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Large N expansion of the moments and free energy of Sachdev-Ye-Kitaev model, and the enumeration of intersection graphs

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ABSTRACT: In this paper we explain the relation between the free energy of the SYK model for N Majorana fermions with a random q-body interaction and the moments of its spectral density. The high temperature expansion of the free energy gives the cumulants of the spectral density. Using that the cumulants are extensive we find the p dependence of the $1/N^2$ correction of the $2p$-th moments obtained in [1]. Conversely, the $1/N^2$ corrections to the moments give the $1/q^3$ correction of the high temperature expansion of the free energy obtained by Tarnopolsky using a mean field expansion. These considerations also lead to a more powerful method for solving the moment problem and intersection-graph enumeration problems. We take advantage of this and push the moment calculation to $1/N^3$ order and find surprisingly simple enumeration identities for intersection graphs. The $1/N^3$ corrections to the moments, give $O(1/q^4)$ corrections to high temperature expansion of the free energy which have not been calculated before.

Contents

1 Introduction

The statistical fluctuations of nuclear levels have been successfully described by the Gaussian Orthogonal Ensemble (GOE). However, the average spectral density of the GOE is a semicircle which is very different from the Bethe formula [2]. In agreement with experimental observations [3], this formula predicts an $\exp(c\sqrt{E - E_0})$ dependence (with E_0 the ground state energy) on the excitation energy E. Moreover, the nuclear interaction is mostly a two-body interaction while for the GOE all the many-body states interact. To address these shortcomings, French and co-workers [4–7] introduced the two-body random ensemble which is now known as the four-body complex SYK model. One of the first results for this ensemble was that the level density is a Gaussian [6] which is closer to the expectation of realistic many-body systems than the semicircular behavior. However, in the nuclear physics community it was not realized that this model actually reproduces the Bethe formula [8]. One of the reasons for missing this opportunity was the custom [9–13] to study this model as the sum of a two-body and a four-body interaction (in the nuclear physics convention, a one-body and a two-body interaction). The reason is that the nuclear interaction was seen as a residual random four-body interaction on top of a mean field which can be represented as a two-body interaction. Note that the four-body interaction is a irrelevant term with regard to the two-body interaction [14–16]. It was also understood early on that the two-body random ensemble still has the level correlations of the Gaussian Orthogonal Ensemble [7]. However, it was also realized that the model was not ergodic [16–19] in the sense that ensemble average of a spectral correlator is not equal to the spectral average of a spectral correlator, the latter given by the result of the Wigner-Dyson ensembles up to much larger distances. For more discussions of the two-body random ensemble in nuclear and many-body physics we refer to [9, 18, 20, 21].

In condensed matter, the model was introduced independently as a random quantum spin model [22] where a mean field is not natural. In this context, Sachdev and co-workers discovered [23] a remarkable property of this model, namely that its zero temperature entropy is extensive which then was identified as the black hole entropy [24]. This property is directly related to its exponentially large level density starting with the ground state region, which is a non-Fermi liquid. The states are characterized by highly entangled states [25] which are very different from the particle-hole excitations of a Fermi liquid.

In the past two years the interest in this model was rekindled because it is possibly dual to 1+1 dimensional gravity [8, 26–46]. The properties that made this model attractive as a model of a compound nucleus are exactly those which are required for the existence of a black-hole dual: it is maximally chaotic [47] with spectral correlations given by random matrix theory [16, 33, 48], it has level density given by the Bethe formula which also implies that the specific heat is linear in temperature for low temperature, the zero temperature entropy is extensive showing that the low-lying states of the model are strongly entangled. What is particularly important in this context is the existence of a conformal limit [26, 27] where the action of the SYK model reduces to the Schwarzian action [27, 42, 49, 50].

Much of the recent progress on the SYK model was made possible because of the path integral formulation [22, 26, 27], from which one can derive an exact result for large N limit after averaging over the random interaction. This was not possible for a formulation that started from the generating function for the resolvent [51]. Although it was straightforward to average over the randomness, it resulted in a complicated theory that was not amenable to taking a large N limit. The disadvantage of the path integral which at the same time is its strength is that it provides access to the Green's function rather than the level density. Since our main interest has been in the level density and the level correlations [1, 8, 14, 16, 33] of the SYK model, we have used the moment method [52] which also proved effective in the early applications to nuclear physics. Two limiting cases for N fermions fermions with a random q body interaction were easily recognized: $q \sim N$ and $q \ll N$. In the first case [18, 53] the SYK model is in the universality class of the Wigner-Dyson ensembles with a semicircular spectral density. In the second case, the spectral density is a Gaussian [6]. This suggests the existence of a double scaling limit which converges to a spectral density in between a semicircle and a Gaussian. Indeed this happens when $q \sim \sqrt{N}$ for $N \to \infty$ [54]. This scaling limit which reveals itself in the path integral formulation of the SYK model was not noted before in the nuclear physics literature either. In the moment method, this limit arises naturally in Wick contractions when treating all intersections as independent which gives moments that are exact to order $1/N$ [8, 33, 38, 54]. Remarkably, these moments turn out to give the spectral density of the weight function of the Q-Hermite polynomials with a nontrivial double scaling limit [8, 38, 54, 55].

The $1/N^2$ corrections to all moments can be calculated analytically as well [1], with a result that has as a simple geometric interpretation. The p-dependence of the $2p$-th moment also turns out to be relatively simple. One of the main goals of this paper is to explain this p-dependence of the moments. Since the high temperature expansion of the free energy is the cumulant expansion of the spectral density, the extensivity of the free energy puts strong constraints on the moments. In fact, we will show that the p-dependence of the moments follows almost entirely from this condition, and that it is determined by a few low-order moments only. Secondly, we study the way large N corrections to the moments contribute to the free energy. We already know that the Q-Hermite moments give the free energy for all temperatures to order $1/q^2$ [8, 56]. In this paper we will show that $1/q^3$ corrections to the free energy follow from the $1/N^2$ corrections to moments. Thirdly, using the inter-relationship between the free energy and the moments, we obtain the $1/N^3$ corrections to the moments which are responsible for the $1/q^4$ correction to the high temperature expansion of the free energy. Using these results we find new enumerative identities for intersection graphs. Results for the supersymmetric SYK model [57–59] which can be derived in a similar way are also given in this paper.

This paper is organized as follows. In section 2, we give a brief review of the SYK model including the moment method. The relation between moments and the free energy is discussed in section 3. In this section we show that at a given order in $1/N$ the p-dependence of the moments follows from a few low-order moments . We also show that large N corrections to the

moments give large q corrections to the free energy. Results are obtained for both even q and odd q. As a new result we obtain the $1/N^3$ corrections to the moments and $1/q^4$ corrections to the free energy. In section 4, we use the results for the moments to derive new enumerative identities for intersection graphs. Some technical results are deferred to two appendices. In appendix A, we evaluate the high temperature expansion to the free energy from the results of Tarnopolsky [56]. We obtain the high temperature expansion of Tarnopolsky's result to all orders and show that it is a convergent series with no singularities on the positive real axis. Finally, we derive in appendix B the $1/N^3$ corrections for $q = 1$ and $q = 2$ with an independent method.

2 Review of SYK model and moment method

In this section we introduce the Sachdev-Ye-Kitaev (SYK) model and the moment method.

2.1 The SYK Hamiltonian

The SYK model is a system of N Majorana particles with the q-body interaction represented by

$$H = \sum J_\alpha \Gamma_\alpha \tag{2.1}$$

with

$$\Gamma_\alpha = (i)^{q(q-1)/2} \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_q}, \tag{2.2}$$

and γ_α are the Euclidean gamma matrices with commutation relations:

$$\{\gamma_k, \gamma_l\} = 2\delta_{kl}. \tag{2.3}$$

The factors of i in the definition of Γ_α have been included so that H is Hermitian. The sum is over all $\binom{N}{q}$ q-particles states denoted by the collective index $\alpha = \{i_1, i_2, \ldots, i_q\}$, with $1 \leq i_1 < i_2 < \cdots < i_q \leq N$. The couplings J_α are Gaussian distributed:

$$P(J_\alpha) = \sqrt{\frac{2^{q-1}N^{q-1}}{(q-1)!\pi J^2}} \exp\left(-\frac{2^{q-1}N^{q-1}J_\alpha^2}{(q-1)!J^2}\right), \tag{2.4}$$

where the parameter J sets a physical scale.[1] For even q the Hamiltonian $\mathcal{H}$ of the SYK model is simply given by

$$\mathcal{H} = H. \tag{2.5}$$

With the large N scaling of the variance in (2.4) this Hamiltonian has a negative-energy ground state energy that is proportional to N for large N.

---

[1]A factor of 2^q has been included in the variance so that our results coincide with the Majorana convention $\{\gamma_k, \gamma_l\} = \delta_{kl}$.

For odd q the operator H is still a well-defined Hermitian operator, but because it has a fermionic grading, it is not a Hamiltonian but rather the supercharge of a supersymmetric theory with the Hamiltonian [57]:

$$\mathcal{H} = H^2. \tag{2.6}$$

This Hamiltonian is positive definite with a ground state energy approaching zero in the thermodynamic limit with the scaling of the variance as in (2.4).

2.1.1 Moments and spectral density

One of the reasons for which we are interested in moments is to study the spectral density $\rho(E)$ of the SYK model:[2]

$$\rho(\lambda) := \left\langle \sum_{k=1}^{2^{\lfloor \frac{N}{2} \rfloor}} \delta(\lambda - \lambda_k) \right\rangle, \tag{2.7}$$

where $\langle \cdots \rangle$ denotes the Gaussian average using the distribution (2.4). By a Fourier transform, we can express $\rho(\lambda)$ as

$$
\begin{aligned}
\rho(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t e^{-iEt} \left\langle \mathrm{Tr} e^{iHt} \right\rangle \\
&= 2^{N/2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t e^{-i\lambda t} \sum_{k=0}^{\infty} \frac{(it)^k}{(k)!} M_k \\
&= 2^{N/2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t e^{-i\lambda t} \sum_{p=0}^{\infty} \frac{(it)^{2p}}{(2p)!} M_{2p},
\end{aligned}
\tag{2.8}
$$

where we have defined the k-th moment M_k to be

$$M_k := \left\langle \mathrm{Tr} H^k \right\rangle / 2^{N/2} = 2^{-N/2} \int d\lambda \rho(\lambda) \lambda^k. \tag{2.9}$$

The third equality of eq. (2.8) used the fact that $M_k = 0$ when k is odd, due to the $J_\alpha \to -J_\alpha$ symmetry of the distribution (2.4). Therefore, we may focus on the calculation of M_{2p}. Notice that since $\langle \cdots \rangle$ is an Gaussian integration over J_α, M_{2p} is given by a sum over all possible Wick contractions among $2p$ Γ's. By convention, we have factored out the Hilbert space dimensionality $2^{N/2}$, but M_{2p} still grows like N^p as $N \to \infty$, as can be seen from the N dependence of M_2:

$$M_2 = \sigma^2 = \binom{N}{q} \frac{(q-1)! J^2}{2^q N^{q-1}}. \tag{2.10}$$

---

[2]As we will see, the spectral density will not play an important role in the higher-order free energy calculation, which is the main subject of this paper. Nevertheless, the spectral density provides a natural introduction of the moments.

