Equation (3.2).

5.2.2 Solution Procedure using $AWNT(n, k)$, A019538:

Therefore, to find the unknown coefficients of a polynomial sequence generated with integers starting at 0, first construct the difference triangle until the elements of a row are all constant. The degree, d, of the polynomial is the row number, as defined in Definition 4, of the difference triangle with the constant values. Then start with column $k = d$ and row $n = d$ in A019538, and for each $c_n, n \in 1..d$, assign the value of $AWNT(n, k)$ as a multiplier to c_n (moving up the column is perhaps easiest) and equate it to the main diagonal value for row $k = n = d$ in the difference triangle. $AWNT(n, d)$ will have multipliers of 0 for all c_n except for c_d, allowing for easy calculation of c_d.

Next, move back to column $k = d-1$ and starting at row $n = d$ assign the value of multipliers to each c_n, again moving up the column. Since the value of c_d is known, equating the coefficients and multipliers to the main diagonal value in row $d - 1$ will leave c_{d-1} as the only unknown. Continue this process backwards to column $k = 1$ to solve for the coefficients down to c_1. The value of c_0 is equal to the value in row 0 of the difference triangle (the first term of the sequence), and the solution is complete. See Example 1 for a worked example using this procedure.

5.3 Using the Main Diagonal with the Mirrored Worpitzky Number Triangle

5.3.1 Mathematical Derivation:

In this section we derive the expressions linking the main diagonal of the difference triangle of a polynomial sequence and the polynomial's coefficient multipliers to the equation for the Mirrored

Worpitzky Number Triangle, $MWNT(n, k)$. This leads to the method of solving for the polynomial's unknown coefficients as shown in Example 2. In this case, the starting integer used to generate the polynomial is $x = 1$, and we conveniently refer to the terms of the sequence as a_1, a_2, a_3, etc. If we look at the mth term of the main diagonal from Equation (5.1), we get:

$$D1_m = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} a_{j+1} \qquad m \geq 0$$

where:

$$a_{j+1} = \sum_{q=0}^{d} c_q (j+1)^q$$

and d is the degree of the polynomial, as before. Substituting, we get:

$$D1_m = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \sum_{q=0}^{d} c_q (j+1)^q \qquad m \geq 0$$

Since both sums have a finite number of terms, and due to the commutative property of multiplication and addition, and the distributive property of multiplication over addition, we may rearrange the above equation into:

$$D1_m = \sum_{q=0}^{d} c_q \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (j+1)^q \qquad \text{Let } i = j+1 \Rightarrow j = i-1:$$

$$D1_m = \sum_{q=0}^{d} c_q \sum_{i=1}^{m+1} (-1)^{m-i+1} \binom{m}{i-1} i^q \qquad \text{Let } k = m+1 \Rightarrow m = k-1:$$

$$D1_{k-1} = \sum_{q=0}^{d} c_q \sum_{i=1}^{k} (-1)^{k-i} \binom{k-1}{i-1} i^q \qquad k \geq 1 \qquad (5.3)$$

We now need to show that the following relationship involving the right hand sum of Equation (5.3) is valid:

$$\sum_{i=1}^{k} (-1)^{k-i} \binom{k-1}{i-1} i^q = \frac{1}{k} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^{q+1} \qquad q \geq 0, \; k \geq 1 \qquad (5.4)$$

First, on the right hand side, the $i = 0$ term is 0 since $0^{q+1} = 0$. The rest of the terms, with $i \geq 1$, are equal on a term by term basis, shown as follows:

$$(-1)^{k-i}\binom{k-1}{i-1}i^q \overset{?}{=} \frac{1}{k}(-1)^{k-i}\binom{k}{i}i^{q+1}$$

$$\binom{k-1}{i-1}i^q \overset{?}{=} \frac{1}{k}\binom{k}{i}i^{q+1}$$

$$\frac{(k-1)!\ i^q}{(k-1-(i-1))!\ (i-1)!} \overset{?}{=} \frac{1}{k}\frac{k!\ i^{q+1}}{(k-i)!\ i!}$$

