COUNTING WALKS WITH LARGE STEPS IN AN ORTHANT

ALIN BOSTAN, MIREILLE BOUSQUET-MÉLOU, AND STEPHEN MELCZER

Abstract. In the past fifteen years, the enumeration of lattice walks with steps taken in a prescribed set $\mathcal{S}$ and confined to a given cone, especially the first quadrant of the plane, has been intensely studied. As a result, the generating functions of quadrant walks are now well-understood, provided the allowed steps are *small*, that is $\mathcal{S} \subset \{-1, 0, 1\}^2$. In particular, having small steps is crucial for the definition of a certain group of bi-rational transformations of the plane. It has been proved that this group is finite if and only if the corresponding generating function is D-finite (that is, it satisfies a linear differential equation with polynomial coefficients). This group is also the key to the uniform solution of 19 of the 23 small step models possessing a finite group.

In contrast, almost nothing is known for walks with arbitrary steps. In this paper, we extend the definition of the group, or rather of the associated orbit, to this general case, and generalize the above uniform solution of small step models. When this approach works, it invariably yields a D-finite generating function. We apply it to many quadrant problems, including some infinite families.

After developing the general theory, we consider the 13 110 two-dimensional models with steps in $\{-2, -1, 0, 1\}^2$ having at least one -2 coordinate. We prove that only 240 of them have a finite orbit, and solve 231 of them with our method. The 9 remaining models are the counterparts of the 4 models of the small step case that resist the uniform solution method (and which are known to have an algebraic generating function). We conjecture D-finiteness for their generating functions, but only two of them are likely to be algebraic. We also prove non-D-finiteness for the 12 870 models with an infinite orbit, except for 16 of them.

1. Introduction

The enumeration of planar lattice walks confined to the quadrant has received a lot of attention over the past fifteen years. The basic question reads as follows: given a finite step set $\mathcal{S} \subset \mathbb{Z}^2$ and a starting point $P \in \mathbb{N}^2$, what is the number q_n of n-step walks, starting from P and taking their steps in $\mathcal{S}$, that remain in the non-negative quadrant $\mathbb{N}^2$? This is a versatile question, since such walks encode in a natural fashion many discrete objects (systems of queues, Young tableaux and their generalizations, among others). More generally, the study of these walks fits in the larger framework of walks confined to cones. These walks are also much studied in probability theory, both in a discrete [36, 38] and in a continuous [31, 42] setting. From a technical point of view, counting walks in the quadrant is part of a general program aiming at solving functional equations that involve *divided differences with respect to several variables* (or *discrete* partial differential equations): see Equation (2) below for a typical example, and [22, Sec. 2] for a general discussion on these equations.

On the combinatorics side, much attention has focused on the *nature* of the associated generating function $Q(t) = \sum_n q_n t^n$. Is it rational in t, as for unconstrained walks? Is it algebraic over $\mathbb{Q}(t)$, as for walks confined to a (rational) half-space? More generally, is $Q(t)$ the solution of a linear differential equation with polynomial coefficients in $\mathbb{Q}[t]$? (in short: is it *D-finite*?) The answer depends on the step set and, to a lesser extent, on the starting point.

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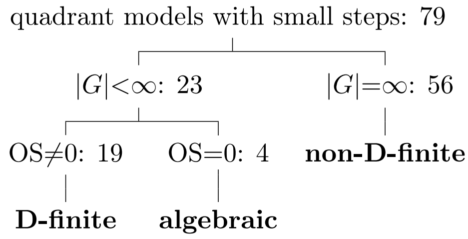


FIGURE 1. Classification of quadrant walks with small steps. The group of the walk is denoted by G, and OS stands for the *orbit sum*, a rational function which vanishes precisely for algebraic models. The 4 algebraic models are those of Figure 2.

A systematic study was initiated in [61, 26] for walks starting at the origin $(0, 0)$ and taking only *small* steps (that is, $\mathcal{S} \subset \{-1, 0, 1\}^2$). For these walks, a complete classification is now available (Figure 1). In particular, the trivariate generating function $Q(x, y; t)$ that also records the coordinates of the endpoint of the walk is D-finite if and only if a certain group G of birational transformations is finite. The proof involves an attractive variety of tools, ranging from elementary power series algebra [21, 61, 26, 22] to complex analysis [55, 65], computer algebra [17, 52], probability theory [32, 36] and number theory [19]. The most recent results on this topic discriminate, among non-D-finite models, those that are still *D-algebraic* (that is, satisfy polynomial differential equations) from those that are not [8, 7, 34, 33]. Remarkably, a new tool then comes into play: differential Galois theory.

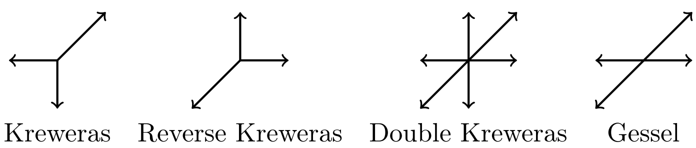


FIGURE 2. The four algebraic small step models in the quadrant, with their usual names.

Contrasting with the precision of this classification is the case of quadrant walks with *arbitrary steps*, for which it is fair to say that almost nothing is known. Indeed, the small step assumption is crucial in all methods used in the small step case, aside from two of them: the computer algebra approach of [17, 52] can in principle be adapted to any steps, provided one is able to *guess* differential or algebraic equations for the solution; and the asymptotic estimates of [32] do not require assumptions on the size of the steps. But even the definition of the group that is central in the classification requires small steps. The complex analytic approach of [55] that has proved very powerful for small steps seems difficult to extend, and the first attempts have not yet led to any explicit solution, nor indications on the nature of the generating functions [39]. The classical reflection principle [45] requires that no walk crosses the x- or y-axis without actually touching it, which is equivalent to a small step condition.

The study of quadrant walks with arbitrary steps is not only a natural mathematical challenge. It is also motivated by "real life" examples. For instance, certain orientations of planar maps were recently shown by Kenyon et al. [53] to be in bijection with quadrant walks taking their steps in $\{(-p, 0), (-p + 1, 1), \ldots, (0, p), (1, -1)\}$. In the forthcoming paper [24] it is shown that the method of the current article solves all these models. Other examples can be found in queuing theory, where several clients may arrive, or be served, at the same time (think of ski-lifts in a ski resort!). Also, a problem as innocuous as counting walks on the square lattice confined to the cone bounded by the x-axis (for x positive) and the line $y = 2x$ becomes, after a linear transformation, a quadrant problem with large steps (Figure 3). Moreover, our study raises intriguing combinatorial questions, which can be seen as an *a posteriori* motivation of this work. For instance, some walks with large steps turn out to be counted by simple hypergeometric numbers, for reasons that remain combinatorially mysterious (see for instance Propositions 24 and 26). Furthermore, our study gives rise to attractive conjectures involving nine large step analogues of the four algebraic models of Figure 2 (Section 8.4). We hope that this paper will have a progeny as rich as its small step counterpart [26].

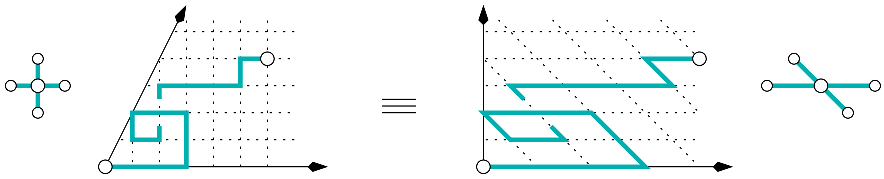


FIGURE 3. A square lattice walk confined to a wedge becomes a quadrant walk with large steps.

Our aim here is primarily to extend to arbitrary steps (and arbitrary dimension, for walks confined to the orthant $\mathbb{N}^d$) a power series approach that was introduced in [26] to solve the 19 easiest small step models, namely those of the leftmost branch of Figure 1. The group is lost, but the associated orbit survives. When the method works, it yields an expression of the generating function as the non-negative part of an algebraic series — a form which implies D-finiteness. On the negative side, we give a criterion that simultaneously implies that the orbit of a 2-dimensional model is infinite and that its generating function is not D-finite. We provide evidence that in 2D, the finiteness of the orbit may still be related to the D-finiteness of the solution. This is based in particular on the systematic exploration of quadrant walks with steps in $\{-2,-1,0,1\}^2$.

Before we give more details on our results, let us examine the solution of a simple small step model, as presented in [26].

1.1. A basic example: $\mathcal{S} = \{\searrow, \leftarrow, \uparrow\}$

We denote by $q(i,j;n)$ the number of walks with steps in $\mathcal{S}$ that start at $(0,0)$, end at (i,j) and remain in the non-negative quadrant $\mathbb{N}^2$. The associated generating function is

$$Q(x,y;t) := \sum_{i,j,n\geq 0} q(i,j;n) x^i y^j t^n.$$

We will find an explicit expression for this power series using a four-step approach, sometimes called the *algebraic kernel method* and borrowed from [26], which we then generalize in the rest of the paper.

A functional equation. A step-by-step construction of quadrant walks with steps in $\{\searrow, \leftarrow, \uparrow\}$ yields the functional equation

$$Q(x,y) = 1 + t(x\bar{y} + \bar{x} + y)Q(x,y) - tx\bar{y}Q(x,0) - t\bar{x}Q(0,y), \tag{1}$$

where we write $\bar{x} := 1/x$, $\bar{y} := 1/y$ and replace $Q(x,y;t)$ by $Q(x,y)$ to lighten notation. In this equation the constant term 1 stands for the empty walk. The next term counts quadrant walks extended by one of our three steps. The final two terms remove the contributions of the two "forbidden moves": either we have extended a walk ending on the x-axis by a $\searrow$ step (term $-tx\bar{y}Q(x,0)$) or we have extended a walk ending on the y-axis by a $\leftarrow$ step (term $-t\bar{x}Q(0,y)$). Observe that the above equation can also be written in a form that involves two divided differences, one in x and the other in y:

$$Q(x,y) = 1 + tyQ(x,y) + tx\,\frac{Q(x,y) - Q(x,0)}{y} + t\,\frac{Q(x,y) - Q(0,y)}{x}. \tag{2}$$

We refer to [22, Sec. 2] for a general discussion on equations involving divided differences with respect to two variables (those that involve divided differences with respect to one variable only are known to have algebraic solutions [25]). We rewrite (1) as

$$K(x,y)xyQ(x,y) = xy - R(x) - S(y), \tag{3}$$

where $R(x) = tx^2Q(x,0)$, $S(y) = tyQ(0,y)$, and $K(x,y) = 1 - t(x\bar{y} + \bar{x} + y)$ is the *kernel* of the equation. Observe the decoupling of the x and y variables in the right-hand side. We call the bivariate series $R(x)$ and $S(y)$ *sections.*

The group of the walk. We now define two bi-rational transformations Φ and Ψ, acting on pairs (u,v) of coordinates (which will be, typically, algebraic functions of x and y):

$$\Phi : (u,v) \mapsto (\bar{u}v, v) \qquad \text{and} \qquad \Psi : (u,v) \mapsto (u, u\bar{v}).$$

Each transformation fixes one coordinate, and transforms the other *so as to leave the step polynomial $u\bar{v} + \bar{u} + v$ unchanged.* Both transformations are involutions, and the orbit of (x,y) under the action of Φ and Ψ consists of 6 elements:

$$(x,y) \overset{\Phi}{\longleftrightarrow} (\bar{x}y, y) \overset{\Psi}{\longleftrightarrow} (\bar{x}y, \bar{x}) \overset{\Phi}{\longleftrightarrow} (\bar{y}, \bar{x}) \overset{\Psi}{\longleftrightarrow} (\bar{y}, x\bar{y}) \overset{\Phi}{\longleftrightarrow} (x, x\bar{y}) \overset{\Psi}{\longleftrightarrow} (x,y).$$

The group generated by Φ and Ψ is thus the dihedral group of order 6.

A section-free equation. We now write, for each element (x', y') of the orbit, the functional equation (3) with (x,y) replaced by (x', y'):

$$\begin{aligned}
K(x,y)\ xyQ(x,y) &= xy - R(x) - S(y),\\
K(x,y)\ \bar{x}y^2Q(\bar{x}y,y) &= \bar{x}y^2 - R(\bar{x}y) - S(y),\\
K(x,y)\ \bar{x}^2yQ(\bar{x}y,\bar{x}) &= \bar{x}^2y - R(\bar{x}y) - S(\bar{x}),\\
\vdots \qquad &= \qquad \vdots\\
K(x,y)\ x^2\bar{y}Q(x,x\bar{y}) &= x^2\bar{y} - R(x) - S(x\bar{y}).
\end{aligned} \tag{4}$$

Due to the definition of Φ and Ψ, two consecutive equations have one section $R(\cdot)$ or $S(\cdot)$ in common. Thus, the alternating sum of our 6 equations has a right-hand side *free from sections*:

$$\begin{aligned}
K(x,y)\Big(xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y})\Big)&\\
= xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}.&
\end{aligned} \tag{5}$$

The right-hand side of this equation is the *orbit sum* occurring in the classification of Figure 1. Equivalently,

$$\begin{aligned}
xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y})&\\
= \frac{xy\,(1-\bar{x}\bar{y})\left(1-\bar{x}^2y\right)\left(1-x\bar{y}^2\right)}{1-t(y+\bar{x}+x\bar{y})}.&
\end{aligned}$$

Extracting $Q(x,y)$. The last equation, combined with the fact that $Q(x,y)$ is a power series in t with polynomial coefficients in x and y, characterizes $Q(x,y)$ uniquely: indeed, the series $xyQ(x,y)$ has coefficients in $xy\mathbb{Q}[x,y]$, and thus involves only positive powers of x and y. But the monomials occurring in each of the five other terms of the left-hand side involve either a negative power of x, or a negative power of y (or both). Hence the series $xyQ(x,y)$ is obtained by expanding the right-hand side as a series in t with coefficients in $\mathbb{Q}[x,\bar{x},y,\bar{y}]$, and then collecting terms with positive powers of x and y. We denote this extraction by:

$$xyQ(x,y) = [x^>y^>]\frac{xy\,(1-\bar{x}\bar{y})\left(1-\bar{x}^2y\right)\left(1-x\bar{y}^2\right)}{1-t(y+\bar{x}+x\bar{y})}.$$

Equivalently, upon dividing by xy, the series $Q(x,y)$ is obtained by collecting the *non-negative part* in x and y of a rational function:

$$Q(x,y) = [x^{\geq}y^{\geq}]\frac{(1-\bar{x}\bar{y})\left(1-\bar{x}^2y\right)\left(1-x\bar{y}^2\right)}{1-t(\bar{x}+y+x\bar{y})}.$$

This explicit expression has strong consequences. First, it guarantees that $Q(x,y)$ is D-finite [56]. Second, expanding $(\bar{x}+y+x\bar{y})^n$ in powers of x and y, it delivers a hypergeometric expression for the number of walks of length $n=3m+2i+j$ ending at (i,j):

$$q(i,j;n) = \frac{(i+1)(j+1)(i+j+2)(3m+2i+j)!}{m!(m+i+1)!(m+i+j+2)!}.$$

We conclude this example with a remark for the combinatorially inclined readers: since walks with steps in $\mathcal{S}=\{\searrow,\leftarrow,\uparrow\}$ give a simple encoding of Young tableaux of height at most 3, the above formula is just the translation in terms of walks of the classical hook formula [66, §3.10].

1.2. Outline of the paper

Based on the above example, we can now describe our results more precisely. The next four sections present the extension to arbitrary steps (and dimension) of the four stages involved in the above solution. The principles of our approach are robust enough to be applicable to the enumeration of *weighted* walks, which can be especially interesting in a probabilistic context. We give many examples to illustrate these stages, but also to show how they can fail: indeed, since our method only solves 19 of the 79 small step models in the quadrant, we know in advance that it *has* to fail for some models. Two obstacles can already be seen in the classification of Figure 1: the group (or what is left of it, namely its orbit) can be infinite, and the orbit sum can vanish. Interestingly, we provide in Section 3.3 a criterion that implies simultaneously the infiniteness of the orbit and the non-D-finiteness of the generating function.

In Sections 6 and 7, we show that our approach applies systematically in dimension 1 (walks on a half-line) and for the so-called *Hadamard models* in dimension 2. Working in dimension 1 is the least one can ask for, as walks on a half-line are very well understood [6, 27, 43]. It is worth noting that the form of our solution is not exactly the standard form obtained by earlier approaches. The second result, dealing with Hadamard models, is more interesting as it seems that many models with finite orbit are Hadamard. In the small step case for instance, 16 of the 19 models solvable by our approach (that is, 16 of the 23 D-finite models) are of Hadamard type.

In Section 8 we apply these principles to the classification of models with steps in $\{-2,-1,0,1\}^2$. Several results are still conjectural, but in a sense we obtain a perfect analogue of the small step classification shown in Figure 1: our approach solves all 231 models with a finite orbit and a non-vanishing orbit sum (Figure 7). For each of them, we express $Q(x,y;t)$ as the non-negative part in x and y of an explicit rational function. Exactly 227 of these 231 solved models are in fact Hadamard. This leaves out 9 models with a finite group but orbit sum zero, for which we state several attractive conjectures. Finally, we establish non-D-finiteness for the 12 870 models with an infinite orbit, except for 16 of them, which we still conjecture to be non-D-finite.

In Section 9 we show that the form of the solutions that we obtain is well-suited to the asymptotic analysis of their coefficients, and we work out explicitly the analysis for the 4 non-Hadamard models with a finite orbit solved in Section 8.

We conclude in Section 10 with a number of remarks and open questions.

Notation and definitions. For the sake of compactness we often encode a step into a word consisting of its coordinates, with a bar above negative coordinates: for example, the step $(-2,3,-5)\in\mathbb{Z}^3$ will be denoted $\bar{2}3\bar{5}$. Similarly, as used above, we use a bar over variables to denote their reciprocals, so that $\bar{x}=1/x$. A *small forward step* has its coordinates in $\{1,0,-1,-2,\ldots\}$ while a *large forward step* has at least one coordinate larger than 1. We define similarly small and large backward steps. A *small step* has only coordinates in $\{-1,0,1\}$.

In two dimensions, small steps can be identified by the compass directions, and we sometimes draw them pictorially with arrows: for instance, $(1,1)$ can be denoted $\nearrow$.

For a ring R, we denote by $R[x]$ (resp. $R[[x]]$) the ring of polynomials (resp. formal power series) in x with coefficients in R. If R is a field, then $R(x)$ stands for the field of rational functions in x, and $R((x))$ is the field of Laurent series in x (that is, series of the form $\sum_{n\geq n_0} a_n x^n$, with $n_0 \in \mathbb{Z}$). This notation is generalized to several variables in the usual way. For instance, the generating function $Q(x,y;t)$ of walks restricted to the first quadrant is a series in $\mathbb{Q}[x,y][[t]]$. We shall also consider *fractional power series*, namely power series in a (positive) fractional power of x, and finally Puiseux series, which are Laurent series in a fractional power of the variable. We recall that if R is an algebraically closed field, then Puiseux series in x with coefficients in R form an algebraically closed field (see [1] or [69, Chap. 6]).

If $F(u;t)$ is a power series in t whose coefficients are Laurent series in u,

$$F(u;t) = \sum_{n\geq 0} t^n \left(\sum_{i \geq i(n)} u^i f(i;n) \right),$$

we denote by $[u^>]F(u;t)$ the *positive part of* F *in* u:

$$[u^>]F(u;t) = \sum_{n\geq 0} t^n \left(\sum_{i>0} u^i f(i;n) \right).$$

We define the *non-negative part* $[u^\geq]F(u;t)$ in a similar fashion, by retaining as well the constant term in u.

We recall that a series $Q(x,y;t)$ is *algebraic* if there exists a non-zero polynomial $P \in \mathbb{Q}[x,y,t,s]$ such that $P(x,y,t,Q(x,y;t)) = 0$. It is *D-finite* (with respect to the variable t) if the vector space over $\mathbb{Q}(x,y,t)$ spanned by the iterated derivatives $\partial_t^m Q(x,y;t)$, for $m \geq 0$, has finite dimension (here ∂_t denotes differentiation with respect to t). The latter definition can be adapted to D-finiteness in several variables, for instance x, y and t: in this case we require D-finiteness with respect to *each* variable separately [57]. Every algebraic series is D-finite [57, Prop. 2.3]. If $Q(x,y;t)$ is D-finite in its three variables, then so are $Q(0,0;t)$ and $Q(1,1;t)$. For a one-variable series $F(t) = \sum f_n t^n$, D-finiteness is equivalent to the existence of a linear recurrence relation with polynomial coefficients in n satisfied by the coefficients sequence (f_n).

We often denote by F_t the derivative $\partial_t F$ of a series $F(t)$. This notation is generalized to several variables. For instance, $F_{t,u}$ stands for $\partial_t \partial_u F$.

2. A functional equation

Let $d \geq 1$ and let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^d$. We would like to count walks that take their steps in $\mathcal{S}$, start from the origin and are confined to the orthant $\mathbb{N}^d$. We denote by $q(i_1,\ldots,i_d;n)$ the number of such walks consisting of n steps and ending at $(i_1,\ldots,i_d)$, and by $Q(x_1,\ldots,x_d;t)$ the associated generating function:

$$Q(x_1,\ldots,x_d;t) \equiv Q(x_1,\ldots,x_d) := \sum_{(i_1,\ldots,i_d,n)\in\mathbb{N}^{d+1}} q(i_1,\ldots,i_d;n) x_1^{i_1}\cdots x_d^{i_d} t^n.$$

Note that we often omit the dependence of Q in t. The notation Q refers to the two-dimensional case (walks in a *quadrant*), from which we will borrow most of our examples. In that case, we use the variables x and y instead of x_1 and x_2.

We use bold notation for multivariate quantities, so that $\mathbf{x} = (x_1,\ldots,x_d)$, and for a d-tuple $\mathbf{i} = (i_1,\ldots,i_d) \in \mathbb{Z}^d$ we use the abbreviation $\mathbf{x}^\mathbf{i} = x_1^{i_1}\cdots x_d^{i_d}$. The *step polynomial* of a model (also called the *characteristic polynomial*) is

$$S(x_1,\ldots,x_d) = S(\mathbf{x}) = \sum_{\mathbf{s}\in\mathcal{S}} \mathbf{x}^\mathbf{s}. \tag{6}$$

The step polynomial is a Laurent polynomial in the variables x_i; here every step has weight 1, but our approach can be adapted to the enumeration of weighted walks with weights in some field $\mathbb{F}$ (for instance $\mathbb{F} = \mathbb{R}$ in a probabilistic context).

One can always write for the generating function $Q(x_1, \dots, x_d; t)$ a functional equation defining this series, based on a step-by-step construction of walks confined to $\mathbb{N}^d$, as was done in (1). This functional equation is linear in the main series $Q(x_1, \dots, x_d; t)$ and, when the terms are grouped on one side of the equation, the coefficient in front of $Q(x_1, \dots, x_d; t)$ is the *kernel*

$$K(x_1, \dots, x_d) = 1 - tS(x_1, \dots, x_d).$$

The equation also involves unknown series that only depend on *some* of the variables $x_1, \dots, x_d$ (and on t), such as, for instance, the series $Q(x, 0)$ and $Q(0, y)$ in (1). These series are called *sections* (of Q). Let us consider a few examples.

Example A. Take $d = 1$, and $\mathcal{S} = \{\bar{1}, 2\}$. The equation satisfied by the series $Q(x; t) \equiv Q(x)$ reads

$$Q(x) = 1 + t(\bar{x} + x^2)Q(x) - t\bar{x}Q(0),$$

where the term $-t\bar{x}Q(0)$ removes forbidden moves from position 0 to position -1. Equivalently, with $K(x) = 1 - t(\bar{x} + x^2)$, the previous equation reads

$$K(x)Q(x) = 1 - t\bar{x}Q(0). \tag{7}$$

□

Example B. Still with $d = 1$, we now reverse the steps of the previous example so as to have a long backward step, and study $\mathcal{S} = \{\bar{2}, 1\}$. Extending a walk w by the step -2 is now forbidden as soon as w ends at position 0 or 1. Hence, denoting by $Q_i \equiv Q_i(t)$ the length generating function of walks ending at position i, the equation satisfied by $Q(x)$ reads

$$Q(x) = 1 + t(\bar{x}^2 + x)Q(x) - t\bar{x}^2 Q_0 - t\bar{x}Q_1,$$

or equivalently, with $K(x) = 1 - t(\bar{x}^2 + x)$,

$$K(x)Q(x) = 1 - t\bar{x}^2 Q_0 - t\bar{x}Q_1. \tag{8}$$

Observe that $Q_0 = Q(0)$ and $Q_1 = \partial_x Q(0)$. The occurrence of a large backward step results in one more section on the right-hand side. □

Example C: Gessel's walks. We return to two-dimensional models, now with the step set $\mathcal{S} = \{\rightarrow, \nearrow, \leftarrow, \swarrow\}$. Appending a south-west step is forbidden as soon as the walk ends at abscissa or ordinate zero. The functional equation thus reads:

$$Q(x, y) = 1 + t(x + xy + \bar{x} + \bar{x}\bar{y})Q(x, y) - t\bar{x}Q(0, y) - t\bar{x}\bar{y}(Q(x, 0) + Q(0, y) - Q(0, 0)).$$

The term in $Q(0, 0)$ avoids removing twice walks that end at $(0, 0)$. Equivalently, with $K(x, y) = 1 - t(x + xy + \bar{x} + \bar{x}\bar{y})$,

$$K(x, y)Q(x, y) = 1 - t\bar{x}(1 + \bar{y})Q(0, y) - t\bar{x}\bar{y}(Q(x, 0) - Q(0, 0)).$$

□

Example D: A model with a large forward step and a large backward step. We now take $\mathcal{S} = \{\bar{2}0, \bar{1}1, 02, 1\bar{1}\}$. Quadrant walks formed of these steps, starting and ending at the origin, are known to be in bijection with *bipolar orientations of quadrangulations* [53, 24]. The functional equation reads

$$Q(x, y) = 1 + t(\bar{x}^2 + \bar{x}y + y^2 + x\bar{y})Q(x, y) - t\bar{x}^2 \left(Q_{0,-}(y) + xQ_{1,-}(y)\right) - t\bar{x}yQ_{0,-}(y) - tx\bar{y}Q(x, 0),$$

where $x^i Q_{i,-}(y)$ counts quadrant walks ending at x-coordinate i. Note that $Q_{0,-}(y) = Q(0, y)$. We can rewrite the functional equation, using $K(x, y) = 1 - t(\bar{x}^2 + \bar{x}y + y^2 + x\bar{y})$, as

$$K(x, y)Q(x, y) = 1 - t\bar{x}(\bar{x} + y)Q_{0,-}(y) - t\bar{x}Q_{1,-}(y) - tx\bar{y}Q(x, 0). \tag{9}$$

□

Example E: A model in three dimensions. We now take $\mathcal{S} = \{\bar{1}\bar{1}\bar{1}, \bar{1}\bar{1}1, \bar{1}10, 100\}$. As for Gessel's walks (Example C), the functional equation involves inclusion-exclusion so as to avoid excluding several times the same move, and one obtains:

$$\begin{aligned} K(x,y,z)Q(x,y,z) = 1 - t\bar{x}(\bar{y}\bar{z} + \bar{y}z + y)Q(0,y,z) - t\bar{x}\bar{y}(\bar{z}+z)Q(x,0,z) - t\bar{x}\bar{y}\bar{z}Q(x,y,0) \\ + t\bar{x}\bar{y}(\bar{z}+z)Q(0,0,z) + t\bar{x}\bar{y}\bar{z}Q(0,y,0) + t\bar{x}\bar{y}\bar{z}Q(x,0,0) - t\bar{x}\bar{y}\bar{z}Q(0,0,0), \end{aligned} \tag{10}$$

where the kernel is

$$K(x,y,z) = 1 - t(\bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y + x).$$

□

After seeing all these examples, the reader should be convinced that a functional equation can be written for any model $\mathcal{S}$. We only give its general form in two cases: first in dimension two, and then for models with small backward steps. In dimension two, the equation reads:

$$K(x,y)Q(x,y) = 1 - t \sum_{(k,\ell)\in\mathcal{S}} x^k y^\ell \left(\sum_{0\le i<-k} x^i Q_{i,-}(y) + \sum_{0\le j<-\ell} y^j Q_{-,j}(x) - \sum_{\substack{0\le i<-k \\ 0\le j<-\ell}} x^i y^j Q_{i,j} \right), \tag{11}$$

where $K(x,y) = 1 - tS(x,y)$ is the kernel, $x^i Q_{i,-}(y)$ (resp. $y^j Q_{-,j}(x)$) counts quadrant walks ending at abscissa i (resp. at ordinate j), and $Q_{i,j}$ is the length generating function of walks ending at (i,j).

For a model of walks with *small backward steps* confined to the orthant $\mathbb{N}^d$ in arbitrary dimension d, the functional equation reads:

$$K(\mathbf{x})Q(\mathbf{x}) = 1 + t \sum_{\emptyset\neq I\subset[\![1,d]\!]} \left((-1)^{|I|} Q_I(\mathbf{x}) \sum_{\substack{\mathbf{s}\in\mathcal{S}: \\ s_i=-1\ \forall i\in I}} \mathbf{x}^{\mathbf{s}} \right), \tag{12}$$

where $\mathbf{x} = (x_1, \dots, x_d)$, $K(\mathbf{x}) = 1 - tS(\mathbf{x})$, and $Q_I(\mathbf{x})$ is the specialization of $Q(\mathbf{x})$ where each $x_i, i \in I$, is set to 0 (for instance, if $I = \{2,3\}$ then $Q_I(x) = Q(x_1, 0, 0, x_4, \dots, x_d)$). The proof is an inclusion-exclusion argument generalizing the proof of (10).

3. The orbit of $(x_1, \dots, x_d)$

In Section 1.1, we have shown on one example how to associate a group to a 2D model with small steps. We now describe, for a general step set $\mathcal{S}$ in arbitrary dimension d, how to define the counterpart of this group, or more precisely of its orbit. To avoid trivial cases, we only consider models that have both positive and negative steps in each direction.

3.1. Definition and first examples

We denote by $\mathbb{K}$ the field $\mathbb{C}(x_1, \dots, x_d)$, and by $\overline{\mathbb{K}}$ an algebraic closure of $\mathbb{K}$. We first define two relations $\approx$ and $\sim$ on elements of $\left(\overline{\mathbb{K}} \setminus \{0\}\right)^d$; recall that $S(\mathbf{x})$ denotes the step polynomial of $\mathcal{S}$, defined by (6).