Hence, to formulate a useful large N expansion for moments, we consider instead the scaled moments:

$$\tilde{M}_{2p} := \frac{M_{2p}}{M_2^p} \sim O(1).\tag{2.11}$$

We distinguish the moments of the SYK operator H, which will be denoted by M_{2p} and the moments of the Hamiltonian $\mathcal{H}$ which will be denoted by μ_p. So we have that

$$\mu_p = M_p, \qquad \text{for even} \quad q, \tag{2.12}$$

$$\mu_p = M_{2p}, \qquad \text{for odd} \quad q. \tag{2.13}$$

3 Free energy, cumulants and high temperature expansions

3.1 Moments and cumulants

The partition function is given by

$$Z = \text{Tr} e^{-\beta \mathcal{H}} \equiv e^{-\beta F}\tag{3.1}$$

with the free energy denoted by F. In terms of high temperature expansion of the free energy, we have

$$-\beta F - \frac{N}{2}\log 2 = \sum_{n=1}^{\infty} \frac{\kappa_n}{n!}(-\beta)^n.\tag{3.2}$$

The quantity κ_n is called the n-th cumulant. Note that the summation starts from $n = 1$, this is because the energy is finite, and at infinite temperature we expect only the entropy to contribute to βF. Since free energy is extensive, we obtain

$$\kappa_n \sim O(N)\tag{3.3}$$

for all n. We will see later that this has important consequences for the N-dependence of the moments.

Alternatively, the partition function can also be expressed in the moments μ_n of $\mathcal{H}$,

$$Z = \sum_{n=0}^{\infty} \frac{\mu_n}{n!}(-\beta)^n.\tag{3.4}$$

One may ask why we do not consider the "free energy" $e^{-\beta F} := \langle \text{Tr}(e^{-\beta H}) \rangle$ also for odd q so that we can have a uniform treatment of the moments for all values of q, instead of two separate cases eqs. (2.12) and (2.13). The problem with this "free energy" is that it is not extensive. Extensivity will turn out to be vital for the application of our method.

The relation between μ_n and κ_n is well studied [60]. Consider the following partition of

an integer n:

$$n = \underbrace{k_1 + \cdots k_1}_{m_1 \text{ times}} + \underbrace{k_2 + \cdots + k_2}_{m_2 \text{ times}} + \cdots + \underbrace{k_l + \cdots k_l}_{m_l \text{ times}} = \sum_{i=1}^{l} m_i k_i, \tag{3.5}$$

where by convention we demand the ordering $k_1 \geq k_2 \geq \cdots \geq k_l$,[3] then we have

$$\mu_n = \sum_{P_n} \left(\frac{n!}{\prod_{i=1}^{l} m_i!(k_i!)^{m_i}} \prod_{j=1}^{l} (\kappa_{k_j})^{m_j} \right), \tag{3.6}$$

$$\kappa_n = \sum_{P_n} \left((-1)^{\sum_i m_i - 1} \left(\sum_{i=1}^{l} m_i - 1 \right)! \frac{n!}{\prod_{i=1}^{l} m_i!(k_i!)^{m_i}} \prod_{j=1}^{l} (\mu_{k_j})^{m_j} \right), \tag{3.7}$$

where $\sum_{P_n}$ denotes sum over all partitions of n. As examples, we list some low-order relations:

$$\begin{aligned} \mu_1 &= \kappa_1, \quad \mu_2 = \kappa_2 + \kappa_1^2, \quad \mu_3 = \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3; \\ \kappa_1 &= \mu_1, \quad \kappa_2 = \mu_2 - \mu_1^2, \quad \kappa_3 = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^2. \end{aligned} \tag{3.8}$$

We remark that since moments are computed by contracting Γ matrices, all the N-dependence comes from counting the subscripts of those Γ matrices. This implies the μ_n must be rational functions of N, and then eq. (3.7) tells us the cumulants κ_n must also be rational functions of N. Hence there can be no factors such as $\log N$ or $\sqrt{N}$ in the large N expansion of moments or cumulants. For even q all odd cumulants vanish, but they enter in the calculations for odd q.

3.2 Even q case

3.2.1 N dependence of cumulants and moments

As discussed, for even q we have $M_k = \mu_k$ and hence $\mu_k = 0$ when k is an odd number. It follows from eq. (3.7) that $\kappa_k = 0$ when k is odd. We are interested in the scaled moments $\tilde{M}_{2p}$, so let us also define the scaled cumulants to be

$$\tilde{\kappa}_{2k} := \frac{\kappa_{2k}}{\kappa_2^k} = \frac{\kappa_{2k}}{M_2^k} = \frac{\kappa_{2k}}{\sigma^{2k}}. \tag{3.9}$$

According to eq. (2.10), $M_2 = \sigma^2 = \binom{N}{q} \frac{J^2(q-1)!}{2^q N^{q-1}}$ which is $\sim N$, together with the free energy extensivity eq. (3.3), we deduce

$$\tilde{\kappa}_{2k} \sim O(N^{-k+1}). \tag{3.10}$$

---

[3]Alternatively we can say, for example, $3 = 2 + 1$ and $3 = 1 + 2$ are the same partition of 3.

We also have the trivial identity

$$\tilde{\kappa}_2 = 1. \tag{3.11}$$

Since $\kappa_2 = \mu_2$ it is clear that eq. (3.6) is also valid for the rescaled moments and cumulants. Because of the N dependence eq. (3.10), the corrections of order $1/N^{k-1}$ to *all* scaled moments only receive contributions from cumulants up to $\tilde{\kappa}_{2k}$, which by themselves are completely determined by the moments up to order $2k$ due to eq. (3.7). Hence we conclude:

To order N^{-k+1}, all $\tilde{M}_{2p}$ are determined by a finite number of moments up to $\tilde{M}_{2l}$ ($l \leq k$), expanded to N^{-k+1}.

For example, the $1/N^2$ expansion of all moments is completely determined by $\tilde{\kappa}_4$ and $\tilde{\kappa}_6$ which in turn are determined by $\tilde{M}_4$ and $\tilde{M}_6$, while at order $1/N^3$ the moments receive only contributions from $\tilde{\kappa}_4$, $\tilde{\kappa}_6$ and $\tilde{\kappa}_8$ and are thus determined by $\tilde{M}_4$, $\tilde{M}_6$ and $\tilde{M}_8$. This is very surprising, because for large p, the Wick contractions contributing to the $2p$-th moment become rather complicated, whereas to calculate up to $\tilde{M}_8$ we only need to consider a small set of contraction diagrams. This will have important implications when it comes to the enumeration of intersection graphs, which is the subject of section 4.

The scaled cumulant $\tilde{\kappa}_{2k}$ is $O(1/N^{k-1})$, but after being rescaled back to κ_{2k} its leading term contributes to the thermodynamic limit of the free energy. The leading term of $\tilde{\kappa}_{2k}$ is determined by the $1/N^{k-1}$ corrections of the moments up to order $2k$. We thus emphasize:

The leading term of the $2k$-th scaled cumulant $\tilde{\kappa}_{2k}$ is determined by the moments $\tilde{M}_{2l}$ ($l \leq k$), expanded to order N^{-k+1}.

This in particular implies that even if we want the complete information of only the thermodynamic limit (leading in $1/N$) of the free energy, we would still need all-order knowledge of scaled cumulants and hence of the scaled moments.

We will see below that the q-dependence of the leading term of the sixth cumulant follows from the $1/N^2$ corrections to the fourth and sixth moment. The correction factor is simply given by $1 - 1/3q$ with no other corrections from higher moments. So the large q expansion of this cumulant terminates at this order.

The discussion in this section is general and also applies to the odd q case, where only some minor changes of notations are needed.

3.2.2 Explicit results to $1/N^3$

We will derive the expansion of $\tilde{M}_{2p}$ to $1/N^3$ in this paper. To the relevant order, eq. (3.6) can be explicitly written as

$$\frac{\tilde{M}_{2p}}{(2p-1)!!} = 1 + \frac{1}{3}\binom{p}{2}\tilde{\kappa}_4 + \frac{1}{15}\binom{p}{3}\tilde{\kappa}_6 + \frac{1}{3}\binom{p}{4}\tilde{\kappa}_4^2 + \frac{1}{105}\binom{p}{4}\tilde{\kappa}_8 + \frac{2}{9}\binom{p}{5}\tilde{\kappa}_6\tilde{\kappa}_4 + \frac{5}{9}\binom{p}{6}\tilde{\kappa}_4^3 + O(1/N^4). \tag{3.12}$$

We emphasize again the fact that only a finite number of terms need to be considered on the right-hand side of the above equation is due to the extensivity of the free energy, see eqs.

(3.3) and (3.10). The first eight moments were calculated exactly in [1], and their expansion up to order $1/N^3$ thus gives the expansion of all moments to this order.

Using the results of [1] we obtain

$$
\begin{aligned}
\tilde{M}_4 =& 3 - \frac{2q^2}{N} + \frac{2q^2(q-1)^2}{N^2} - \frac{2q^2(q-1)^2(2q^2-8q+5)}{3N^3} + O\left(\frac{1}{N^4}\right), \\
\tilde{M}_6 =& 15 - \frac{30q^2}{N} + 2q^2\left(27q^2 - 34q + 15\right)\frac{1}{N^2} \\
& - 2q^2(q-1)^2\left(38q^2 - 56q + 25\right)\frac{1}{N^3} + O(N^{-4}), \\
\tilde{M}_8 =& 105 - \frac{420q^2}{N} + 28q^2\left(44q^2 - 38q + 15\right)\frac{1}{N^2} \\
& - 4q^2\left(716q^4 - 1712q^3 + 1743q^2 - 854q + 175\right)\frac{1}{N^3} + O(N^{-4}).
\end{aligned}
\tag{3.13}
$$

This results in the cumulants,

$$
\begin{aligned}
\tilde{\kappa}_4 =& -3 + \tilde{M}_4 \\
=& -\frac{2q^2}{N} + \frac{2q^2(q-1)^2}{N^2} - \frac{2q^2(q-1)^2(2q^2-8q+5)}{3N^3} + O\left(N^{-4}\right), \\
\tilde{\kappa}_6 =& 30 - 15\tilde{M}_4 + \tilde{M}_6 \\
=& \frac{8q^3(3q-1)}{N^2} - \frac{8q^3(q-1)^2(7q-4)}{N^3} + O\left(N^{-4}\right), \\
\tilde{\kappa}_8 =& -630 + 420\tilde{M}_4 - 35\tilde{M}_4^2 - 28\tilde{M}_6 + \tilde{M}_8 \\
=& -\frac{16q^4\left(46q^2 - 36q + 7\right)}{N^3} + O\left(N^{-4}\right),
\end{aligned}
\tag{3.14}
$$

which have exactly the leading order N dependence of eq. (3.10) required for an extensive free energy.