$$\frac{(k-1)!\ i^q}{(k-i)!\ (i-1)!} \overset{?}{=} \frac{(k-1)!\ i^{q+1}}{(k-i)!\ i!}$$

$$\frac{i^q}{(i-1)!} \overset{?}{=} \frac{i^q\ i}{i!}$$

$$\frac{i^q}{(i-1)!} \overset{\checkmark}{=} \frac{i^q}{(i-1)!}$$

which confirms the relationship (again, with $i \geq 1$). We now substitute the right side expression of Equation (5.4) for the right side sum of Equation (5.3) and get:

$$D1_{k-1} = \sum_{q=0}^{d} c_q\ \frac{1}{k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^{q+1} \qquad q \geq 0,\ k \geq 1 \qquad \text{Let } n = q+1 \Rightarrow q = n-1$$

$$D1_{k-1} = \sum_{n=1}^{d+1} c_{n-1}\ \underbrace{\frac{1}{k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^{n}}_{MWNT(n,k)} \qquad n \geq 1,\ k \geq 1 \tag{5.5}$$

This equation shows that for $k \geq 1$, the $(k-1)^{th}$ main diagonal element value is composed of c_{n-1}, $n \in 1..(d+1)$, multiplied by $\frac{1}{k}\sum_{i=0}^{k}(-1)^{(k-i)}\binom{k}{i}i^n$. This is column, k, with corresponding row, n, in A028246, as seen by comparison of Equation (5.5) with Equation (3.1).

5.3.2   Solution Procedure using $MWNT(n,k)$, A028246:

Therefore, to find the unknown coefficients of a polynomial sequence generated with integers starting at 1, first construct the difference triangle until the elements of a row are all constant. The degree, d, of the polynomial is the row number, as defined in Definition 4, of the difference triangle with the constant values. Then start with column $k = d+1$ and row $n = d+1$ in A028246, and for each $c_{n-1}, n \in 1..(d+1)$, assign the value of $MWNT(n,k)$ as a multiplier to c_{n-1} (moving up the column) and equate it to the main diagonal value for row $k-1 = n-1 = d$ in the difference triangle. $MWNT(n, d+1)$ will have multipliers of 0 for all c_{n-1} except for c_d, allowing for easy calculation of c_d.

Next, move back to column $k = d$ and starting at row $n = d+1$ assign the value of multipliers to each c_{n-1}, again moving up the column. Since the value of c_d is known, equating the coefficients and multipliers to the main diagonal value in row $d-1$ will leave c_{d-1} as the only unknown. Continue this process backwards to column $k = 1$ to solve for the coefficients down to c_0, and the solution is complete. See Example 2 for a worked example using this procedure.

5.4   Gaining Insight Using Euler's Finite Difference Theorem

Euler's Finite Difference Theorem as presented by Gould and Quaintance, (2016) [14] states that given $f(x) = \sum_{j=0}^{d} c_j x^j$ then:

$$\sum_{i=0}^{k}(-1)^i \binom{k}{i} f(i) = \begin{cases} 0, & 0 \le d < k \\ (-1)^k k! \ c_k, & d = k \end{cases}$$

The authors use Euler's Finite Difference Theorem and let:

$$f(x) = (z - bx)^n \ \Rightarrow \ f(i) = (z - bi)^n, \ n \in \mathbb{Z}_{\ge 0}$$

to derive the following equation [15]:

$$\sum_{i=0}^{k}(-1)^i \binom{k}{i} (z - bi)^n = \begin{cases} 0, & n < k \\ b^k \ k!, & n = k \end{cases} \tag{5.6}$$

By setting $b = -1$ and $z = 0$ we can derive Equation (5.7) below as follows:

$$\sum_{i=0}^{k}(-1)^i \binom{k}{i} i^n = \begin{cases} 0, & n < k \\ (-1)^k k!, & n = k \end{cases}$$