Definition 1. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two distinct* d*-tuples in* $\left(\overline{\mathbb{K}} \setminus \{0\}\right)^d$*, and let* $1 \le i \le d$*. Then* $\mathbf{u}$ *and* $\mathbf{v}$ *are* i-adjacent*, denoted* $\mathbf{u} \overset{i}{\approx} \mathbf{v}$*, if* $S(\mathbf{u}) = S(\mathbf{v})$ *and* $\mathbf{u}$ *and* $\mathbf{v}$ *differ only by their* i*th coordinate. They are* adjacent*, denoted* $\mathbf{u} \approx \mathbf{v}$*, if they are* i*-adjacent for some* i*.*

Clearly, the relation $\approx$ *is symmetric. We denote by* $\sim$ *its reflexive and transitive closure. The* orbit *of* $\mathbf{u}$ *is its equivalence class for this relation.*

The $\mathbf{u}$-length *of an element* $\mathbf{v}$ *in the orbit of* $\mathbf{u}$ *is the smallest* ℓ *such that there exists* $\mathbf{u}^{(0)} = \mathbf{u}, \mathbf{u}^{(1)}, \dots, \mathbf{u}^{(\ell)} = \mathbf{v}$ *with* $\mathbf{u}^{(0)} \approx \mathbf{u}^{(1)} \approx \cdots \approx \mathbf{u}^{(\ell)}$*.*

Note that the value of S is constant over the orbit of $\mathbf{u}$. We will often refer to the orbit of $\mathbf{x} = (x_1, \dots, x_d)$ (or (x, y) in two dimensions) as *the* orbit of the model $\mathcal{S}$, and to the *length* of an element of this orbit as its $\mathbf{x}$-length. We use the word *orbit* even though we have not defined any underlying group: this terminology comes from the case of small steps, as justified by Proposition 5 below. Before we proceed, let us check that the structure of the orbit does not depend on the choice of the algebraic closure of $\mathbb{K}$.

Lemma 2. *Let $\overline{\mathbb{K}}$ and $\widehat{\mathbb{K}}$ be two algebraic closures of $\mathbb{K}$ and $\tau : \overline{\mathbb{K}} \to \widehat{\mathbb{K}}$ a field automorphism fixing $\mathbb{K}$. For any $\mathbf{u} = (u_1, \dots, u_d) \in \left(\overline{\mathbb{K}} \setminus \{0\}\right)^d$, we denote by $\tau(\mathbf{u})$ the element of $\left(\widehat{\mathbb{K}} \setminus \{0\}\right)^d$ obtained by applying τ to $\mathbf{u}$ component-wise. Then τ preserves adjacencies, and sends the orbit of $\mathbf{u}$ onto the orbit of $\tau(\mathbf{u})$.*

Proof. (sketch) Clearly, if $S(\mathbf{v}) = S(\mathbf{w})$ then $S(\tau(\mathbf{v})) = S(\tau(\mathbf{w}))$, because S has rational coefficients. And if $\mathbf{v}$ and $\mathbf{v}'$ differ by their ith coordinate, the same holds for their images by τ. This shows that adjacencies are preserved. The isomorphism of orbits then follows by induction on the length. ∎

The next proposition tells that two models that are equivalent up to a symmetry of the hypercube have isomorphic orbits. Since these symmetries are generated by a reflection and adjacent transpositions, it suffices to examine these two cases.

Proposition 3. *Let $\mathcal{S} \subset \mathbb{Z}^d$ be a model with step polynomial $S(x_1, \dots, x_d)$, and let $\tilde{\mathcal{S}}$ be the model obtained by swapping the first two coordinates, with step polynomial $S(x_2, x_1, x_3, \dots, x_d)$. Then the orbits of $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are* isomorphic *(there is a bijection from one to the other that preserves adjacencies).*

The same holds if $\tilde{\mathcal{S}}$ is obtained from $\mathcal{S}$ by a reflection in the hyperplane $x_1 = 0$; that is, if its step polynomial is $S(1/x_1, x_2, \dots, x_d)$.

Proof. To lighten notation, we prove this result in two dimensions. The proof is similar in higher dimensions.

In the first case, let us construct the orbit of $\mathcal{S}$ in the field $\overline{\mathbb{K}}$ of iterated Puiseux series in x and y (Puiseux series in x whose coefficients are Puiseux series in y). We shall construct the orbit of $\tilde{\mathcal{S}}$ in the field $\widehat{\mathbb{K}}$ of iterated Puiseux series in y and x (note the inversion). If $u \in \overline{\mathbb{K}}$, let $\delta(u) \in \widehat{\mathbb{K}}$ be obtained from u by swapping x and y. We claim that, if (u, v) is in the orbit of $\mathcal{S}$, then the pair $(\delta(v), \delta(u))$ is in the orbit of $\tilde{\mathcal{S}}$, and vice-versa. First, if $(u, v) = (x, y)$, then $(\delta(v), \delta(u)) = (x, y)$. Then, if (u, v) is 2-adjacent to (u, w) in the orbit of $\mathcal{S}$, then $(\delta(v), \delta(u))$ is 1-adjacent to $(\delta(w), \delta(u))$ in the orbit of $\tilde{\mathcal{S}}$, because

$$\tilde{S}(\delta(w), \delta(u)) = S(\delta(u), \delta(w)) = S(\delta(u), \delta(v)) = \tilde{S}(\delta(v), \delta(u)).$$

(The second equality comes from the 2-adjacency of (u, v) and (u, w) for $\mathcal{S}$.) One proves similarly that 1-adjacencies for $\mathcal{S}$ become 2-adjacencies for $\tilde{\mathcal{S}}$. The isomorphism between the orbits of $\mathcal{S}$ and $\tilde{\mathcal{S}}$ then follows by induction on the length.

The proof is similar in the second case, upon constructing the orbit of $\tilde{\mathcal{S}}$ in the field $\widehat{\mathbb{K}}$ of iterated Puiseux series in $\bar{x}$ and y. Denoting by δ the transformation from $\overline{\mathbb{K}}$ to $\widehat{\mathbb{K}}$ that sends x to $\bar{x}$, a pair (u, v) is in the orbit of $\mathcal{S}$ if and only if $(1/\delta(u), \delta(v))$ is in the orbit of $\tilde{\mathcal{S}}$. ∎

We will now examine examples. One important observation is the following.

Lemma 4. *If the coordinates of $\mathbf{u}$ are algebraically independent over $\mathbb{Q}$, then the same holds for any $\mathbf{v}$ in the orbit of $\mathbf{u}$. Moreover, the number of elements $\mathbf{v}$ that are i-adjacent to $\mathbf{u}$ is $M_i + m_i - 1$, where M_i (resp. $-m_i$) is the largest (resp. smallest) move in the ith direction among the steps of $\mathcal{S}$. In particular, for small step models ($M_i = m_i = 1$ for all i) there is one adjacent element in every direction.*

Proof. Let $\mathbf{v} = (u_1, \dots, u_{i-1}, v, u_{i+1}, \dots, u_d)$ be i-adjacent to $\mathbf{u} = (u_1, \dots, u_d)$, and let us prove that the coordinates of $\mathbf{v}$ are independent over $\mathbb{Q}$. Assume that there exists a non-trivial polynomial $P(\mathbf{a})$ with rational coefficients such that $P(\mathbf{v}) = 0$. Since the u_i's are algebraically independent, $P(\mathbf{a})$ must depend on a_i. Hence v is algebraic over $\mathbb{Q}(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_d)$. The same holds for $S(\mathbf{v})$, and hence for $S(\mathbf{u})$. Since $S(\mathbf{a})$ actually depends on a_i, this means that u_i is algebraic over $\mathbb{Q}(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_d)$, which contradicts the algebraic independence of the u_j's. The first statement of the lemma follows, by induction on the length.

Then, by expanding $S(\mathbf{v})$ in powers of v, we have

$$S(\mathbf{v}) := P_{M_i}(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_d) v^{M_i} + \cdots + P_{-m_i}(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_d) v^{-m_i} = S(\mathbf{u}).$$

As the coordinates of $\mathbf{u}$ are algebraically independent, this equation has $M_i + m_i$ solutions in v. One of them is the trivial solution $v = u_i$. Each root v gives rise to an element $\mathbf{v}$ whose coordinates are algebraically independent. In particular, $S_{a_i}(\mathbf{v}) \neq 0$, which means that v is not a multiple root of $S(\mathbf{v}) - S(\mathbf{u})$. Hence this polynomial (in v) has distinct roots. Removing the trivial root $v = u_i$ gives $m_i + M_i - 1$ distinct elements $\mathbf{v}$ that are i-adjacent to $\mathbf{u}$. ∎

Example ($d = 1$). In dimension 1 the orbit of x consists of all solutions x' of the equation $S(x) = S(x')$. It is thus finite. □

Example: small steps with $d = 2$. Let us get back to the example of Section 1.1. Then it can be checked that two elements are adjacent if and only of one is obtained from the other by applying Φ or Ψ. This will be generalized to all small step models (in arbitrary dimension) in the proposition below.

Note however that in dimension 2, and beyond, the orbit may be infinite. This happens for 56 of the 79 small step quadrant models [26], for instance when $\mathcal{S} = \{\uparrow, \rightarrow, \swarrow, \leftarrow\}$, in which case $S(x, y) = y + x + \bar{x}\bar{y} + \bar{x}$.

For models with small steps, the orbit of $\mathbf{x}$ is indeed its orbit under the action of a certain group, as in the example of Section 1.1.

Proposition 5. *Assume that $\mathcal{S}$ consists of small steps, that is, $\mathcal{S} \subset \{-1, 0, 1\}^d$. Define d bi-rational transformations $\Phi_1, \dots, \Phi_d$ by:*

$$\Phi_i(a_1, \dots, a_d) = \left(a_1, \dots, a_{i-1}, \frac{1}{a_i} \frac{S_i^-(\mathbf{a})}{S_i^+(\mathbf{a})}, a_{i+1}, \dots, a_d \right),$$

where $\mathbf{a} = (a_1, \dots, a_d)$ and $S_i^-(\mathbf{a})$ (resp. $S_i^+(\mathbf{a})$) is the coefficient of $1/a_i$ (resp. a_i) in $S(\mathbf{a})$. Then the Φ_i's are involutions. If the a_j's are algebraically independent over $\mathbb{Q}$, then $\mathbf{a}$ and $\Phi_i(\mathbf{a})$ are i-adjacent.

Conversely, let $\mathbf{x} = (x_1, \dots, x_d)$ and let $\mathbf{u} = (u_1, \dots, u_d)$ be in the orbit of $\mathbf{x}$. An element $\mathbf{v}$ of $\left(\mathbb{K} \setminus \{0\}\right)^d$ is i-adjacent to $\mathbf{u}$ if and only if $\mathbf{v} = \Phi_i(\mathbf{u})$. Consequently, the orbit of $\mathbf{x}$ is indeed its orbit under the action of a group, namely the group generated by the involutions Φ_i.

Finally, the length of two adjacent elements in the orbit of $\mathbf{x}$ differ by ± 1.

Proof. To prove that Φ_i is an involution, we first observe that $S_i^+(\mathbf{a})$ and $S_i^-(\mathbf{a})$ are independent of a_i. Hence, denoting $\mathbf{a}' = \Phi_i(\mathbf{a})$, the ith coordinate of $\Phi_i(\mathbf{a}')$ is

$$\frac{1}{a_i'} \frac{S_i^-(\mathbf{a}')}{S_i^+(\mathbf{a}')} = a_i \frac{S_i^+(\mathbf{a})}{S_i^-(\mathbf{a})} \frac{S_i^-(\mathbf{a})}{S_i^+(\mathbf{a})} = a_i.$$

If the a_j's are algebraically independent over $\mathbb{Q}$, then $\mathbf{a}$ and $\mathbf{a}'$ are distinct, differ in their ith coordinate only, and, upon writing

$$S(\mathbf{x}) = \frac{1}{x_i} S_i^-(\mathbf{x}) + S_i^0(\mathbf{x}) + x_i S_i^+(\mathbf{x}),$$

we can check that $S(\mathbf{a}) = S(\mathbf{a}')$. Hence $\mathbf{a}$ and $\Phi_i(\mathbf{a})$ are i-adjacent.

Now let $\mathbf{u}$ be in the orbit of $\mathbf{x}$. Write $S(\mathbf{x})$ as above. Note that $S_i^-(\mathbf{x})$, $S_i^0(\mathbf{x})$ and $S_i^+(\mathbf{x})$ are unchanged if we only modify the ith coordinate of $\mathbf{x}$. So if $\mathbf{v} \overset{i}{\approx} \mathbf{u}$, the fact that $S(\mathbf{u}) = S(\mathbf{v})$ gives

$$\frac{1}{u_i} S_i^-(\mathbf{u}) + u_i S_i^+(\mathbf{u}) = \frac{1}{v_i} S_i^-(\mathbf{u}) + v_i S_i^+(\mathbf{u}), \quad \text{that is,} \quad S_i^-(\mathbf{u}) = u_i v_i S_i^+(\mathbf{u}).$$

By the above lemma, the coordinates of $\mathbf{u}$ are algebraically independent, hence $S_i^+(\mathbf{u}) \neq 0$ and $\mathbf{v}$ must be $\Phi_i(\mathbf{u})$. Conversely, we have proved above that $\Phi_i(\mathbf{u})$ is i-adjacent to $\mathbf{u}$. This concludes the description of the orbit of $\mathbf{x}$.

The proof of the final result was communicated to us by Andrew Elvey Price and Michael Wallner, whom we thank for their great help. Clearly, if $\mathbf{u}$ and $\mathbf{v}$ are two adjacent elements in the orbit of $\mathbf{x}$, their lengths differ by $0, +1$ or -1. We want to exclude the value 0, which amounts to saying that in the graph whose vertices are the elements of the orbit, with edges between adjacent elements, there is no odd cycle. Equivalently, this graph is bipartite. In order to prove this, we define a sign $\varepsilon(\mathbf{u}) \in \{-1, +1\}$ on elements $\mathbf{u}$ of the orbit of $\mathbf{x}$, which changes when an involution Φ_i is applied. The sign is defined by

$$\varepsilon(\mathbf{u}) = \left(\prod_{i=1}^{d} x_i \right) \det M(\mathbf{u}), \qquad \text{where} \qquad M(\mathbf{u}) = \left(\frac{1}{u_i} \frac{\partial u_i}{\partial x_j} \right)_{1 \leq i,j \leq d}.$$

It is readily checked that $\varepsilon(\mathbf{x}) = 1$, and this implies $\varepsilon(\mathbf{u}) = (-1)^{\text{length}(\mathbf{u})}$. Let us then take $\mathbf{v} = \Phi_i(\mathbf{u})$, and prove that $\varepsilon(\mathbf{v}) = -\varepsilon(\mathbf{u})$. The matrix $M(\mathbf{v})$ only differs from $M(\mathbf{u})$ in the ith row. Let us denote

$$\Phi_i(\mathbf{a}) = \left(a_1, \ldots, a_{i-1}, \frac{1}{a_i} R_i(\mathbf{a}), a_{i+1}, \ldots, a_d \right),$$

where $R_i(\mathbf{a}) = S_i^-(\mathbf{a})/{S_i}^+(\mathbf{a})$ only depends on the variables $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d$. Then for $1 \leq j \leq d$, the (i, j) entry of $M(\mathbf{v})$ is

$$\begin{aligned} \frac{1}{v_i} \frac{\partial v_i}{\partial x_j} &= \frac{u_i}{R_i(\mathbf{u})} \left(-\frac{1}{u_i^2} R_i(\mathbf{u}) \frac{\partial u_i}{\partial x_j} + \sum_{k \neq i} \frac{\partial R_i}{\partial a_k}(\mathbf{u}) \frac{\partial u_k}{\partial x_j} \right) \\ &= -\frac{1}{u_i} \frac{\partial u_i}{\partial x_j} + \frac{u_i}{R_i(\mathbf{u})} \sum_{k \neq i} \frac{\partial R_i}{\partial a_k}(\mathbf{u}) \frac{\partial u_k}{\partial x_j}. \end{aligned}$$

Upon subtracting from the ith row of $M(\mathbf{v})$ its kth row, multiplied by $u_i u_k \partial R_i / \partial a_k(\mathbf{u})/R_i(\mathbf{u})$, for $1 \leq k \neq i \leq d$, we see that $\det M(\mathbf{v})$ is also the determinant of the matrix obtained from $M(\mathbf{u})$ by changing the sign of all elements of the ith row, which concludes the proof. ∎

Example D (continued): large steps with $d = 2$. Let us take $\mathcal{S} = \{\bar{2}0, \bar{1}1, 02, 1\bar{1}\}$, so that

$$S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y}.$$

We will incrementally construct the orbit of (x, y). This example should provide the intuition for the algorithm given in the next subsection.

We start from (x, y) and want to determine which elements (X, y) are 1-adjacent to it; that is, to find the solutions to $S(X, y) = S(x, y)$ with $X \neq x$. We have

$$S(X, y) - S(x, y) = \frac{(X - x)\left(x^2 X^2 - y(1 + xy)X - xy\right)}{x^2 y X^2}.$$

Hence the two elements that are 1-adjacent to (x, y) are (x_1, y) and (x_2, y), where x_1 and x_2 are the two roots of $P_1(X, x, y) := x^2 X^2 - y(1 + xy)X - xy$ (when solved for X). The x_i's can be taken as Laurent series in x with coefficients in $\mathbb{Q}[y, \bar{y}]$:

$$x_1 = yx^{-2} + y^2 x^{-1} - x_2 \qquad \text{and} \qquad x_2 = -x + yx^2 - y^2 x^3 + \left(y^3 + \bar{y}\right) x^4 + O(x^5).$$

Similarly, we find that the two elements that are 2-adjacent to (x, y) are (x, y_1) and (x, y_2), where y_1 and y_2 are the roots of $Q_1(Y, x, y) := xyY^2 + y(1 + xy)Y - x^2$. But $Q_1(Y, x, y)$ coincides with $P_1(1/Y, x, y)$ (up to a factor of Y^2), thus we take $y_1 = \bar{x}_1 := 1/x_1$ and $y_2 = \bar{x}_2 := 1/x_2$. We have now obtained five elements in the orbit of (x, y) (one can follow the construction on Figure 4).

Now we want to find the elements (x_1, Y) that are 2-adjacent to (x_1, y). In principle, we should thus solve $S(x_1, Y) = S(x_1, y)(= S(x, y))$, but we prefer not to handle equations with algebraic coefficients (like x_1). So instead, we consider the *polynomial system*

$$P_1(X, x, y) = 0, \qquad S(X, Y) = S(x, y),$$

whose solutions (X, Y) are the pairs (x_i, Y) belonging to the orbit. Upon eliminating X between these two equations, we find that Y is necessarily either y, or $\bar{x}$, or one of the series $\bar{x}_i$. Upon checking that $S(x_1, \bar{x}_1) \neq S(x, y)$, we conclude that the two elements that are 2-adjacent to (x_1, y) are $(x_1, \bar{x})$ and $(x_1, \bar{x}_2)$. Symmetrically, $(x_2, \bar{x})$ and $(x_2, \bar{x}_1)$ are 2-adjacent to (x_2, y). We now have 9 elements in the orbit.

In order to find the elements that are 1-adjacent to $(x, \bar{x}_i)$, for $i = 1, 2$, we study similarly the polynomial system

$$Q_1(Y, x, y) = 0, \qquad S(X, Y) = S(x, y)$$

and conclude that $(\bar{y}, \bar{x}_1)$ and $(x_2, \bar{x}_1)$ are 1-adjacent to $(x, \bar{x}_1)$ while $(\bar{y}, \bar{x}_2)$ and $(x_1, \bar{x}_2)$ are 1-adjacent to $(x, \bar{x}_2)$. We have reached 11 elements.

At this stage, we still need one element that would be 1-adjacent to $(x_1, \bar{x})$ and $(x_2, \bar{x})$, and one element that would be 2-adjacent to $(\bar{y}, \bar{x}_1)$ and $(\bar{y}, \bar{x}_2)$. We address the first problem by solving $S(X, \bar{x}) = S(x, y)$, and find that $(\bar{y}, \bar{x})$ in fact solves both problems. The orbit is now complete, and contains 12 elements.

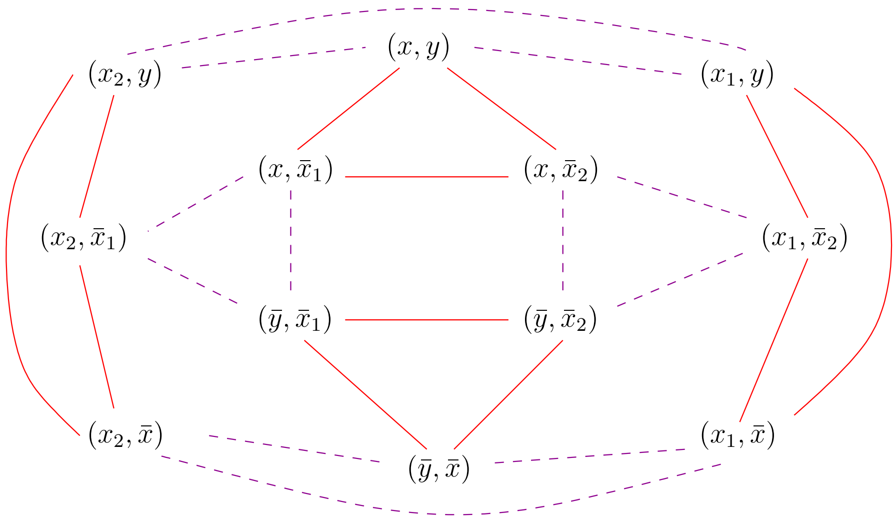


FIGURE 4. The orbit of $\mathcal{S} = \{\bar{2}0, \bar{1}1, 02, 1\bar{1}\}$. The values x_1 and x_2 are the roots of $P_1(X, x, y) = x^2X^2 - y(1 + xy)X - xy$. The values $\bar{x}_1$ and $\bar{x}_2$ are their reciprocals. The dashed (resp. solid) edges join 1-adjacent (resp. 2-adjacent) elements.

3.2. An algorithm that detects finite orbits (case $d = 2$)

Given a model in dimension $d = 2$ we now describe a (semi-)algorithm that stops if and only if the orbit is finite. This algorithm constructs incrementally two sets $\mathcal{P}$ and $\mathcal{Q}$ of irreducible polynomials in X and Y, respectively, with coefficients in $\mathbb{Q}(x, y)$. It starts with $\mathcal{P} = \{X - x\}$ and $\mathcal{Q} = \{Y - y\}$, and both polynomials are declared non-treated. At each stage, the algorithm chooses a non-treated polynomial in $\mathcal{P} \cup \mathcal{Q}$, say $Q \in \mathcal{Q}$, and constructs a new polynomial

$P'(X,x,y)$, which is the resultant in Y of $Q(Y,x,y)$ and the numerator of the Laurent polynomial $S(x,y)-S(X,Y)$ (namely $(xX)^{m_1}(yY)^{m_2}(S(x,y)-S(X,Y))$, where $-m_1$ is the smallest move in the x-direction and similarly for m_2). Then the algorithm adds to $\mathcal{P}$ every irreducible factor of P', and the new factors are declared non-treated. The algorithm treats symmetrically polynomials of $\mathcal{P}$. These stages are repeated as long as there are non-treated polynomials.

We recall that $\overline{\mathbb{K}}$ denotes an algebraic closure of $\mathbb{K} := \mathbb{Q}(x,y)$.

Proposition 6. *The following two properties hold at each stage of the algorithm:*

- (i) *the set $\mathcal{P}$ contains no element of $\mathbb{Q}[X]$; moreover, for $P \in \mathcal{P}$ and $x' \in \overline{\mathbb{K}}$ such that $P(x',x,y)=0$, there exists $Q \in \mathcal{Q}$ and $y' \in \overline{\mathbb{K}}$ such that $Q(y',x,y)=0$ and (x',y') is in the orbit of (x,y),*
- (ii) *symmetrically, the set $\mathcal{Q}$ contains no element of $\mathbb{Q}[Y]$; moreover, for $Q \in \mathcal{Q}$ and $y' \in \overline{\mathbb{K}}$ such that $Q(y',x,y)=0$, there exists $P \in \mathcal{Q}$ and $x' \in \overline{\mathbb{K}}$ such that $P(x',x,y)=0$ and (x',y') is in the orbit of (x,y).*

The algorithm stops if and only if the orbit of (x,y) is finite. In this case, the converse of (i) and (ii) holds, that is:

- (iii) *for every (x',y') in the orbit of (x,y), the minimal polynomials of x' and y' over $\mathbb{Q}(x,y)$ belong respectively to $\mathcal{P}$ and $\mathcal{Q}$.*

Note that the sets $\mathcal{P}$ and $\mathcal{Q}$ do not determine completely the orbit: one still has to decide, for each x' that solves a polynomial of $\mathcal{P}$, which y' (taken from the roots of the polynomials of $\mathcal{Q}$) go with it in the orbit, as was done in Example D above.

Proof. Let us first prove (i) and (ii), by induction on the number of stages performed by the algorithm. Both properties obviously hold at the initialization step, where $\mathcal{P}=\{X-x\}$ and $\mathcal{Q}=\{Y-y\}$.

Now assume that they hold at some stage, and that we treat a polynomial $Q \in \mathcal{Q}$ as described at the beginning of Section 3.2. Let us prove that the extended collections of polynomials still satisfy (i) and (ii). Clearly (ii) still holds, since we have not extended $\mathcal{Q}$. So let us check (i). It suffices to check it for the factors of P' that we have added to $\mathcal{P}$. So let us take one of these factors, and let x' be one of its roots. Then x' is a root of $P'(X,x,y)$. The properties of the resultant imply that there exists y' such that $Q(y',x,y)=0$ and

$$x'^{m_1}y'^{m_2}\left(S(x,y)-S(x',y')\right)=0, \tag{13}$$

where $-m_1$ (resp. $-m_2$) is the smallest move along the x-axis (resp. the y-axis). By Property (ii) applied to Q and y', there exists an element $x'' \in \overline{\mathbb{K}}$ such that (x'',y') is in the orbit. By Lemma 4, x'' and y' are algebraically independent over $\mathbb{Q}$, and in particular y' is not an algebraic number. If $x'=0$, then (13) tells us that the coefficient of x^{-m_1} in $S(x,y)$, evaluated at $y=y'$, vanishes, which would make y' algebraic, a contradiction. Thus $x' \neq 0$, $y' \neq 0$, and $S(x,y)=S(x',y')$. Hence $S(x',y')=S(x'',y')$, which shows that (x',y') is adjacent to (x'',y), and thus is in the orbit of (x,y). In particular, x' and y' are algebraically independent over $\mathbb{Q}$, thus $x' \notin \overline{\mathbb{Q}}$, which means that its minimal polynomial P is not in $\mathbb{Q}[X]$.

Now assume that the algorithm stops; that is, that there are no more non-treated polynomials. Let us prove (iii) by induction of the length of (x',y'). If $\ell=0$, then $(x',y')=(x,y)$ and we have precisely initialized $\mathcal{P}$ and $\mathcal{Q}$ with the minimal polynomials of x and y. Now assume that (iii) holds for length $\ell-1$, and that (x',y') has length ℓ. Without loss of generality, we can assume that $(x',y') \approx (x'',y')$, where (x'',y') has length $\ell-1$. By the induction hypothesis, the minimal polynomial Q of y' belongs to $\mathcal{Q}$, so we only need to consider x'. The polynomials (in Y) $Q(Y,x,y)$ and $X^{m_1}Y^{m_2}\left(S(x,y)-S(X,Y)\right)$ have a common root (namely y') when $X=x'$. Hence their resultant $P'(X,x,y)$ must have x' as a root. This implies that one of the factors of P' is the minimal polynomial of x', and this factor is added to $\mathcal{P}$ when the algorithm treats the polynomial Q (unless it was already in $\mathcal{P}$).

We have thus established (iii), assuming the algorithm stops. In this case $\mathcal{P}$ and $\mathcal{Q}$ are finite so (iii) implies that the orbit is finite.

Conversely, assume that the orbit is finite. By (i), every $P \in \mathcal{P}$ must be the minimal polynomial of some $x' \in \overline{\mathbb{K}}$ such that (x', y') is in the orbit for some y'. Hence $\mathcal{P}$ cannot grow indefinitely. A similar argument applies to $\mathcal{Q}$, and the algorithm has to stop. ■

3.3. Infinite orbits and the excursion exponent

We now describe an approach, of wide applicability, to prove that a model has an infinite orbit. It generalizes a fixed point argument applied to quadrant walks with small steps in [26, Thm. 3] (see also [35] for an application to 3D walks with small steps). It also constructs a group of transformations which generates part of the orbit of $\mathbf{x}$. In the 2-dimensional case, it establishes a connection with the asymptotic proof of non-D-finiteness developed in [19]. One outcome will be the following convenient criterion for 2-dimensional models.

Theorem 7. *Let $\mathcal{S} \subset \mathbb{Z}^2$ be a step set that is not contained in a half-plane, and contains an element of $\mathbb{N}^2$. Then the step polynomial $S(x,y)$ has a unique* critical point *(a,b) in $\mathbb{R}^2_{>0}$ (that is, a solution of $S_x(a,b) = S_y(a,b) = 0$), which satisfies $S_{xx}(a,b) > 0$ and $S_{yy}(a,b) > 0$. Define*

$$c = \frac{S_{xy}(a,b)}{\sqrt{S_{xx}(a,b)S_{yy}(a,b)}}.$$

Then $c \in [-1,1]$ can be written as $\cos\theta$. If θ is not a rational multiple of π, then the orbit of $\mathcal{S}$ is infinite, and the series $Q(x,y;t)$ is not D-finite.

Note that this result is algorithmic: the quantities a, b, c are algebraic over $\mathbb{Q}$ and one can compute their minimal polynomials. Saying that θ is a rational multiple of π amounts to saying that the solutions of $z + 1/z = 2c$ are roots of unity, so that their minimal polynomials are cyclotomic. This can be checked algorithmically. In Section 8 we apply this theorem systematically to the 13 110 models having steps in $\{-2,-1,0,1\}^2$ and at least one large step. Combined with the algorithm that detects finite orbits, it determines the size of the orbit for all but 16 models. (These 16 models turn out to have an infinite orbit, see Section 8.2.3).