Substituting these results for cumulants back to eq. (3.12), we obtain the following large N expansion for the scaled moments to $1/N^3$:

$$
\begin{aligned}
\frac{\tilde{M}_{2p}}{(2p-1)!!} =& 1 - \frac{2}{3}\binom{p}{2}\frac{q^2}{N} + \left[\binom{p}{2}\left(\frac{2}{3}q^2(q-1)^2\right) + \binom{p}{3}\left(\frac{8}{15}q^3(3q-1)\right) + \binom{p}{4}\frac{4}{3}q^4\right]\frac{1}{N^2} \\
& - \left[\binom{p}{2}\left(\frac{2}{9}q^2(q-1)^2(2q^2-8q+5)\right) + \binom{p}{3}\left(\frac{8}{15}q^3(q-1)^2(7q-4)\right)\right. \\
& \left. + \binom{p}{4}\left(\frac{8}{105}q^4(127q^2 - 142q + 49)\right) + \binom{p}{5}\left(\frac{32}{9}q^5(3q-1)\right)\right. \\
& \left. + \binom{p}{6}\frac{40}{9}q^6\right]\frac{1}{N^3} + O\left(\frac{1}{N^4}\right).
\end{aligned}
\tag{3.15}
$$

In [1] $\tilde{M}_{2p}$ was expanded to $1/N^2$ by Q-Hermite approximation and triangle counting, and
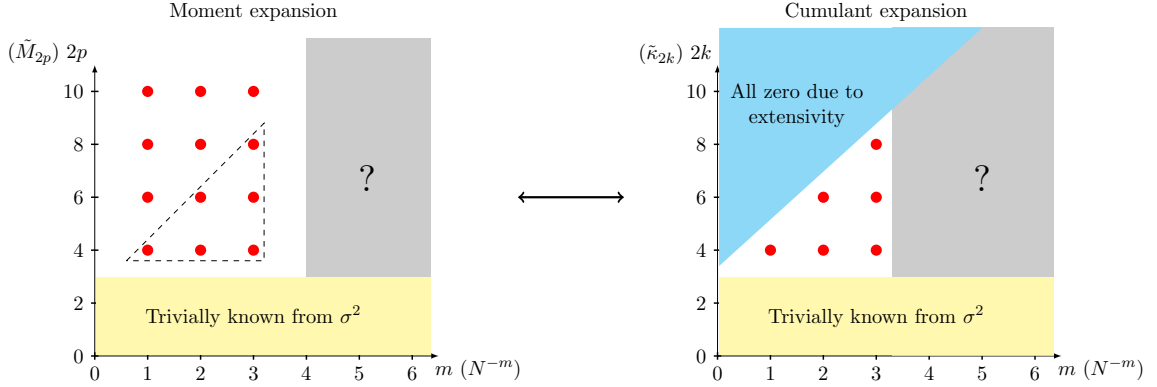
Figure 1. Relation between two expansions (for even q). The horizontal axes denote orders in $1/N$, while the vertical axes denote orders in (scaled) moments and cumulants. The red dots are where the corresponding $1/N$ coefficients are known, and the gray region is unknown. It is interesting to see that on the moment side infinitely many coefficients are known, but they are all determined by the finite number of dots in the dashed triangle. This is ultimately because on cumulant side we easily know an infinite number of coefficients due to extensivity (the cyan region).

it agrees with (3.15), which we just obtained by a completely independent method. The discussion above suggests an interesting "map of knowledge" between the moment expansion and cumulant expansion, as shown in figure 1.

3.2.3 Free energy

We can substitute eq. (3.14) into eq. (3.2) to obtain the high temperature expansion of the free energy:

$$ -\beta F = -\frac{N}{2}\log 2 + \frac{1}{2!}\sigma^2\beta^2 + \frac{1}{4!}\tilde{\kappa}_4\sigma^4\beta^4 + \frac{1}{6!}\tilde{\kappa}_6\sigma^6\beta^6 + \frac{1}{8!}\tilde{\kappa}_8\sigma^8\beta^8 + \cdots \tag{3.16} $$

Since $\sigma^2 \sim N$ we obtain in the thermodynamic limit

$$ \frac{-\beta F}{N} = \frac{1}{2}\log 2 + \frac{1}{4q^2}\beta^2 - \frac{1}{2q^2 4!}\beta^4 + \frac{8q^3(3q-1)}{8q^6 6!}\beta^6 - \frac{16q^4(46q^2 - 36q + 7)}{16q^8 8!}\beta^8 + O(\beta^{10}), \tag{3.17} $$

where we have set $J^2 = 2^{q-1}/q$. Using the $1/N$ corrections to the cumulants and the variance, it is straightforward to calculate $1/N$ corrections to the free energy. We have sufficient data to obtain terms up to $1/N^3$, each expanded to β^8, but since they are of limited physical relevance, we have not written them down. To summarize, we have obtained the high temperature expansion of the free energy to order β^8.

Using a completely different method, a recent publication [56] computes the large q ex-

pansion to order $1/q^3$ for the free energy at leading order of $1/N$.[4] In addition to reproducing the large q expansion calculated by [56] to order $1/q^3$ (see appendix A), we also obtain the $1/q^4$ correction to the free energy at order β^8 which is given by

$$-N\frac{7}{8!q^4}\beta^8.\tag{3.18}$$

Our results also show that there are no further large q corrections for terms up to order β^8.

We also remark that, if all one wants is the high temperature expansion of the free energy to a certain order in β and $1/N$, a general expression for the large N expansion of $\tilde{M}_{2p}$ such as eq. (3.15) is not necessary, as only equations such as (3.14) are used to calculate the high temperature expansion, where only a finite number of moments are needed. However, the scope of this paper is wider than just computing the high temperature expansion, and a general expression for all $\tilde{M}_{2p}$ will be needed to solve the enumeration problem for intersection graphs, which will be discussed in section 4.

3.3 Odd q case

We can repeat the same calculation for supersymmetric SYK models. However in this case we need to define the scaled cumulants as

$$\tilde{\kappa}_p := \frac{\kappa_p}{\kappa_1^p} = \frac{\kappa_p}{M_2^p},\tag{3.19}$$

because $M_2 = \mu_1 = \kappa_1$ for the odd q case. This means

$$\tilde{\kappa}_k \sim O(N^{-k+1}).\tag{3.20}$$

Repeating the same calculation that led to eq. (3.12) we get

$$\tilde{M}_{2p} = \frac{\mu_p}{\mu_1^p} = 1 + \binom{p}{2}\tilde{\kappa}_2 + \binom{p}{3}\tilde{\kappa}_3 + 3\binom{p}{4}\tilde{\kappa}_2^2 + \binom{p}{4}\tilde{\kappa}_4 + 10\binom{p}{5}\tilde{\kappa}_2\tilde{\kappa}_3 + 15\binom{p}{6}\tilde{\kappa}_2^3 + O(N^{-4}).\tag{3.21}$$

There is no $(2p-1)!!$ factor this time. To calculate $\tilde{\kappa}_2$, $\tilde{\kappa}_3$ and $\tilde{\kappa}_4$ we will need $\tilde{M}_4$, $\tilde{M}_6$ and $\tilde{M}_8$ as we did for the even q case. They can be again calculated from the analytical results in [1] resulting in

$$\begin{aligned}
\tilde{M}_4 &= 1 + \frac{2q^2}{N} - 2q^2(q-1)^2\frac{1}{N^2} + \frac{2}{3}q^2(q-1)^2\left(2q^2 - 8q + 5\right)\frac{1}{N^3} + O\left(N^{-4}\right),\\
\tilde{M}_6 &= 1 + \frac{6q^2}{N} - 2q^2(q-3)(3q-1)\frac{1}{N^2} + 2q^2(q-1)^2\left(6q^2 - 24q + 5\right)\frac{1}{N^3} + O\left(N^{-4}\right),\\
\tilde{M}_8 &= 1 + \frac{12q^2}{N} + 4q^2(14q-3)\frac{1}{N^2} - 4q^2\left(4q^4 + 32q^3 - 93q^2 + 50q - 5\right)\frac{1}{N^3} + O\left(N^{-4}\right).
\end{aligned}\tag{3.22}$$

---

[4]Their result, however, is valid for all temperatures.

Using the relation between cumulants and moments, eq. (3.7), we obtain

$$
\begin{aligned}
\tilde{\kappa}_2 &= \tilde{\mu}_2 - 1 = \tilde{M}_4 - 1 \\
&= \frac{2q^2}{N} - 2q^2(q-1)^2 \frac{1}{N^2} + \frac{2}{3}q^2(q-1)^2 \left(2q^2 - 8q + 5\right) \frac{1}{N^3} + O\left(N^{-4}\right), \\
\tilde{\kappa}_3 &= \tilde{M}_6 - 3\tilde{M}_4 + 2 \\
&= \frac{8q^3}{N^2} + 8q^3(q-4)(q-1)^2 \frac{1}{N^3} + O\left(N^{-4}\right), \\
\tilde{\kappa}_4 &= \tilde{M}_8 - 4\tilde{M}_6 - 3\tilde{M}_4^2 + 12\tilde{M}_4 - 6 \\
&= -16q^4 \left(2q^2 - 4q - 1\right) \frac{1}{N^3} + O\left(N^{-4}\right).
\end{aligned}
\tag{3.23}
$$

Together with (3.21), we conclude that the p-dependence of the moments is given by

$$
\begin{aligned}
\tilde{M}_{2p} ={}& 1 + \binom{p}{2}\frac{2q^2}{N} - \left[\binom{p}{2}2q^2(q-1)^2 - \binom{p}{3}8q^3 - \binom{p}{4}12q^4\right]\frac{1}{N^2} \\
&+ \left[\binom{p}{2}\frac{2}{3}q^2(q-1)^2\left(2q^2 - 8q + 5\right) + \binom{p}{3}8q^3(q-1)^2(q-4)\right. \\
&\left. - \binom{p}{4}8q^4\left(7q^2 - 14q + 1\right) + \binom{p}{5}160q^5 + \binom{p}{6}120q^6\right]\frac{1}{N^3} + O(N^{-4}).
\end{aligned}
\tag{3.24}
$$

The high temperature expansion of free energy has the form:

$$
-\beta F = -\frac{N}{2}\log 2 - \sigma^2\beta + \frac{1}{2!}\sigma^4\tilde{\kappa}_2\beta^2 - \frac{1}{3!}\sigma^6\tilde{\kappa}_3\beta^3 + \frac{1}{4!}\sigma^8\tilde{\kappa}_4\beta^4 + O(\beta^5).
\tag{3.25}
$$

More explicitly, we have in the thermodynamic limit

$$
\frac{-\beta F}{N} = \frac{1}{2}\log 2 - \frac{1}{2q^2}\beta + \frac{1}{2!(2q^2)}\beta^2 - \frac{1}{3!q^3}\beta^3 + \frac{-(2q^2 - 4q - 1)}{4!q^4}\beta^4 + O(\beta^5),
\tag{3.26}
$$

where again we have set $J^2 = 2^{q-1}/q$. We only displayed the coefficient of the leading term in $1/N$, but the $1/N$ corrections can be calculated in a straightforward way as well.