If we multiply each side by $(-1)^k$, we get:

$$\sum_{i=0}^{k}(-1)^{k+i} \binom{k}{i} i^n = \begin{cases} 0, & n < k \\ (-1)^{2k} k!, & n = k \end{cases}$$

If we multiply the left side by $(-1)^{-2i} = 1$, for each i in turn, and since $(-1)^{2k} = 1$, we get:

$$\sum_{i=0}^{k}(-1)^{k-i} \binom{k}{i} i^n = \begin{cases} 0, & n < k \\ k!, & n = k \end{cases} \tag{5.7}$$

We can use Equation (5.6) and Equation (5.7) to gain insight into the structure of the contents of the Mirrored Worpitzky Number Triangle (Table 1), the Alternate Worpitzky Number Triangle (Table 2) and the difference triangle of a sequence. Equation (5.7) shows why the factorials of numbers appear on the right diagonal of both triangles, starting from 0! in Table 1, and from 1! in Table 2. It also explains the zeros in the tables when $n < k$, and the rows of 0 that one would get by continuing the difference triangle past the constant row, as $n < k$.

Equation (5.6) may be used to explain the constant row of the difference triangle. In that equation, z may be taken to be any number (not just 0), so along with having $b = -1$, this explains why successive terms in the row are all equal (and related to the factorial), and that the method of isolating the coefficients could be used with any row of the difference triangle because the coefficient multipliers would also remain 0 for $n < k$.[2] However, different tables of numbers would need to be used for the multipliers (the factorials and zeros would still be in place), or the multipliers could just be calculated from the terms in Equation (5.6) with the appropriate values for z and b – see Section 6 for the terms used in Example 1 with A019538 ($z = 0, \ b = -1$).

6 Example 1 Revisited

In this section, we will show Example 1 in full, per Equation (5.2) with all terms shown. Along with Equation (5.7), this will hopefully provide a more complete view of how the method works. So we have, with the binomial coefficients shown in bold:

---

[2] A multiplication by $(-1)^k$ is assumed as was done in proceeding from Equation (5.6) to Equation (5.7) in order to match the form of the diagonal terms where the multipliers of the $i = k$ term are positive (so that the factorials are all positive).

$$\begin{aligned}
2880 &= c_6 \cdot (\mathbf{1} \cdot 6^6 - \mathbf{6} \cdot 5^6 + \mathbf{15} \cdot 4^6 - \mathbf{20} \cdot 3^6 + \mathbf{15} \cdot 2^6 - \mathbf{6} \cdot 1^6 + \mathbf{1} \cdot 0^6) = c_6 \cdot AWNT(6,6) = c_6 \cdot 720 \\
&+ c_5 \cdot (\mathbf{1} \cdot 6^5 - \mathbf{6} \cdot 5^5 + \mathbf{15} \cdot 4^5 - \mathbf{20} \cdot 3^5 + \mathbf{15} \cdot 2^5 - \mathbf{6} \cdot 1^5 + \mathbf{1} \cdot 0^5) = c_5 \cdot AWNT(5,6) = c_5 \cdot 0 \\
&+ c_4 \cdot (\mathbf{1} \cdot 6^4 - \mathbf{6} \cdot 5^4 + \mathbf{15} \cdot 4^4 - \mathbf{20} \cdot 3^4 + \mathbf{15} \cdot 2^4 - \mathbf{6} \cdot 1^4 + \mathbf{1} \cdot 0^4) = c_4 \cdot AWNT(4,6) = c_4 \cdot 0 \\
&+ c_3 \cdot (\mathbf{1} \cdot 6^3 - \mathbf{6} \cdot 5^3 + \mathbf{15} \cdot 4^3 - \mathbf{20} \cdot 3^3 + \mathbf{15} \cdot 2^3 - \mathbf{6} \cdot 1^3 + \mathbf{1} \cdot 0^3) = c_3 \cdot AWNT(3,6) = c_3 \cdot 0 \\
&+ c_2 \cdot (\mathbf{1} \cdot 6^2 - \mathbf{6} \cdot 5^2 + \mathbf{15} \cdot 4^2 - \mathbf{20} \cdot 3^2 + \mathbf{15} \cdot 2^2 - \mathbf{6} \cdot 1^2 + \mathbf{1} \cdot 0^2) = c_2 \cdot AWNT(2,6) = c_2 \cdot 0 \\
&+ c_1 \cdot (\mathbf{1} \cdot 6^1 - \mathbf{6} \cdot 5^1 + \mathbf{15} \cdot 4^1 - \mathbf{20} \cdot 3^1 + \mathbf{15} \cdot 2^1 - \mathbf{6} \cdot 1^1 + \mathbf{1} \cdot 0^1) = c_1 \cdot AWNT(1,6) = c_1 \cdot 0
\end{aligned}$$