The above theorem also shows that the calculations performed in [26] to prove that 51 small step models have an infinite group are equivalent to those performed in [19] to prove that these 51 models have a non-D-finite generating function.

3.3.1. **A group acting on the orbit.** We begin with the part of the above theorem that deals with the size of the orbit. In fact, we have a more general result that holds for models in d dimensions. So let $\mathcal{S} \in \mathbb{Z}^d$, and assume that there exists a point $\mathbf{a} := (a_1, \ldots, a_d)$ such that $S_{x_1}(\mathbf{a}) = \partial S/\partial x_1(a_1, \ldots, a_d) = 0$. If $I(X, \mathbf{x})$ denotes the Laurent polynomial

$$I(X, \mathbf{x}) = \frac{S(X, x_2, \ldots, x_d) - S(x_1, \ldots, x_d)}{X - x_1}$$

(after normalizing the rational function) then $I(a_1, \mathbf{a}) = S_{x_1}(\mathbf{a}) = 0$. Assume now that $S_{x_1x_1}(\mathbf{a}) \neq 0$, so that $I_X(a_1, \mathbf{a}) = S_{x_1x_1}(\mathbf{a})/2 \neq 0$. By the implicit function theorem (in its analytic form), there exists a unique analytic function $\mathcal{X}_1(x_1, \ldots, x_d)$ defined in a neighborhood of $\mathbf{a}$, satisfying $\mathcal{X}_1(\mathbf{a}) = a_1$ and

$$I(\mathcal{X}_1(\mathbf{x}), \mathbf{x}) = \frac{S(\mathcal{X}_1(\mathbf{x}), x_2, \ldots, x_d) - S(\mathbf{x})}{\mathcal{X}_1(\mathbf{x}) - x_1} = 0. \tag{14}$$

The expansion of $\mathcal{X}_1(\mathbf{x})$ around $\mathbf{a}$ can be computed inductively. Writing $\mathbf{x} = \mathbf{a} + \mathbf{u}$, we have

$$\mathcal{X}_1(\mathbf{x}) = a_1 - u_1 - \frac{2}{S_{x_1x_1}(\mathbf{a})} \sum_{i=2}^{d} S_{x_1x_i}(\mathbf{a}) u_i + \cdots, \tag{15}$$

the missing terms being of degree at least 2 in the u_i's. We define the transformation Φ_1 by $\Phi_1(\mathbf{x}) = (\mathcal{X}_1(\mathbf{x}), x_2, \ldots, x_d)$. Clearly, $\Phi_1(\mathbf{x})$ is 1-adjacent to $\mathbf{x}$ and thus lies in the orbit of $\mathbf{x}$

(which we construct in an algebraic closure of $\mathbb{Q}(\mathbf{x})$ containing power series in the u_i's). Since $\Phi_1(\mathbf{a}) = \mathbf{a}$, we can iterate Φ_1. In particular,

$$\Phi_1 \circ \Phi_1(\mathbf{x}) = (\mathcal{X}_1(\mathcal{X}_1(\mathbf{x}), x_2, \dots, x_d), x_2, \dots, x_d)$$

satisfies

$$S(\Phi_1 \circ \Phi_1(\mathbf{x})) = S(\Phi_1(\mathbf{x})) = S(\mathbf{x}),$$

by (14). Hence either $\Phi_1 \circ \Phi_1$ is the identity, or

$$I(\Phi_1 \circ \Phi_1(\mathbf{x})) = \frac{S(\Phi_1 \circ \Phi_1(\mathbf{x})) - S(\mathbf{x})}{\mathcal{X}_1(\mathcal{X}_1(\mathbf{x}), x_2, \dots, x_d) - x_1} = 0,$$

which means that the function $\tilde{\mathcal{X}_1} : \mathbf{x} \mapsto \mathcal{X}_1(\mathcal{X}_1(\mathbf{x}), x_2, \dots, x_d)$ satisfies the same conditions as $\mathcal{X}_1$. By uniqueness of $\mathcal{X}_1$, this would imply that $\tilde{\mathcal{X}_1} = \mathcal{X}_1$: but this is impossible as $\mathcal{X}_1(\mathbf{x})$ has linear part $a_1 - u_1 + \cdots$ while $\tilde{\mathcal{X}_1}$ has linear part $a_1 + u_1 + \cdots$ (by (15)). Hence Φ_1 is an involution.

Assume now that $\mathbf{a}$ is a critical point of S, that is, $S_{x_i}(\mathbf{a}) = 0$ for $i = 1, \dots, d$. Assume moreover that $S_{x_i x_i}(\mathbf{a}) \neq 0$ for all i. We then define similarly the transformations Φ_i for $i = 1, \dots, d$. Still writing $\mathbf{x} = \mathbf{a} + \mathbf{u}$, each Φ_i leaves the constant term of $\mathbf{x}$ unchanged, so we can compose them and they form a group G. For any Θ in this group, $\Theta(\mathbf{x})$ lies in the orbit of $\mathbf{x}$. If the orbit of $\mathbf{x}$ is finite, G is finite as well, and every $\Theta \in G$ has finite order. The expansion of Θ around $\mathbf{a}$ reads:

$$\Theta(\mathbf{a} + \mathbf{u}) = \mathbf{a} + \mathbf{u}\, J(\mathbf{a}) + \text{quadratic terms in the } u_i,$$

hence the Jacobian matrix $J(\mathbf{a})$ must have finite order. This means that its eigenvalues are roots of unity, which, once again, can be checked algorithmically.

We now restrict the discussion to the 2-dimensional case, in order to lighten notation. We denote $\Phi := \Phi_1$ and $\Psi := \Phi_2$, $\mathbf{x} = (x, y)$, $\mathbf{a} = (a, b)$ and $\mathbf{u} = (u, v)$. For $\Theta := \Psi \circ \Phi$, we have

$$J(a, b) := \begin{pmatrix} -1 & -\eta \\ \nu & \eta\nu - 1 \end{pmatrix}$$

where

$$\eta = \frac{2S_{xy}(a, b)}{S_{xx}(a, b)} \qquad \text{and} \qquad \nu = \frac{2S_{xy}(a, b)}{S_{yy}(a, b)}.$$

The eigenvalues of J are the roots of

$$\lambda^2 - (\eta\nu - 2)\lambda + 1$$

and, as the orbit is finite, they must equal $e^{\pm 2i\theta}$ for θ a rational multiple of π. That is,

$$\lambda^2 - (\eta\nu - 2)\lambda + 1 = (\lambda - e^{2i\theta})(\lambda - e^{-2i\theta}).$$

Extracting the coefficient of λ gives the following proposition.

Proposition 8. *Consider a two-dimensional model $\mathcal{S}$, and a critical point (a, b) of $S(x, y)$ such that $S_{xx}(a, b)S_{yy}(a, b) \neq 0$. Then one can define involutions Φ and Ψ as described above. If the orbit is finite, then $\Theta := \Psi \circ \Phi$ has finite order. In particular, there exists a rational multiple of π, denoted θ, such that*

$$\frac{S_{xy}(a, b)^2}{S_{xx}(a, b)S_{yy}(a, b)} = \cos^2\theta.$$

We can now prove the part of Theorem 7 that deals with the orbit size. Since $\mathcal{S}$ is not contained in a half-plane, there exists a unique *positive* critical point (a, b) (an argument is given in the proof of [19, Thm. 4]). The derivatives S_{xx} and S_{yy} are positive at this point (because every monomial $x^i y^j$ gives a non-negative contribution, and one of them at least gives a positive contribution), and thus the above proposition applies. ∎

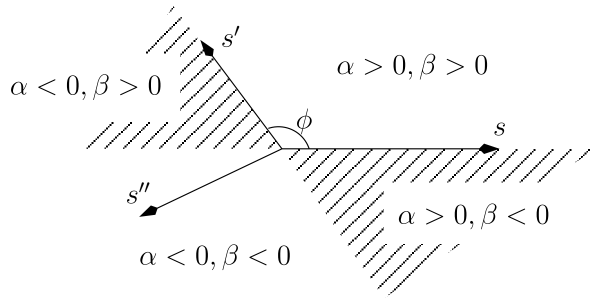


Figure 5. On the existence of excursions.

3.3.2. **The excursion exponent.** We will now show that in the 2-dimensional case, the above criterion is closely related to an asymptotic result that has been used as a criterion for the non-D-finiteness of $Q(x, y; t)$ in [19]. This result originally applies to *strongly aperiodic models* only, and it will take us a bit of work to obtain a version that is valid for periodic models as well. Given a model $\mathcal{S}$, we denote by Λ the lattice of $\mathbb{Z}^2$ spanned by its steps. Then $\mathcal{S}$ is *strongly aperiodic* if for any point $x \in \Lambda$, the lattice Λ_x spanned by the points $x + s$ for $s \in \mathcal{S}$, which is clearly a sublattice of Λ, coincides with Λ. For instance, Kreweras' model $\{\nearrow, \leftarrow, \downarrow\}$ is *not* strongly aperiodic: one has $\Lambda = \mathbb{Z}^2$, but for $x = (1, 0)$, the lattice Λ_x only contains points (i, j) such that $i + j$ is a multiple of 3.

Given a model $\mathcal{S}$, and a point (i, j) in $\mathbb{Z}^2$, we denote by $w(i, j; n)$ the number of n-step walks going from $(0, 0)$ to (i, j) consisting of steps taken in $\mathcal{S}$ *without the quadrant condition*. We call any walk starting and ending at the same point an *excursion*.

Proposition 9. *Let $\mathcal{S} \subset \mathbb{Z}^2$ be a model that is not contained in a half-plane, and denote by Λ the lattice of $\mathbb{Z}^2$ generated by $\mathcal{S}$. Then there exists an integer p, called the* period *of $\mathcal{S}$, such that for any $(i, j) \in \Lambda$, there exists $r \in [\![0, p-1]\!]$ with $w(i, j; n) = 0$ if $n \not\equiv r \mod p$ and $w(i, j; n) > 0$ if $n = mp + r$ and m is large enough.*

The model $\mathcal{S}$ is strongly aperiodic if and only if $p = 1$.

Proof. Several ingredients of the proof are borrowed from Spitzer [68, Sec. I.5], who deals with recurrent random walks and only considers the case $\Lambda = \mathbb{Z}^2$. The fact that all points (i, j) of Λ can be reached from $(0, 0)$ is closely related to Farkas' Lemma [67, Sec. 7.3].

Let $\mathcal{N} = \{n \geq 0 : w(0, 0; n) \neq 0\}$. Since one can concatenate two walks starting and ending at the origin, $\mathcal{N}$ is an additive semi-group of $\mathbb{N}$. Our first objective is to prove that it is not reduced to $\{0\}$, that is, that there exist non-empty excursions.

Let s be a non-zero vector of $\mathcal{S}$. Since $\mathcal{S}$ is not contained in a half-plane, there exists another non-zero vector of $\mathcal{S}$, say s', such that the wedge formed by the pair s, s' forms an angle $\phi \in (0, \pi)$. Let us choose s' so as to maximize ϕ in this interval (Figure 5). Since s and s' form a basis of $\mathbb{R}^2$, any other vector s'' of $\mathcal{S}$ can be written as $s'' = \alpha s + \beta s'$ for a unique pair $(\alpha, \beta) \in \mathbb{Q}^2$. Since $\mathcal{S}$ is not contained in a half-plane, there must exist a vector s'' in $\mathcal{S}$ such that α is negative. By maximality of ϕ, this vector is such that $\beta \leq 0$. Writing $\alpha = -a/d$ and $\beta = -b/d$ with a, d positive integers and b a non-negative integer, we conclude that $as + bs' + ds'' = 0$, which shows that the walk starting at the origin and formed of a copies of s, b copies of s' and d copies of s'' ends at the origin as well. Thus there exist non-empty excursions. Moreover, we have

$$-s = (a - 1)s + bs' + ds''.$$

Since a is a positive integer, and s is an arbitrary element of $\mathcal{S}$, this proves that the set of endpoints of walks starting at the origin is not only a semi-group of $\mathbb{Z}^2$ (again, by a concatenation argument), but in fact the entire lattice Λ.

We have established that $\mathcal{N} \neq \{0\}$. Let p be the greatest common divisor of the elements of $\mathcal{N}$. The structure of semi-groups of $\mathbb{N}$ are well-understood: $\mathcal{N} \subset p\mathbb{N}$, and $pm \in \mathcal{N}$ for any large enough m. We have thus proved the first statement of the proposition for $(i, j) = (0, 0)$, with $r = 0$.

Now let $(i, j) \in \Lambda$. We have proved above that there exists a walk going from $(0, 0)$ to (i, j). Assume that there are two such walks w and w', and choose a walk w'' from (i, j) to $(0, 0)$. Then both ww'' and $w'w''$ are excursions, hence they must have length 0 modulo p. Consequently, w and w' must have the same length modulo p, say r. Finally, by concatenating a large excursion to a walk ending at (i, j), we see that for m large enough, there is a walk of length $pm + r$ from the origin to (i, j).

The equivalence between strong aperiodicity and $p = 1$ can be proved by mimicking the corresponding part of the proof of Proposition P1 in [68, Sec. I.5]. ∎

In the following theorem, we assume that each step s of $\mathcal{S}$ is weighted by a positive weight ω_s. This means that the "number" $q(i, j; n)$ is actually the sum of the weights of all quadrant walks from $(0, 0)$ to (i, j), the weight of a walk being the product of the weights of its steps. In this context, the step polynomial is

$$S(x, y) = \sum_{s=(s_1,s_2)\in\mathcal{S}} \omega_s x^{s_1} y^{s_2}.$$

Definition 10. *Given a model $\mathcal{S} \subset \mathbb{Z}^2$, a point $(i, j) \in \mathbb{N}^2$ is* reachable from infinity *if there exists a quadrant walk that starts from a point $(k, \ell) \in (i, j) + \mathbb{Z}^2_{>0}$ and ends at (i, j).*

Note that in this case, (k, ℓ) itself is reachable from infinity. Moreover, upon concatenating several copies of the walk, we can find a starting point (k', ℓ') with arbitrarily large coordinates, and a quadrant walk from this point to (i, j). Finally, Proposition 9 implies that if $\mathcal{S}$ is not contained in a half-plane, then any point with large enough coordinates is reachable from infinity.

We can now complete the asymptotic result of Denisov and Wachtel [32] with a statement that holds in the periodic case.

Theorem 11. *Let $\mathcal{S} \subset \mathbb{Z}^2$ be a model that is not contained in a half-plane and contains an element of $\mathbb{N}^2$. Then the step polynomial $S(x, y)$ has a unique critical point (a, b) in $\mathbb{R}^2_{>0}$, which satisfies $S_{xx}(a, b) > 0$ and $S_{yy}(a, b) > 0$. Define*

$$\mu = S(a, b), \qquad c = \frac{S_{xy}(a, b)}{\sqrt{S_{xx}(a, b)S_{yy}(a, b)}} \qquad \text{and} \qquad \alpha = -1 - \pi/\arccos(-c).$$

Assume first that $\mathcal{S}$ is strongly aperiodic. Then if (i, j) is reachable from infinity, there exists a positive constant κ such that, as n goes to infinity,

$$q(i, j; n) \sim \kappa\, \mu^n n^\alpha.$$

If $\mathcal{S}$ is not strongly aperiodic and has period $p > 1$, define

$$\overline{\mathcal{S}} = \{s_1 + \cdots + s_p,\ (s_1, \ldots, s_p) \in \mathcal{S}^p\},$$

and let $\overline{\Lambda}$ be the lattice spanned by the vectors of $\overline{\mathcal{S}}$. Then if $(i, j) \in \overline{\Lambda}$ is reachable from infinity for $\mathcal{S}$, there exist positive constants κ_1 and κ_2 such that for $n = pm$ and m large enough,

$$\kappa_1\, \mu^n n^\alpha \le q(i, j; n) \le \kappa_2\, \mu^n n^\alpha. \tag{16}$$

We call α the excursion exponent.

Remarks
1. It is very likely that an asymptotic estimate holds as well in the periodic case (see [37, p. 3/4]), but the proof does not seem to be written down, and we will content ourselves with the above bounds.
2. The reachability condition, which is somewhat implicit in [32], is important. Consider for instance the (strongly aperiodic) model $\mathcal{S} = \{10, 01, 1\bar{1}, \bar{1}1, \bar{3}2, 2\bar{3}\}$. Then for $n > 0$,

$$q(0, 0; n) = 0 \quad \text{and} \quad q(1, 0; n) = 1,$$

while

$$q(1,1;n) \sim \kappa\, 6^n n^\alpha$$

with $\alpha = -1 - \pi/\arccos(7/8)$. The reason for these different asymptotic behaviours is that the points $(0,0)$ and $(1,0)$ are not reachable from infinity, while $(1,1)$ is. Similarly, any asymptotic result for quadrant walks starting from a given point (k,ℓ) should require that there exists a quadrant walk that starts from (k,ℓ) and ends in $(k,\ell)+\mathbb{Z}^2_{>0}$ (we say that (k,ℓ) *reaches infinity*). Given that we have assumed that $\mathcal{S}$ contains a point of $\mathbb{N}^2$ and is not included in a half-plane, this condition holds here for any (k,ℓ).

Proof of Theorem 11. In the aperiodic case, the proof can be copied verbatim from the proof of Theorem 4 in [19]. One considers an underlying random walk and normalizes it into a walk whose projections on the x- and y-axes are centered, reduced, and of covariance 0. The key result is then a local limit theorem of Denisov and Wachtel that applies to such walks [32, Thm. 6] (note that one should assume in that theorem that $V(x) > 0$ and $V'(y) > 0$, which holds if x reaches infinity and y is reached from infinity).

We thus focus on the periodic case. The idea is to consider p consecutive steps of a walk as a single generalized step to obtain a strongly aperiodic walk. More precisely, let us define $\overline{\mathcal{S}}$ as above, and define the weight of a step s of $\overline{\mathcal{S}}$ to be

$$\bar\omega_s = \sum_{\substack{(s_1,\ldots,s_p)\in\mathcal{S}^p\\ s_1+\cdots+s_p=s}} \omega_{s_1}\cdots\omega_{s_p}.$$

We denote with bars all quantities that deal with the model $\overline{\mathcal{S}}$. For instance, $\bar w(i,j;n)$ is the (weighted) number of walks going from $(0,0)$ to (i,j) in n steps taken from $\overline{\mathcal{S}}$. By the definition of p, we have $w(0,0;pn) = \bar w(0,0;n) > 0$ for n large enough, hence $\overline{\mathcal{S}}$ is strongly aperiodic (on the lattice $\overline{\Lambda}$ that it generates). Observe that

$$\bar S(x,y) = S(x,y)^p, \qquad (\bar a,\bar b) = (a,b), \qquad \bar\mu = \mu^p, \qquad \bar c = c \qquad \text{and} \qquad \bar\alpha = \alpha.$$

Note that if $(i,j) \in \overline{\Lambda}$ is reachable from infinity in the model $\mathcal{S}$, then it is also reachable from infinity in the model $\overline{\mathcal{S}}$. Since we will consider both models $\mathcal{S}$ and $\overline{\mathcal{S}}$ at the same time, we will often refer to a walk with steps in $\mathcal{S}$ as an *$\mathcal{S}$-walk.*

Upper bound. A quadrant walk from $(0,0)$ to (i,j) consisting of $n = pm$ steps of $\mathcal{S}$ can be seen as a quadrant walk from $(0,0)$ to (i,j) consisting of m steps of $\overline{\mathcal{S}}$ (the converse is not true in general: for instance, taking a step $(1,0)$ in $\overline{\mathcal{S}}$ may correspond to a sequence $(-1,0),(2,0)$ of steps of $\mathcal{S}$ and involve crossing the y-axis). Hence

$$q(i,j;pm) \le \bar q(i,j;m).$$

Since $\overline{\mathcal{S}}$ is strongly aperiodic, and (i,j) reachable from infinity in $\overline{\mathcal{S}}$, the right hand-side is asymptotic to $\kappa\,(\mu^p)^m m^\alpha$ for some positive κ, which gives the desired upper bound on $q(i,j;n)$.

Lower bound. Since $(0,0)$ reaches infinity, and (i,j) is reachable from infinity, we can pick two quadrant walks w_1 and w_2 satisfying the following conditions:

- w_1 goes from $(0,0)$ to a point $x = (i_1,j_1)$, whose coordinates are larger than pM, where M is the maximal norm of a step of $\mathcal{S}$. Moreover, w_1 has length pm_1;
- w_2 goes from some point $y = (i_2,j_2)$ to (i,j), and the coordinates of y are large enough for $y-x$ to be reachable from infinity in the model $\overline{\mathcal{S}}$ (in particular, $i_2 \ge i_1$ and $j_2 \ge j_1$). Moreover, w_2 has length pm_2.

Now take a quadrant walk w from $(0,0)$ to $y-x$ consisting of m elements of $\overline{\mathcal{S}}$: if we replace every step $\sigma = s_1+\cdots+s_p$ (with each $s_k \in \mathcal{S}$), by the sequence $s_1,\ldots,s_p$, the resulting walk $\tilde w$ may exit the quadrant. But it will remain in the translated quadrant $[-pM,\infty)^2$. Thus, if we translate $\tilde w$ so that it starts at x, it will remain in the quadrant $\mathbb{N}^2$, and end at y. Adding w_1 as a prefix and w_2 as a suffix gives a quadrant walk of length $n = p(m_1 + m_2 + m)$ ending at (i, j). Consequently,

$$q(i, j; n) \geq c\,\bar{q}(i_2 - i_1, j_2 - j_1; m)$$

for some positive constant c that depends on the weights of w_1 and w_2. Since $\overline{\mathcal{S}}$ is strongly aperiodic, and $y - x$ is reachable from infinity in this model, the right-hand side is asymptotic to some $\kappa\,(\mu^p)^m m^\alpha$, which gives the desired lower bound on $q(i, j; n)$. ∎

We can now conclude the proof of Theorem 7.

Proof of Theorem 7. We have already established the part that deals with the orbit size, so we focus on the nature of the series $Q(x, y; t)$. We assign weight $\omega_s = 1$ to every step of $\mathcal{S}$. Let $(i, j) \in \overline{\Lambda}$, with i and j large enough for (i, j) to be reachable from infinity. Then the bounds (16) on $q(i, j; n)$ hold (whether the model is periodic or not) with α irrational. The generating function $\sum_n q(i, j; n)t^n$ is the coefficient of $x^i y^j$ in $Q(x, y; t)$, and it is D-finite if $Q(x, y; t)$ is D-finite. In this case it must be a G-function [16, Sec. 2]. But the properties of these functions are incompatible with the existence of such bounds [16, Thm. 2], and thus $Q(x, y; t)$ cannot be D-finite. Indeed, it follows from the Katz-Chudnovsky-André theorem [2, 40] on the local structure of G-functions, combined with classical transfer theorems, that $q(i, j; n)$ needs to be asymptotically equivalent to a sum of terms of the form $\kappa \rho^n n^a (\log n)^b$ with only *rational* exponents a, and our exponent α must be one of these a's. ∎

3.3.3. **Examples.** We now illustrate the above results with five examples.

Example D (continued): a model with rational exponent and finite orbit. Let us take $S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y}$. The unique positive critical pair is $(a, b) = (3^{1/4}, 3^{-1/4})$. We have seen that the orbit of $\mathcal{S}$ is finite (Figure 4), and indeed,

$$c := \frac{S_{12}(a, b)}{\sqrt{S_{11}(a, b) S_{22}(a, b)}} = -\frac{1}{2} = \cos\frac{2\pi}{3}.$$

With the notation of Theorem 7, we have $\theta = 2\pi/3$, and by Theorem 11 the excursion exponent is $\alpha = -4$. The involutions Φ and Ψ defined in Section 3.3.1 satisfy

$$\Phi(a+u, b+v) = (a-u+\sqrt{3}v+\cdots, b+v) \qquad \Psi(a+u, b+v) = (a+u, b+u/\sqrt{3}-v+\cdots),$$

so that

$$\Theta(a + u, b + v) = (a - u + \sqrt{3}v + \cdots, b - u/\sqrt{3} + \cdots).$$

The matrix J is

$$J = \begin{pmatrix} -1 & \sqrt{3} \\ -1/\sqrt{3} & 0 \end{pmatrix},$$

its eigenvalues are $e^{\pm 2i\pi/3}$, and J^3 is the identity matrix. In fact, it can be checked that $\Theta^3 = \mathrm{id}$. This is reflected in Figure 4 by the existence of bicoloured hexagons. □

Example: a model with irrational exponent and infinite orbit. Now take $S(x, y) = \bar{x}^2 + y + x\bar{y}$. The unique positive critical pair is $(a, b) = (2^{2/5}, 2^{1/5})$. We have

$$c := \frac{S_{12}(a, b)}{\sqrt{S_{11}(a, b) S_{22}(a, b)}} = -\frac{1}{\sqrt{6}}.$$

Let us prove that this is not the cosine of a rational multiple θ of π. With $z = e^{i\theta}$, this would mean that $z + 1/z = -\sqrt{2/3}$, so that the minimal polynomial of z (and $1/z$) would be $z^4 + 4z^2/3 + 1$. This is not a cyclotomic polynomial, hence c is not of the requested form. We conclude from Theorem 7 that the orbit is infinite, and the series $Q(x, y; t)$ not D-finite. The excursion exponent is $\alpha = -1 - \pi/\arccos(1/\sqrt{6}) \sim -3.73\ldots$, and it is an irrational number.

The involutions Φ and Ψ satisfy

$$\Phi(a+u, b+v) = (a-u+2^{6/5}v/3+\cdots, b+v) \qquad \Psi(a+u, b+v) = (a+u, b+u/2^{1/5}-v+\cdots),$$

so that
$$\Theta(a+u,b+v) = (a-u+2^{6/5}v/3+\cdots, b-u/2^{1/5}-v/3+\cdots).$$
The matrix J is
$$J = \begin{pmatrix} -1 & 2^{6/5}/3 \\ -1/2^{1/5} & -1/3 \end{pmatrix}.$$
Its eigenvalues are the roots of $\lambda^2+4\lambda/3+1$, and thus are not roots of unity. In particular, the group generated by Φ and Ψ is infinite. □

The same argument proves that the walks of Figure 3 have an irrational excursion exponent $\alpha = -1-\pi/\arccos(1/\sqrt{5})$, and thus a non-D-finite generating function.

We will now consider three models that have a rational excursion exponent, but still an infinite orbit. We will prove this using the approach of Section 3.3.1, either by taking for (a,b) the positive critical point and pushing further the expansion of Θ, or by considering another critical point.

Example: a model with rational exponent but infinite orbit. Take $S(x,y) = x+y+\bar{x}+\bar{y}+x\bar{y}^2+\bar{x}^2y$. This is model #13 in Table 2 (Section 8.2.3). The unique positive critical pair is $(a,b) = (\sqrt{2},\sqrt{2})$. We have
$$c := \frac{S_{12}(a,b)}{\sqrt{S_{11}(a,b)S_{22}(a,b)}} = -\frac{1}{2} = \cos\frac{2\pi}{3}.$$
With the notation of Theorem 7, we have $\theta = 2\pi/3$. The excursion exponent is $\alpha = -4$.

If we start from the positive critical point $(a,b) = (\sqrt{2},\sqrt{2})$ to define the involutions Φ and Ψ, we find
$$\Phi(a+u,b+v) = (a-u+v+\cdots, b+v) \qquad \Psi(a+u,b+v) = (a+u, b+u-v+\cdots),$$
so that
$$\Theta(a+u,b+v) = (a-u+v+\cdots, b-u+\cdots).$$
The matrix J is
$$J = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$
Its eigenvalues are $e^{\pm 2i\pi/3}$, and $J^3 = \mathrm{id}$, so we cannot use the criterion of Theorem 7 to prove that the orbit is infinite. But let us push further the expansion of Θ. We have
$$\Phi(a+u,b+v) = \left(a-u+v+\frac{5}{8}au^2-\frac{5}{8}auv-\frac{1}{8}av^2-\frac{25}{32}u^3+\frac{7}{8}u^2v+\frac{1}{16}uv^2+\frac{1}{32}v^3+\cdots, b+v\right),$$
where the missing terms are of order 4 or more. A symmetric formula holds for Ψ. Hence
$$\Theta(a+u,b+v) = \left(a-u+v+\frac{5}{8}au^2-\frac{5}{8}auv-\frac{1}{8}av^2-\frac{25}{32}u^3+\frac{7}{8}u^2v+\frac{1}{16}uv^2+\frac{1}{32}v^3+\cdots,\right.$$
$$\left. b-u+\frac{1}{2}au^2+\frac{1}{4}auv-\frac{1}{4}av^2-\frac{1}{2}u^3-\frac{3}{8}u^2v+\frac{7}{16}v^3+\cdots\right).$$
We have already seen that Θ^3 comes close to being the identity – at least, it is the identity at first order. But in fact,
$$\Theta^3(a+u,b+v) = \left(a+u+\frac{1}{8}(u-2v)(u^2-uv+v^2)+\cdots, b+v-\frac{1}{8}(v-2u)(u^2-uv+v^2)+\cdots\right),$$
so that
$$\Theta^{3k}(a+u,b+v) = \left(a+u+\frac{k}{8}(u-2v)(u^2-uv+v^2)+\cdots, b+v-\frac{k}{8}(v-2u)(u^2-uv+v^2)+\cdots\right).$$
Thus Θ has infinite order, and by Proposition 8 the orbit is infinite. The nature of $Q(x,y;t)$ remains unknown.

An alternative way to prove infiniteness of the orbit for this model is to start from another critical point and use a first order argument rather than the above longer expansion. Let us take $(a,b) = (e^{5i\pi/6}, e^{i\pi/6})$. Then the involutions Φ and Ψ satisfy

$$\Phi(a+u,b+v) = \left(a - u - \frac{2-6i\sqrt{3}}{7}v + \cdots, b+v\right),$$

$$\Psi(a+u,b+v) = \left(a+u, b - v - \frac{2+6i\sqrt{3}}{7}u + \cdots\right),$$

so that

$$\Theta(a+u,b+v) = \left(a - u - \frac{2-6i\sqrt{3}}{7}v + \cdots, b + \frac{2+6i\sqrt{3}}{7}u + \frac{9}{7}v + \cdots\right).$$

The characteristic polynomial of the corresponding matrix J is $\lambda^2 - 2\lambda/7 + 1$, and its roots are not roots of unity. By Proposition 8, the orbit is infinite. $\square$

In the next example, the excursion exponent is again rational, and only the first method above (expanding Θ to higher order) works to prove infiniteness of the orbit.