4 Enumerative identities of intersection graphs

In this section we derive enumerative identities for intersection graph, some of which have not appeared in the literature. We start with a short review of the graphical calculation of the moments, and the details can be found in [1].

4.1 Graphical calculation of moments

The moments of the SYK model are given by the expectation value

$$
\left\langle 2^{-N/2}\mathrm{Tr}(J_\alpha\Gamma_\alpha)^{2p}\right\rangle.
\tag{4.1}
$$
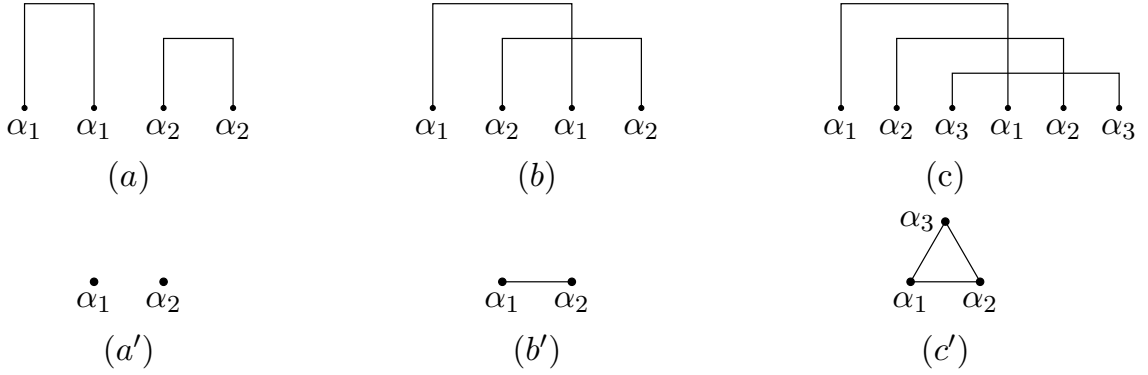
Figure 2. Three examples of chord diagrams $(a),(b),(c)$, and the corresponding intersection graphs $(a'),(b')$ and (c'). In particular (b) and (b') represent the value of η in eq. (4.3).

Because the probability is Gaussian, the average is given by the sum over all possible Wick contractions, which can be represented by *rooted chord diagrams*, and some examples of such chord diagrams are given in the first row of figure 2. For large N and finite p, the indices of the Γ_α are mostly different, so that they can be commuted to pairs of $\Gamma_\alpha \Gamma_\alpha = 1$ (no implicit summation over α). Since in this limit all contractions contribute equally for even q, this results in a Gaussian spectral density. For odd q, when two Γ_α and Γ_β with no common indices anti-commute, the contractions are alternately positive and negative, leaving only one net contraction for all moments which results in the moments of two delta functions in this limit. We always consider the scaled moments $\tilde{M}_{2p} := M_{2p}/M_2^p$, so that the variance cancels in the ratio, and the values of chord diagrams always refer to the values of Wick contractions normalized by M_2^p.

To calculate $1/N$ corrections, we have to take into account that Γ_α and Γ_β commute or anti-commute depending on how many indices they have in common,

$$\Gamma_\alpha \Gamma_\beta + (-1)^{q+1+k} \Gamma_\beta \Gamma_\alpha = 0, \tag{4.2}$$

where k is the number of common indices in α and β. Taking this into account, we obtain the value of two intersecting contraction lines:

$$\eta = (-1)^q \binom{N}{q}^{-1} \sum_{k=1}^{q} (-1)^k \binom{N-q}{q-k}\binom{q}{k}. \tag{4.3}$$

We can further translate chord diagrams into *intersection graphs*, which are obtained by representing each chord by a vertex, and connecting two vertices if and only if the two chords they represent intersect each other in the chord diagram. We give some examples of intersection graphs in the second row of figure 2. We will denote a generic intersection graph

by G. An important approximation to the moments is when all intersecting contraction lines are treated as being independent. In the language of intersection graphs, if an intersection graph G has E edges, its value η_G is approximated by

$$\eta_G \approx \eta^E. \tag{4.4}$$

This approximation gives moments that are correct up to order $1/N$. The corresponding moments are the moments of the weight function of the Q-Hermite polynomials. That is why this approximation is known as the Q-Hermite approximation. The Q-Hermite approximation to moments is thus the sum of η^E over all the $(2p-1)!!$ intersection graphs.

For more details of the graphical calculations including the graph-theoretic identities needed to sum all graphs we refer to [1]. To conclude this review, we remark that the method of [1] relies on a back-and-forth interplay between the moment expansion and intersection graphs: intersection graphs inform us on how to calculate moments, and moment calculations for the exactly solvable $q = 1$ and $q = 2$ SYK model prove enumerative identities about intersection graphs, which in turn feed back to the moment calculation for general q. In the current paper the relation is more one-way: we have obtained results for general moments in previous sections without relying on the enumeration of intersection graphs, and the graphs at best could play a minor role as a book keeping device. However, now we can use the results for the moment expansion to prove identities for intersection graphs, which will be discussed in this section.

4.2   Structure of contributions

In section 3, we have calculated the large N expansions of $\tilde{M}_{2p}$ to order $1/N^3$. We did not use any enumerative identities like the ones in [1]. Nevertheless, we see that various binomial factors arise in eqs. (3.12) and (3.21) simply from the relations between the moments and the cumulants. This suggests we can turn around and use the calculations we just presented to generate enumerative identities for intersection graphs. To see what type of graph-theoretic objects are to be enumerated, we first state the main result of the next two sections:

$$\eta_G - \eta^E = (-1)^{Eq} \frac{-8q^3}{N^2} T \tag{4.5}$$
$$+ (-1)^{Eq}[16ETq^5 + (-72T - 80f_6 - 16f_5 + 16f_4)q^4 + 32Tq^3]\frac{1}{N^3} + O(N^{-4}),$$

where E is the numbers of edges, T is the number of triangles, and f_6, f_5, f_4 are the number of the four-point structures depicted in figure 3, in an intersection graph G. We can check this formula for a few nontrivial graphs that contribute to moments up to $\tilde{M}_8$ denoted by η, T_6, T_{44}, T_{66} and T_8 in tables 1 and 2. Expanding the results for these graphs obtained in
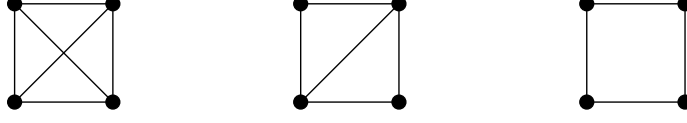
Figure 3. The four-point structures appearing in a general intersection graph G. Their numbers are denoted by f_6, f_5 and f_4 from the left to the right, respectively.

Intersection graph	$\cdot\ \cdot$ $\cdot\ \cdot$	$\cdot$ $\bullet\!-\!\bullet$	$\diagdown$	$\triangle$
Value	1	η	η^2	T_6
Multiplicity	5	6	3	1

Table 1. All the intersection graphs contributing to the sixth moment.

Intersection graph	$\cdot\ \cdot$ $\cdot\ \cdot$	$\cdot\ \cdot$ $\bullet\!-\!\bullet$	$\bullet\!-\!\bullet$ $\bullet\!-\!\bullet$	$\cdot$ $\bullet\!-\!\bullet$	$\diagdown$	$\sqsupset$	$\diagup\!|$	$|\!\diagdown$	$\sqcap$	$\diagup\!|$	$\boxtimes$
Value	1	η	η^2	η^2	η^3	η^3	T_6	ηT_6	T_{44}	T_{66}	T_8
Multiplicity	14	28	4	24	4	8	8	8	2	4	1

**Table 2**. All the intersection graphs contributing to the eighth moment.

[1] to order $1/N^3$ we obtain,

$$(-1)^q \eta = 1 - \frac{2q^2}{N} + \frac{2q^2(q-1)^2}{N^2} - \frac{2q^2(q-1)^2(2q^2 - 8q + 5)}{3N^3} + O\left(\frac{1}{N^4}\right)$$

$$(-1)^{3q} T_6 = (-1)^{3q} \eta^3 - \frac{8q^3}{N^2} + (48q^5 - 72q^4 + 32q^3)\frac{1}{N^3} + O\left(\frac{1}{N^4}\right),$$

$$(-1)^{4q} T_{44} = (-1)^{4q} \eta^4 + \frac{16q^4}{N^3} + O\left(\frac{1}{N^4}\right), \tag{4.6}$$

$$(-1)^{5q} T_{66} = (-1)^{5q} \eta^5 - \frac{16q^3}{N^2} + (160q^5 - 160q^4 + 64q^3)\frac{1}{N^3} + O\left(\frac{1}{N^4}\right),$$

$$(-1)^{6q} T_8 = (-1)^{6q} \eta^6 - \frac{32q^3}{N^2} + (384q^5 - 368q^4 + 128q^3)\frac{1}{N^3} + O\left(\frac{1}{N^4}\right).$$

Using the graphs in tables 1 and 2 one can easily verify that the above results satisfy eq. (4.5). Note that triangles (whose value is T_6) made their first appearance as a complete intersection graph for the sixth moment (table 1), and in the eighth moment, they become substructures of various graphs. The same is true for the four-point structures whose values are T_{44}, T_{66} and T_8: they first appear as complete graphs for the eighth moment (table 2), and will become substructures for higher moments. Some examples are given in figure 4.

How do we proceed to prove eq. (4.5) in its full generality? A rigorous proof of eq. (4.5) to order $1/N^2$ was given in [1], which is essentially a very tedious brute-force calculation. The
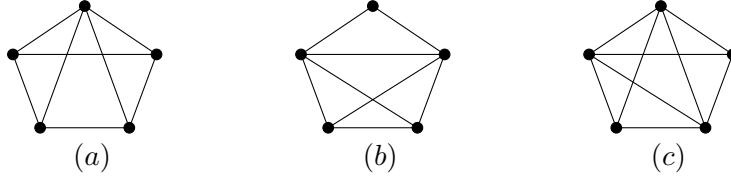
Figure 4. Some examples of four-point structure counting in intersection graphs with five vertices: (a) has $E = 8$, $T = 4$ and $f_6 = 0$, $f_5 = 4$, $f_4 = 1$; (b) has $E = 8$, $T = 5$ and $f_6 = 1$, $f_5 = 2$, $f_4 = 0$; (c) has $E = 9$, $T = 7$ and $f_6 = 2$, $f_5 = 3$, $f_4 = 0$.