$c_6 \Rightarrow 4$

$$\begin{aligned}
7800 &= 4 \cdot (\mathbf{1} \cdot 5^6 - \mathbf{5} \cdot 4^6 + \mathbf{10} \cdot 3^6 - \mathbf{10} \cdot 2^6 + \mathbf{5} \cdot 1^6 - \mathbf{1} \cdot 0^6) &&= 4 \cdot AWNT(6,5) = 4 \cdot 1800 \\
&+ c_5 \cdot (\mathbf{1} \cdot 5^5 - \mathbf{5} \cdot 4^5 + \mathbf{10} \cdot 3^5 - \mathbf{10} \cdot 2^5 + \mathbf{5} \cdot 1^5 - \mathbf{1} \cdot 0^5) &&= c_5 \cdot AWNT(5,5) = c_5 \cdot 120 \\
&+ c_4 \cdot (\mathbf{1} \cdot 5^4 - \mathbf{5} \cdot 4^4 + \mathbf{10} \cdot 3^4 - \mathbf{10} \cdot 2^4 + \mathbf{5} \cdot 1^4 - \mathbf{1} \cdot 0^4) &&= c_4 \cdot AWNT(4,5) = c_4 \cdot 0 \\
&+ c_3 \cdot (\mathbf{1} \cdot 5^3 - \mathbf{5} \cdot 4^3 + \mathbf{10} \cdot 3^3 - \mathbf{10} \cdot 2^3 + \mathbf{5} \cdot 1^3 - \mathbf{1} \cdot 0^3) &&= c_3 \cdot AWNT(3,5) = c_3 \cdot 0 \\
&+ c_2 \cdot (\mathbf{1} \cdot 5^2 - \mathbf{5} \cdot 4^2 + \mathbf{10} \cdot 3^2 - \mathbf{10} \cdot 2^2 + \mathbf{5} \cdot 1^2 - \mathbf{1} \cdot 0^2) &&= c_2 \cdot AWNT(2,5) = c_2 \cdot 0 \\
&+ c_1 \cdot (\mathbf{1} \cdot 5^1 - \mathbf{5} \cdot 4^1 + \mathbf{10} \cdot 3^1 - \mathbf{10} \cdot 2^1 + \mathbf{5} \cdot 1^1 - \mathbf{1} \cdot 0^1) &&= c_1 \cdot AWNT(1,5) = c_1 \cdot 0
\end{aligned}$$