Example: one more model with rational exponent but infinite orbit. Take $S(x,y) = xy + x\bar{y}^2 + \bar{x}^2 y$. This is model #2 in Table 2. The positive critical point is $(a,b) = (1,1)$, and $c = -1/2 = \cos(2\pi/3)$. The excursion exponent is again -4. Note that there is no quadrant excursion from $(0,0)$ to $(0,0)$, because this point is not reachable from infinity. But the asymptotic bounds (16) apply for instance for $(i,j) = (1,1)$ (with period $p = 3$).

We can prove that the orbit is infinite by expanding $\Theta := \Psi \circ \Phi$ up to order 3:

$$\Theta(1+u,1+v) = \Big(1 - u + v + \frac{4}{3}u^2 - \frac{4}{3}uv - \frac{2}{3}v^2 - \frac{16}{9}u^3 + 2u^2v + \frac{2}{3}uv^2 - \frac{1}{9}v^3 + \cdots,$$
$$1 - u + \frac{2}{3}u^2 + \frac{4}{3}uv - \frac{4}{3}v^2 + \frac{1}{9}u^3 - 3u^2v + \frac{1}{3}uv^2 + \frac{22}{9}v^3 + \cdots\Big)$$

which gives

$$\Theta^3(1+u,1+v) = \left(1 + u + \frac{2}{3}(u-2v)(u^2 - uv + v^2) + \cdots, 1 + v - \frac{2}{3}(v-2u)(u^2 - uv + v^2) + \cdots\right).$$

We conclude as above that all series $\Theta^{3k}(1+u,1+v)$ are distinct, and that the orbit is infinite.

Starting from another critical pair (a,b) does not make the argument shorter: for all possible choices, the transformation Θ^3 is the identity at linear order. $\square$

We conclude with a third model with a rational exponent but an infinite orbit. This one is symmetric in both coordinate axes. Recall that highly symmetric models *with small steps* behave nicely in any dimension: they have a finite orbit, a D-finite generating function, and explicit asymptotic enumeration is known [59]. But the large step, highly symmetric model of the next example has an infinite orbit. This cannot be proved starting from the positive critical point, because the corresponding involutions Φ and Ψ *do* generate a finite group. But taking another critical point works.

Example: a highly symmetric model with an infinite orbit. Take

$$S(x,y) = (x + \bar{x})(y + \bar{y}) + (x^2 + \bar{x}^2)(y^2 + \bar{y}^2).$$

The positive critical point is $(a,b) = (1,1)$, and $c = 0$. The excursion exponent is -3. The transformations Φ and Ψ defined from $a = b = 1$ are respectively $(x,y) \mapsto (\bar{x}, y)$ and $(x,y) \mapsto (x, \bar{y})$ and they *do* generate a finite group, of order 4. But let us consider instead the critical point $(a,b) = (i,i)$. Then

$$\Theta(a+u, b+b) = (a - u + v/2 + \cdots, b + u/2 - v + \cdots),$$

and the Jacobian matrix J has characteristic polynomial $\lambda^2 + 7\lambda/4 + 1$, which is not cyclotomic. Hence the orbit is infinite.

We do not know about the nature of the associated generating function, but the first 70 000 terms of the series $Q(0, 0)$ (modulo a prime) did not allow us to guess any recurrence relation for its coefficients. □

4. Section-free functional equations

In this section we consider step sets $\mathcal{S} \subset \mathbb{Z}^d$ such that the orbit of $\mathbf{x} = (x_1, \ldots, x_d)$ is finite. For every element $\mathbf{x}'$ of this orbit we can replace $\mathbf{x}$ by $\mathbf{x}'$ in the main functional equation defining $Q(\mathbf{x})$, as we did in (4). The resulting equation will be called an *orbit equation*[1]. As the left-hand side of the original functional equation is $K(\mathbf{x})Q(\mathbf{x})$, where $K(\mathbf{x}) = 1 - tS(\mathbf{x})$ is the kernel, the orbit equation associated with $\mathbf{x}'$ has left-hand side $K(\mathbf{x})Q(\mathbf{x}')$, because the kernel takes the same value for all elements in the orbit. On the right-hand side of the orbit equations are several specializations of the generating function Q, which we call *sections*. Due to the construction of the orbit, every section occurs at least in two orbit equations.

The next step in our approach is to form a linear combination of the orbit equations that is free from sections, if one exists, as was the case for (5). Once the main functional equation is written, and the (finite) orbit determined, section-free equations can be found by solving a linear system with coefficients in the algebraic closure of $\mathbb{Q}(x_1, \ldots, x_d)$. In all cases that we have examined, we find that a section-free equation exists (and sometimes several). However, we have not been able to find a generic form for section-free equations. Let us examine two simple examples; the first one shows that there can be multiple section-free combinations.

Example A (continued). We return to the one-dimensional step set $\mathcal{S} = \{\bar{1}, 2\}$. The step polynomial is $S(x) = \bar{x}+x^2$, and the elements x' of the orbit of x are the solutions of $S(x) = S(x')$. Hence the orbit is $\{x, x_1, x_2\}$, with

$$x_{1,2} = \frac{-x^2 \pm \sqrt{x\,(x^3+4)}}{2x}.$$

Substituting the three orbit elements into the functional equation (7) gives three orbit equations, each involving only one section (namely, $Q(0)$). There are several section-free linear combinations of the orbit equations. One of them is

$$K(x)\,(xQ(x) - x_1 Q(x_1)) = x - x_1, \tag{17}$$

another one is

$$K(x)\,(xQ(x) - x_2 Q(x_2)) = x - x_2, \tag{18}$$

and in fact any section-free equation is a linear combination of these two. □

Example B (continued). We now reverse the steps of the previous example and consider $\mathcal{S} = \{\bar{2}, 1\}$. The orbit of x consists of x, x_1 and x_2 with

$$x_{1,2} = \frac{1 \pm \sqrt{4\,x^3+1}}{2x^2}.$$

Substituting the orbit elements into (8) gives three orbit equations containing two sections, Q_0 and Q_1. There is, up to a multiplicative factor, a *unique* section-free linear combination of these three equations:

$$K(x)\left(\frac{x^2}{(x-x_1)(x-x_2)}Q(x) + \frac{x_1^2}{(x_1-x)(x_1-x_2)}Q(x_1) + \frac{x_2^2}{(x_2-x)(x_2-x_1)}Q(x_2)\right)$$
$$= \frac{x^2}{(x-x_1)(x-x_2)} + \frac{x_1^2}{(x_1-x)(x_1-x_2)} + \frac{x_2^2}{(x_2-x)(x_2-x_1)} = 1.$$

□

[1]In other papers, like [11], the *orbit equation* is what we call here the *section-free* equation. We hope that this change in the terminology will not cause any trouble.

The above two examples are instances of a more general result that applies to any 1-dimensional model.

Proposition 12. *Assume $d = 1$. Let $-m$ (resp. M) be the smallest (resp. largest) element in $\mathcal{S}$, and assume that $m \geq 0$ and $M > 0$. Then the orbit of x has cardinality $m + M$. The vector space of section-free equations consists of all linear combinations of*

$$K(x) \sum_{i=0}^{m} \frac{u_i^m}{\prod_{j \neq i}(u_i - u_j)} Q(u_i) = 1,$$

where $u_0, u_1, \dots, u_m$ are any distinct elements in the orbit of x.

This proposition will proved in Section 6. The number of ways of choosing the u_i's is $\binom{m+M}{m+1}$. These section-free equations are not always linearly independent (in Example A, the third equation of this type, which involves x_1 and x_2, is the difference of (17) and (18)). However, if the largest step is 1 (that is, $M = 1$), then Proposition 12 tells that there is a unique section-free equation (up to a multiplicative factor). This was observed in Example B, and seems to generalize to dimension 2.

Conjecture 13. *When $d = 2$ and the orbit is finite, there always exist non-trivial section-free linear combinations of the orbit equations. Moreover, if there is no large forward step, then there is a unique section-free combination, up to a multiplicative factor.*

Example. In some x/y-symmetric quadrant models, like Kreweras' model $\mathcal{S} = \{\nearrow, \leftarrow, \downarrow\}$, the orbit of (x, y) contains (y, x), and we want to clarify what we mean with the uniqueness of the section-free equation. The functional equation reads

$$K(x, y)Q(x, y) = 1 - t\bar{x}Q(0, y) - t\bar{y}Q(x, 0).$$

The orbit of (x, y) consists of 6 pairs:

$$(x, y), \quad (\bar{x}\bar{y}, y), \quad (\bar{x}\bar{y}, x), \quad (y, x), \quad (y, \bar{x}\bar{y}), \quad (x, \bar{x}\bar{y}).$$

A linear combination of the 6 orbit equations, with indeterminate weights $\alpha_1, \dots, \alpha_6$, involves 6 sections: three specializations of $Q(x, 0)$, and three of $Q(0, y)$. If we require the contribution of each to vanish, we find (up to a multiplicative factor) a unique solution for the α_i's, and thus a unique section-free equation:

$$K(x, y)\big(xyQ(x, y) - \bar{x}Q(\bar{x}\bar{y}, y) + \bar{y}Q(\bar{x}\bar{y}, x) - xyQ(y, x) + \bar{x}Q(y, \bar{x}\bar{y}) - \bar{y}Q(x, \bar{x}\bar{y})\big) = 0. \quad (19)$$

Note that the right-hand side (the so-called *orbit sum*) vanishes. The x/y-symmetry makes this equation trivial.

However, it makes sense to exploit the symmetry of the model in the functional equation, and to write:

$$K(x, y)Q(x, y) = 1 - t\bar{x}Q(y, 0) - t\bar{y}Q(x, 0).$$

Now a linear combination of the 6 orbit equations involves only 3 sections. If we want the contribution of each to vanish, we find a vector space of dimension 3 of solutions, generated by all equations of the form

$$K(x, y) \left(Q(x', y') - Q(y', x') \right) = 0,$$

for (x', y') in the orbit. Again, these equations are trivial. □

We now prove that Conjecture 13 holds in the case of small steps — and in fact, in arbitrary dimension.

Proposition 14. *If $\mathcal{S} \subset \{-1, 0, 1\}^d$ has positive and negative steps in every direction, and the associated orbit is finite, then there is a unique section-free linear combination of orbit equations, up to a multiplicative factor. It reads*

$$\sum_{\mathbf{u}} (-1)^{\ell(\mathbf{u})} K(\mathbf{u}) Q(\mathbf{u}) \prod_{i=1}^{d} u_i = \sum_{\mathbf{u}} (-1)^{\ell(\mathbf{u})} \prod_{i=1}^{d} u_i, \quad (20)$$

where the sum runs over all elements $\mathbf{u} = (u_1, \dots, u_d)$ *of the orbit and* $\ell(\mathbf{u})$ *is the length of* $\mathbf{u}$.

Proof. We consider the result of multiplying the functional equation (12) by the product of all variables $\prod_i x_i$:

$$K(\mathbf{x})Q(\mathbf{x}) \prod_i x_i = \prod_i x_i + t \sum_{\emptyset \neq I \subset [\![1,d]\!]} \left((-1)^{|I|} Q_I(\mathbf{x}) \sum_{\mathbf{s} \in \mathcal{S} : s_i = -1 \forall i \in I} \mathbf{x}^{\mathbf{s}} \prod_i x_i \right). \quad (21)$$

Note that, since the last sum is over all $\mathbf{s}$ such that $s_i = -1$ for $i \in I$, the monomial $\mathbf{x}^{\mathbf{s}} \prod_i x_i$ does not involve any of the x_i's for $i \in I$. The same holds for $Q_I(\mathbf{x})$. We now call any version of (21) instantiated at an orbit element an *orbit equation*.

Take $I = \{i\}$, with $1 \leq i \leq d$. For $\mathbf{u}$ in the orbit of $\mathbf{x}$, the section $Q_I(\mathbf{u})$ occurs in exactly two orbit equations: the equation obtained from $\mathbf{u}$, and the one obtained from $\mathbf{v} := \Phi_i(\mathbf{u})$, with Φ_i defined as in Proposition 5. Moreover, the coefficient of $Q_I(\mathbf{u})$ is the same in both equations (it does not depend on the ith coordinate of $\mathbf{u}$). Hence in a section-free linear combination of orbit equations, the weights of the equations associated with $\mathbf{u}$ and $\Phi_i(\mathbf{u})$ must be opposite. By transitivity, there cannot be more than one section-free equation. Moreover, in the small step case, the lengths of two adjacent elements differ by ± 1 (Proposition 5), and thus the only possible section-free equation is (20).

So let us form the linear combination of orbit equations having the same left-hand side as (20). For $\mathbf{u}$ in the orbit and $I \subset [\![1, d]\!]$, the section $Q_I(\mathbf{u})$ occurs (with the same weight) in all orbit equations obtained from elements $\mathbf{v}$ that only differ from $\mathbf{u}$ at positions of I. We can define on these elements an involution that changes the parity of the length (for instance $\Phi_{\min(I)}$). This implies that the coefficient of $Q_I(\mathbf{u})$ in the signed sum vanishes, and that we have indeed constructed a section-free equation. ∎

Of course, all the examples of this paper support Conjecture 13. The next example shows that the number of sections occurring in the orbit equations can be larger than the number of orbit equations, which makes the existence of section-free equations more surprising.

Example F: a model with small forward steps. Take $\mathcal{S} = \{10, \bar{1}0, \bar{2}1, 0\bar{1}\}$. Then the orbit of (x, y) consists of the following pairs:

$$\begin{array}{ccc} (x, y) & (x_1, y) & (x_2, y) \\ (x, x^2\bar{y}) & (-\bar{x}_1, x^2\bar{y}) & (-\bar{x}_2, x^2\bar{y}) \\ (x_1, x_1^2\bar{y}) & (-\bar{x}, x_1^2\bar{y}) & (-\bar{x}_2, x_1^2\bar{y}) \\ (x_2, x_2^2\bar{y}) & (-\bar{x}, x_2^2\bar{y}) & (-\bar{x}_1, x_2^2\bar{y}) \end{array} \quad (22)$$

where

$$x_{1,2} = \frac{x + y \pm \sqrt{(x+y)^2 + 4x^3y}}{2x^2}$$

and $\bar{x}_i = 1/x_i$. The structure of this orbit is the first shown in Figure 10 (Section 8). The functional equation reads

$$K(x, y)Q(x, y) = 1 - t\bar{x}(1 + \bar{x}y)Q_{0,-}(y) - t\bar{x}yQ_{1,-}(y) - t\bar{y}Q(x, 0). \quad (23)$$

The 12 orbit equations involve in total $6 + 4 + 4 = 14$ distinct sections: 6 specializations of $Q(x, 0)$, 4 specializations of $Q_{0,-}(y)$ and 4 specializations of $Q_{1,-}(y)$. Hence in order to find a section-free equation, we need to solve a linear system with 14 equations but only 12 unknowns. Still, we find a solution (and only one, up to a multiplicative factor). The weight of the orbit equation associated with the pair (x', y') is

$$\pm x'^2(x'_1 - x'_2)\sqrt{yy'},$$

where $(x_i', y') \approx (x', y')$ for $i = 1, 2$, and $x_1' \neq x_2'$. More precisely, the weights associated with the above 12 orbit elements are

$$\begin{array}{ccc} x^2(x_1 - x_2)y & x_1^2(x_2 - x)y & -x_2^2(x_1 - x)y \\ x^2(\bar{x}_2 - \bar{x}_1)x & -\bar{x}_1^2(x + \bar{x}_2)x & \bar{x}_2^2(x + \bar{x}_1)x \\ x_1^2(\bar{x} - \bar{x}_2)x_1 & \bar{x}^2(x_1 + \bar{x}_2)x_1 & -\bar{x}_2^2(x_1 + \bar{x})x_1 \\ -x_2^2(\bar{x} - \bar{x}_1)x_2 & -\bar{x}^2(x_2 + \bar{x}_1)x_2 & \bar{x}_1^2(x_2 + \bar{x})x_2. \end{array} \tag{24}$$

□

Example D (continued): a model with large forward and backward step. Let us take $\mathcal{S} = \{\bar{2}0, \bar{1}1, 02, 1\bar{1}\}$. Recall that the orbit of (x, y) is shown in Figure 4, with

$$x_{1,2} = \frac{xy^2 + y \pm \sqrt{y\,(x^2y^3 + 4\,x^3 + 2\,xy^2 + y)}}{2x^2}.$$

The functional equation for this model is given by (9), and the 12 orbit equations involve $4 + 4 + 4 = 12$ sections. The vector space of section-free linear combinations has dimension 2; it is generated by two linear combinations of 9 orbit equations:

$$x^2y\,Q(x,y) - \frac{{x_1}^2y\,(x - x_2)\,Q(x_1,y)}{x_1 - x_2} + \frac{{x_2}^2y\,(x - x_1)\,Q(x_2,y)}{x_1 - x_2} - x^2\bar{x}_1\,Q(x,\bar{x}_1)$$
$$+ \frac{{x_2}^2\,(xy - 1)\,Q(x_2,\bar{x}_1)}{x_1\,(x_2\,y - 1)} - \frac{(x - x_2)\,Q(\bar{y},\bar{x}_1)}{yx_1\,(x_2\,y - 1)} + \frac{{x_1}^2\,(x - x_2)\,Q(x_1,\bar{x})}{x\,(x_1 - x_2)}$$
$$- \frac{{x_2}^2\,(x_1\,y - 1)\,(x - x_2)\,Q(x_2,\bar{x})}{x\,(x_1 - x_2)\,(x_2\,y - 1)} + \frac{(x - x_2)\,Q(\bar{y},\bar{x})}{xy\,(x_2\,y - 1)} = \frac{(1 - xy)\,(1 - x_1y)\,(x - x_1)\,(x - x_2)}{xyx_1K(x,y)},$$

and the same equation with x_1 and x_2 exchanged. We refer to [24] for the solution of a family of models with arbitrarily large steps which generalizes this one.

5. Extracting the main generating function

We now assume that, for a step set $\mathcal{S}$ with a finite orbit, we have obtained one (or several) section-free functional equations. Can we extract from these equations the main generating function $Q(x_1, \dots, x_d)$, as we did in Section 1.1? Not systematically, as we already learnt from some small step models.

Example C: Gessel's walks (continued). The orbit of (x, y) consists of 8 elements. The steps are small, hence the unique section-free equation is the alternating sum (20). Remarkably, its right-hand side vanishes:

$$xyQ(x,y) - \bar{x}Q(\bar{x}\bar{y},y) + xQ(\bar{x}\bar{y},x^2y) - xyQ(\bar{x},x^2y)$$
$$+ \bar{x}\bar{y}Q(\bar{x},\bar{y}) - xQ(xy,\bar{y}) + \bar{x}Q(xy,\bar{x}^2\bar{y}) - \bar{x}\bar{y}Q(x,\bar{y}\bar{x}^2) = 0.$$

This homogeneous equation does not characterize $Q(x, y)$. For instance, 1, x, xy, and $y - x^2$ are solutions. The space of solutions is actually infinite dimensional, as it clearly contains all monomials x^iy^i. □

Among the 23 quadrant models with small steps that have a finite orbit, exactly 4 have a section-free equation that does not characterize $Q(x, y)$: Gessel's model, as just shown, and the three Kreweras like models: $\mathcal{S} = \{\nearrow, \leftarrow, \downarrow\}$, its reverse $\overline{\mathcal{S}} = \{\swarrow, \rightarrow, \uparrow\}$ and the union $\mathcal{S} \cup \overline{\mathcal{S}}$ [26]. For those three, the orbit of (x, y) contains (y, x), and the section-free equation is (19). Clearly, any symmetric series in x and y satisfies this equation.

For these four models, the *orbit sum*, that is, the right-hand side of the section-free equation, vanishes. However, there exist as well (weighted) models with a non-vanishing orbit sum, for which the section-free equation does not characterize $Q(x, y)$. Let us recall an example taken from [11, Sec. 8.2].

Example. Take $\mathcal{S} = \{\bar{1}\bar{1}, \bar{1}1, \bar{1}0, \bar{1}0, 10, 11\}$ (note the repeated West step). The step polynomial is

$$S(x, y) = (1 + y)\left(\bar{x}(1 + \bar{y}) + x\right).$$

The orbit of (x, y) contains 6 elements, and the unique section-free equation reads:

$$\begin{aligned} xyQ(x,y) - \bar{x}(1+y)Q(\bar{x}(1+\bar{y}), y) + \frac{x(1+y)}{(1+y)^2+x^2y^2}\, Q\left(\bar{x}(1+\bar{y}), \frac{x^2y}{(1+y)^2+x^2y^2}\right) \\ - \frac{xy(1+y+x^2y)}{(1+y)^2+x^2y^2}\, Q\left(\bar{x}(1+y)+xy, \frac{x^2y}{(1+y)^2+x^2y^2}\right) \\ + \frac{\bar{x}\bar{y}(1+y+x^2y)}{1+x^2} Q\left(\bar{x}(1+y)+xy, \frac{\bar{y}}{1+x^2}\right) - \frac{x\bar{y}}{1+x^2}\, Q\left(x, \frac{\bar{y}}{1+x^2}\right) \\ = \frac{\left(1+y(1-x^2)\right)\left(1-y^2(1+x^2)\right)\left(1-x^2+y(1+x^2)\right)}{xy(1+x^2)K(x,y)\left((1+y)^2+x^2y^2\right)}. \end{aligned}$$

The right-hand side is non-zero, but this equation does not define $Q(x, y)$ uniquely in the ring $\mathbb{Q}[x, y][[t]]$. In fact, the associated homogeneous equation (in $Q(x, y)$) seems to have an infinite dimensional space of solutions. It includes at least the following polynomials in x and y:

$$x, \quad 2xy + x^3y, \quad x^2y + x^2 + y + 2, \quad x^3y^2 - x^3y + x^3 + 2xy^2.$$

□

We now consider examples where the series $Q(\mathbf{x})$ is indeed characterized by a section-free equation, but for which the extraction is not as simple as in Section 1.1. Our first example is one-dimensional.

Example A (continued). It can be seen that (17) (or (18)) characterizes $Q(x)$, but how can we extract it effectively? Here is one solution.

Take the first of these two linear combinations, written as

$$Q(x) - \bar{x}x_1Q(x_1) = \frac{1 - \bar{x}x_1}{K(x)}$$

with $K(x) = 1 - t(\bar{x} + x^2)$, and choose for the algebraic closure of $\mathbb{Q}(x)$ the set of Puiseux series in $\bar{x}$ (not in x!). Then

$$x_1 = \frac{\sqrt{4\,\bar{x}^3 + 1} - 1}{2\bar{x}} = \bar{x}^2 - \bar{x}^5 + O(\bar{x}^8)$$

is a formal power series in $\bar{x}$. Now both sides of the above section-free equation are series in t whose coefficients are Laurent series in $\bar{x}$. Extracting the non-negative part in x gives:

$$Q(x) = [x^{\geq}]\frac{1 - \bar{x}x_1}{K(x)},$$

where the right-hand side is first expanded in t, then in $\bar{x}$. This will be generalized to arbitrary one-dimensional models in Section 6 (Proposition 19). □

In our next example, one simply has to extract the positive part of a rational series to obtain $Q(x, y)$, but justifying why is a bit delicate.

Example F (continued). Let $\mathcal{S} = \{10, \bar{1}0, \bar{2}1, 0\bar{1}\}$. The functional equation is given by (23), the orbit by (22) and the weights in the section-free linear combination by (24). Let us divide this linear combination by $x^2y(x_1 - x_2)K(x, y)$, so as to isolate $Q(x, y)$. The resulting equation reads

$$Q(x, y) + x\bar{x}_1\bar{x}_2\bar{y}\, Q(x, x^2\bar{y}) + A_1 + A_2 + A_3 + A_4 + A_5 = R(x, y) \tag{25}$$

with

$$
\begin{aligned}
A_1 &= \bar{x}^2\,\frac{x_1^2(x_2-x)Q(x_1,y)-x_2^2(x_1-x)Q(x_2,y)}{x_1-x_2}\\
A_2 &= -\bar{x}^2\bar{y}\,\frac{\bar{x}_1^2(x+\bar{x}_2)xQ(-\bar{x}_1,x^2\bar{y})-\bar{x}_2^2(x+\bar{x}_1)xQ(-\bar{x}_2,x^2\bar{y})}{x_1-x_2}\\
A_3 &= \bar{x}^2\bar{y}\,\frac{x_1^3(\bar{x}-\bar{x}_2)Q(x_1,x_1^2y)-x_2^3(\bar{x}-\bar{x}_1)Q(x_2,x_2^2y)}{x_1-x_2}\\
A_4 &= \bar{x}^2\bar{y}\,\frac{\bar{x}^2(x_1+\bar{x}_2)x_1Q(-\bar{x},x_1^2\bar{y})-\bar{x}^2(x_2+\bar{x}_1)x_2Q(-\bar{x},x_2^2\bar{y})}{x_1-x_2}\\
A_5 &= -\bar{x}^2\bar{y}\,\frac{\bar{x}_2^2(x_1+\bar{x})x_1Q(-\bar{x}_2,x_1^2\bar{y})-\bar{x}_1^2(x_2+\bar{x})x_2Q(-\bar{x}_1,x_2^2\bar{y})}{x_1-x_2}
\end{aligned}
\tag{26}
$$

and

$$
R(x,y)=\frac{\left(x^2+1\right)(x+y)(y-x)\left(x^2y-2\,x-y\right)\left(x^3-x-2\,y\right)}{x^7y^3\left(1-t(x+\bar{x}+\bar{x}^2y+\bar{y})\right)}.
$$

Each term in (25) is written as a power series in t whose coefficients are Laurent polynomials in x, y, x_1 and x_2, symmetric in x_1 and x_2 (because the numerators of the series A_i are *anti-symmetric* in x_1 and x_2). Observe that the symmetric functions of x_1 and x_2 are Laurent polynomials in x and y, and more precisely, polynomials in $\bar{x}\mathbb{Q}[\bar{x},y,\bar{y}]$ (we say that they are *x-negative*):

$$
x_1+x_2=\bar{x}(1+\bar{x}y)\quad\text{and}\quad x_1x_2=-\bar{x}y. \tag{27}
$$

The symmetric functions of their reciprocals are Laurent polynomials in x and y, and more precisely, polynomials in $\mathbb{Q}[x,\bar{x},\bar{y}]$ (we say that they are *y-non-positive*):

$$
\bar{x}_1+\bar{x}_2=-\bar{x}-\bar{y}\quad\text{and}\quad \bar{x}_1\bar{x}_2=-x\bar{y}. \tag{28}
$$

Hence every term of (25) is a series in t whose coefficients are Laurent polynomials in x and y. We claim that extracting from the left-hand side of (25) the non-negative part in x and y gives $Q(x,y)$. First, the second term of (25) is y-negative, and hence does not contribute. Then

$$
A_1=\bar{x}\,\frac{x_1^2(\bar{x}x_2-1)Q(x_1,y)-x_2^2(\bar{x}x_1-1)Q(x_2,y)}{x_1-x_2},
$$

and is x-negative by (27). Using $xx_1x_2=-y$, we see that the same holds for

$$
A_3=\bar{x}\,\bar{y}\,\frac{x_1^3(\bar{x}^2+x_1\bar{y})Q(x_1,x_1^2y)-x_2^3(\bar{x}^2+x_2\bar{y})Q(x_2,x_2^2y)}{x_1-x_2},
$$

and for

$$
A_4=\bar{x}^2\bar{y}\,\frac{\bar{x}x_1^2(\bar{x}-\bar{y})Q(-\bar{x},x_1^2\bar{y})-\bar{x}x_2^2(\bar{x}-\bar{y})Q(-\bar{x},x_2^2\bar{y})}{x_1-x_2}.
$$

We are left with two terms. One is

$$
A_2=-\bar{x}\,\bar{y}^2\,\frac{\bar{x}_1^2(x+\bar{x}_2)xQ(-\bar{x}_1,x^2\bar{y})-\bar{x}_2^2(x+\bar{x}_1)xQ(-\bar{x}_2,x^2\bar{y})}{\bar{x}_1-\bar{x}_2},
$$

which is y-negative by (28). The other is A_5, which looks more challenging because the variables in the series Q mix positive and negative powers of the x_i's. Its analysis requires the following lemma.

Lemma 15. *For $a\geq 0$, the expression*

$$
E_a:=\frac{x_1^{a+1}-x_2^{a+1}}{x_1-x_2} \tag{29}
$$

is a polynomial in $\bar{x}$ and y. Every monomial $\bar{x}^ey^f$ that occurs in it satisfies $f\leq e$.

Proof. By induction on $a\geq 0$, using $E_{-1}=0$, $E_0=1$, $E_a=(x_1+x_2)E_{a-1}-x_1x_2E_{a-2}$ and (27). ∎

Let us return to the expression (26) of A_5. Since $Q(x,y)$ is a series in t with polynomial coefficients in x and y, it suffices to prove that, for $i,j \geq 0$, the term obtained by replacing $Q(x,y)$ by $x^i y^j$, namely

$$\pm \bar{x}^2 \bar{y}\, \frac{\bar{x}_2^2(x_1+\bar{x})x_1\bar{x}_2^i x_1^{2j}\bar{y}^j - \bar{x}_1^2(x_2+\bar{x})x_2\bar{x}_1^i x_2^{2j}\bar{y}^j}{x_1-x_2},$$

has no non-negative part in x and y. By splitting the sum and using $xx_1x_2=-y$, it suffices to prove this for

$$\bar{x}^2\bar{y}^{j+1}\,\frac{\bar{x}_2^{2+i}x_1^{2+2j}-\bar{x}_1^{2+i}x_2^{2+2j}}{x_1-x_2} = (-1)^i x^i \bar{y}^{i+j+3}\,\frac{x_1^{4+i+2j}-x_2^{4+i+2j}}{x_1-x_2} \tag{30}$$

and for

$$\bar{x}^3\bar{y}^{j+1}\,\frac{\bar{x}_2^{i+2}x_1^{1+2j}-\bar{x}_1^{i+2}x_2^{1+2j}}{x_1-x_2} = (-1)^{i+1} x^{i-1} \bar{y}^{i+j+3}\,\frac{x_1^{3+i+2j}-x_2^{3+i+2j}}{x_1-x_2}. \tag{31}$$

By Lemma 15, any monomial $x^a y^b$ that occurs in (30) satisfies

$$a = i-e, \qquad b = f-i-j-3,$$

with $f \leq e$. Saying that a and b are both non-negative means that $e \leq i$ and $f \geq i+j+3$, so that

$$e+j+3 \leq f \leq e,$$

which is impossible for $j \geq 0$. A similar argument proves that (31) contains no monomial that would be non-negative in x and in y. So the non-negative part of the left-hand side of (25) is indeed $Q(x,y)$. This tricky extraction deserves a proposition.