$1/N^3$ term can be obtained rigorously by the same method, but the calculation turns out to be extremely tedious and uninstructive. So instead, in the next two sections, we will present a less rigorous but more instructive method to obtain the result (4.5). As a warm-up we will start with the $1/N^2$ term in the next section, and then continue with the $1/N^3$ corrections.

4.3 Graphical calculation of the $1/N^2$ term

We first take note of a rather trivial fact that if an intersection graph G is disconnected with two components G_1 and G_2, then $\eta_G = \eta_{G_1}\eta_{G_2}$. Moreover, this factorization property also holds in a less trivial situation, where an intersection graph can be made disconnected by cutting a vertex, which was proved in [1].[5] To be concrete, let us first look at the $1/N^2$ order corrections. We have argued that the $1/N^2$ coefficient for $\tilde{M}_{2p}$ is completely determined by $\tilde{M}_4$ and $\tilde{M}_6$, which only depend on the values of edges and triangles (η and T_6) according to table 1 and factorization. However, this fact appears rather mysterious if one thinks about the moments in terms of the Wick contractions

$$\tilde{M}_{2p} = \sum_{i=1}^{(2p-1)!!} \eta_{G_i}, \tag{4.7}$$

where G_i denotes the i-th intersection graph and all the intersection graphs have p vertices. An intersection graph can get quite complicated when p is large, for example imagine a complete graph with p vertices, which has $\binom{p}{2}$ edges. On top of that one then needs to sum over all these complicated graphs. How can one tame such a complex beast by only using $\tilde{M}_4$ and $\tilde{M}_6$? The only plausible way to reconcile these two seemingly conflicting view points is that the $1/N^2$ coefficient of each η_{G_i} ought to be a function of only the number of edges and triangles in that intersection graph, not something more global or any other property of the graph. So the fact that we are only looking at a fixed order in $1/N$ allows a huge

---

[5]In a nutshell, the cut-vertex factorization property holds because the subscripts α (vertices in intersection graphs) for the Γ_α live in a "isotropic" space, so that if one first sums over all vertices on one side of the cut-vertex, the result no longer depends on the cut-vertex. A graph that cannot be made disconnected by cutting a single vertex is called "2-connected" in mathematics literature, here we just call it "irreducible".

simplification to happen. We may summarize this plausible result as

$$\eta_G - \eta^E = \frac{1}{N^2} A(E, T) + O(1/N^3), \tag{4.8}$$

where $A(E, T)$ is some function of E and T.[6] Note that since η^E is a function of E, the plausibility argument loses no generality by considering $\eta_G - \eta^E$ instead of η_G on the left-hand side of eq. (4.8), and this is motivated by the Q-Hermite approximation eq. (4.4). Let us pause here and summarize the reasons for the necessity for such simplicity:

- *The cut-vertex factorization property of η_G, which together with table 1 implies $\tilde{M}_4$ and $\tilde{M}_6$ depend only on the values of edges and triangles (η and T_6).*

- *The extensivity of the free energy/cumulants, which forces the $1/N^2$ coefficient of $\tilde{M}_{2p}$ to depend on only $\tilde{M}_4$ and $\tilde{M}_6$, for any value of p.*

Now the task is to fix the explicit form of $A(E, T)$. It is sufficient to consider those special graphs G with only disconnected irreducible structures, so that there is a complete factorization. Since the relevant irreducible structures can only be edges or triangles, let us take G to be an intersection graph with E edges and T triangles where all the triangles are disconnected, and the edges other than the ones that make up triangles are also disconnected, which means there are $E - 3T$ of them. This implies

$$\eta_G = T_6^T \eta^{E-3T}, \tag{4.9}$$

where T_6 is the value of the Wick contraction represented by a triangle. We now have

$$\eta_G - \eta^E = \eta^E \left(\left[\frac{T_6}{\eta^3} \right]^T - 1 \right). \tag{4.10}$$

We define

$$\delta T_6 := \frac{T_6}{\eta^3} - 1. \tag{4.11}$$

By the expansion of T_6 in eq. (4.6), we have $\delta T_6 \sim O(1/N^2)$ so that

$$\eta_G - \eta^E = T\eta^E \delta T_6 + O(1/N^3). \tag{4.12}$$

---

[6]Eq. (4.8) takes for granted that $\eta_G - \eta^E$ is $1/N$-exact. It will be clear that the $1/N$-exactness can be proved by the method we are unfolding, but since $1/N$-exactness of $\eta_G - \eta^E$ was extensively discussed in [1, 8, 33], we choose not to prove it in this discussion.

Using that

$$\eta = (-1)^q + O(1/N), \tag{4.13}$$

$$\delta T_6 = \frac{-8q^3}{N^2} + O(1/N^3), \tag{4.14}$$

we finally obtain

$$\eta_G - \eta^E = (-1)^{Eq} \left(\frac{-8q^3}{N^2} \right) T + O(1/N^3). \tag{4.15}$$

Hence by considering special graphs we have fixed the form of $A(E,T)$ to be $A(E,T) = (-1)^{Eq}(-8q^3)T$, which must also be true for a general graph. This proves eq. (4.5) to order $1/N^2$.

4.4 Graphical calculation of the $1/N^3$ term

We may continue with this strategy to order $1/N^3$. Since $\tilde{M}_{2p}$ to this order is completely determined by $\tilde{M}_4$, $\tilde{M}_6$ and $\tilde{M}_8$, by the same argument as in previous subsection, we conclude for any intersection graph G we have

$$\eta_G - \eta^E = (-1)^{Eq} \left(\frac{-8q^3}{N^2} \right) T + \frac{1}{N^3} B(E,T,f_6,f_5,f_4) + O(1/N^4), \tag{4.16}$$

because E, T, f_6, f_5 and f_4 count all the irreducible structures that appear for moments up to $\tilde{M}_8$. Again consider an intersection graph G with E edges, T triangles, and the four-point structures counted by f_6, f_5 and f_4 (see figure 3), where all the irreducible structures are disconnected, so we have $T - 4f_6 - 2f_5$ disconnected triangles and $E - 3(T - 4f_6 - 2f_5) - 6f_6 - 5f_5 - 4f_4 = E - 3T + 6f_6 + f_5 - 4f_4$ disconnected edges. Thus

$$\eta_G = \eta^{E-3T+6f_6+f_5-4f_4} T_6^{T-4f_6-2f_5} T_8^{f_6} T_{66}^{f_5} T_{44}^{f_4}$$

$$= \eta^E \left[\frac{T_6}{\eta^3} \right]^{T-4f_6-2f_5} \left[\frac{T_8}{\eta^6} \right]^{f_6} \left[\frac{T_{66}}{\eta^5} \right]^{f_5} \left[\frac{T_{44}}{\eta^4} \right]^{f_4}, \tag{4.17}$$

where T_8, T_{66} and T_{44} are the values of the intersection graphs counted by f_6, f_5 and f_4, see table 2. We have that

$$\eta = (-1)^q(1 + \delta\eta) + O(1/N^2),$$

$$\frac{T_6}{\eta^3} = 1 + \delta T_6,$$

$$\frac{T_8}{\eta^6} = 1 + 4\delta T_6 + \delta T_8,$$

$$\frac{T_{66}}{\eta^5} = 1 + 2\delta T_6 + \delta T_{66},$$

$$\frac{T_{44}}{\eta^4} = 1 + \delta T_{44}. \tag{4.18}$$

The above equations should be understood as definitions of the symbols $\delta\eta$, δT_6, δT_8 δT_{66} and δT_{44} from the left-hand sides. From eq. (4.6) we see δT_8, δT_{66} and δT_{44} are of order $1/N^3$ and δT_6 contains both $1/N^2$ and $1/N^3$ terms. To order $1/N^3$ the only mixed contribution is of the form $\delta\eta\delta T_6$, while the other corrections only contribute by their leading orders. We thus obtain

$$
\begin{aligned}
\eta_G - \eta^E &= (-1)^{qE}(1 + E\delta\eta)(T\delta T_6 + f_6\delta T_8 + f_5\delta T_{66} + f_4\delta T_{44}) \\
&= (-1)^{qE}(1 + ET\delta\eta\delta T_6 + f_6\delta T_8 + f_5\delta T_{66} + f_4\delta T_{44}) + O(1/N^4).
\end{aligned}
\tag{4.19}
$$

If we use the explicit expressions for the irreducible structures given in eq. (4.6), we finally find

$$
\begin{aligned}
\eta_G - \eta^E = (-1)^{Eq}\left(\frac{-8q^3}{N^2}\right)T + (-1)^{Eq}\Big[16ETq^5 \\
+(-72T - 80f_6 - 16f_5 + 16f_4)q^4 + 32Tq^3\Big]\frac{1}{N^3} + O(N^{-4}).
\end{aligned}
\tag{4.20}
$$

This fixes the form of $B(E, T, f_6, f_5, f_4)$ and proves eq. (4.5).

To calculate the correction to the Q-Hermite moments we sum over all intersection graphs:

$$
\tilde{M}_{2p} - \tilde{M}_{2p}^{\mathrm{QH}} = \sum_{i=1}^{(2p-1)!!} \left(\eta_{G_i} - \eta^{E_i}\right).
\tag{4.21}
$$

It is now clear that the new objects to be enumerated at order $1/N^3$ are $(-1)^{Eq}ET$ and $(-1)^{Eq}(5f_6 + f_5 - f_4)$. We will discuss their enumerations for the even q and the odd q cases in the following sections.