$c_5 \Rightarrow 5$

$$\begin{aligned}
7584 &= 4 \cdot (\mathbf{1} \cdot 4^6 - \mathbf{4} \cdot 3^6 + \mathbf{6} \cdot 2^6 - \mathbf{4} \cdot 1^6 + \mathbf{1} \cdot 0^6) &&= 4 \cdot AWNT(6,4) = 4 \cdot 1560 \\
&+ 5 \cdot (\mathbf{1} \cdot 4^5 - \mathbf{4} \cdot 3^5 + \mathbf{6} \cdot 2^5 - \mathbf{4} \cdot 1^5 + \mathbf{1} \cdot 0^5) &&= 5 \cdot AWNT(5,4) = 5 \cdot 240 \\
&+ c_4 \cdot (\mathbf{1} \cdot 4^4 - \mathbf{4} \cdot 3^4 + \mathbf{6} \cdot 2^4 - \mathbf{4} \cdot 1^4 + \mathbf{1} \cdot 0^4) &&= c_4 \cdot AWNT(4,4) = c_4 \cdot 24 \\
&+ c_3 \cdot (\mathbf{1} \cdot 4^3 - \mathbf{4} \cdot 3^3 + \mathbf{6} \cdot 2^3 - \mathbf{4} \cdot 1^3 + \mathbf{1} \cdot 0^3) &&= c_3 \cdot AWNT(3,4) = c_3 \cdot 0 \\
&+ c_2 \cdot (\mathbf{1} \cdot 4^2 - \mathbf{4} \cdot 3^2 + \mathbf{6} \cdot 2^2 - \mathbf{4} \cdot 1^2 + \mathbf{1} \cdot 0^2) &&= c_2 \cdot AWNT(2,4) = c_2 \cdot 0 \\
&+ c_1 \cdot (\mathbf{1} \cdot 4^1 - \mathbf{4} \cdot 3^1 + \mathbf{6} \cdot 2^1 - \mathbf{4} \cdot 1^1 + \mathbf{1} \cdot 0^1) &&= c_1 \cdot AWNT(1,4) = c_1 \cdot 0
\end{aligned}$$

$c_4 \Rightarrow 6$

$$\begin{aligned}
3168 &= 4 \cdot (\mathbf{1} \cdot 3^6 - \mathbf{3} \cdot 2^6 + \mathbf{3} \cdot 1^6 - \mathbf{1} \cdot 0^6) &&= 4 \cdot AWNT(6,3) = 4 \cdot 540 \\
&+ 5 \cdot (\mathbf{1} \cdot 3^5 - \mathbf{3} \cdot 2^5 + \mathbf{3} \cdot 1^5 - \mathbf{1} \cdot 0^5) &&= 5 \cdot AWNT(5,3) = 5 \cdot 150 \\
&+ 6 \cdot (\mathbf{1} \cdot 3^4 - \mathbf{3} \cdot 2^4 + \mathbf{3} \cdot 1^4 - \mathbf{1} \cdot 0^4) &&= 6 \cdot AWNT(4,3) = 6 \cdot 36 \\
&+ c_3 \cdot (\mathbf{1} \cdot 3^3 - \mathbf{3} \cdot 2^3 + \mathbf{3} \cdot 1^3 - \mathbf{1} \cdot 0^3) &&= c_3 \cdot AWNT(3,3) = c_3 \cdot 6 \\
&+ c_2 \cdot (\mathbf{1} \cdot 3^2 - \mathbf{3} \cdot 2^2 + \mathbf{3} \cdot 1^2 - \mathbf{1} \cdot 0^2) &&= c_2 \cdot AWNT(2,3) = c_2 \cdot 0 \\
&+ c_1 \cdot (\mathbf{1} \cdot 3^1 - \mathbf{3} \cdot 2^1 + \mathbf{3} \cdot 1^1 - \mathbf{1} \cdot 0^1) &&= c_1 \cdot AWNT(1,3) = c_1 \cdot 0
\end{aligned}$$

$c_3 \Rightarrow 7$

$$\begin{aligned}
540 &= 4 \cdot (\mathbf{1} \cdot 2^6 - \mathbf{2} \cdot 1^6 + \mathbf{1} \cdot 0^6) &&= 4 \cdot AWNT(6,2) = 4 \cdot 62 \\
&+ 5 \cdot (\mathbf{1} \cdot 2^5 - \mathbf{2} \cdot 1^5 + \mathbf{1} \cdot 0^5) &&= 5 \cdot AWNT(5,2) = 5 \cdot 30 \\
&+ 6 \cdot (\mathbf{1} \cdot 2^4 - \mathbf{2} \cdot 1^4 + \mathbf{1} \cdot 0^4) &&= 6 \cdot AWNT(4,2) = 6 \cdot 14 \\
&+ 7 \cdot (\mathbf{1} \cdot 2^3 - \mathbf{2} \cdot 1^3 + \mathbf{1} \cdot 0^3) &&= 7 \cdot AWNT(3,2) = 7 \cdot 6 \\
&+ c_2 \cdot (\mathbf{1} \cdot 2^2 - \mathbf{2} \cdot 1^2 + \mathbf{1} \cdot 0^2) &&= c_2 \cdot AWNT(2,2) = c_2 \cdot 2 \\
&+ c_1 \cdot (\mathbf{1} \cdot 2^1 - \mathbf{2} \cdot 1^1 + \mathbf{1} \cdot 0^1) &&= c_1 \cdot AWNT(1,2) = c_1 \cdot 0
\end{aligned}$$