Proposition 16. *The generating function $Q(x,y)$ of quadrant walks with steps in $\mathcal{S}=\{10,\bar{1}0,0\bar{1},\bar{2}1\}$ is the non-negative part (in x and y) of the rational series*

$$R(x,y) = \frac{\left(x^2+1\right)(x+y)(y-x)\left(x^2y-2\,x-y\right)\left(x^3-x-2\,y\right)}{x^7y^3\left(1-t(x+\bar{x}+\bar{x}^2y+\bar{y})\right)},$$

seen as a power series in t with coefficients in $\mathbb{Q}[x,\bar{x},y,\bar{y}]$.

From this, one can derive interesting results for the specialization $Q(0,0)$ counting excursions.

Corollary 17. *For $\mathcal{S}=\{10,\bar{1}0,0\bar{1},\bar{2}1\}$, the sequence $e_n := q(0,0;2n)$ counting excursions satisfies a linear recurrence relation of order 2:*

$$(n+3)(n+2)(n+1)e_n = 12(2\,n-1)(2\,n-3)(n-1)e_{n-2} + 4(2\,n-1)(n+2)(n+1)e_{n-1},$$

with $e_0=e_1=1$. It is not hypergeometric.

The associated generating function admits an expression in terms of hypergeometric series:

$$Q(0,0) = \frac{3}{4t} + \frac{9t-2}{2t^2}\int \frac{(1+4t)^{3/2}}{(9t-2)^2}\left({}_2F_1\left(\begin{matrix}-\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\, \frac{16\,t}{1+4t}\right) + 2\times {}_2F_1\left(\begin{matrix}-\frac{1}{2} & \frac{3}{2}\\ & 3\end{matrix}\,\middle|\, \frac{16\,t}{1+4t}\right)\right).$$

Proof. (sketched) The recurrence relation is easily guessed from the first few values of e_n. It can be proved using computer algebra and the approach of [14]. The idea is to write $Q(0,0)$ as the constant coefficient (w.r.t. x and y) of the rational function $R(x,y)$, then to apply creative telescoping techniques. This proves that $Q(0,0)$ satisfies an explicit linear differential equation of order 4, from which the validity of the above linear recurrence relation for e_n is easily deduced. The fact that the sequence (e_n) is not hypergeometric follows from Petkovšek's algorithm [63].

The use of ${}_2F_1$ solving algorithms [15, 48, 14] then provides a closed-form expression of $Q(0,0)$.

∎

6. The one-dimensional case revisited

So far we have only studied sporadic models. We now consider a family of models, namely general one-dimensional models. We take $\mathcal{S} \subset \mathbb{Z}$ and denote by $-m$ (resp. M) the smallest (resp. largest) step of $\mathcal{S}$; to avoid trivial cases we assume $m \geq 0$ and $M > 0$. Finally, we allow step weights taken in some algebraically closed field $\mathbb{F}$ of characteristic zero. The indeterminates t and x are algebraically independent over $\mathbb{F}$. The step polynomial is then

$$S(x) = \sum_{i \in \mathcal{S}} w_i x^i,$$

where w_i is the weight of the step i. The weight of a walk is the product of the weights of its steps.

Let us first recall the standard solution, originally obtained by Gessel [43] (see also [27, Ex. 3] and [6]). It involves auxiliary series X_i, which are fractional series in the length variable t, algebraic over $\mathbb{F}(t)$.

Proposition 18. *The kernel $K(x) = 1 - tS(x)$, when solved for x, admits $m + M$ roots, which are Puiseux series in t with coefficients in $\mathbb{F}$. Exactly m of these roots, denoted $X_1, \dots, X_m$, are finite at $t = 0$ (and in fact, vanish at $t = 0$). Let us denote by $X_{m+1}, \dots, X_{m+M}$ the other ones.*

The generating function $Q(x;t) \equiv Q(x)$ is

$$Q(x) = \frac{\prod_{i=1}^{m}(1 - \bar{x}X_i)}{K(x)} = -\frac{1}{tw_M} \prod_{i=m+1}^{m+M} \frac{1}{x - X_i}. \tag{32}$$

We recall the proof given in [27, Ex. 3] or [6], for comparison with the approach of this paper. Roughly speaking, the standard solution is obtained by *canceling the kernel* by appropriate specializations of x, while the approach of this paper is more algebraic and consists in playing with certain invariance properties of the kernel.

Proof. The statements of the proposition dealing with the roots of the kernel come from the fact that the equation $K(x) = 0$, once written as a polynomial equation in x (that is, as $x^m K(x) = 0$), has degree $m + M$ in x, reducing to m when $t = 0$ (see [69, Prop. 6.1.8]).

Let us write $Q(x) = \sum_{i \geq 0} x^i Q_i$, where Q_i counts walks ending at abscissa i. The functional equation reads

$$K(x)Q(x) = 1 - \sum_{k=-m}^{-1} x^k G_k, \tag{33}$$

where

$$G_k = t \sum_{i \in \mathcal{S}, i \leq k} w_i Q_{k-i}.$$

So we have m unknown series $G_{-1}, \dots, G_{-m}$ (or equivalently, $Q_0, \dots, Q_{m-1}$) on the right-hand side of the functional equation. When we replace x by X_i in (33), for $1 \leq i \leq m$, both the left and right-hand sides vanish (we only use the "small" roots $X_1, \dots, X_m$, because the substitution by a root involving negative powers of t may be undefined). But the right-hand side is a polynomial in $\bar{x}$, of degree m and constant term 1. Hence it must be equal to $\prod_{i=1}^{m}(1 - \bar{x}X_i)$, and this gives the first expression of $Q(x)$. The second one follows by factoring $K(x)$ as

$$K(x) = -tw_M \prod_{i=1}^{m}(1 - \bar{x}X_i) \prod_{i=m+1}^{m+M} (x - X_i). \tag{34}$$

(The factor $-tw_M$ is obtained by extracting the coefficient of x^M in $K(x)$.) ∎

We now present the expression provided by the method of this paper. Rather than algebraic series in t (the X_i's), it involves algebraic series in $\bar{x}$ (denoted by x_i), and then the extraction of a non-negative part. Admittedly, it is not as attractive as the standard solution. In particular, it does not make the algebraicity of $Q(x)$ clear, unless the largest step is 1. But we show later how to recover the standard solution from it. One surprising feature of this solution is that, as foreseen in Example A, it involves expansions in $\bar{x}$ rather than x.

Proposition 19. *The equation $S(X) = S(x)$ (when solved for X) admits $m + M$ roots, which can be taken in the field of Puiseux series in $\bar{x} := 1/x$ with coefficients in $\mathbb{F}$. Exactly m of these roots, denoted $x_1, \dots, x_m$, contain no positive power of x (and, in fact, have no constant term either).*

The generating function $Q(x;t) \equiv Q(x)$ is

$$Q(x) = [x^{\geq}] \frac{\prod_{j=1}^{m}(1 - \bar{x}x_j)}{K(x)}, \tag{35}$$

where the right-hand side is expanded first in t, then in $\bar{x}$.

If the largest step of $\mathcal{S}$ is $M = 1$ the right-hand side of (35) is rational, and

$$Q(x) = [x^{\geq}] \frac{S'(x)}{w_1 K(x)}. \tag{36}$$

We will use the following lemma, which is a simple application of the Lagrange interpolation formula [23, Lemma 13].

Lemma 20. *Let $u_0, u_1, \dots, u_m$ be $m + 1$ variables. Then*

$$\sum_{i=0}^{m} \frac{{u_i}^d}{\prod_{j \neq i}(u_i - u_j)} = \begin{cases} 1 & \text{if } d = m, \\ 0 & \text{if } 0 \leq d < m. \end{cases}$$

Proof of Propositions 12 and 19. We first establish the section-free equation of Proposition 12. The equation $S(X) = S(x)$ has $m + M$ solutions (counted with multiplicity), including $X = x$, which form the orbit of x. These solutions are in fact distinct: a solution of $S'(X) = 0$ belongs to the ground field $\mathbb{F}$, and cannot satisfy $S(X) = S(x)$ since x is an indeterminate.

Let $u_0, \dots, u_m$ be $m + 1$ distinct orbit elements. For $0 \leq i \leq m$, the functional equation (33) specializes into

$$K(x)Q(u_i) = 1 - \sum_{k=-m}^{-1} u_i^k G_k.$$

Note that $K(u_i) = K(x)$ since $K(x) = 1 - tS(x)$. We can eliminate the m series G_k by taking an appropriate linear combination of our $m + 1$ equations, namely:

$$\begin{aligned} K(x) \sum_{i=0}^{m} \frac{u_i^m}{\prod_{j \neq i}(u_i - u_j)} Q(u_i) &= \sum_{i=0}^{m} \frac{u_i^m}{\prod_{j \neq i}(u_i - u_j)} - \sum_{k=-m}^{-1} G_k \sum_{i=0}^{m} \frac{u_i^{k+m}}{\prod_{j \neq i}(u_i - u_j)} \\ &= 1 \end{aligned} \tag{37}$$

by Lemma 20.

We have thus exhibited $\binom{m+M}{m+1}$ section-free equations, each involving $m + 1$ orbit equations, but we still need to prove that they generate all section-free equations. So let us take a generic section-free equation, say

$$\sum_{i=0}^{m+M-1} \alpha_i K(x) Q(u_i) = \sum_{i=0}^{m+M-1} \alpha_i \left(1 - \sum_{k=-m}^{-1} u_i^k G_k\right) = \sum_{i=0}^{m+M-1} \alpha_i,$$

where $u_0, u_1, \dots, u_{m+M-1}$ are now all orbit elements. By subtracting a number of versions of (37) (with well chosen u_i's and well chosen weights), we can assume that this equation only involves (at most) m of the u_i's, say $u_1, \dots, u_m$. Then saying that this equation is section-free means that for all k in $[\![-m, -1]\!]$,

$$\sum_{i=1}^{m} \alpha_i u_i^k = 0.$$

But the determinant of this system is not zero (since the u_i's are distinct), and thus all α_i's must be zero.

We now go on with the proof of Proposition 19. The equation $S(X) = S(x)$, written as a polynomial in $\bar{x}$ and X, reads

$$\bar{x}^M \sum_{i\in\mathcal{S}} w_i X^{m+i} = X^m \sum_{i\in\mathcal{S}} w_i \bar{x}^{M-i}.$$

The number of solutions X that are fractional power series in $\bar{x}$ is the degree in X of the above polynomial, once evaluated at $\bar{x} = 0$ (see again [69, Prop. 6.1.8]), hence m. From now on we denote these roots by $x_1, \dots, x_m$, and it is clear that x is not among them so we denote $x_0 = x$.

We now write the section-free equation (37) with $u_i = x_i$, and isolate $Q(x_0) = Q(x)$:

$$Q(x) + \prod_{j=1}^{m}(1-\bar{x}x_j) \sum_{i=1}^{m} \frac{x_i^m}{\prod_{0\le j\ne i\le m}(x_i - x_j)} Q(x_i) = \frac{\prod_{j=1}^{m}(1-\bar{x}x_j)}{K(x)}. \tag{38}$$

Comparing with (35) shows that we have to prove that the second term in the left-hand side, once expanded as a series in t, only contains negative powers of x. In the coefficient of $Q(x_i)$, the term $(1-\bar{x}x_i)$ coming from the numerator gets simplified with the term $(x_i - x_0) = (x_i - x) = -x(1-\bar{x}x_i)$ coming from the denominator. Hence the least common denominator of the coefficients of all $Q(x_i)$ is the Vandermonde determinant in $x_1, \dots, x_m$. We can thus rewrite the second term as follows:

$$\prod_{j=1}^{m}(1-\bar{x}x_j) \sum_{i=1}^{m} \frac{x_i^m}{\prod_{0\le j\ne i\le m}(x_i - x_j)} Q(x_i) = -\bar{x} \sum_{i=1}^{m} \left(x_i^m Q(x_i) \prod_{1\le j\ne i\le m} \frac{1-\bar{x}x_j}{x_i - x_j} \right)$$

$$= \frac{\bar{x}}{\prod_{1\le k<\ell\le m}(x_k - x_\ell)} \sum_{i=1}^{m} \left((-1)^i x_i^m Q(x_i) \prod_{1\le j\ne i\le m} (1-\bar{x}x_j) \prod_{\substack{1\le k<\ell\le m\\ k,\ell\ne i}} (x_k - x_\ell) \right). \tag{39}$$

The sum over i is easily checked to be an antisymmetric expression in $x_1, \dots, x_m$. More precisely, if we exchange in this sum x_a and x_{a+1}, the summands involving $Q(x_a)$ and $Q(x_{a+1})$ are exchanged, and their signs change (because of the factor $(-1)^i$), and for $i \notin \{a, a+1\}$ the sign of the summand involving $Q(x_i)$ changes (because of the factor $(x_a - x_{a+1})$ occurring in the rightmost product). Thus, dividing the sum over i by the Vandermonde determinant in $x_1, \dots, x_m$ gives a series in t with *polynomial* coefficients in $\bar{x}, x_1, \dots, x_m$. Hence, once expanded in t and $\bar{x}$, the right-hand side of (39) contains only negative powers of x (because the x_i's contain no positive power of x and there is a factor $\bar{x}$). We now return to (38), which we expand in powers of t and $\bar{x}$. The expression (35) of $Q(x)$ follows.

Now assume $M = 1$. Then $x_1, \dots, x_m$ are *all* roots of $S(X) = S(x)$ except $X = x$. That is,

$$\frac{S(X) - S(x)}{X - x} = w_1 \prod_{j=1}^{m}(1 - x_j/X).$$

Taking the limit as $X \to x$ gives

$$S'(x) = w_1 \prod_{j=1}^{m}(1-\bar{x}x_j). \tag{40}$$

Substituting into (35) gives (36). ∎

Why Proposition 19 implies Proposition 18. We now derive from (35) the standard expression (32). We start from the factorization (34) of the kernel. It gives the following partial fraction decomposition in x:

$$\frac{1}{K(x)} = -\frac{1}{tw_M}\sum_{i=1}^{m} \frac{\bar{x}X_i^m}{(1-\bar{x}X_i)\prod_{j\neq i}(X_i-X_j)} + \frac{1}{tw_M}\sum_{i=m+1}^{m+M} \frac{X_i^{m-1}}{(1-x/X_i)\prod_{j\neq i}(X_i-X_j)}.$$

The expansion in $\bar{x}$ of the term $A(\bar{x}) := \prod_{j=1}^m (1-\bar{x}x_j)$ only involves non-positive powers of x, hence (35) implies

$$Q(x) = \frac{1}{tw_M}[x^{\geq}]A(\bar{x})\sum_{i=m+1}^{m+M} \frac{X_i^{m-1}}{(1-x/X_i)\prod_{j\neq i}(X_i-X_j)}. \qquad (41)$$

Recall that, as $x_1, \dots, x_m$ themselves, $A(\bar{x})$ is a fractional power series in $\bar{x}$ with coefficients in $\mathbb{F}$, say $A(\bar{x}) = \sum_{n\geq 0} a_n \bar{x}^{n/p}$, for a positive integer p. In fact we can take $p=1$. Indeed, by [69, Prop. 6.1.6], for $1 \leq i \leq m$, every conjugate of x_i over the field $\mathbb{F}((\bar{x}))$ of Laurent series in $\bar{x}$ is one of the x_j's, with $1 \leq j \leq m$; hence $\prod_{i=1}^m (u - x_i)$ is a product of minimal polynomials over $\mathbb{F}((\bar{x}))$, and thus only involves integer powers of $\bar{x}$ in its expansion.

Now let us return to (41), and focus on the term $A(\bar{x})/(1-x/X_i)$. Recall that for $i > m$, X_i is a Puiseux series in t, infinite at $t=0$. Thus $1/X_i$ is a fractional power series in t, vanishing at $t=0$, and hence $A(1/X_i)$ is also a fractional series in t. Moreover, in the ring of fractional series in t with coefficients in $\mathbb{F}[[\bar{x}]]$, we have

$$\begin{aligned}
[x^{\geq}]\frac{A(\bar{x})}{1-x/X_i} &= [x^{\geq}]\left(\sum_{m\geq 0}\frac{x^m}{X_i^m}\sum_{n\geq 0} a_n \bar{x}^n\right) \\
&= \sum_{n\geq 0} a_n \sum_{m\geq n} \frac{x^{m-n}}{X_i^m} \\
&= \sum_{n\geq 0} a_n \frac{1}{X_i^n(1-x/X_i)} \\
&= \frac{A(1/X_i)}{1-x/X_i}.
\end{aligned}$$

Returning to (41), this gives:

$$Q(x) = \frac{1}{tw_M}\sum_{i=m+1}^{m+M} \frac{X_i^{m-1}A(1/X_i)}{(1-x/X_i)\prod_{j\neq i}(X_i-X_j)}.$$

Thus it remains to determine $A(1/X_i)$ when X_i is one of the roots of the kernel that diverges at $t=0$. That is, we have to know the values of $x_1, \dots, x_m$ when x is X_i. Recall the definition of these x_j: they are power series in (a rational power of) $\bar{x}$, satisfying $S(x_j) = S(x)$. Specializing this at $\bar{x} = 1/X_i$ shows that when $x = X_i$, the series $x_1, \dots, x_m$ are power series in (a fractional power of) t, satisfying $S(X_i) = S(x_j)$. But X_i cancels the kernel $1-tS$, hence the x_j are also roots of the kernel, and since they must be finite at $t=0$, they are $X_1, \dots, X_m$. This holds for any X_i with $i > m$.

Hence,

$$\begin{aligned}
Q(x) &= \frac{1}{tw_M}\sum_{i=m+1}^{m+M} \frac{X_i^{m-1}}{(1-x/X_i)\prod_{j\neq i}(X_i-X_j)}\prod_{j=1}^m (1-X_j/X_i) \\
&= \frac{1}{tw_M}\sum_{i=m+1}^{m+M} \frac{1}{(X_i-x)\prod_{j>m, j\neq i}(X_i-X_j)}
\end{aligned}$$

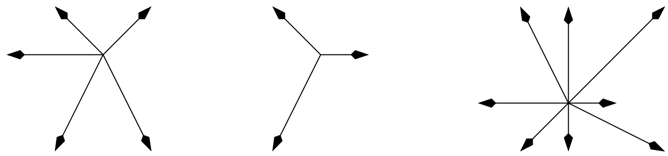


FIGURE 6. Some Hadamard models. The series $Q(x,y)$ is D-finite for all of them, and given as the non-negative part of a rational function for those that have small forward steps (the leftmost two).

where we recognize the partial fraction expansion of

$$-\frac{1}{tw_M}\prod_{i=m+1}^{m+M}\frac{1}{x-X_i}.$$

This gives the second expression in (32).

7. Two-dimensional Hadamard walks

Following [11], we say that a 2-dimensional model $\mathcal{S}$ is *Hadamard* if its step polynomial can be written as:

$$S(x,y)=U(x)+V(x)T(y), \tag{42}$$

for some Laurent polynomials U, V and T. Some examples are shown in Figure 6. When $T(y)=y+\bar{y}$, the model has small variations along the y-axis and is symmetric with respect to the x-axis. It was proved in [20, 28] that the associated generating function $Q(x,y)$ is always D-finite. This holds in fact for *all* two-dimensional Hadamard models, whatever $T(y)$ is. We provide two proofs, one based on a simple projection argument, the other on the method of this paper.

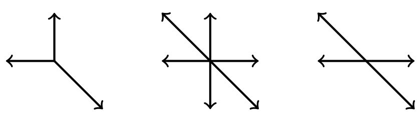
Proposition 21. *Consider a Hadamard model with step polynomial given by* (42)*, and let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{T}$ be the subsets of $\mathbb{Z}$ with generating polynomials $U(x)$, $V(x)$ and $T(y)$ respectively. Let $C_1(x,v;t)$ be the generating function of walks on $\mathbb{N}$, starting from* 0 *and taking steps in the multiset $\mathcal{U}\cup\mathcal{V}$ (steps in $\mathcal{U}\cap\mathcal{V}$ occur twice), counted by the length (variable t), the position of the endpoint (x), and the number of steps in $\mathcal{V}$ (v). Let $C_2(y;v)$ be the generating function of walks on $\mathbb{N}$, starting from* 0 *and taking steps in $\mathcal{T}$, counted by the length (v) and the endpoint (y). Then $C_1(x,v;t)$ and $C_2(y;v)$ are algebraic, and the generating function of quadrant walks with steps in $\mathcal{S}$ is*

$$Q(x,y;t)=\left. C_1(x,v;t)\odot_v C_2(y;v)\right|_{v=1},$$

where $\odot_v$ denotes the Hadamard product in v, defined by $\sum a_n v^n \odot_v \sum b_n v^n=\sum a_n b_n v^n$. In particular, $Q(x,y;t)$ is D-finite.

Proof. The proof is the same as in [11, Sec. 5], but generalized (in a harmless fashion) to walks with arbitrary steps. It goes by projecting quadrant walks along the x-axis, and "decorating" steps of $\mathcal{V}$ in this 1D walk with steps of a "vertical" walk with steps in $\mathcal{T}$; we omit the details. The Hadamard product of algebraic (and in fact, of D-finite) series is known to be D-finite [56]. ■

The approach of this paper works systematically in the Hadamard case, and provides the solution as the positive part of an algebraic (sometimes rational) series, often more explicitly than the above solution. In the case of small steps, 16 of the 19 models solvable by the method of this paper (the leftmost branch in Figure 1) are Hadamard. The three remaining ones are shown below.



Consider a Hadamard model $\mathcal{S}$. Let $-m$ (resp. M) be the valuation (resp. degree) of $S(x,y)$ in x, and write similarly $-m'$ and M' for the valuation and degree in y. In other words, $-m$ (resp. M) is the smallest (resp. largest) move in the x-direction, and similarly for m' and M'. We assume $m, m' \geq 0$ and $M, M' > 0$. The solution given below has strong analogies with the 1-dimensional case of Proposition 19.

Proposition 22. *The equation $S(x,y) = S(X,y)$, solved for X, admits $m+M$ solutions (including x itself), which can be seen as Puiseux series in $\bar{x}$ with coefficients in an algebraic closure of $\mathbb{Q}(y)$ (below we take Puiseux series in $\bar{y}$). We denote them by $x_0(y), \dots, x_{m+M-1}(y)$, with $x_0(y) = x$. Exactly m of them, say $x_1(y), \dots, x_m(y)$, do not involve positive powers of x.*

The equation $S(x,y) = S(x,Y)$, now solved for Y, reads $T(y) = T(Y)$. It admits $m'+M'$ solutions (including y itself), which can be seen as Puiseux series in $\bar{y}$ with coefficients in $\mathbb{C}$. We denote them by $y_0, \dots, y_{m'+M'-1}$, with $y_0 = y$. Exactly m' of them, say $y_1, \dots, y_{m'}$, do not involve positive powers of y.

The orbit of (x,y) consists of all pairs (x_i, y_j), for $i \in [\![0, m+M-1]\!]$ and $j \in [\![0, m'+M'-1]\!]$.

The series $Q(x,y)$ reads

$$Q(x,y) = [x^{\geq} y^{\geq}] \frac{\prod_{i=1}^{m}(1-\bar{x}x_i(y)) \prod_{j=1}^{m'}(1-\bar{y}y_j)}{K(x,y)}, \tag{43}$$

where the right-hand side is expanded first in powers of t, then $\bar{x}$, and finally $\bar{y}$. The extraction of the non-negative part in x can be done explicitly, and yields:

$$Q(x,y) = [y^{\geq}] \frac{\prod_{i=1}^{m}(1-\bar{x}X_i(y)) \prod_{j=1}^{m'}(1-\bar{y}y_j)}{K(x,y)} = -[y^{\geq}] \frac{\prod_{j=1}^{m'}(1-\bar{y}y_j)}{tB_M(y)\prod_{i=m+1}^{m+M}(x-X_i)}, \tag{44}$$

where $X_1(y), \dots, X_m(y)$ are the roots (in x) of $1 - tS(x,y)$, seen as Puiseux series in t with coefficients in the algebraic closure of $\mathbb{Q}(y)$, that are finite at $t=0$, and $X_{m+1}, \dots, X_{m+M}$ are the other ones. The polynomial $B_M(y)$ is the coefficient of x^M in $S(x,y)$.

If $M = 1$, then the derivative of $S(x,y)$ with respect to x factors as

$$S_x(x,y) = B_1(y) \prod_{i=1}^{m} (1 - \bar{x}x_i(y)).$$

Similarly, if $M' = 1$, then

$$T_y(y) = \prod_{j=1}^{m'} (1 - \bar{y}y_j).$$

This simplifies the above expressions. In particular, when all forward steps are small ($M = M' = 1$), we can write $Q(x,y)$ as the non-negative part of a simple rational function:

$$Q(x,y) = [x^{\geq} y^{\geq}] \frac{S_x(x,y)T_y(y)}{B_1(y)K(x,y)}. \tag{45}$$

Proof. The statements dealing with roots are in essence one-dimensional, and follow from Proposition 19 since we allowed weights in the previous section.

We next want to build the orbit of (x,y). By definition of the x_i's and y_j's we have $(x_i, y) \approx (x,y) \approx (x, y_j)$ for all i and j. Now

$$\begin{aligned} S(x_i, y_j) &= U(x_i) + V(x_i)T(y_j) \\ &= U(x_i) + V(x_i)T(y) \quad \text{by definition of } y_j \\ &= S(x_i, y). \end{aligned}$$

Thus $(x_i, y_j) \approx (x_i, y)$, and all pairs (x_i, y_j) are in the orbit. In this collection, every element (x_i, y_j) is 1-adjacent to $m+M-1$ other elements, and 2-adjacent to $m'+M'-1$ other elements, hence the orbit is complete (Lemma 4).

The functional equation has the following general form (see (11)):

$$K(x,y)Q(x,y) = 1 - \sum_{k=1}^{m} \bar{x}^k R_k(y) - \sum_{\ell=1}^{m'} \bar{y}^\ell S_\ell(x),$$

for some series $R_k(y)$ and $S_\ell(x)$. A similar equation holds with (x,y) replaced by any element (x_i,y_j) of the orbit. The fact that the orbit is a Cartesian product allows us to construct a section-free equation by mimicking the argument that led to (37):

$$K(x,y)\left(\sum_{i=0}^{m}\sum_{j=0}^{m'} \frac{x_i^m y_j^{m'} Q(x_i,y_j)}{\prod_{0\le k\ne i\le m}(x_i-x_k)\prod_{0\le \ell\ne j\le m'}(y_j-y_\ell)}\right) = 1.$$

Equivalently, after isolating $Q(x_0,y_0)=Q(x,y)$:

$$Q(x,y) - \bar{y}\sum_{j=1}^{m'} y_j^{m'} Q(x,y_j) \prod_{1\le \ell\ne j\le m'} \frac{1-\bar{y}y_\ell}{y_j-y_\ell} - \bar{x}\sum_{i=1}^{m} x_i^m Q(x_i,y) \prod_{1\le k\ne i\le m} \frac{1-\bar{x}x_k}{x_i-x_k}$$
$$+ \bar{x}\bar{y}\sum_{i=1}^{m}\sum_{j=1}^{m'} x_i^m y_j^{m'} Q(x_i,y_j) \prod_{1\le k\ne i\le m} \frac{1-\bar{x}x_k}{x_i-x_k} \prod_{1\le \ell\ne j\le m'} \frac{1-\bar{y}y_\ell}{y_j-y_\ell}$$
$$= \frac{\prod_{i=1}^{m}(1-\bar{x}x_i)\prod_{j=1}^{m'}(1-\bar{y}y_j)}{K(x,y)}.$$

We now expand the coefficient of t^n in this identity in powers of $\bar{x}$ (with coefficients in the field of Puiseux series in $\bar{y}$), and extract the non-negative powers of x. The coefficients of the first two terms on the first line (those involving $Q(x,y)$ and $Q(x,y_j)$) are clearly non-negative in x. By recycling our analysis of (39), we see that the coefficient of t^n in the third term (involving $Q(x_i,y)$) is a *polynomial* in y, $\bar{x}$, $x_1,\dots,x_m$, multiplied by $\bar{x}$, and thus only involves negative powers of x and does not contribute. A similar argument shows that the second line does not contribute either. We are thus left with

$$Q(x,y) - \bar{y}\sum_{j=1}^{m'} y_j^{m'} Q(x,y_j) \prod_{\ell\ne j\in[\![1,m']\!]} \frac{1-\bar{y}y_\ell}{y_j-y_\ell} = [x^{\ge}]\frac{\prod_{i=1}^{m}(1-\bar{x}x_i)\prod_{j=1}^{m'}(1-\bar{y}y_j)}{K(x,y)}.$$

The symmetry argument applied earlier to (39) shows that the sum over j is a series in t whose coefficients are polynomials in $x,\bar{y},y_1,\dots,y_m$. Hence a final expansion in powers of $\bar{y}$, followed by the extraction of non-negative powers of y gives the first expression (43) of $Q(x,y)$. The second one, that is (44), follows by combining the one-dimensional results of Propositions 18 and 19. Indeed, Proposition 19 shows that

$$[x^{\ge}]\frac{\prod_{i=1}^{m}(1-\bar{x}x_i(y))}{K(x,y)}$$

counts walks with steps in $\mathcal{S}$ confined to the half-plane $\{(i,j): i\ge 0\}$, and Proposition 18 gives an alternative expression for this series.