4.4.1 Even q case

In the same way that $\tilde{M}_{2p}$ to order $1/N^3$ is determined by $\tilde{M}_4$, $\tilde{M}_6$ and $\tilde{M}_8$, the Q-Hermite moment $\tilde{M}_{2p}^{\mathrm{QH}}$ to order $1/N^3$ is determined by $\tilde{M}_4^{\mathrm{QH}}, \tilde{M}_6^{\mathrm{QH}}$ and $\tilde{M}_8^{\mathrm{QH}}$. These Q-Hermite moments can be calculated most easily calculated from the definition

$$
\tilde{M}_{2p}^{\mathrm{QH}} = \sum_{i=1}^{(2p-1)!!} \eta^{E_i},
\tag{4.22}
$$

the large N expansion of η in eq. (4.6), and tables 1 and 2. Alternatively, the sum in (4.22) may be evaluated by the Riordan-Touchard formula [1]. We find the moments,

$$
\begin{aligned}
\tilde{M}_4^{\mathrm{QH}} =&\, 3 - \frac{2q^2}{N} + \frac{2q^2(q-1)^2}{N^2} - \frac{2q^2(q-1)^2(2q^2-8q+5)}{3N^3} + O\left(\frac{1}{N^4}\right), \\
\tilde{M}_6^{\mathrm{QH}} =&\, 15 - \frac{30q^2}{N} + \frac{6q^2\left(9q^2-10q+5\right)}{N^2} \\
&\, - \frac{2q^2\left(38q^4-108q^3+139q^2-90q+25\right)}{N^3} + O(N^{-4}), \\
\tilde{M}_8^{\mathrm{QH}} =&\, 105 - \frac{420q^2}{N} + \frac{28q^2\left(44q^2-30q+15\right)}{N^2} \\
&\, - \frac{4q^2\left(716q^4-1232q^3+1211q^2-630q+175\right)}{N^3} + O(N^{-4}).
\end{aligned}
\tag{4.23}
$$

In the same manner as in eq. (3.14), we get the following Q-Hermite cumulants:

$$
\begin{aligned}
\tilde{\kappa}_4^{\mathrm{QH}} =&\, -3 + \tilde{M}_4^{\mathrm{QH}} \\
=&\, -\frac{2q^2}{N} + \frac{2q^2(q-1)^2}{N^2} - \frac{2q^2(q-1)^2(2q^2-8q+5)}{3N^3} + O\left(N^{-4}\right), \\
\tilde{\kappa}_6^{\mathrm{QH}} =&\, 30 - 15\tilde{M}_4^{\mathrm{QH}} + \tilde{M}_6^{\mathrm{QH}} \\
=&\, \frac{24q^4}{N^2} - \frac{8q^4\left(7q^2-12q+6\right)}{N^3} + O\left(N^{-4}\right), \\
\tilde{\kappa}_8^{\mathrm{QH}} =&\, -630 + 420\tilde{M}_4^{\mathrm{QH}} - 35\left(\tilde{M}_4^{\mathrm{QH}}\right)^2 - 28\tilde{M}_6^{\mathrm{QH}} + \tilde{M}_8^{\mathrm{QH}} \\
=&\, -\frac{736q^6}{N^3} + O\left(N^{-4}\right).
\end{aligned}
\tag{4.24}
$$

The p-dependence of the moments is then obtained from eq. (3.12):

$$
\begin{aligned}
\frac{\tilde{M}_{2p}^{\mathrm{QH}}}{(2p-1)!!} =&\, 1 - \frac{2}{3}\binom{p}{2}\frac{q^2}{N} + \left[\binom{p}{2}\left(\frac{2}{3}q^2(q-1)^2\right) + \binom{p}{3}\frac{8}{5}q^4 + \binom{p}{4}\frac{4}{3}q^4\right]\frac{1}{N^2} \\
&\, - \left[\binom{p}{2}\left(\frac{2}{9}q^2(q-1)^2(2q^2-8q+5)\right) + \binom{p}{3}\left(\frac{8}{15}q^4\left(7q^2-12q+6\right)\right)\right. \\
&\, + \binom{p}{4}\left(\frac{8}{105}q^4\left(127q^2-70q+35\right)\right) + \binom{p}{5}\frac{32}{3}q^6 \\
&\, \left. + \binom{p}{6}\frac{40}{9}q^6\right]\frac{1}{N^3} + O\left(\frac{1}{N^4}\right).
\end{aligned}
\tag{4.25}
$$

In fact, using the definition of the Q-Hermite moments (4.22) and

$$
\begin{aligned}
(-1)^{qE}\eta^E = & 1 - \frac{2Eq^2}{N} + \frac{2E^2q^4 - 4Eq^3 + 2Eq^2}{N^2} \\
& - \frac{4E^3q^6 - 24E^2q^5 + 12E^2q^4 + 34Eq^4 - 36Eq^3 + 10Eq^2}{3N^3} + O\left(\frac{1}{N^4}\right),
\end{aligned}
\tag{4.26}
$$

we can match the coefficients of powers of q and $1/N$ in eqs. (4.25) and (4.26), and this already gives several enumerative identities for E:

$$
\sum_{i=1}^{(2p-1)!!} E_i = \frac{(2p-1)!!}{3}\binom{p}{2},
\tag{4.27}
$$

$$
\sum_{i=1}^{(2p-1)!!} E_i^2 = \frac{(2p-1)!!}{90}\binom{p}{2}(5p^2 - p + 12),
\tag{4.28}
$$

$$
\sum_{i=1}^{(2p-1)!!} E_i^3 = \frac{(2p-1)!!}{3780}\binom{p}{2}\left(35p^4 + 14p^3 + 235p^2 - 188p + 24\right).
\tag{4.29}
$$

In fact the above three identities are already known in mathematics literature [61]. However the proof therein is based on an analytical-combinatorial method quite distinct from ours.

Now we proceed to calculate $\tilde{M}_{2p} - \tilde{M}_{2p}^{\mathrm{QH}}$ for even q from eqs. (3.15) and (4.25):

$$
\begin{aligned}
\frac{\tilde{M}_{2p} - \tilde{M}_{2p}^{\mathrm{QH}}}{(2p-1)!!} = & -\frac{8}{15}\binom{p}{3}q^3\frac{1}{N^2} + \left[\frac{8}{315}\binom{p}{3}(7p^2 + 5p + 48)q^5\right. \\
& + \left.\left(-\frac{24}{5}\binom{p}{3} - \frac{16}{15}\binom{p}{4}\right)q^4 + \frac{32}{15}\binom{p}{3}q^3\right]\frac{1}{N^3} + O(N^{-4}).
\end{aligned}
\tag{4.30}
$$

Matching the coefficients of the expansion in $1/N$ and q with graph-theoretic result eq. (4.5), we obtain the following enumerative identities:

$$
\sum_{i=1}^{(2p-1)!!} T_i = \frac{(2p-1)!!}{15}\binom{p}{3},
\tag{4.31}
$$

$$
\sum_i E_i T_i = \frac{(2p-1)!!}{630}\binom{p}{3}(7p^2 + 5p + 48),
\tag{4.32}
$$

$$
\sum_{i=1}^{(2p-1)!!} (5f_{6i} + f_{5i} - f_{4i}) = \frac{(2p-1)!!}{15}\binom{p}{4},
\tag{4.33}
$$

where f_{6i}, f_{5i}, f_{4i} are the numbers of the three four-point structures in the intersection graph G_i. These are the truly new identities that could not have been obtained by the analytical-combinatorial method of [61], since our identities involve higher structures such as triangles and four-point structures.

4.4.2 Odd q case

We repeat the same calculation for odd q case. The Q-Hermite moments up to eighth order are

$$
\begin{aligned}
\tilde{M}_4^{\mathrm{QH}} &= 1 + \frac{2q^2}{N} - 2q^2(q-1)^2 \frac{1}{N^2} + \frac{2}{3}q^2(q-1)^2 \left(2q^2 - 8q + 5\right) \frac{1}{N^3} + O\left(N^{-4}\right), \\
\tilde{M}_6^{\mathrm{QH}} &= 1 + \frac{6q^2}{N} - \frac{6(q-1)^2 q^2}{N^2} + \frac{2q^2\left(6q^4 - 12q^3 + 23q^2 - 18q + 5\right)}{N^3} + O(N^{-4}), \qquad (4.34) \\
\tilde{M}_8^{\mathrm{QH}} &= 1 + \frac{12q^2}{N} + \frac{12q^2(2q-1)}{N^2} - \frac{4q^2(q-1)^2(2q-1)(2q+5)}{N^3} + O(N^{-4}).
\end{aligned}
$$

Together with eq. (3.23) this gives the Q-Hermite cumulants

$$
\begin{aligned}
\tilde{\kappa}_2^{\mathrm{QH}} &= \tilde{M}_4^{\mathrm{QH}} - 1 = \frac{2q^2}{N} - 2q^2(q-1)^2 \frac{1}{N^2} + \frac{2}{3}q^2(q-1)^2 \left(2q^2 - 8q + 5\right) \frac{1}{N^3} + O\left(N^{-4}\right), \\
\tilde{\kappa}_3^{\mathrm{QH}} &= \tilde{M}_6^{\mathrm{QH}} - 3\tilde{M}_4^{\mathrm{QH}} + 2 = \frac{8q^6}{N^3} + O\left(N^{-4}\right), \qquad (4.35) \\
\tilde{\kappa}_4^{\mathrm{QH}} &= \tilde{M}_8^{\mathrm{QH}} - 4\tilde{M}_6^{\mathrm{QH}} - 3\left(\tilde{M}_4^{\mathrm{QH}}\right)^2 + 12\tilde{M}_4^{\mathrm{QH}} - 6 = -\frac{32q^6}{N^3} + O\left(N^{-4}\right).
\end{aligned}
$$

Substituting this into eq. (3.21) and we find

$$
\begin{aligned}
\tilde{M}_{2p}^{\mathrm{QH}} =\ & 1 + 2q^2 \binom{p}{2} \frac{1}{N} + \left[\binom{p}{4} 12q^4 - \binom{p}{2} 2(q-1)^2 q^2\right] \frac{1}{N^2} \\
& + \left[\binom{p}{2}\frac{2}{3}q^2(q-1)^2\left(2q^2 - 8q + 5\right) + \binom{p}{3}8q^6 - \binom{p}{4}8q^4\left(7q^2 - 6q + 3\right) + \binom{p}{6}120q^6\right] \frac{1}{N^3} \\
& + O\left(\frac{1}{N^4}\right).
\end{aligned} \qquad (4.36)
$$

Again, given $\tilde{M}_{2p}^{\mathrm{QH}} = \sum_i \eta^{E_i}$ and the form of η in eq. (4.26), we have the following enumerative identities by matching the coefficients of the expansion in $1/N$ and q :

$$
\sum_{i=1}^{(2p-1)!!} (-1)^{E_i} E_i = -\binom{p}{2}, \qquad (4.37)
$$

$$
\sum_{i=1}^{(2p-1)!!} (-1)^{E_i} E_i^2 = \frac{1}{2}\binom{p}{2}(p-1)(p-4), \qquad (4.38)
$$

$$
\sum_{i=1}^{(2p-1)!!} (-1)^{E_i} E_i^3 = -\frac{1}{4}\binom{p}{2}\left(p^4 - 14p^3 + 57p^2 - 76p + 24\right). \qquad (4.39)
$$

These edge identities could also have been obtained by the analytical-combinatorial approach used in [61]. Now we calculate the difference between $\tilde{M}_{2p}$ and $\tilde{M}_{2p}^{\mathrm{QH}}$:

$$\tilde{M}_{2p} - \tilde{M}_{2p}^{\mathrm{QH}} = 8q^3 \binom{p}{3} \frac{1}{N^2} + \left[8\binom{p}{3}p(p-5)q^5 + \left(72\binom{p}{3} + 16\binom{p}{4} \right)q^4 - 32\binom{p}{3}q^3 \right] \frac{1}{N^3}$$
$$+ O\left(\frac{1}{N^4} \right).$$
(4.40)

By matching with eq. (4.5), we obtain the following graded identities:

$$\sum_{i=1}^{(2p-1)!!} (-1)^{E_i} T_i = -\binom{p}{3},$$
(4.41)

$$\sum_{i=1}^{(2p-1)!!} (-1)^{E_i} E_i T_i = \frac{1}{2}\binom{p}{3}p(p-5),$$
(4.42)

$$\sum_{i=1}^{(2p-1)!!} (-1)^{E_i} (5f_{6i} + f_{5i} - f_{4i}) = -\binom{p}{4}.$$
(4.43)

These identities could not have been obtained by using the method in [61].