$c_2 \Rightarrow 8$

$$\begin{aligned}
39 &= 4 \cdot (\mathbf{1} \cdot 1^6 - \mathbf{1} \cdot 0^6) &&= 4 \cdot AWNT(6,1) = 4 \cdot 1 \\
&+ 5 \cdot (\mathbf{1} \cdot 1^5 - \mathbf{1} \cdot 0^5) &&= 5 \cdot AWNT(5,1) = 5 \cdot 1 \\
&+ 6 \cdot (\mathbf{1} \cdot 1^4 - \mathbf{1} \cdot 0^4) &&= 6 \cdot AWNT(4,1) = 6 \cdot 1 \\
&+ 7 \cdot (\mathbf{1} \cdot 1^3 - \mathbf{1} \cdot 0^3) &&= 7 \cdot AWNT(3,1) = 7 \cdot 1 \\
&+ 8 \cdot (\mathbf{1} \cdot 1^2 - \mathbf{1} \cdot 0^2) &&= 8 \cdot AWNT(2,1) = 8 \cdot 1 \\
&+ c_1 \cdot (\mathbf{1} \cdot 1^1 - \mathbf{1} \cdot 0^1) &&= c_1 \cdot AWNT(1,1) = c_1 \cdot 1
\end{aligned}$$

$c_1 \Rightarrow 9$

We know that $c_0 = 10$ for the reasons stated before in Example 1. This completes the solution.

Further notes: Obviously, c_0 is absent from any row number greater than 0 in the difference triangle as it is subtracted out from each term in row 1. It should also be noted that c_1 will be absent from any row number greater than 1 in the difference triangle because in row 1 there will be a difference of $c_1 \cdot 1x$, which will subtracted out from each term in row 2. This is not the case for the higher order terms.

7 Example 2 (Partially) Revisited

In this section, we will show a partial working of Example 2. Although only partial, this presentation of the solution will hopefully clarify how the method works by going into more detail in the areas that are covered, especially when compared to the full solution to Example 1 given above. Our focus will be on Equation (5.3) and Equation (5.5), for the calculation of c_6.

Equation (5.3) gives the expression derived for the main diagonal values and the coefficients per Equation (5.1), given that the starting integer used to generate the sequence is $x = 1$. Since $k = 7$, and with q taken appropriately for each coefficient, we have, (partially):

$$
\begin{aligned}
1440 = & c_6 \cdot (\mathbf{1} \cdot 7^6 - \mathbf{6} \cdot 6^6 + \mathbf{15} \cdot 5^6 - \mathbf{20} \cdot 4^6 + \mathbf{15} \cdot 3^6 - \mathbf{6} \cdot 2^6 + \mathbf{1} \cdot 1^6) = c_6 \cdot 720 \\
& + c_5 \cdot (\mathbf{1} \cdot 7^5 - \mathbf{6} \cdot 6^5 + \mathbf{15} \cdot 5^5 - \mathbf{20} \cdot 4^5 + \mathbf{15} \cdot 3^5 - \mathbf{6} \cdot 2^5 + \mathbf{1} \cdot 1^5) = c_5 \cdot 0 \\
& \vdots
\end{aligned}
$$