The rest of the proof follows the same lines as the end of the proof of Proposition 19 (see in particular (40)). ■

Example: a Hadamard model with small forward steps. Take $\mathcal{S} = \{10, \bar{1}1, \bar{1}\bar{2}\}$. The step polynomial is

$$S(x,y) = x + \bar{x}(y+\bar{y}^2) = U(x) + V(x)T(y)$$

with $U(x)=x$, $V(x)=\bar{x}$ and $T(y)=y+\bar{y}^2$, so this is a Hadamard model. Moreover, the forward steps are small, so that the simple formula (45) holds:

$$Q(x,y) = [x^{\ge}y^{\ge}]\frac{(1-\bar{x}^2(y+\bar{y}^2))(1-2\bar{y}^3)}{1-t(x+\bar{x}(y+\bar{y}^2))}.$$

The number of walks of length n ending at (i, j) is non-zero if and only if $n = i + 2j + 6m$ for some m, in which case

$$q(i, j; n) = \frac{(i+1)(j+1)n!}{m!(2m+j+1)!(3m+i+j+1)!}. \tag{46}$$

□

Example: a Hadamard model with a large forward step. Let us now reverse the above steps. The step polynomial becomes

$$S(x, y) = \bar{x} + x(\bar{y} + y^2)$$

and is of course still Hadamard. With the notation of Proposition 22, $m = m' = 1$,

$$x_1(y) = \frac{\bar{x}}{\bar{y} + y^2} \qquad \text{and} \qquad y_1 = \frac{-1 + \sqrt{4\bar{y}^3 + 1}}{2\bar{y}}.$$

Indeed y_1 is a power series in $\bar{y}$, while its conjugate root y_2 contains a term $-y$ in its expansion. The two expressions of Proposition 22 read:

$$Q(x, y) = [x^{\geq} y^{\geq}] \frac{(1 - \bar{x}x_1(y))(1 - \bar{y}y_1)}{K(x, y)} = -[y^{\geq}] \frac{\bar{x}(1 - \bar{y}y_1)}{t(\bar{y} + y^2)(1 - \bar{x}X_2)},$$

with

$$X_2 = \frac{1 + \sqrt{1 - 4t^2(\bar{y} + y^2)}}{2t(\bar{y} + y^2)}.$$

As before, we expand the right-hand side first in t, then $\bar{x}$, then $\bar{y}$.

8. Quadrant walks with steps in $\{-2, -1, 0, 1\}^2$

In this section, we explore systematically all models obtained by taking $\mathcal{S}$ in $\{-2, -1, 0, 1\}^2 \setminus (0, 0)$, with the (ultimate) objective of reaching a classification similar to that of quadrant walks with small steps (Figure 1). Our results are summarized in Figure 7. In Section 10 we discuss the classification of orbits (not of generating functions!) for models in $\{-1, 0, 1, 2\}^2 \setminus (0, 0)$.

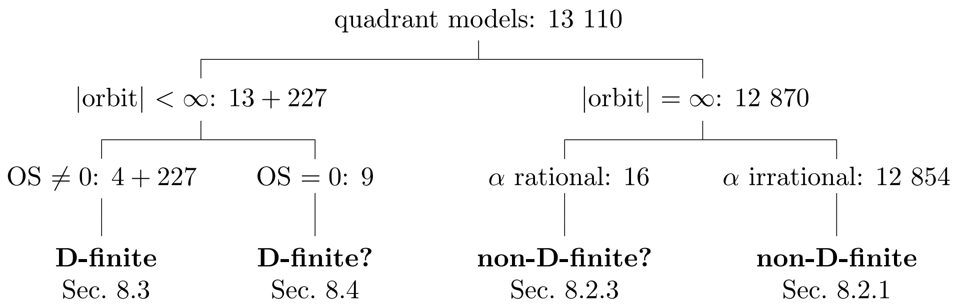


FIGURE 7. Partial classification of quadrant walks with steps in $\{-2, -1, 0, 1\}^2$, when at least one large backward step is allowed. The approach of this paper solves the 231 models on the leftmost branch, including 227 Hadamard models.

8.1. The number of relevant models

We first proceed as in [26, Sec. 2] in order to count, among the 2^{15} possible models (Figure 8), those that are really distinct and relevant. Clearly, we do not want to consider separately two models that only differ by an x/y-symmetry, as such models are isomorphic. Moreover, for certain models, forcing walks to lie in some half-plane automatically forces them to remain in the first quadrant. This happens, for instance, for $\mathcal{S} = \{\nearrow, \uparrow, \swarrow\}$ and the right half-plane. Half-plane models are essentially 1-dimensional and thus have an algebraic generating function, which can be determined in an automatic fashion (Proposition 18).

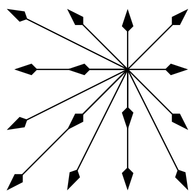


Figure 8. The 15 allowed steps.

Using the same arguments as in [26, Sec. 2], we first determine the number of step sets $\mathcal{S}$ that contain at least an x-positive, an x-negative, a y-positive and a y-negative step. More precisely, we count such sets by their cardinality. An inclusion-exclusion argument gives their generating polynomial as:

$$P_1(z) = (1+z)^{15}-2(1+z)^{11}-2(1+z)^7+2(1+z)^3+(1+z)^8+2(1+z)^5+(1+z)^3-2(1+z)^2-2(1+z)+1.$$

The term $(1+z)^{15}$ counts all step sets, while $(1+z)^{11}$ counts those that contain no x-positive step, $(1+z)^7$ those that contain no x-negative step, $(1+z)^3$ those that contain no x-positive nor x-negative step, and so on. We refer to [26, Sec. 2] for a detailed argument. Then, we must exclude sets in which no step belongs to $\mathbb{N}^2$. This leaves fewer step sets, counted by:

$$P_2(z) = P_1(z) - \big((1+z)^{12} - 2(1+z)^{10} + (1+z)^8\big).$$

We also do not wish to consider step sets such that all walks confined to the right half plane $x \geq 0$ are automatically quadrant walks. As in the case of small steps, this means that all steps (i, j) of $\mathcal{S}$ satisfy $j \geq i$. That is, we have an upper diagonal model. The generating polynomial of such sets, satisfying the above conditions (steps in all directions, at least one step in $\mathbb{N}^2$) is

$$z\left((1+z)^8 - (1+z)^5\right),$$

where the factor z accounts for the step $(1, 1)$, which is necessarily in such a set. Symmetrically, we need to exclude lower diagonal models, and avoid excluding twice the models that are both upper and lower diagonal. We are left with a collection of step sets counted by

$$P_3(z) = P_2(z) - 2z\left((1+z)^8 - (1+z)^5\right) + z(2z + z^2).$$

Finally, if two models differ only by a diagonal symmetry, we do not want to consider them both. We thus have to count separately the models counted by P_3 that have an x/y symmetry. Mimicking the above argument, and including the symmetry constraint, gives:

$$P_1^{\text{sym}}(z) = (1+z)^3(1+z^2)^6 - (1+z)^2(1+z^2)^3 - (1+z)(1+z^2) + 1,$$

$$P_2^{\text{sym}}(z) = P_1^{\text{sym}}(z) - \big((1+z)^2(1+z^2)^5 - (1+z)^2(1+z^2)^3\big),$$

and

$$P_3^{\text{sym}} = P_2^{\text{sym}} - z(2z + z^2).$$

We have thus restricted the collection of models that we have to study to 13 189 models, with generating polynomial

$$\frac{1}{2}\left(P_3(z)+P_3^{\rm sym}(z)\right) = z^{15}+9z^{14}+57z^{13}+236z^{12}+691z^{11}+1481z^{10}+2374z^9+2872z^8 + 2610z^7+1749z^6+826z^5+248z^4+35z^3.$$

Among these, we know from [26] that those with small steps are counted by

$$7\,z^3+23\,z^4+27\,z^5+16\,z^6+5\,z^7+z^8,$$

and we are thus left with 13 110 models with at least one large backward step, counted by

$$z^{15}+9z^{14}+57z^{13}+236z^{12}+691z^{11}+1481z^{10}+2374z^9+2871z^8+2605z^7+1733z^6+799z^5+225z^4+28z^3.$$

Note that no model in our collection is included in a half-plane. This will allow us to apply Theorem 7 systematically.

8.2. The size of the orbit

8.2.1. **The excursion exponent.** Consider a model $\mathcal{S}$ in our collection. Recall that if the quantity c defined in Theorem 7 cannot be written as $\cos\theta$ with $\theta\in\pi\mathbb{Q}$, then the orbit of $\mathcal{S}$ is infinite and $Q(x,y;t)$ is not D-finite. In order to decide if c is of the requested form, we apply the following procedure, borrowed from [19].

(1) Compute a polynomial $P(C)$ that admits c as a root. This is done by eliminating the variables x,y and u from the polynomial system comprised of (the numerators of):

$$S_x(x,y),\quad S_y(x,y),\quad C^2-\frac{S_{xy}(x,y)}{S_{xx}(x,y)^2S_{yy}(x,y)^2},\quad 1-uxy.$$

The final equation forces $x,y\neq 0$. This is done via a Gröbner basis computation.

(2) Identify the irreducible factor $I(C)$ of $P(C)$ which admits c as a root. To do this it is sufficient to determine the critical pair (a,b), and thus c, to sufficient numerical precision.

(3) Decide whether c can be written as $\cos\theta$, with $\theta\in\pi\mathbb{Q}$. Equivalently, decide if the solutions of $2c=z+1/z$ are roots of unity. To do this it is sufficient to examine whether the polynomial $R(z):=z^{\deg I}I(\frac{z+1/z}{2})$ has cyclotomic factors.

The polynomials $R(z)$ which are constructed by running this algorithm on the 13 110 step sets in our collection are all irreducible and have degree less than 72. Thus, as the degree of the kth cyclotomic polynomial is

$$\phi(k)>\frac{k}{e^\gamma\log\log k+\frac{3}{\log\log k}},$$

where $\gamma\approx 0.577$ is Euler's constant [5, Thm. 8.8.7], to prove that the excursion exponent is irrational it is sufficient to show that $R(z)$ is not divisible by any of the first 349 cyclotomic polynomials; constructing cyclotomic polynomials is a routine task in computer algebra [4].

After performing this filtering step we conclude that 12 854 models have an irrational excursion exponent, and thus an infinite orbit and a non-D-finite generating function. They form the rightmost branch in Figure 7.

8.2.2. **Detecting finite orbits.** We are thus left with 256 step sets, each of which having a rational exponent α. Among them we find 227 Hadamard models. Proposition 22 tells us that they have a finite orbit, of cardinality 6 or 9 depending on the sizes of the steps (Figure 9). For each of them the excursion exponent is found to be $\alpha=-3$.

There are 29 models remaining. We apply to them the semi-algorithm of Section 3.2, which detects 13 more models with a finite orbit, of cardinality 12 or 18. They are listed in Table 1. Three distinct orbit structures arise, shown in Figure 10.

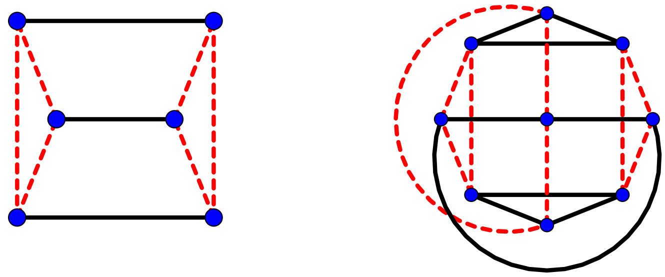


FIGURE 9. The possible orbits for two-dimensional Hadamard models with long steps in $\{-2,-1,0,1\}^2$, depending on whether there are steps with -2 in only one coordinate (left) or both coordinates (right). The convention for dashed and solid edges is the same as in Figure 4.

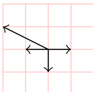
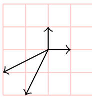
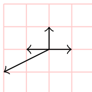
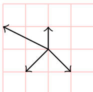
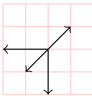
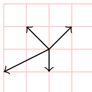
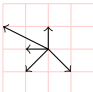
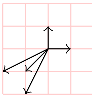
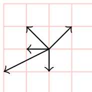
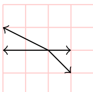
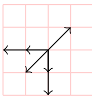
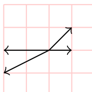
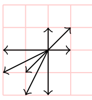
g	steps	orbit	α	g	steps	orbit	α	g	steps	orbit	α
1		O_{12}	-4	2		$\tilde{O}_{12}$	$-5/2$	1		O_{12}	$-5/2$
2		O_{12}	-4	3		$\tilde{O}_{12}$	$-5/2$	2		O_{12}	$-5/2$
2		O_{12}	-4	2		$\tilde{O}_{12}$	$-5/2$	2		O_{12}	$-5/2$
2		O_{18}	-4	3		$\tilde{O}_{12}$	$-5/2$	2		O_{18}	$-7/3$
				4		$\tilde{O}_{12}$	$-5/2$				

TABLE 1. The 13 non-Hadamard models with a finite orbit. Our method solves the ones on the left, proving that their generating function is D-finite (and transcendental). We conjecture that the 9 others are D-finite too, two of them being possibly algebraic (the second and third in the last column). We also give the excursion exponent α, and the genus g of the curve $K(x, y)$, which is 0 or 1 for small step models.

8.2.3. **Sixteen models with a rational exponent but an infinite orbit.** For each of the remaining 16 models, listed in Table 2, we ran our semi-algorithm by specializing $x = 1$ and $y = 2$ until we found at least 200 distinct orbit elements (the sum of the degrees of the polynomials in $\mathcal{P}$ — or $\mathcal{Q}$ — gives a lower bound on the size of the orbit). We found in each case minimal polynomials of degree over 100. The following proposition explains why.

Proposition 23. *The* 16 *models of Table 2 have an infinite orbit.*

Proof. The proof is based on Proposition 8, and mimics the proof used in the third example of Section 3.3.3. For each model, we start from the positive critical point (a, b), define Φ and

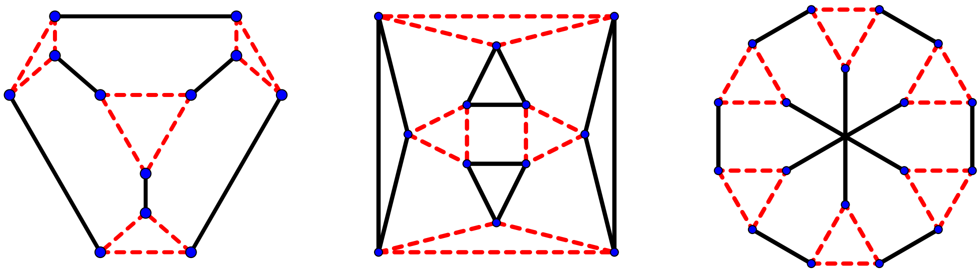


Figure 10. The three finite orbit types which arise from non-Hadamard models in $\{-2,-1,0,1\}^2$. Two have cardinality 12, the third one has cardinality 18. We call these orbit structures, from left to right, O_{12}, $\tilde{O}_{12}$ and O_{18}.

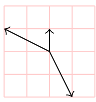
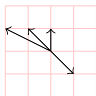
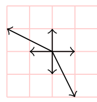
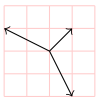
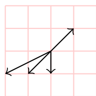
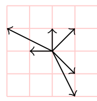
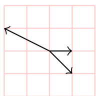
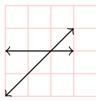
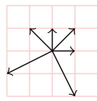
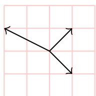
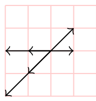
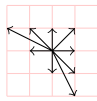
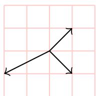
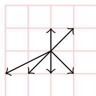
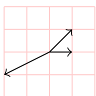
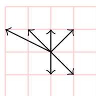
	steps	α		steps	α		steps	α
#1		-5	#7		-7	#13		-4
#2		-4	#8		-11/5	#14		-4
#3		-7	#9		-7/3	#15		-3
#4		-5	#10		-7/3	#16		-4
#5		-7/3	#11		-5/2			
#6		-11/5	#12		-4			

Table 2. Sixteen models with a rational excursion exponent α and an infinite orbit.

Ψ as in Section 3.3.1, and compute the expansion of $\Theta := \Psi \circ \Phi$ to cubic order. There exists some integer $m > 0$ such that Θ^m is the identity at first order (otherwise the excursion exponent would be irrational). Moreover, we observe that the quadratic term in Θ^m vanishes, but there is a non-zero cubic term. This implies that all elements Θ^{km} are distinct, so that the orbit is infinite.

We give below the values of a, b, and m (for model #10 the value of a is the positive root of $a^3 = a + 2$). Since models #5, #6, #8 and #12 are obtained from another model in the table by a reflection in the x-axis, we omit them (their orbits are infinite by Proposition 3). However, the method works as well for them (with b replaced by $1/b$, and a and m unchanged).

model	#1	#2	#3	#4	#7
(a,b)	$(3^{1/2}, 3^{1/2}/2^{1/3})$	$(1,1)$	$(1,1)$	$(2^{-1/3}, 3^{-1/2})$	$(1, 3^{-1/2})$
m	4	3	6	4	6

model	#9	#10	#11	#13	#14	#15	#16
(a,b)	$(2^{1/3}, 1)$	$(a,1)$	$(1,\sqrt{2})$	$(\sqrt{2},\sqrt{2})$	$(1,1)$	$(1,1)$	$(1,1)$
m	4	4	3	3	3	2	3

■

For each model, we have tested D-finiteness experimentally, by generating 10 000 coefficients of the series $Q(0,0)$, and trying to guess from them a linear recurrence relation for the coefficients or a linear differential equation for the generating function. The guessing procedure is detailed in Section 8.4. We could not find any recurrence nor differential equation, and are tempted to believe that $Q(x,y;t)$ is not D-finite for these 16 models. However, it must be noted that, for some models of Section 8.4, it takes more than 10 000 coefficients to guess a differential equation.

8.3. Solving models with a finite orbit

As written above, 227 of the 240 models that have a finite orbit are Hadamard. Our method applies systematically to them, as proved in Section 7. In particular, all these models have a D-finite generating function, and, because they make small forward moves, $Q(x,y)$ is expressed as the non-negative part of a simple rational function (see (45)). The excursion exponent being -3 in all cases, these series are transcendental [41].

We are left with the 13 models shown in Table 1. For each of them, there exists a unique section-free equation, in agreement with Conjecture 13 (up to a multiplicative factor, as usual). For the 4 models shown in the first column, this equation defines $Q(x,y)$ uniquely and we are able to extract it as the positive part of a rational series. In particular, these four series are D-finite (but transcendental, because of the exponent -4). Details are given below, and we work out detailed asymptotic behaviour of their coefficients in Section 9. For the remaining 9 models, the right-hand side of the section-free equation vanishes, so that this equation does not characterize $Q(x,y)$ (for a start, any constant is a solution). These models are the counterparts of the 4 algebraic models from the small step case, shown in the second branch of Figure 1. Clearly they deserve a specific study, and we state conjectures regarding the nature of their generating functions in Section 8.4.

8.3.1. Case $\mathcal{S} = \{10, \bar{1}0, 0\bar{1}, \bar{2}1\}$.

This is model F, which we have studied as one of our examples in this paper. Our main result is stated in Proposition 16:

$$Q(x,y) = [x^{\geq}y^{\geq}]\frac{\left(x^2+1\right)(x+y)(y-x)\left(x^2y-2\,x-y\right)\left(x^3-x-2\,y\right)}{x^7y^3\left(1-t(x+\bar{x}+\bar{x}^2y+\bar{y})\right)}.$$

The excursion exponent $\alpha \equiv \alpha_e$, given by Theorem 11, is -4. The exponent of all quadrant walks – that is, the exponent associated with the coefficients of $Q(1,1)$ – can be determined using multivariate singularity analysis, and is found to be $\alpha_w = -4$. This is detailed for all four models shown on the left of Table 1 in Section 9.

For comparison with the next cases, we recall that the orbit, given by (22), has type O_{12}.

8.3.2. Case $\mathcal{S} = \{01, 1\bar{1}, \bar{1}\bar{1}, \bar{2}1\}$.

Proposition 24. *For $\mathcal{S} = \{01, 1\bar{1}, \bar{1}\bar{1}, \bar{2}1\}$, we have*

$$Q(x,y) = [x^{\geq}y^{\geq}]\frac{\left(x^3-2\,y^2-x\right)\left(y^2-x\right)\left(x^2y^2-y^2-2\,x\right)}{x^5y^4\left(1-t(y+x\bar{y}+\bar{x}\bar{y}+\bar{x}^2y)\right)},$$

where the right-hand side is seen as a power series in t with coefficients in $\mathbb{Q}[x,\bar{x},y,\bar{y}]$. The coefficients are hypergeometric: $q(i,j;n)$ is zero unless n is of the form $n=2i+j+4m$, in which case

$$q(i,j;n)=\frac{(i+1)(j+1)(i+j+2)n!(n+2)!}{m!(3m+2i+j+2)!(2m+i+1)!(2m+i+j+2)!}.$$

The excursion exponent α_e and the walk exponent α_w are both -4.

Proof. The proof is very similar to the solution of Example F. The step polynomial is $S(x,y)=y+x\bar{y}+\bar{x}\bar{y}+\bar{x}^2y$. All elements of the orbit belong to the extension of $\mathbb{Q}(x,y)$ generated by $\sqrt{(x+y^2)^2+4x^3y^2}$. More precisely, denoting

$$x_{1,2}=\frac{x+y^2\pm\sqrt{(x+y^2)^2+4x^3y^2}}{2x^2},$$

the orbit has type O_{12} and consists of the following 12 pairs:

$$\begin{array}{ccc} (x,y) & (x_1,y) & (x_2,y)\\ (x,x\bar{y}) & (-\bar{x}_1,x\bar{y}) & (-\bar{x}_2,x\bar{y})\\ (x_1,x_1\bar{y}) & (-\bar{x},x_1\bar{y}) & (-\bar{x}_2,x_1\bar{y})\\ (x_2,x_2\bar{y}) & (-\bar{x},x_2\bar{y}) & (-\bar{x}_1,x_2\bar{y}). \end{array} \tag{47}$$

Note the similarities with the orbit (22) obtained for model F. The functional equation reads:

$$(1-tS(x,y))Q(x,y)=1-tx\bar{y}Q(x,0)-t\bar{x}\bar{y}(Q_{0,-}(y)+Q(x,0)-Q(0,0))-t\bar{x}^2y\left(Q_{0,-}(y)+xQ_{1,-}(y)\right),$$

where as before $x^iQ_{i,-}(y)$ is the generating function of walks ending at abscissa i. There is a unique section-free equation. To form it, the orbit equation associated with (x',y') must be weighted by $\pm x'^2(x_1'-x_2')$, where $(x_1',y')\approx(x',y')\approx(x_2',y')$ and $x_1'\neq x_2'$. More precisely, the weights associated with the 12 above orbit elements are:

$$\begin{array}{rrr} x^2(x_1-x_2) & x_1^2(x_2-x) & -x_2^2(x_1-x)\\ x^2(\bar{x}_2-\bar{x}_1) & -\bar{x}_1^2(x+\bar{x}_2) & \bar{x}_2^2(x+\bar{x}_1)\\ x_1^2(\bar{x}-\bar{x}_2) & \bar{x}^2(x_1+\bar{x}_2) & -\bar{x}_2^2(\bar{x}+x_1)\\ -x_2^2(\bar{x}-\bar{x}_1) & -\bar{x}^2(x_2+\bar{x}_1) & \bar{x}_1^2(\bar{x}+x_2). \end{array} \tag{48}$$

Again, note the similarities with (24). We now divide the section-free equation by $x^2(x_1-x_2)$, so as to isolate $Q(x,y)$. This gives an equation similar to the one obtained with Example F (see (25)):

$$Q(x,y)+\bar{x}_1\bar{x}_2Q(x,x\bar{y})+A_1+A_2+A_3+A_4+A_5=R(x,y), \tag{49}$$

where $R(x,y)$ is the rational function occurring in Proposition 24, and each A_i involves two of the series $Q(x',y')$ (again, as in Example F), chosen so that the expression of A_i is symmetric in x_1 and x_2. More precisely, the orbit elements occurring in A_1 (resp. A_2,A_3,A_4,A_4,A_5) are (x_1,y) and (x_2,y) (resp. $(-\bar{x}_1,x\bar{y})$ and $(-\bar{x}_2,x\bar{y})$, $(x_1,x_1\bar{y})$ and $(x_2,x_2\bar{y})$, $(-\bar{x},x_1\bar{y})$ and $(-\bar{x},x_2\bar{y})$, $(-\bar{x}_1,x_2\bar{y})$ and $(-\bar{x}_2,x_1\bar{y})$). We now examine the symmetric functions of the x_i's and of their reciprocals. They are Laurent polynomials in x and y, which are, respectively, negative in x and non-positive in y:

$$x_1+x_2=\bar{x}+\bar{x}^2y^2,\quad x_1x_2=-\bar{x}y^2,$$

while

$$\bar{x}_1+\bar{x}_2=-\bar{x}-\bar{y}^2\quad\text{and}\quad\bar{x}_1\bar{x}_2=-x\bar{y}^2.$$

With this, we conclude that the series A_i also have coefficients in $\mathbb{Q}[x,\bar{y},y,\bar{y}]$, that A_1, A_3 and A_4 are negative in x, and that A_2 is negative in y. As in Example F, the case of A_5 is a bit trickier, due to the mixture of positive and negative powers of the x_i's. Following the same lines as in Example F, one can prove that A_5 contains no monomial that would be non-negative in x and y. The counterpart of Lemma 15 is that every monomial $\bar{x}^ey^f$ occurring in the expression E_a defined by (29), for the values of x_1 and x_2 here, satisfies $f\leq 2e$.

The simplicity of the coefficients $q(i,j;n)$ comes from the fact that the expansion of $S(x,y)^n=(1+\bar{x}^2)^n(y+x\bar{y})^n$ in x and y has simple coefficients.

The excursion exponent can be determined using Theorem 11, but it is more natural to start from the explicit expression of $q(0,0;4m)$, for which we derive:

$$q(0,0;4m) \sim \frac{4\sqrt{3}}{27\pi m^4}\left(\frac{16}{3}\right)^{3m}.$$

The asymptotic behaviour of the number of quadrant walks is determined in Section 9. ■

8.3.3. **Case $\mathcal{S} = \{01, 1\bar{1}, \bar{1}\bar{1}, \bar{2}1, \bar{1}0\}$.**

Proposition 25. *For $\mathcal{S} = \{01, 1\bar{1}, \bar{1}\bar{1}, \bar{2}1, \bar{1}0\}$, we have:*

$$Q(x,y) = [x^{\geq}y^{\geq}]\frac{\left(y^2-x\right)\left(x^2y^2-xy-y^2-2\,x\right)\left(x^3-xy-2\,y^2-x\right)}{x^5y^4\left(1-t(y+x\bar{y}+\bar{x}\bar{y}+\bar{x}^2y+\bar{x})\right)},$$

where the right-hand side is seen as a power series in t with coefficients in $\mathbb{Q}[x,\bar{x},y,\bar{y}]$.

The excursion exponent α_e and the walk exponent α_w are both -4.

Proof. This example is very close to the previous one, from which it only differs by one West step. The orbit is still given by (47), but with different values of x_1 and x_2:

$$x_{1,2} = \frac{x+xy+y^2 \pm \sqrt{(x+xy+y^2)^2+4x^3y^2}}{2x^2}.$$

The symmetric functions of the x_i's, and of their reciprocals, have promising non-negativity properties (the same as in the previous example):

$$x_1+x_2 = \bar{x}+\bar{x}y+\bar{x}^2y^2, \quad x_1x_2 = -\bar{x}y^2, \quad \bar{x}_1+\bar{x}_2 = -\bar{x}-\bar{y}-\bar{y}^2 \quad \text{and} \quad \bar{x}_1\bar{x}_2 = -x\bar{y}^2.$$

The functional equation differs from the previous one by the new term $-t\bar{x}Q_{0,-}(y)$ in the right-hand side. There is a unique section-free equation, with weights again given by (48). Thus this equation is again (49), with the same expressions of the series A_i. The series $Q(x,y)$ is extracted in the same way as in the previous example. In particular, only one series, A_5, raises difficulties in the extraction procedure. They are solved as before by proving that $f \leq 2e$ for every monomial $\bar{x}^e y^f$ occurring in the expression E_a defined by (29).

The excursion exponent is determined from Theorem 11, and the walk exponent in Section 9.
■

8.3.4. **Case $\mathcal{S} = \{10, 1\bar{1}, \bar{2}1, \bar{2}0\}$.**

Proposition 26. *For $\mathcal{S} = \{10, 1\bar{1}, \bar{2}1, \bar{2}0\}$, we have:*

$$Q(x,y) = [x^{\geq}y^{\geq}]\frac{(2-y)(x^3-y^2)(x^6y-3x^3y-x^3-y^2)(x^3-2y)}{x^9y^4(1-t(\bar{x}^2+\bar{x}^2y+x\bar{y}+x))},$$

where the right-hand side is seen as a power series in t with coefficients in $\mathbb{Q}[x,\bar{x},y,\bar{y}]$. The coefficients are hypergeometric: $q(i,j;n)$ is zero unless n is of the form $n = i+3j+3m$, in which case

$$q(i,j;n) = \frac{(i+1)(j+1)(i+3j+4)\big((i+2j+2)(i+2j+3)+m(2i+3j+4)\big)n!(n+3)!}{m!(m+j+1)!(2m+i+2j+3)!(2m+i+3j+4)!}.$$

The excursion exponent α_e and the walk exponent α_w are both -5.

Proof. The step polynomial is $S(x,y) = x + x\bar{y} + \bar{x}^2 + \bar{x}^2y$. All elements of the orbit belong to the extension of $Q(x,y)$ generated by $\sqrt{y(y+4x^3)}$ and $\sqrt{1+4y}$. More precisely, let us define

$$x_{1,2} = \frac{y \pm \sqrt{y(y+4x^3)}}{2x^2} \quad \text{and} \quad u_{3,4} = \frac{1 \pm \sqrt{1+4y}}{2y}. \tag{50}$$

Then the orbit consists of the following 18 pairs:

$$\begin{array}{ccc}
(x,y) & (x_1,y) & (x_2,y)\\
(x,x^3\bar y) & (xu_3,x^3\bar y) & (xu_4,x^3\bar y)\\
(x_1,x_1^3\bar y) & (x_1u_3,x_1^3\bar y) & (x_1u_4,x_1^3\bar y)\\
(x_2,x_2^3\bar y) & (x_2u_3,x_2^3\bar y) & (x_2u_4,x_2^3\bar y)\\
(xu_3,u_3^3y) & (x_1u_3,u_3^3y) & (x_2u_3,u_3^3y)\\
(xu_4,u_4^3y) & (x_1u_4,u_4^3y) & (x_2u_4,u_4^3y)
\end{array}$$

and its structure is shown in Figure 11.