Finally we discuss in what sense the method presented in this paper is more powerful than the method developed in [1] to compute moments of the SYK model. The idea there is to solve for $\tilde{M}_{2p} - \tilde{M}_{2p}^{\mathrm{QH}}$ for $q = 1$ and $q = 2$ models, where moments can be evaluated exactly, and then do a matching with $\sum(\eta_{G_i} - \eta^{E_i})$ expressed in terms of graph-theoretic objects. One can get a flavor of this method from appendix B. The old method works very well to order $1/N^2$, however it becomes problematic starting from order $1/N^3$. If we look at eq. (4.5), we see that at order $1/N^3$, on top of triangles, there are two new types of structures that need to be enumerated: $(-1)^{Eq}ET$ for the q^5 term and $(-1)^{Eq}(5f_6 + f_5 - f_4)$ for the q^4 term. If we compute the moments for $q = 1$ and $q = 2$ models, at best we can obtain the enumeration of a linear combination of the two above-mentioned new structures, that is, $(-1)^E(ET - 5f_6 - f_5 + f_4)$ for $q = 1$ and $2ET - 5f_6 - f_5 + f_4$ for $q = 2$. However to recover the full q dependence, we need separate enumerations of $(-1)^{Eq}ET$ and $(-1)^{Eq}(5f_6 + f_5 - f_4)$. The method presented in this paper faces no such difficulty since the full q dependence is retained at every stage of the calculation, and thus is capable of enumerating the two new structures separately. Nevertheless, the old method provides an independent consistency check of the results obtained in the present paper, which will be demonstrated in appendix B.

5 Conclusions and outlook

We have established the relation between the $1/N$ expansion of SYK moments and the $1/N$ expansion of the high temperature expansion coefficients of the free energy. It turns out that

	Non-SUSY	SUSY
$1/N$	$\sum E_i = \frac{(2p-1)!!}{3}\binom{p}{2}$	$\sum (-1)^{E_i} E_i = -\binom{p}{2}$
$1/N^2$	$\sum T_i = \frac{(2p-1)!!}{15}\binom{p}{3}$	$\sum (-1)^{E_i} T_i = -\binom{p}{3}$
$1/N^3$	$\sum (5f_{6i} + f_{5i} - f_{4i}) = \frac{(2p-1)!!}{15}\binom{p}{4}$	$\sum (-1)^{E_i}(5f_{6i} + f_{5i} - f_{4i}) = -\binom{p}{4}$

Table 3. Some of the enumerative identities for intersection graphs generated at each order of $1/N$. All summation symbols run from $i = 1$ to $i = (2p-1)!!$.

to compute the high temperature expansion of the free energy to a certain order in β and $1/N$, we only need to compute a finite number of moments to an appropriate order in $1/N$. In particular, we have found in the thermodynamic limit the $1/q^3$ correction to the coefficient of β^6 (there are no other large q corrections) and both the $1/q^3$ and $1/q^4$ corrections to the coefficients of β^8 (these are the only large q corrections). The leading order $1/q^2$ results as well as the $1/q^3$ results are in agreement with a large N calculation of the path integral formulation of the SYK model, while the $1/q^4$ correction was not calculated before.

This relation also allows for calculations of moments to higher order in $1/N$, and we have pushed the calculation to order $1/N^3$. More surprisingly, we found that to a given order in $1/N$, all moments are determined by a finite number of moments. This also explains the p-dependence of the moments $\tilde{M}_{2p}$ obtained in a previous work [1]. One important consequence of this is that the SYK model generates elegant enumeration identities, at each consecutive order in $1/N$, as discussed in section 4. We tabulate some of them in table 3. To the best of our knowledge, it seems that only the identities at order $1/N$ are present in mathematical literature [61], and our study of the SYK model suggests they are only the first layer of a hierarchy of identities.

It is clear that the procedure we presented can be extended to even higher orders in $1/N$, and the most computationally burdensome part is to calculate and expand η_G for the irreducible structures, for which we have a general and practical formula [1]. For the large N expansions of generic systems, there is often no powerful simplifying principle that allows the calculation to be pushed to higher and higher order easily. For SYK model, although the large N coefficients of the moments are still somewhat complicated, the enumeration identities generated at each order of $1/N$ (table 3) are incredibly simple. Does this simplicity of enumeration persist to higher and higher orders? If it does, does it imply there is something we can say about the moments to all orders in $1/N$ instead of calculating them order by order? We hope to clarify these questions in future studies. Another issue is that we are in a somewhat awkward situation that on the one hand we have the Q-Hermite expression for the spectral density [1], which is a resummed approximation that is very accurate and gives the leading order free energy for all temperatures, but Q-Hermite moments are only

$1/N$-exact; on the other hand, we have expressions for the moments that are exact to order $1/N^3$, but they only give the high temperature information of the free energy. It would be desirable to find an improved resummed expression for the spectral density that goes beyond the Q-Hermite approximation, which gives a free energy that is both $1/N^3$-exact and accurate at all temperatures.

6   Acknowledgments

A   High temperature expansion of free energy from [56]

In section 3.2.3 we claimed that our result for the free energy for even q reproduces an independent calculation [56] to order $1/q^3$, and in this appendix we justify this claim.

In [56] the large q expansion for the extensive (leading in $1/N$) part of the free energy is given:

$$\frac{-\beta F}{N} = \frac{1}{2}\log 2 + \frac{1}{q^2}\pi u\left(\tan\frac{\pi u}{2} - \frac{\pi u}{4}\right) + \frac{1}{q^3}\pi u\left[\pi u - 2\tan\frac{\pi u}{2}\left(1 - \frac{\pi^2 u^2}{12}\right)\right], \quad (A.1)$$

where

$$\frac{\pi u}{\cos\frac{\pi u}{2}} = \beta J^2 2^{1-q}q. \tag{A.2}$$

If we set $J^2 = 2^{q-1}/q$ as we did in section 3.2.3, we have simply

$$\frac{\pi u}{\cos\frac{\pi u}{2}} = \beta. \tag{A.3}$$

Then by iterating eq. (A.3) we obtain

$$u = \frac{1}{\pi}\left(\beta - \frac{1}{3!}\frac{3}{4}\beta^3 + \frac{1}{5!}\frac{65}{16}\beta^5 - \frac{1}{7!}\frac{3787}{64}\beta^7\right) + O(\beta^9). \tag{A.4}$$

Using this high temperature expansion of u, we obtain for the free energy, eq. (A.1),

$$\frac{-\beta F}{N} = \frac{\log 2}{2} + \frac{1}{2!}\frac{\beta^2}{2q^2} - \frac{1}{4!}\frac{\beta^4}{2q^2} + \frac{1}{6!}\frac{\beta^6(3q-1)}{q^3} + \frac{1}{8!}\frac{\beta^8(36-46q)}{q^3} + O\left(\beta^{10}\right). \tag{A.5}$$

This is consistent with our result eq. (3.26) to order $1/q^3$ at leading order in $1/N$.

The advantage of the result of [56] is that it is valid at all temperatures, however our result, albeit only valid at high temperatures, gives the expansion to even higher orders in $1/q$ and $1/N$.

As a side remark, the coefficients of the high temperature expansion of u are known analytically. If we write

$$u = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! 2^{2n}} a_n \beta^{2n+1}, \tag{A.6}$$

then a_n are given by

$$a_n = \frac{1}{2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} (2k - 2n - 1)^{2n}. \tag{A.7}$$

This is a convergent series with a convergence radius of $\beta_c \approx 1.33$. The numbers a_n are also the number of labeled rooted trees on $2n+1$ nodes with each node having an even number of children [63]. Since there is no singularity on the positive real axis, we do not expect a phase transition as β varies.

B   Consistency check by computing the $q=1$ and $q=2$ SYK models

In the main text we showed that the $1/N^3$ coefficient of $\eta_G - \eta^E$ is

$$\eta_G - \eta^E \cong \left[16ETq^5 + (-72T - 80f_6 - 16f_5 + 16f_4)q^4 + 32Tq^3 \right] \frac{1}{N^3}, \tag{B.1}$$

where the notation "$\cong$" means everything but the $1/N^3$ term is omitted. We will calculate this quantity summed over all intersection graphs for $q=1$ and $q=2$. By Wick's theorem, we need to compute $\tilde{M}_{2p} - \tilde{M}_{2p}^{\mathrm{QH}}$.

B.1 $q=1$ case

As discussed in detail in [1], for $q=1$, $H^{2p} = \left(\sum_{\alpha=1}^N J_\alpha^2 \right)^p \mathbb{1}$, and because J_α is Gaussian distributed, $\left(\sum_{\alpha=1}^N J_\alpha^2 \right)^p$ follows a χ^2 distribution. Hence for the $q=1$ SYK model we have

$$\tilde{M}_{2p} = \frac{\Gamma \left(\frac{N}{2} + p \right)}{\left(\frac{N}{2} \right)^p \Gamma \left(\frac{N}{2} \right)} \cong \frac{(p-1)^2 p^2 \left(p^2 - 5p + 6 \right)}{6N^3}, \tag{B.2}$$

where again we have only kept the $1/N^3$ term.

To evaluate $\tilde{M}^{\mathrm{QH}}$, we first expand η^E to $1/N^3$. For $q=1$ the first equation of (4.6) simplifies to

$$(-1)^q \eta = 1 - \frac{2}{N} \tag{B.3}$$

and the binomial expansion gives the $1/N^3$ correction

$$\eta^E \cong -(-1)^E \frac{4E(E-1)(E-2)}{3N^3}. \tag{B.4}$$

The enumeration of $E(E-1)(E-2)$ for odd q was worked out by our method in eqs. (4.37), (4.38) and (4.39), but as mentioned it can also be obtained by a completely independent method. Using eq. (4.39), we obtain

$$\tilde{M}_{2p}^{\mathrm{QH}} \cong \binom{p}{3}(p^3 - 12p^2 + 39p - 28)\frac{1}{N^3} \tag{B.5}$$

Hence we find

$$\tilde{M}_{2p} - \tilde{M}_{2p}^{\mathrm{QH}} \cong \frac{2}{3}p(p-1)^2\left(2p^2 - 11p + 14\right)\frac{1}{N^3}. \tag{B.6}$$

This result is consistent with eqs. (4.41), (4.42), (4.43) and (B.1).