Equation (5.5), with $n = q + 1$ taken appropriately for each coefficient, gives (partially):

$$
\begin{aligned}
1440 = & c_6 \cdot \tfrac{1}{7}(\mathbf{1} \cdot 7^7 - \mathbf{7} \cdot 6^7 + \mathbf{21} \cdot 5^7 - \mathbf{35} \cdot 4^7 + \mathbf{35} \cdot 3^7 - \mathbf{21} \cdot 2^7 + \mathbf{7} \cdot 1^7 - \mathbf{1} \cdot 0^7) = c_6 \cdot MWNT(7,7) = c_6 \cdot 720 \\
& + c_5 \cdot \tfrac{1}{7}(\mathbf{1} \cdot 7^6 - \mathbf{7} \cdot 6^6 + \mathbf{21} \cdot 5^6 - \mathbf{35} \cdot 4^6 + \mathbf{35} \cdot 3^6 - \mathbf{21} \cdot 2^6 + \mathbf{7} \cdot 1^6 - \mathbf{1} \cdot 0^6) = c_5 \cdot MWNT(6,7) = c_5 \cdot 0 \\
& \vdots
\end{aligned}
$$

8 Closing Remarks

We have presented a method of solving for the unknown coefficients of a polynomial sequence using two of the Worpitzky Number Triangles and the main diagonal of the sequence's difference triangle. However, we are unsure if our description of the method in this paper meets the rigorous requirements for a mathematical proof.[3] Regarding the use of the method described in Section 5.2 and Section 5.3, if we have not met the requirements, we feel that we are fairly close in doing so. We are quite sure that we have not met the requirements of mathematical rigor regarding extending the method as in Example 3. However, we believe that the extension of the method is valid and that any result obtained in practice may be checked for validity on a case by case basis.

We welcome papers that address any shortcomings in this paper, with due credit going to the authors.[4,5] Certainly, we feel that the Worpitzky Number Triangles are worthy of wider recognition and perhaps of standard definition and notation as well.

---

[3]Our experience is in engineering, not in producing ironclad mathematical proofs. The recognition of the method as presented in this paper came about upon investigation of the results of the statistical analysis of an electronics manufacturing process.

[4]We are assuming, perhaps incorrectly, that members of the mathematical community with the necessary skills and experience to write such papers will also deem it worthwhile to do so.

[5]We also note that while we are interested in real numbers, we surmise that mathematicians may wish to extend the method to include complex numbers, if possible.

References

[1] *The On-Line Encyclopedia of Integer Sequences*, published electronically at https://oeis.org, 2010, `https://oeis.org`.

[2] Ibid, Sequence A028246, `https://oeis.org/A028246`.

[3] Ibid, Sequence A019538, `https://oeis.org/A019538`.

[4] Ibid, Sequence A005460, `https://oeis.org/A005460`.

[5] Ibid, Sequence A130850 `https://oeis.org/A130850`.

[6] Ibid, Sequence A008277 `https://oeis.org/A008277`.

[7] Ibid, `https://oeis.org/A000629/a000629.pdf`, p. 3., Austin, R.; Guy, R. K.; Nowakowski, R. (c. 1987) Unpublished notes.

[8] Vandervelde, Sam (2018). The Worpitzky Numbers Revisited. *The American Mathematical Monthly, 125:3*, pp. 198-206. `https://doi.org/10.1080/00029890.2018.1408347`.

[9] Graham, Ronald L.; Knuth, Donald E.; Patashnik, Oren (1994). *Concrete Mathematics: A Foundation for Computer Science* (2nd ed.). Reading, MA, USA: Addison-Wesley, p. 162.

[10] Ibid, pp. 257-258.

[11] Ibid, p. 188.

[12] Ibid.

[13] Gould, Henry W.; Quaintance, Jocelyn (2016). *Combinatorial Identities for Stirling Numbers: The Unpublished Notes of H. W. Gould*, Singapore: World Scientific Publishing, p. 147.

[14] Ibid, Equation 6.16, p. 68.

[15] Ibid, Equation 6.21, p. 70.

[16] Ibid, p. 63.

[17] Ibid, p. 147.

[18] Ibid, Chapter 9, pp. 113-138.