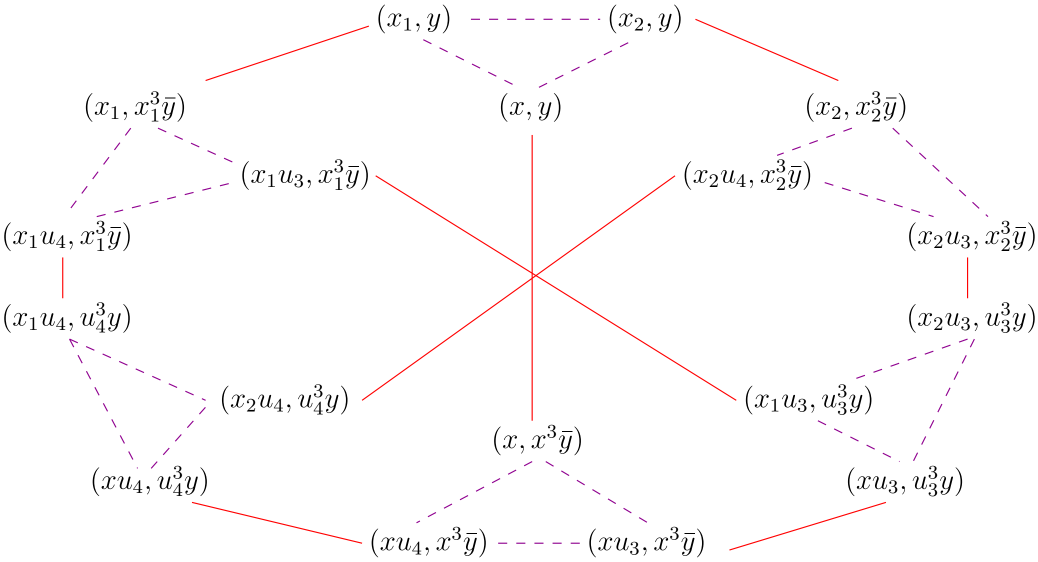


FIGURE 11. The orbit of $\mathcal{S} = \{10, 1\bar 1, \bar 21, \bar 20\}$. The values x_i and u_i are given by (50).

The functional equation reads

$$(1-tS(x,y))Q(x,y) = 1 - tx\bar y Q(x,0) - t\bar x^2(1+y)\left(Q_{0,-}(y) + xQ_{1,-}(y)\right),$$

and there is a unique section-free equation. To form it, the orbit equation associated with (x',y') must be weighted by $\pm x'y'(x_1'-x_2')(x_3'-x_4')$, where

$$\left.\begin{array}{l}(x_1',y'')\\(x_2',y'')\end{array}\right\}\approx (x',y')\approx(x',y'')\approx\left\{\begin{array}{l}(x_3',y'')\\(x_4',y'')\end{array}\right.$$

and both $x_1'\neq x_2'$ and $x_3'\neq x_4'$. More precisely, the weights associated with the 18 above pairs are

$$\begin{array}{rrr}
x^2y(x_1-x_2)(u_3-u_4) & -x_1^2y(x-x_2)(u_3-u_4) & x_2^2y(x-x_1)(u_3-u_4)\\
-x^5\bar y(x_1-x_2)(u_3-u_4) & x^5\bar y u_3^2(1-u_4)(x_1-x_2) & -x^5\bar y u_4^2(1-u_3)(x_1-x_2)\\
x_1^5\bar y(x-x_2)(u_3-u_4) & -x_1^5\bar y u_3^2(1-u_4)(x-x_2) & x_1^5\bar y u_4^2(1-u_3)(x-x_2)\\
-x_2^5\bar y(x-x_1)(u_3-u_4) & x_2^5\bar y u_3^2(x-x_1)(1-u_4) & -x_2^5\bar y u_4^2(x-x_1)(1-u_3)\\
-x^2u_3^5y(1-u_4)(x_1-x_2) & x_1^2u_3^5y(x-x_2)(1-u_4) & -x_2^2u_3^5y(x-x_1)(1-u_4)\\
x^2u_4^5y(1-u_3)(x_1-x_2) & -x_1^2u_4^5y(1-u_3)(x-x_2) & x_2^2u_4^5y(1-u_3)(x-x_1).
\end{array}\tag{51}$$

We now divide the section-free equation by $x^2y(x_1-x_2)(u_3-u_4)$, so as to isolate $Q(x,y)$. This gives:

$$Q(x,y) - x^3\bar y^2Q(x,x^3\bar y) + A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = R(x,y), \tag{52}$$

where $R(x, y)$ is the rational function occurring in Proposition 26 and each A_i involves two or four instances of the series Q, as described below:

A_1	A_2	A_3	A_4	A_5	A_6
(x_1, y)	$(x_1, x_1^3\bar{y})$	$(xu_3, x^3\bar{y})$	(xu_3, u_3^3y)	$(x_1u_3, x_1^3\bar{y})$	(x_1u_3, u_3^3y)
(x_2, y)	$(x_2, x_2^3\bar{y})$	$(xu_4, x^3\bar{y})$	(xu_4, u_4^3y)	$(x_2u_3, x_2^3\bar{y})$	(x_2u_3, u_3^3y)
				$(x_1u_4, x_1^3\bar{y})$	(x_1u_4, u_4^3y)
				$(x_2u_4, x_2^3\bar{y})$	(x_2u_4, u_4^3y)

Then each A_i is a series in t whose coefficients are polynomials in $x, \bar{x}, y, \bar{y}, x_1, x_2, u_3, u_4$. The symmetric functions of the x_i's (resp. u_i's) are Laurent polynomials in x and y, negative in x (resp. in y):

$$x_1 + x_2 = \bar{x}^2 y, \qquad x_1x_2 = -\bar{x}y, \qquad u_3 + u_4 = \bar{y}, \qquad u_3u_4 = -\bar{y}.$$

Hence each A_i is a series in t whose coefficients are Laurent polynomials in x and y. We now want to extract the non-negative part, in x and y, of (52). Clearly the second term is y-negative. Then the above properties, and the form (51) of the weights, imply that

- A_1, A_2, A_5, and A_6 are x-negative;
- A_3 is y-negative.

There remains to examine

$$A_4 = \frac{-u_3^5(1-u_4)Q(xu_3, u_3^3y) + u_4^5(1-u_3)Q(xu_4, u_4^3y)}{u_3 - u_4}.$$

Since $Q(x, y)$ has polynomial coefficients in x and y, it suffices to prove that for any $i, j \geq 0$, the expression

$$\frac{-u_3^5(1-u_4)u_3^iu_3^{3j}y^j + u_4^5(1-u_3)u_4^iu_4^{3j}y^j}{u_3 - u_4},$$

which is a Laurent polynomial in y, is in fact y-negative. This is readily checked, using the fact that $u_3u_4 = -\bar{y}$ and

$$E_a := \frac{u_3^{a+1} - u_4^{a+1}}{u_3 - u_4}$$

is a polynomial in $\bar{y}$ of valuation $\lceil a/2 \rceil$ (this is proved by induction on a, as $E_a = \bar{y}(E_{a-1} + E_{a-2})$). The expression of $Q(x, y)$ follows by extracting the non-negative part, in x and y, of (52).

The simplicity of the coefficients $q(i, j; n)$ comes from the fact that the expansion of $S(x, y)^n = (1 + y)^n(x\bar{y} + x^2)^n$ in x and y has simple coefficients.

The excursion exponent can be computed from Theorem 11, but it is more natural to start from the explicit expression of $q(0, 0; 3m)$, for which we derive:

$$q(0, 0; 3m) \sim \frac{81}{32\,\pi m^5} \left(\frac{27}{4}\right)^{2m}.$$

The asymptotic behaviour of the number of quadrant walks is determined in Section 9. ∎

8.4. Nine interesting models with a finite orbit

For nine models, shown in the second and third columns of Table 1, an interesting phenomenon occurs: the orbit is finite and the right-hand side of the unique section-free equation vanishes. These models come in two types, depending on whether they have an x/y-symmetry or not. They cannot be solved using the method of this paper, and we explore them *experimentally.*

Questions. For each of the nine models, we focus on two important univariate specializations of $Q(x, y) = Q(x, y; t)$, namely the generating function of excursions $Q(0, 0) = \sum_n e_n t^n$ and the generating function of all quadrant walks $Q(1, 1) = \sum_n q_n t^n$. For these 18 power series we address, as before, three types of questions: *qualitative* (are they algebraic? are they D-finite transcendental? are they non-D-finite?), *quantitative* (do they admit closed-form expressions?) and *asymptotic* (what is the growth of the sequences (e_n) and (q_n)?).

Answers. In this section, most answers to these questions are *conjectural*, although with a high degree of confidence. They are obtained by performing computer calculations that take as input a finite amount of information on $Q(0,0)$ and $Q(1,1)$, namely the first terms[2] of the sequences (e_n) and (q_n). The main technique that we use is *automated guessing*, a classical tool in experimental mathematics [16]. In principle, the guessing part could be complemented by an *automated proof* part, which would make the (algebraicity/D-finiteness) results fully rigorous, as in [17] and [11, §8]. This would require, among other things, to consider more general series such as $Q(x,0)$ and $Q(0,y)$. Given that the equations conjectured for $Q(0,0)$ and $Q(1,1)$ are already quite big (see Tables 3 and 4), we have decided to conduct the guessing part only.

Approach. For each model, we have first tried to guess linear recurrence relations with coefficients in $\mathbb{Z}[n]$ satisfied by the sequences (e_n) and (q_n), starting from the integer values of their first terms. When the available terms were not enough to recognize such a recurrence, we have used more terms modulo the prime $p = 2147483647$, and tried to recover recurrences with coefficients in $\mathbb{Z}/p\mathbb{Z}[n]$. In both cases, we used the guessed recurrence relations to produce even more terms, on which we repeated guessing procedures in order to get (hopefully) minimal-order linear differential equations with polynomial coefficients (in $\mathbb{Z}[t]$, resp. in $\mathbb{Z}/p\mathbb{Z}[t]$) for the associated series. On the one hand, such minimal-order equations are hard to guess because they tend to have many apparent singularities and thus coefficients of very large degrees; sometimes, it is necessary to produce them indirectly, e.g., by taking (right) gcd's of equations with higher orders but smaller degrees. On the other hand, they are interesting because they contain a lot of information on their solutions. For instance, minimal-order differential equations with coefficients in $\mathbb{Z}[t]$ are helpful in proving transcendence of their solutions. This is detailed below in Section 8.4.1.

Even when one can only guess differential equations with coefficients in $\mathbb{Z}/p\mathbb{Z}[t]$, for a sufficiently large prime such as $p = 2147483647$, rational reconstruction allows one to predict the small factors of the leading coefficients of plausible differential operators over $\mathbb{Q}[t]$, and thus the growth constant in the asymptotics of (e_n) and (q_n). A similar procedure applied to recurrences instead of differential equations allows one to guess the critical exponents of these sequences. They can also give, via p-curvature computations [12, 13], some insight on the algebraic/transcendental nature of the power series in $\mathbb{Z}[[t]]$ (modulo classical conjectures in the arithmetic theory of G-operators [3]). Examples are provided in [16, 17] and [9, §2.3.3]. However, given the size of our conjectured equations, and especially of the prime number p, we have not applied these algorithms here.

We refer to [16, 9] for more details on guessing techniques, and now describe the results that we have obtained on the nine models.

8.4.1. **Five models of the Kreweras type.** These models are symmetric in the first diagonal, and are shown in the central column of Table 1 and in Table 3 below. Their orbits are all of the same form: they consist of all pairs (x_i, x_j), with $0 \le i \neq j \le 3$, where $x_3 = y$ and $x_0 = x$, and x_1 and x_2 are the three roots of the equation $S(X,y) = S(x,y)$. In particular, for a pair (x', y') in the orbit, the symmetric pair (y', x') also lies in the orbit. The orbit structure is $\tilde{O}_{12}$, as shown in Figure 10.

Theorem 11 gives for each model the growth constant μ of excursions and the associated exponent $\alpha \equiv \alpha_e$, which happens to be $-5/2$ in all cases. The first two models are not strongly aperiodic, but it appears (numerically) that an asymptotic estimate $e_n \sim \kappa\, \mu^n n^{-5/2}$ holds in all cases (provided n is a multiple of 4 in the first case, and of 2 in the second case). The growth constant of the total number q_n of quadrant walks of length n can be determined using the results of [42, 50]: in all five cases, it coincides with the excursion constant μ. Observe that the *drift* $(S_x(1,1), S_y(1,1))$ is always negative. When the model is, in addition, aperiodic (last three

[2]Precisely, 20 000 integer coefficients, and even 100 000 coefficients modulo the prime $p = 2147483647$. For this time- and memory-consuming step, we have appealed to highly efficient implementations due to Axel Bacher.

models), we can apply the result of [36, Ex. 7]: there exists a constant K such that

$$q_n \sim K\mu^n n^{-5/2}.$$

Numerical computations (of two different types: floating point and modulo p) suggest that this also holds for the first two (periodic) models, with a constant K that depends on $n \mod 4$ (first model) and on $n \mod 2$ (second model). Such periodicity phenomena will be established in Section 9 for the four solved models of Section 8.3 (see for instance (56)).

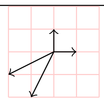
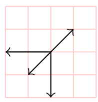
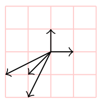
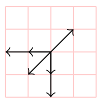
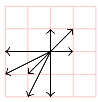
model	m	e_n	$Q(0,0)$	alg.	α_e	q_n	$Q(1,1)$	alg.	α_w
	4	$[2,12]$	$[8,13]$ irred.	no	$-5/2$	$[32,76]$	$[17,296]$ red. min.	no	$-5/2$?
	2	$[4,5]$	$[5,8]$ red. min.	no	$-5/2$	$[19,59]$	$[9,83]$ red. min.	no	$-5/2$?
	1	$[12,37]$	$[9,52]$ red. min.	no	$-5/2$	$[33,266]$	$[17,309]$ red. min.	no	$-5/2$
	1	$[20,75]$	$[13,94]$ red. min.	no	$-5/2$	$[60,118]^\star$	$[25,663]^\star$	?	$-5/2$
	1	$[36,520]^\star$	$[26,573]^\star$	?	$-5/2$	$[99,204]^\star$	$[44,652]^\star$	?	$-5/2$

Table 3. The five Kreweras-like models, with their periods m. For each sequence (e_{mn}) and (q_n), resp. for the associated series $Q(0,0;t^{1/m})$ and $Q(1,1;t)$, a pair $[r,d]$ indicates the order r and the coefficients degree d of a (conjectural) recurrence relation, resp. of a differential equation. A star indicates that we have only guessed recurrences or differential equations modulo $p = 2147483647$.

Algorithmic guessing has succeeded for all 10 sequences in Table 3, but only modulo $p = 2147483647$ for 3 of them. We are extremely confident that the guessed recurrences and differential operators are correct. In particular, they pass with success the filters described in [16, Sec. 2.4]. For instance, the leading coefficients of the differential operators that (conjecturally) annihilate $Q(0,0)$ and $Q(1,1)$, or their rational reconstruction when operators are available modulo p only, vanish at $t = 1/\mu$. Also, the occurrence of $3/2$ among the local exponents of the operators around $t = 1/\mu$ is in agreement with the exponents $\alpha_e = \alpha_w = -5/2$.

Assuming these recurrences and equations correct, we can use them to derive some properties of the sequences (e_n) and (q_n). For instance, guessing already strongly indicates that there is no hypergeometric sequence among the 10 sequences. In cases where recurrences are guessed over the integers (not only modulo p), we have applied Petkovšek's algorithm [64] to them, and obtained a proof that these sequences are indeed not hypergeometric.

Guessing also strongly indicates that there is no algebraic generating function for any of the 10 sequences. In cases where differential equations are guessed over the integers (not only modulo p), we have a proof for this fact, based on the following strategy. Linear differential operators can be factored algorithmically [70]. Those that are irreducible in $\mathbb{Q}(t)\langle\partial_t\rangle$ are necessarily minimal. We have proved minimality of the others using the argument of [11, Prop. 8.4]. Next, we computed the first terms of a local basis of solutions at $t = 0$. At least one basis element contains logarithms, which, combined with minimality, implies that the solution is transcendental [29,

§2]. Note that this cannot be directly deduced from estimates of the form $c\,\mu^n n^{-5/2}$, which *are* compatible with algebraicity [41]. For the excursions of the second model, we were even able to solve the differential equation, thus obtaining a conjectural closed form expression of $Q(0,0;\sqrt{t})$ (Conjecture 27). When we have only guessed differential equations modulo $p = 2147483647$, we still conjecture that the corresponding operators have minimal order.

We now review briefly the five Kreweras-like models and add a few details completing Table 3.

• **Case** $\mathcal{K}_1 = \{\bar{2}\bar{1}, \bar{1}\bar{2}, 01, 10\}$. The excursion generating function $Q(0,0) = \sum_n e_n t^n$ starts

$$Q(0,0) = 1 + 6\,t^4 + 236\,t^8 + 14988\,t^{12} + 1193748\,t^{16} + O(t^{20})$$

and the walk generating function $Q(1,1) = \sum_n q_n t^n$ starts

$$Q(1,1) = 1 + 2\,t + 4\,t^2 + 8\,t^3 + 22\,t^4 + 64\,t^5 + 178\,t^6 + O(t^7).$$

The growth constant is $\mu = 8/(3^{3/4})$ for both sequences.

The model has period $m = 4$, and $e_n = 0$ if n is not a multiple of 4. For the subsequence $(u_n) = (e_{4n})$ we have guessed that

$$\begin{aligned}
&(4608\,n^4 + 37504\,n^3 + 114144\,n^2 + 153992\,n + 77715)\times\\
&\qquad (2\,n+3)\,(2\,n+1)\,(4\,n+5)\,(4\,n+1)\,(n+1)^2\,(4\,n+3)^2\,u_n -\\
&\Big(62208\,n^{12} + 1159488\,n^{11} + 9826272\,n^{10} + 50056248\,n^9 + \frac{341349339}{2}\,n^8 + 410259762\,n^7 +\\
&\quad \frac{22807094283}{32}\,n^6 + \frac{28845939249}{32}\,n^5 + \frac{421694744175}{512}\,n^4 + \frac{1085550761145}{2048}\,n^3 +\\
&\quad \frac{1868027110233}{8192}\,n^2 + \frac{1929023165205}{32768}\,n + \frac{1807811742825}{262144}\Big)\,u_{n+1} +\\
&(n+3)\,(n+2)\,(2\,n+5)^2\,(6\,n+13)^2\,(6\,n+11)^2\times\\
&\qquad \left(\frac{81}{2048}\,n^4 + \frac{1341}{8192}\,n^3 + \frac{8235}{32768}\,n^2 + \frac{22257}{131072}\,n + \frac{44739}{1048576}\right)\,u_{n+2} = 0.
\end{aligned}$$

The leading coefficient of the minimal differential operator L_e annihilating $Q(0,0;t^{1/4})$ is

$$t^7\,(27 - 4096\,t)^2\left(t^4 - \frac{47}{640}\,t^3 - \frac{374489}{125829120}\,t^2 - \frac{23644531}{2319282339840}\,t + \frac{29645}{281474976710656}\right),$$

where the factor $27 - 4096\,t$ vanishes when $t = 27/4096 = 1/\mu^4$.

For walks ending anywhere in the quadrant, the leading coefficient of the operator L_w is

$$t^{13}\,(4t-1)\,(16\,t^3 + 8\,t^2 + 11\,t - 4)^4\,(4096\,t^4 - 27)^4\ \times (\text{irreducible poly. of degree } 254)\,,$$

which is again compatible with the value of μ.

• **Case** $\mathcal{K}_2 = \{\bar{2}0, \bar{1}\bar{1}, 0\bar{2}, 11\}$. The excursion generating function $Q(0,0) = \sum_n e_n t^n$ starts

$$Q(0,0) = 1 + t^2 + 4\,t^4 + 21\,t^6 + 138\,t^8 + 1012\,t^{10} + 8064\,t^{12} + O(t^{14})$$

while

$$Q(1,1) = 1 + t + 2\,t^2 + 5\,t^3 + 12\,t^4 + 32\,t^5 + 86\,t^6 + O(t^7).$$

The growth constant is $\mu = 2\sqrt{3}$ for both sequences.

The model has period $m = 2$, and $e_n = 0$ if n is odd. For the nontrivial subsequence $(u_n) = (e_{2n})$ we have guessed that

$$\begin{aligned}
&(4\,n+9)\,(n+5)^2\,(n+4)^2\,u_{n+4} - 4\,(n+2)\,(16\,n^2 + 100\,n + 153)\,(n+4)^2\,u_{n+3} -\\
&\qquad 4\,(32\,n^5 + 584\,n^4 + 4096\,n^3 + 13909\,n^2 + 22947\,n + 14742)\,u_{n+2} +\\
&\qquad\quad 96\,(2\,n+3)\,(n+2)\,(16\,n^3 + 108\,n^2 + 239\,n + 183)\,u_{n+1} +\\
&\qquad\quad (9216\,n^5 + 76032\,n^4 + 230400\,n^3 + 319680\,n^2 + 201024\,n + 44928)\,u_n = 0.
\end{aligned}$$

The differential operator L_e found for $Q(0,0;t^{1/2}) = \sum_n e_{2n} t^n$ has leading coefficient

$$t^3\,(1+4\,t)^2\,(1-12\,t)^3$$

where the factor $(1-12\,t)$ is compatible with the value of μ. Furthermore, L_e is reducible in $\mathbb{Q}(t)\langle\partial_t\rangle$; one can write $L = L_2^{(1)} L_2^{(2)} L_1$, where L_1 has order 1 and $L_2^{(1)}$ and $L_2^{(2)}$ have order 2. More importantly, L can be written as the least common left multiple of the three following operators:

$$\partial_t + \frac{1}{t}, \qquad \partial_t^2 + \frac{120\,t^2+2\,t-3}{(-1+12\,t)t(4\,t+1)}\,\partial_t + \frac{288\,t^3-48\,t^2+14\,t+1}{(4\,t+1)t^2(-1+12\,t)^2},$$
$$\partial_t^2 + \frac{120\,t^2+2\,t-3}{(-1+12\,t)t(4\,t+1)}\,\partial_t + \frac{24\,t^2-8\,t-1}{t^2(4\,t+1)(-1+12\,t)}.$$

The use of ${}_2F_1$ solving algorithms [15, 48, 14] leads us to the following conjectural expression.

Conjecture 27. *For the model $\mathcal{S} = \{\bar{2}0, \bar{1}\bar{1}, 0\bar{2}, 11\}$, the excursion generating function $Q(0,0;t^{1/2})$ is equal to*

$$\frac{1}{3t} - \frac{\sqrt{1-12t}}{6t}\left({}_2F_1\left(\begin{matrix}\frac{1}{6} & \frac{1}{3}\\ & 1\end{matrix}\,\middle|\,\frac{-108\,t(1+4t)^2}{(1-12t)^3}\right) + {}_2F_1\left(\begin{matrix}-\frac{1}{6} & \frac{2}{3}\\ & 1\end{matrix}\,\middle|\,\frac{-108\,t(1+4t)^2}{(1-12t)^3}\right)\right).$$

Remark. The first hypergeometric term above can be rewritten with a simpler argument, as

$$\frac{1}{\sqrt{1-12t}}\;{}_2F_1\left(\begin{matrix}\frac{1}{6} & \frac{1}{3}\\ & 1\end{matrix}\,\middle|\,\frac{-108\,t(1+4t)^2}{(1-12t)^3}\right) = {}_2F_1\left(\begin{matrix}\frac{1}{6} & \frac{1}{3}\\ & 1\end{matrix}\,\middle|\,108t^2(1+4t)\right).$$

Moreover, the square of this power series is known to count excursions of the face centered cubic lattice [10, Appendix A], see also [51, §4]. This is entry A002899 in the on-line encyclopedia of integer sequences [49]. The guessed operator L_e is the minimal-order operator canceling the conjectured series. The leading coefficient of the operator L_w contains the factor

$$t^5\,(4\,t-1)\,(4\,t^2+1)^2\,(16\,t^3+8\,t^2+11\,t-4)^2\,(12\,t^2-1)^4,$$

which is compatible with $\mu = 2\sqrt{3}$.

- **Case $\mathcal{K}_3 = \{\bar{2}\bar{1}, \bar{1}\bar{2}, \bar{1}\bar{1}, 01, 10\}$.** The excursion generating function starts

$$Q(0,0) = 1 + 2\,t^3 + 6\,t^4 + 16\,t^6 + 122\,t^7 + 236\,t^8 + O(t^9)$$

while

$$Q(1,1) = 1 + 2\,t + 4\,t^2 + 10\,t^3 + 32\,t^4 + 98\,t^5 + 292\,t^6 + O(t^7).$$

The model is strongly aperiodic, with growth constant $\mu \sim 4.03$ for both sequences, where μ is the unique positive root of $4069 + 768\,u - 6\,u^2 + u^3 - 27u^4$.

The leading coefficient of L_e is

$$t^8\,(1+t^2)^2\,(4069\,t^4+768\,t^3-6\,t^2+t-27)^2 \times (\text{irreducible poly. of degree } 32)\,,$$

and vanishes at $t = 1/\mu$. Similarly, the leading coefficient of L_w is

$$t^{13}\,(1-5\,t)\,(t^2+1)^2\,(4069\,t^4+768\,t^3-6\,t^2+t-27)^4\,(23\,t^3+32\,t^2+8\,t-4)^4 \times (\text{irreducible poly. of degree } 263)\,.$$

- **Case $\mathcal{K}_4 = \{\bar{2}0, \bar{1}\bar{1}, \bar{1}0, 0\bar{2}, 0\bar{1}, 11\}$.** The excursion generating function starts

$$Q(0,0) = 1 + t^2 + 2\,t^3 + 4\,t^4 + 24\,t^5 + 37\,t^6 + 276\,t^7 + O(t^8)$$

while

$$Q(1,1) = 1 + t + 4\,t^2 + 11\,t^3 + 42\,t^4 + 148\,t^5 + 576\,t^6 + O(t^7).$$

The model is strongly aperiodic, with growth constant $\mu \sim 4.91$ for both sequences, where μ is the largest positive root of $405 - 108\,u - 72\,u^2 + u^3 + 3u^4$.

The leading coefficient of L_e is

$$t^8\,(65\,t^2+8\,t+16)^2\,(405\,t^4-108\,t^3-72\,t^2+t+3)^4 \times (\text{irreducible poly. of degree } 66)\,,$$

and vanishes at $t = 1/\mu$. Similarly, the leading coefficient of L_w contains the factor

$$t^{17}\,(1-6t)\,(405\,t^4-108\,t^3-72\,t^2+t+3)^8\,(3\,t^3+4\,t^2+20\,t-4)^6\,(65\,t^2+8\,t+16)^2.$$

• **Case $\mathcal{K}_5 = \{\bar2\bar1, \bar20, \bar1\bar2, \bar1\bar1, 0\bar2, 01, 10, 11\}$.** The excursion generating function starts

$$Q(0,0) = 1 + t^2 + 8\,t^3 + 10\,t^4 + 106\,t^5 + 467\,t^6 + 1850\,t^7 + O(t^8)$$

while

$$Q(1,1) = 1 + 3\,t + 10\,t^2 + 51\,t^3 + 260\,t^4 + 1350\,t^5 + 7568\,t^6 + O(t^7).$$

The model is strongly aperiodic, with growth constant $\mu = 2\sqrt{3} + 8/3^{3/4} \approx 6.97$ for both sequences. The value μ is the unique positive root of $208 + 4608\,u + 648\,u^2 - 27\,u^4$.

In this case we only have conjectures modulo $p = 2147483647$ for both e_n and q_n. For excursions, rational reconstruction shows that the leading coefficient of the operator L_e contains the factor

$$t^{23}\,(4\,t^2+1)^4\,(208\,t^4+4608\,t^3+648\,t^2-27)^7$$

which vanishes at $t = 1/\mu$. Similarly, the leading coefficient of L_w contains the factor

$$t^{35}\,(8\,t-1)\,(4\,t^2+1)^4\,(64\,t^3+16\,t^2+11\,t-2)^{10}\,(208\,t^4+4608\,t^3+648\,t^2-27)^{12}$$

which vanishes again at $t = 1/\mu$.

8.4.2. **Four models of the Gessel type.** The remaining 4 models, shown on the right of Table 1 and in Table 4, do not have a symmetry property. They are obtained from the models shown on the left of Table 1 (solved in Section 8.3) by a reflection in a horizontal line. By Proposition 3, their orbit type is O_{12} for the first three, and O_{18} for the last one. More precisely, in the first three cases the orbit consists of

$$\begin{array}{ccc} (x,y) & (x_1,y) & (x_2,y) \\ (x,\bar x^e\bar y) & (-\bar x_1,\bar x^e\bar y) & (-\bar x_2,\bar x^e\bar y) \\ (x_1,\bar x_1^e\bar y) & (-\bar x,\bar x_1^e\bar y) & (-\bar x_2,\bar x_1^e\bar y) \\ (x_2,\bar x_2^e\bar y) & (-\bar x,\bar x_2^e\bar y) & (-\bar x_1,\bar x_2^e\bar y) \end{array}$$

where x_1, x_2 are the two solutions of $S(X,y) = S(x,y)$ (different from x), $e = 2$ for the first model and $e = 1$ for the next two. In the fourth case, the orbit consists of

$$\begin{array}{ccc} (x,y) & (x_1,y) & (x_2,y) \\ (x,\bar x^3\bar y) & (xu_3,\bar x^3\bar y) & (xu_4,\bar x^3\bar y) \\ (x_1,\bar x_1^3\bar y) & (x_1u_3,\bar x_1^3\bar y) & (x_1u_4,\bar x_1^3\bar y) \\ (x_2,\bar x_2^3\bar y) & (x_2u_3,\bar x_2^3\bar y) & (x_2u_4,\bar x_2^3\bar y) \\ (xu_3,\bar u_3^3 y) & (x_1u_3,\bar u_3^3 y) & (x_2u_3,\bar u_3^3 y) \\ (xu_4,\bar u_4^3 y) & (x_1u_4,\bar u_4^3 y) & (x_2u_4,\bar u_4^3 y) \end{array}$$

where

$$x_{1,2} = \frac{1 \pm \sqrt{1+4x^3y}}{2x^2y} \qquad \text{and} \qquad u_{3,4} = \frac{y \pm \sqrt{y^2+4y}}{2}.$$

For each of these four models, there exists a unique section-free equation, and its right-hand side vanishes.