B.2 $\quad q = 2$ case

As explained in detail in [1], for $q = 2$ we have

$$\tilde{M}_{2p} = \left\langle \left(\sum_{k=1}^{N/2} x_k\right)^{2p} \right\rangle \Bigg/ \left(\frac{N}{2}\langle x_1^2\rangle\right)^p \tag{B.7}$$

where the brackets $\langle \cdots \rangle$ on the right-hand side denote the ensemble average with the probability distribution [43, 64],

$$P(x_1, \ldots, x_{N/2})\prod_{l=1}^{N/2} dx_l = ce^{-\sum_k x_k^2}\prod_{i<j}(x_i^2 - x_j^2)^2 \prod_{l=1}^{N/2} dx_l, \tag{B.8}$$

and the constant c normalizes the total probability to one. We can do a multinomial expansion for the right-hand side of eq. (B.7):

$$\left\langle \left(\sum_{k=1}^{N/2} x_k\right)^{2p} \right\rangle = \sum_{m_1+\cdots+m_{N/2}=p} \frac{(2p)!}{(2m_1)!(2m_2)!\cdots(2m_{N/2})!}\left\langle x_1^{2m_1} x_2^{2m_2}\cdots x_{N/2}^{2m_{N/2}} \right\rangle. \tag{B.9}$$

Following the argument laid out in [1], we conclude only the following terms contribute to $\tilde{M}_{2p}$ to $1/N^3$ order:

$$
\begin{aligned}
\tilde{M}_{2p} =& \left(\frac{N}{2}\right)^{-p}\binom{N/2}{p}\frac{(2p)!}{2^p}\frac{W_1}{W_0^p} + \left(\frac{N}{2}\right)^{-p}\binom{N/2}{p-1}\binom{p-1}{1}\frac{(2p)!}{2^{p-2}4!}\frac{W_2}{W_0^p} \\
&+ \left(\frac{N}{2}\right)^{-p}\binom{N/2}{p-2}\left[\binom{p-2}{2}\frac{(2p)!}{2^{p-4}4!4!}\frac{W_3}{W_0^p} + \binom{p-2}{1}\frac{(2p)!}{2^{p-3}6!}\frac{W_4}{W_0^p}\right] \\
&+ \left(\frac{N}{2}\right)^{-p}\binom{N/2}{p-3}\left[\binom{p-3}{3}\frac{(2p)!}{2^{p-6}(4!)^3}\frac{W_5}{W_0^p} + 2\binom{p-3}{2}\frac{(2p)!}{2^{p-5}6!4!}\frac{W_6}{W_0^p}+\right. \\
&\qquad\qquad\qquad\qquad \left.\binom{p-3}{1}\frac{(2p)!}{2^{p-4}8!}\frac{W_7}{W_0^p}\right] + O(1/N^4),
\end{aligned}
\tag{B.10}
$$

where

$$
\begin{aligned}
W_0 &:= \langle x_1^2\rangle, \\
W_1 &:= \langle x_1^2 x_2^2 \cdots x_p^2\rangle, \\
W_2 &:= \langle x_1^4 x_2^2 \cdots x_{p-1}^2\rangle, \\
W_3 &:= \langle x_1^4 x_2^4 x_3^2 \cdots x_{p-2}^2\rangle, \\
W_4 &:= \langle x_1^6 x_2^2 \cdots x_{p-2}^2\rangle, \\
W_5 &:= \langle x_1^4 x_2^4 x_3^4 x_4^2 \cdots x_{p-3}^2\rangle, \\
W_6 &:= \langle x_1^6 x_2^4 x_3^2 \cdots x_{p-3}^2\rangle, \\
W_7 &:= \langle x_1^8 x_2^2 \cdots x_{p-3}^2\rangle.
\end{aligned}
\tag{B.11}
$$

Note that all W_i are of the form of a Selberg-like integral. Before evaluating W_i, we first expand the prefactors to $1/N^3$:

$$
\begin{aligned}
\tilde{M}_{2p} =& (2p-1)!! \left\{\left[1 - 2\binom{p}{2}\frac{1}{N} + (3p-1)\binom{p}{3}\frac{1}{N^2} - 8\binom{p}{4}\binom{p}{2}\frac{1}{N^3}\right]\frac{W_1}{W_0^p}\right. \\
&+ \left[\frac{2}{3}\binom{p}{2}\frac{1}{N} - 2(p-1)\binom{p}{3}\frac{1}{N^2} + \frac{4}{3}(p-1)(3p-4)\binom{p}{4}\frac{1}{N^3}\right]\frac{W_2}{W_0^p} \\
&+ \left[\frac{4}{3}\binom{p}{4}\frac{1}{N^2} - \frac{8}{3}\binom{p}{4}\binom{p-2}{2}\frac{1}{N^3}\right]\frac{W_3}{W_0^p} \\
&+ \left[\frac{4}{15}\binom{p}{3}\frac{1}{N^2} - \frac{16}{15}(p-2)\binom{p}{4}\frac{1}{N^3}\right]\frac{W_4}{W_0^p} + \frac{40}{9}\binom{p}{6}\frac{1}{N^3}\frac{W_5}{W_0^p} \\
&\left.+ \frac{16}{9}\binom{p}{5}\frac{1}{N^3}\frac{W_6}{W_0^p} + \frac{8}{105}\binom{p}{4}\frac{1}{N^3}\frac{W_7}{W_0^p}\right\} + O\left(1/N^4\right).
\end{aligned}
\tag{B.12}
$$

We can work out the values of the W_i's by employing a set of recursion relations for Selberg integrals developed in [65] resulting in

$$
W_0 = \frac{N-1}{2},
$$

$$
W_1 = \prod_{k=0}^{p-1}\left(\frac{N}{2}-p+k+\frac{1}{2}\right),
$$

$$
W_2 = \left(N-p+\frac{3}{2}\right)\prod_{k=0}^{p-2}\left(\frac{N}{2}-p+k+\frac{3}{2}\right),
$$

$$
W_3 = \left(N-p+\frac{3}{2}\right)\left(N-p+\frac{5}{2}\right)\prod_{k=0}^{p-3}\left(\frac{N}{2}-p+k+\frac{5}{2}\right),
$$

$$
W_4 = \left(N+\frac{1}{2}\right)\left(N-p+\frac{5}{2}\right)\prod_{k=0}^{p-3}\left(\frac{N}{2}-p+k+\frac{5}{2}\right)
$$
$$
+ \left(\frac{N}{2}-p+2\right)\prod_{k=0}^{p-2}\left(\frac{N}{2}-p+k+\frac{3}{2}\right),
$$

$$
W_5 = \prod_{k=0}^{2}\left(N-p+\frac{3}{2}+k\right)\prod_{l=0}^{p-4}\left(\frac{N}{2}-p+l+\frac{7}{2}\right), \qquad\text{(B.13)}
$$

$$
W_6 = \left(N-\frac{1}{2}\right)\prod_{k=0}^{1}\left(N-p+\frac{5}{2}+k\right)\prod_{l=0}^{p-4}\left(\frac{N}{2}-p+l+\frac{7}{2}\right)
$$
$$
+ \left(\frac{N}{2}-p+3\right)\left(N-p+\frac{5}{2}\right)\prod_{k=0}^{p-3}\left(\frac{N}{2}-p+k+\frac{5}{2}\right),
$$

$$
W_7 = \left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\left(N-p+\frac{7}{2}\right)\prod_{l=0}^{p-4}\left(\frac{N}{2}-p+l+\frac{7}{2}\right)
$$
$$
+ \left(N+\frac{3}{2}\right)\left(\frac{N}{2}-p+3\right)\prod_{k=0}^{p-3}\left(\frac{N}{2}-p+k+\frac{5}{2}\right)
$$
$$
+ (p-4)\prod_{k=0}^{1}\left(N-p+\frac{5}{2}+k\right)\prod_{l=0}^{p-4}\left(\frac{N}{2}-p+l+\frac{7}{2}\right)
$$
$$
+ (N-2p+6)\left(N-p+\frac{5}{2}\right)\prod_{l=0}^{p-3}\left(\frac{N}{2}-p+l+\frac{5}{2}\right).
$$

To relevant order, we have

$$\frac{W_1}{W_0^p} = 1 - 2\binom{p}{2}\frac{1}{N} + \frac{1}{3}\binom{p}{2}(3p^2 - 7p - 4)\frac{1}{N^2} + \frac{1}{3}\binom{p}{2}(p^2 - 4p - 2)(p^2 - 2p - 1)\frac{1}{N^3},$$

$$\frac{W_2}{W_0^p} = 2 - (2p^2 - 4p - 1)\frac{1}{N} + \frac{1}{3}(3p^4 - 16p^3 + 18p^2 + 7p - 3)\frac{1}{N^2},$$

$$\frac{W_3}{W_0^p} = 4 - 4(p^2 - 3p)\frac{1}{N},$$

$$\frac{W_4}{W_0^p} = 5 - (5p^2 - 17p + 1)\frac{1}{N}, \tag{B.14}$$

$$\frac{W_5}{W_0^p} = 8,$$

$$\frac{W_6}{W_0^p} = 10,$$

$$\frac{W_7}{W_0^p} = 14.$$

For $q = 2$ we finally arrive at

$$\tilde{M}_{2p} \cong (2p-1)!! \left(-\frac{32p^6}{81} + \frac{32p^5}{27} - \frac{16p^4}{405} - \frac{128p^3}{135} - \frac{436p^2}{405} + \frac{172p}{135} \right) \frac{1}{N^3}, \tag{B.15}$$

where we have omitted all but the $1/N^3$ term. The results up to order $1/N^2$ can be found in [1]. To evaluate $\tilde{M}_{2p}^{\mathrm{QH}}$, the $1/N$ expansion of η in the first equation of (4.6) simplifies for $q = 2$ to

$$\eta = 1 - \frac{8}{N} + \frac{8}{N^2} + \frac{8}{N^3} + O(1/N^4), \tag{B.16}$$

and hence

$$\eta^E \cong -\frac{256E^3 - 576E^2 + 296E}{3N^3} \tag{B.17}$$

for $q = 2$. Using eqs. (4.27), (4.28) and (4.29), we obtain

$$\tilde{M}_{2p}^{\mathrm{QH}} \cong (2p-1)!! \left(-\frac{32p^6}{81} + \frac{32p^5}{135} + \frac{8048p^4}{2835} - \frac{512p^3}{315} - \frac{14092p^2}{2835} + \frac{740p}{189} \right) \frac{1}{N^3}. \tag{B.18}$$

Finally,

$$\tilde{M}_{2p} - \tilde{M}_{2p}^{\mathrm{QH}} \cong (2p-1)!! \left(\frac{128p^5}{135} - \frac{544p^4}{189} + \frac{128p^3}{189} + \frac{736p^2}{189} - \frac{832p}{315} \right) \frac{1}{N^3} \tag{B.19}$$

which agrees with eqs. (4.31), (4.32), (4.33) and (B.1).

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