Theorem 11 gives for each model the excursion constant μ and the corresponding exponent, which is $\alpha_e = -5/2$ for the first three models, and $\alpha_e = -7/3$ for the last one. Only the third model is strongly aperiodic, the other models having respectively period $m = 2$ (first model), $m = 4$ (second model) and $m = 3$ (last model). But it appears numerically that an asymptotic estimate $q(0,0;n) \sim \kappa\,\mu^n n^{\alpha_e}$ holds in all cases (provided n is a multiple of m in the periodic cases). The growth constant $\bar\mu$ of the sequence (q_n) can be determined using the results of [42, 50]: in all four cases, it is larger than the excursion constant μ. Observe that the *drift* $(S_x(1,1), S_y(1,1))$ is always of the form $(-\delta, 0)$ with δ positive. The second component being 0, we cannot apply the result of [36, Ex. 7], and indeed, the walk exponent α_w, that we conjecture numerically, turns out to differ from α_e. In fact, we believe that for each of the four models,

$$q_n \sim K\bar{\mu}^n n^{-3/2}.$$

with a constant K that depends on $n \mod m$ in the periodic cases.

What we have done. We have applied to these four models the same guessing procedures as for the Kreweras-like models. Remarkably, we discovered two possibly algebraic models among them. More precisely, for the second and third models, the series $Q(0,0)$ seems to be algebraic of degree 32. But it must be noted that in contrast with Kreweras-like models, for three of the four models we could not guess any recurrence for the sequence (q_n), even modulo the prime $p = 2147483647$.

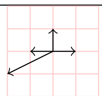
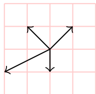
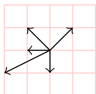
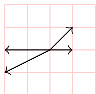
model	m	e_n	$Q(0,0)$	alg.	α_e	q_n	$Q(1,1)$	alg.	α_w
	2	$[8,5]$	$[9,18]$ red. min.	no	$-5/2$	$[46,176]$	$[17,400]$ red. min.	no	$-3/2$?
	4	$[2,12]$	$[8,13]$ irred.	$[32,14]$	$-5/2$	?	?	?	$-3/2$?
	1	$[12,37]$	$[9,52]$ irred.	$[32,57]$	$-5/2$	?	?	?	$-3/2$?
	3	$[23,572]^\star$	$[48,589]^\star$	?	$-7/3$	?	?	?	$-3/2$?

TABLE 4. The four Gessel-like models, with their periods m. The table gives, for each sequence (e_{mn}) and (q_n), and for the associated series $Q(0,0;t^{1/m})$ and $Q(1,1;t)$, the order and degree of the guessed recurrence relation or differential equation (in the two algebraic cases, the first/second value is the degree in the series/variable). A star indicates that we have only guessed recurrences or differential equations modulo $p = 2147483647$.

Our results are summarized in Table 4, and completed with a few details below.

• **Case $\mathcal{G}_1 = \{\bar{2}\bar{1}, \bar{1}0, 01, 10\}$.** The excursion generating function starts

$$Q(0,0) = 1 + t^2 + 5\,t^4 + 27\,t^6 + 188\,t^8 + 1414\,t^{10} + O(t^{12})$$

while

$$Q(1,1) = 1 + 2\,t + 5\,t^2 + 13\,t^3 + 38\,t^4 + 112\,t^5 + 346\,t^6 + 1071\,t^7 + O(t^8).$$

The model has period $m = 2$, and $e_n = 0$ if n is odd. The growth constant is $\mu = 2\sqrt{3}$ for the excursion sequence, and

$$\bar{\mu} = \frac{\sqrt[3]{6371 + 624\sqrt{78}}}{12} + \frac{217}{12\sqrt[3]{6371 + 624\sqrt{78}}} + \frac{11}{12} \sim 3.61 \tag{53}$$

for all quadrant walks. The value $\bar{\mu}$ is the unique (positive) real root of $16 + 8u + 11u^2 - 4u^3$.

The leading coefficient of the operator L_e annihilating $Q(0,0;t^{1/2})$ is

$$t^5\,(1+4\,t)^3\,(1-12\,t)^5\,(279936\,t^5 - 62208\,t^4 + 13608\,t^3 - 5796\,t^2 + 675\,t - 20)$$

where the factor $(1-12\,t)$ is compatible with the growth constant μ. The leading coefficient of L_w is

$$t^{10}\,(1+4\,t)\,(1-4\,t)^4\,(1+4\,t^2)^5\,(1-12\,t^2)^9\,\,(16\,t^3+8\,t^2+11\,t-4)^4\times(\text{irreducible poly. of degree } 345)\,,$$

which is compatible with the value of $\bar{\mu}$.

• **Case $\mathcal{G}_2 = \{\bar{2}\bar{1}, \bar{1}1, 0\bar{1}, 11\}$.** The excursion generating function starts

$$Q(0,0) = 1 + 5\,t^4 + 190\,t^8 + 11892\,t^{12} + 939572\,t^{16} + O(t^{20})$$

while

$$Q(1,1) = 1 + t + 3\,t^2 + 8\,t^3 + 24\,t^4 + 65\,t^5 + 211\,t^6 + 649\,t^7 + O(t^8).$$

The model has period $m = 4$, and $e_n = 0$ if n is not a multiple of 4. The growth constant of excursions is $\mu = 8/3^{3/4}$, while the constant for all quadrant walks is again given by (53).

For the nontrivial subsequence $(u_n) = (e_{4n})$ we have guessed that

$$\begin{aligned}
&3\,(6\,n+11)\,(18\,n+41)\,(2\,n+5)\,(3\,n+7)\,(18\,n+35)\,(6\,n+13)\,(n+2)\,(18\,n+29)\\
&\quad(41472\,n^4+150144\,n^3+200864\,n^2+117704\,n+25491)\,u_{n+2}-\\
&(47552535724032\,n^{12}+\ 798266178404352\,n^{11}+6092888790269952\,n^{10}+27954969361514496\,n^9\\
&\quad+85850716160655360\,n^8+185860480394330112\,n^7+290753615920332800\,n^6+\\
&\quad331020927507759104\,n^5+272073153165252608\,n^4+157356059182977536\,n^3+\\
&\quad60749526504280448\,n^2+14046784950077600\,n+1470033929525700)\,u_{n+1}+\\
&1048576\,(12\,n+5)\,(4\,n+1)\,(3\,n+2)\,(2\,n+1)\,(12\,n+11)\,(4\,n+3)\,(6\,n+7)\,(n+1)\\
&\quad(41472\,n^4+316032\,n^3+900128\,n^2+1135752\,n+535675)u_n = 0.
\end{aligned}$$

Remarkably, the series $E(t) := Q(0,0;t^{1/4})$ appears to be algebraic, of degree 32. More precisely, $E(t)^2$ seems to have degree 16 and to satisfy an equation $P(t, E(t)^2) = 0$ with coefficients of degree at most 14 in t. The guessed polynomial $P(t,z)$ seems plausible because: it has a small bitsize compared to the bitsize of the expansion of $E(t)$ that we used to produce it; we have then checked, using more terms of $E(t)$, that it annihilates $E(t)^2$ to much higher orders; its discriminant factors as

$$t^{418}\,(268435456\,t^3+57671680\,t^2-69632\,t-27)^2\,(4096\,t-27)^{48}\times(\text{irreducible poly. of degree } 31)^4\,,$$

which is compatible with the value of μ. Moreover, $P(t,z)$ defines a rational curve, parametrized by

$$t = \frac{U\,(1-2\,U)^3\,(1-3\,U)^3\,(1-6\,U)^9}{(1-4\,U)^4}, \qquad z = \frac{(1-4\,U)^2\,(1-24\,U+120\,U^2-144\,U^3)^2}{(1-3\,U)^2\,(1-2\,U)^3\,(1-6\,U)^9}.$$

This leads to the following conjectural statement.

Conjecture 28. *For the model $\mathcal{S} = \{\bar{2}\bar{1}, \bar{1}1, 0\bar{1}, 11\}$, the excursion generating function $Q(0,0;t)$ is equal to*

$$\frac{(1-4\,U)\,(1-24\,U+120\,U^2-144\,U^3)}{(1-3\,U)\,(1-2\,U)^{3/2}\,(1-6\,U)^{9/2}},$$

where $U = t^4 + 53\,t^8 + 4363\,t^{12} + \cdots$ is the unique power series in $\mathbb{Q}[[t]]$ satisfying

$$U\,(1-2\,U)^3\,(1-3\,U)^3\,(1-6\,U)^9 = t^4\,(1-4\,U)^4.$$

As mentioned above, we could not guess any differential nor algebraic equation for $Q(1,1;t)$, even with 100 000 terms and modulo p= 2147483647.

• **Case $\mathcal{G}_3 = \{\bar{2}\bar{1}, \bar{1}0, \bar{1}1, 0\bar{1}, 11\}$.** The excursion generating function starts

$$Q(0,0) = 1 + 2\,t^3 + 5\,t^4 + 16\,t^6 + 107\,t^7 + 190\,t^8 + O(t^9)$$

while

$$Q(1,1) = 1 + t + 4\,t^2 + 12\,t^3 + 39\,t^4 + 133\,t^5 + 485\,t^6 + 1746\,t^7 + O(t^8).$$

This model is strongly aperiodic. The growth constants are $\mu \sim 4.03$, the unique positive root of $4069 + 768\,u - 6\,u^2 + u^3 - 27\,u^4$, and

$$\bar{\mu} = \frac{\sqrt[3]{1261 + 57\sqrt{57}}}{6} + \frac{56}{3\,\sqrt[3]{1261 + 57\sqrt{57}}} + \frac{2}{3} \sim 4.22.$$

This is the unique (positive) real root of $4\,u^3 - 8\,u^2 - 32\,u - 23$.

Again, $Q(0,0;t)$ appears to be algebraic of degree 32, this time with coefficients of degree 57. The guessed polynomial seems plausible for various reasons, including the nice factorization of its discriminant as

$$t^{1732}\,(t^2+1)^{32}\,(4069\,t^4 + 768\,t^3 - 6\,t^2 + t - 27)^{48} \\ \times (\text{irreducible poly. of degree } 13)^2 \times (\text{irreducible poly. of degree } 31)^4,$$

which vanishes at $t = 1/\mu$.

Remark. There are analogies between excursions of this model and those of the Kreweras-type model $\mathcal{K}_3 = \{\bar{2}\bar{1}, \bar{1}\bar{2}, \bar{1}\bar{1}, 01, 10\}$. Indeed, the sizes of the recurrence relation and of the differential equation match. The growth constant and the singular exponent are also the same for both models.

• **Case $\mathcal{G}_4 = \{\bar{2}\bar{1}, \bar{2}0, 10, 11\}$.** The excursion generating function starts

$$Q(0,0) = 1 + 3\,t^3 + 41\,t^6 + 850\,t^9 + 21538\,t^{12} + 614530\,t^{15} + O(t^{16})$$

while

$$Q(1,1) = 1 + 2\,t + 4\,t^2 + 15\,t^3 + 45\,t^4 + 121\,t^5 + 471\,t^6 + 1533\,t^7 + O(t^8).$$

The model has period $m = 3$, and $e_n = 0$ if n is not a multiple of 3. The growth constants are $\mu = 9/2^{4/3}$ and $\bar{\mu} = 3 \cdot 2^{1/3}$.

The leading coefficient of L_e contains the factor $t^{42}\,(729\,t - 16)^{23}$ which is compatible with the value of μ.

9. A glimpse at asymptotics

The method that we develop in this paper provides expressions for generating functions of walks confined to an orthant, as positive parts of certain rational or algebraic series. We now demonstrate that these expressions are often well suited to a multivariate singularity analysis. The use of analytic techniques in this fashion is the domain of *analytic combinatorics in several variables* (ACSV) [62]; recent work has shown the strength of this approach, proving conjectures in lattice path asymptotics [60], generalizations in higher dimensions [59], and handling families of models with weighted steps [30]. Much of the singularity analysis is effective [58] when the multivariate generating function under consideration is represented in the form $Q(x,y;t) = [x^{\geq}y^{\geq}]R(x,y;t)$ for a *rational* function $R(x,y;t)$. Although some asymptotic techniques have been developed to perform a singularity analysis on multivariate functions with algebraic singularities [46], this is a more difficult task. For the purposes of this paper, we show how dominant asymptotics for the number of walks in the four models of Section 8.3 can be determined through the simple use of analytic techniques. We focus on the series $Q(1,1)$ counting all quadrant walks. Future work could extend this argument to deal with the multivariate algebraic functions which arise, for instance, in the generating functions for 2D Hadamard models given by Proposition 22.

The first step is to convert our expression of the form $Q(x,y;t) = [x^{\geq}y^{\geq}]R(x,y;t)$ for the multivariate generating function $Q(x,y;t)$ into an expression for the univariate generating function $Q(1,1;t)$ which is amenable to asymptotic computations. Given an element

$$R(x,y;t) = \sum_{n\geq 0}\left(\sum_{i,j} r(i,j;n)x^i y^j t^n\right) \in \mathbb{Q}[x,\bar{x},y,\bar{y}][[t]], \tag{54}$$

the *diagonal* operator Δ takes $R(x,y;t)$ and returns the univariate power series $(\Delta R)(t) := \sum_{n\geq 0} r(n,n;n)t^n$. The relationship between positive parts and diagonals is given by the following lemma.

Lemma 29. *Given $R(x,y;t)$ as in* (54)*, and $(a,b)\in\{0,1\}^2$, one has*

$$[x^{\geq}y^{\geq}]R(x,y;t)\Big|_{x=a,y=b} = \Delta\left(\frac{R(\bar{x},\bar{y};xyt)}{(1-x)^a(1-y)^b}\right).$$

The proof follows from basic formal series manipulations; see Proposition 2.6 of [59] for details. In particular, this lemma, combined with the expressions obtained for $Q(x,y;t)$ in Section 8.3, gives us diagonal representations for the generating functions of quadrant walks ending anywhere ($a=b=1$), returning to the origin (excursions, $a=b=0$), or returning to the x- or y-axes ($a=1$, $b=0$ or $a=0$, $b=1$).

At its most basic level, the theory of ACSV takes a multivariate Cauchy residue integral representation for power series coefficients and reduces it to an integral expression where saddle-point techniques can be used to determine asymptotics. Because of the simple rational functions which are obtained for many lattice path models, the usual analysis can be greatly simplified. In particular, for each of the four models detailed in Section 8.3 we obtain the generating function $Q(1,1;t)$ as a diagonal of the form

$$Q(1,1;t) = \Delta\left(\frac{P(x,y)}{(1-x)(1-y)(1-txyS(\bar{x},\bar{y}))}\right),$$

where $P(x,y)$ is a Laurent polynomial which is coprime with $1-x$ and $1-y$. Expanding the rational function on the right-hand side of this equation as a power series in t then gives

$$q_n = [t^n]Q(1,1;t) = [x^n y^n t^n]\left(\frac{P(x,y)}{(1-x)(1-y)(1-txyS(\bar{x},\bar{y}))}\right) = [x^0y^0]\frac{P(x,y)S(\bar{x},\bar{y})^n}{(1-x)(1-y)},$$

and the multivariate Cauchy integral formula [62, Prop. 7.2.6] implies

$$\begin{aligned} q_n &= \frac{1}{(2\pi i)^2}\int_{|x|=r_1,|y|=r_2} \frac{P(x,y)S(\bar{x},\bar{y})^n}{(1-x)(1-y)}\cdot\frac{\mathrm{d}x\,\mathrm{d}y}{xy} \\ &= \frac{1}{(2\pi i)^2}\int_{|x|=r_1,|y|=r_2} \frac{P(x,y)}{xy(1-x)(1-y)}e^{n\log S(\bar{x},\bar{y})}\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

for any $0<r_1,r_2<1$. Making the substitutions $x=r_1e^{i\theta_1}$ and $y=r_2e^{i\theta_2}$ converts this integral into a *Fourier-Laplace* integral; that is, an integral of the form

$$\int_T A(\theta_1,\theta_2)e^{-n\phi(\theta_1,\theta_2)}\mathrm{d}\theta_1\mathrm{d}\theta_2.$$

Here $T=[-\pi,\pi]^2$, while

$$A(\theta_1,\theta_2) = \frac{1}{(2\pi)^2}\frac{P(r_1e^{i\theta_1},r_2e^{i\theta_2})}{(1-r_1e^{i\theta_1})(1-r_2e^{i\theta_2})},$$

and

$$\phi(\theta_1,\theta_2) := -\log S\left(r_1^{-1}e^{-i\theta_1}, r_2^{-1}e^{-i\theta_2}\right).$$

The asymptotics of Fourier-Laplace integrals have been well studied. In particular, suppose the *amplitude* A and *phase* ϕ are analytic functions on the domain T. If ϕ admits a non-empty finite set of critical points[3], at which the Hessian of ϕ is non-singular and the real part of ϕ is locally minimized, then explicit asymptotic formulas in terms of the Taylor coefficients of A and ϕ are known [47, Theorem 7.7.5] (see also [58, Prop. 53] for the explicit formulas used in our calculations). Each critical point of ϕ has an asymptotic contribution, and one simply sums up the contributions of all critical points to determine dominant asymptotics of the Fourier-Laplace integral.

For the above value of $\phi(\theta_1, \theta_2)$, the chain rule shows that in order to find real values r_1 and r_2 such that ϕ admits critical points, it is sufficient to find the complex points (x, y) such that

$$S_x(\bar{x}, \bar{y}) = S_y(\bar{x}, \bar{y}) = 0 \tag{55}$$

and take r_1 and r_2 to be their moduli. One then determines the corresponding critical pairs (θ_1, θ_2), that is, the arguments of x and y satisfying (55), and computes the Hessian of ϕ at these points. In each of the four cases that we consider there are critical points with $0 < r_1, r_2 < 1$, and the Hessian is never singular. The next step is to show that the real part of $\phi(\theta_1, \theta_2)$,

$$\Re\phi(\theta_1, \theta_2) = -\log |S(r_1^{-1}e^{-i\theta_1}, r_2^{-1}e^{-i\theta_2})|,$$

is locally minimized at critical values of (θ_1, θ_2). Minimizing this quantity means maximizing $|S(\bar{x}, \bar{y})|$ on $\{(x, y) : |x| = r_1, |y| = r_2\}$. Since $S(\bar{x}, \bar{y})$ is a (Laurent) polynomial with non-negative coefficients, when $|x|$ and $|y|$ are fixed then $|S(\bar{x}, \bar{y})|$ is maximized (in particular) when x and y are positive and real (that is, $x = r_1, y = r_2$). The triangle inequality then shows that the maximizers of $|S(\bar{x}, \bar{y})|$ occur when the arguments of all monomials occurring in $S(\bar{x}, \bar{y})$ are equal. When this holds for all critical values (θ_1, θ_2), explicit asymptotics can be obtained by direct computation. In particular, the exponential growth associated with the critical point (θ_1, θ_2) is $e^{-\phi(\theta_1,\theta_2)} = S(\bar{x}, \bar{y})$.

We now list our results; full details of the computations can be found in an accompanying Maple worksheet, available on the authors' webpages[4].

9.1. Case $\mathcal{S} = \{10, \bar{1}0, 0\bar{1}, \bar{2}1\}$

Specializing Lemma 29 to Proposition 16 gives the diagonal representation

$$Q(1, 1; t) = \Delta\left(\frac{(x^2+1)(x^2+2xy-1)(2x^3+x^2y-y)(x^2-y^2)}{x^2y(1-x)(1-y)(1-t(x^3+x^2y+xy^2+y))}\right).$$

Solving (55) for x and y gives two solutions with coordinates of modulus less than 1,

$$(x, y) = \left(3^{-1/2}, 3^{-1/2}\right) \quad \text{and} \quad (x, y) = \left(-3^{-1/2}, -3^{-1/2}\right),$$

along with solutions $(i, -i)$, and $(-i, i)$ which are irrelevant to asymptotics. Taking $r_1 = r_2 = 3^{-1/2}$ in the argument above, one gets a Fourier-Laplace integral with critical points at $(\theta_1, \theta_2) = (0, 0)$ and (π, π). A direct computation shows that the Hessian of ϕ is non-singular at these critical points. Following the above lines, we then check that

$$|S(\bar{x}, \bar{y})| = \left|\bar{x} + x + y + x^2\bar{y}\right|$$

is indeed maximal on the integration domain for angles $(0, 0)$ and (π, π), as desired.

The exponential growth of the resulting Fourier-Laplace integral is given by the value of $e^{-\phi(\theta_1,\theta_2)} = S(\bar{x}, \bar{y})$ at the critical points, in this case $S(\sqrt{3}, \sqrt{3}) = 2\sqrt{3}$ and $S(-\sqrt{3}, -\sqrt{3}) = -2\sqrt{3}$. One then computes successively higher order terms in an asymptotic expansion

$$(2\sqrt{3})^n\left(A_0 + \frac{A_1}{n} + \frac{A_2}{n^2} + \cdots\right) + (-2\sqrt{3})^n\left(A_0' + \frac{A_1'}{n} + \frac{A_2'}{n^2} + \cdots\right)$$

[3]For the purposes of this discussion, points in T where the gradient of ϕ vanishes.

[4]For lattice path examples with more exotic critical point behaviour, see [58, Ch. 10 and 11].

until finding terms which are non-zero (see [58, Prop. 53]). The vanishing of the highest order terms is related to, but not completely determined by, the order of vanishing of the amplitude $A(\theta_1, \theta_2)$ at the critical points under consideration. Ultimately, we obtain the asymptotic expansion

$$q_n = \frac{(2\sqrt{3})^n}{\pi n^4} \left(C_n + O\left(\frac{1}{n}\right)\right),$$

where

$$C_n = \begin{cases} 5616\sqrt{3} & : n \text{ even} \\ 9720 & : n \text{ odd} \end{cases}. \tag{56}$$

9.2. Case $\mathcal{S} = \{01, 1\bar{1}, \bar{1}\bar{1}, \bar{2}1\}$

Applying Lemma 29 to the generating function expression in Proposition 24 gives a diagonal representation

$$Q(1,1;t) = \Delta\left(\frac{(2xy^2 + x^2 - 1)(x - y^2)(x^2y^2 + 2x^3 - y^2)}{xy^2(1-x)(1-y)(1 - t(x^2y^2 + x^3 + y^2 + x))}\right).$$

This time the system of equations (55) admits four solutions whose coordinates have moduli less than 1,

$$\left(3^{-1/2}, 3^{-1/4}\right), \left(3^{-1/2}, -3^{-1/4}\right), \left(-3^{-1/2}, i3^{-1/4}\right), \left(-3^{-1/2}, -i3^{-1/4}\right),$$

all of which have coordinate-wise moduli $(r_1, r_2) = \left(3^{-1/2}, 3^{-1/4}\right)$. A similar analysis to the first case gives

$$q_n = \frac{\left(8 \cdot 3^{-3/4}\right)^n}{\pi n^4} \left(C_n + O\left(\frac{1}{n}\right)\right),$$

where

$$C_n = \begin{cases} 5120\sqrt{3} & : n \equiv 0 \mod 4 \\ 6656 \cdot 3^{1/4} & : n \equiv 1 \mod 4 \\ 26624/3 & : n \equiv 2 \mod 4 \\ 3840 \cdot 3^{3/4} & : n \equiv 3 \mod 4 \end{cases}.$$

9.3. Case $\mathcal{S} = \{01, 1\bar{1}, \bar{1}\bar{1}, \bar{2}1, \bar{1}0\}$

Specializing Lemma 29 to Proposition 25 gives a diagonal representation

$$Q(1,1;t) = \Delta\left(\frac{(x - y^2)(2xy^2 + x^2 + xy - 1)(x^2y^2 + 2x^3 + x^2y - y^2)}{xy^2(1-x)(1-y)(1 - t(x^2y^2 + x^3 + x^2y + y^2 + x)))}\right).$$

Here the system (55) has four solutions (x, y) with coordinates of modulus less than 1, which make up the set

$$\left\{(y^2, y) : 3y^4 + y^3 - 1 = 0\right\}. \tag{57}$$

The polynomial $3y^4 + y^3 - 1$ has a unique positive root, $y_c \simeq 0.688...$, and we consider the solution (y_c^2, y_c). None of the three other solutions has the same coordinate-wise moduli, hence our only critical point associated with moduli $(r_1, r_2) = (y_c^2, y_c)$ is $(\theta_1, \theta_2) = (0, 0)$. The Hessian of ϕ is not singular at $(0, 0)$, and by positivity, this point maximizes the modulus of S in $[-\pi, \pi]^2$. In the end, one obtains asymptotics

$$q_n = \frac{(8y_c^3 + 3y_c^2)^n}{2313\pi n^4} \left(C + O\left(\frac{1}{n}\right)\right) \approx (1112.183\cdots)\frac{(4.03164\cdots)^n}{n^4},$$

where

$$C = \sqrt{3}\left(2527386y_c^3 + 2727881y_c^2 + 1805111y_c + 1306017\right).$$

It can be checked that the three other solutions in (57) are *not* local maximizers of $|S(\bar{x}, \bar{y})|$ among points with the same coordinate-wise moduli.

9.4. Case $\mathcal{S} = \{10, 1\bar{1}, \bar{2}1, \bar{2}0\}$

Specializing Lemma 29 to Proposition 26 gives a diagonal representation

$$Q(1,1;t) = \Delta\left(\frac{(1-2y)(x^3-y^2)(x^6+x^3y^2+3x^3y-y)(2x^3-y)}{x^3y^2(1-x)(1-y)(1-t(x^3y+x^3+y^2+y))}\right).$$

Here there are three solutions to (55) with moduli less than 1:

$$\left(4^{-1/3}, 1/2\right), \quad \left(e^{2\pi i/3}4^{-1/3}, 1/2\right), \quad \left(e^{-2\pi i/3}4^{-1/3}, 1/2\right).$$

All of them have moduli $(r_1, r_2) = \left(4^{-1/3}, 1/2\right)$. They give rise to three critical points of ϕ, where the Hessian is non-singular and $|S(\bar{x}, \bar{y})|$ is maximized. An analysis similar to those above gives another periodic asymptotic expansion

$$q_n = \frac{\left(9 \cdot 4^{-2/3}\right)^n}{\pi n^5}\left(C_n + O\left(\frac{1}{n}\right)\right),$$

where

$$C_n = \begin{cases} 216513/2 & : n \equiv 0 \mod 3 \\ 1358127 \cdot 2^{-11/3} & : n \equiv 1 \mod 3 \\ 124659 \cdot 2^{-1/3} & : n \equiv 2 \mod 3 \end{cases}.$$

10. Final questions and comments

We have outlined above the first general approach to count walks confined to an orthant with arbitrary steps, and demonstrated its efficacy across several families and a large number of sporadic cases. In addition to the examples presented here, the power of this method is illustrated by the fact that it solves another family of quadrant models, with steps $\mathcal{S} = \{(-p, 0), (-p+1, 1), \dots, (0, p), (1, -1)\}$, which arose naturally in other applications; the details of this family (containing both large forward and large backward steps) are given in [24]. The current work attempts to lay a basis for the systematic study of lattice walks with longer steps, and we suggest here some possible research directions.

- **Uniqueness of the section-free equation.** Is it true that, for a model with no large forward step and a finite orbit, there exists a unique section-free equation (Conjecture 13)? Can one describe it generically?
- **Walks with steps in $\{\bar{2}, \bar{1}, 0, 1\}^2$.** In our study of these walks (Section 8) we have left open the case of nine models which have analogies with the four tricky-but-algebraic small step models of Figure 2 (see Tables 3 and 4). Can one apply to them some of the techniques used for the small step algebraic models [7, 17, 18, 21, 22, 26, 44, 52, 54, 61]? In particular, are the associated series D-finite? Which ones are algebraic?

  Can one prove the non-D-finiteness of the 16 models of Table 2, which have a rational excursion exponent but an infinite orbit?
- **Walks with steps in $\{\bar{1}, 0, 1, 2\}^2$.** Symmetrically, one can examine the 14 268 interesting (non-isomorphic and non-trivial) models with steps in $\{-1, 0, 1, 2\}^2$, having at least one large forward step. Proceeding as in Section 8 reveals that 1 189 of them are included in a half-space, and thus analogous to the 5 half-space models with small steps. Of the remaining 13 079 models, 12 828 have an irrational excursion exponent, and hence a non-D-finite generating function and an infinite orbit (Section 3.3). The 251 that have a rational exponent split in three families:
  - 11 have yet an infinite orbit. They are the reverses of the 11 models of Table 2 that contain a step in $\mathbb{Z}_-^2$ (for the other 5 models in this table, there is no non-trivial walk starting at the origin after reversing steps);
  - 227 are Hadamard, and thus solvable by Proposition 22 and D-finite. They are the reverses of the 227 Hadamard models of Section 8.3;

– 13 are the reverses of the models in Table 1, and thus share their orbit structure: O_{12}, $\tilde{O}_{12}$ or O_{18}. They also share their excursion generating function, which we have either proved or conjectured to be D-finite in all 13 cases.

- It has been proved [14] that for the 19 small step models in the quadrant that are D-finite but transcendental, the series $Q(x, y; t)$ has an explicit expression involving integrals and specializations of the hypergeometric series ${}_2F_1$. For which models with larger steps is this still true? Corollary 17 and Conjecture 27 show that a similar property may indeed hold in some cases.
- We have focussed in this paper on 2D examples, because the quadrant is already a rich source of interesting problems. But the four stages of the method, described in Sections 2 to 5, apply just as well to higher dimensional models. In fact, they were already successfully applied to 3D models with small steps in [11].

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AB: INRIA Saclay, 1 rue Honoré d'Estienne d'Orves, F-91120 Palaiseau, France
E-mail address: Alin.Bostan@inria.fr

MBM: CNRS, LaBRI, Université de Bordeaux, 351 cours de la Libération, F-33405 Talence Cedex, France
E-mail address: bousquet@labri.fr

SM: Department of Mathematics, University of Pennsylvania, 209 S. 33rd Street, Philadelphia, PA 19104, USA.
E-mail address: smelczer@sas.upenn.edu