

An Identity for the Partition Function Involving Parts of k Different Magnitudes[☆]

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Abstract

Using previous work by Merca, we show the partition function involving parts of k different magnitudes, shifted by the triangular numbers $\binom{k+1}{2}$, equals the self convolution of the unrestricted partition function. We also provide a combinatorial proof of this result.

Keywords: Integer partition, Partition identity

1. Introduction

Recall that a *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n . The partition function $p(n)$ counts the number of such partitions of n . For example, since the partitions of 4 are

$$4, \quad 3 + 1, \quad 2 + 2, \quad 2 + 1 + 1, \quad \text{and} \quad 1 + 1 + 1 + 1,$$

$p(4) = 5$. We also take $p(0) = 1$.

Euler [1] began the mathematical theory of partitions in 1748 and MacMahon [2] in 1921 was the first to study the number of partitions of n that have exactly k different values for the parts. Let $p(k, n)$ denote this function. For example, $p(3, 8) = 5$ since the five partitions in question are

$$5 + 2 + 1, \quad 4 + 3 + 1, \quad 4 + 2 + 1 + 1, \quad 3 + 2 + 2 + 1, \quad 3 + 2 + 1 + 1 + 1.$$

Let $q(k, n)$ denote the number of partitions of n into exactly k distinct parts. So $q(3, 8) = 2$ since we have the two partitions

$$5 + 2 + 1, \quad 4 + 3 + 1.$$

Clearly $q(k, n) \leq p(k, n)$ and $p(k, n) = 0$ when $n < \binom{k+1}{2}$. We also take $q(0, 0) = p(0, 0) = 1$ and $p(0, n) = p(k, 0) = 0$ for $k, n > 0$.

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Due to Merca [4, Corollary 1.2], we have the following: Let $k, n \geq 0$ be integers. Then

$$p(k, n) = \sum_{m=\binom{k+1}{2}}^n a(k, m) p(n - m) \quad (1)$$

where

$$a(k, m) = \sum_{j=k}^{\infty} (-1)^{j-k} \binom{j}{k} q(j, m). \quad (2)$$

In this paper, we use this result to prove the following:

Theorem 1. *Let $k, n \geq 0$ be integers with $k \geq n$. Then*

$$p\left(k, n + \binom{k+1}{2}\right) = \sum_{m=0}^n p(m) p(n - m).$$

Remark 2. *The sequence of integers, $(A_n)_{n \geq 0}$, given by the self convolution*

$$A_n = \sum_{m=0}^n p(m) p(n - m),$$

is listed in [6, sequence number A000712] and may be thought of as the number of partitions of n into parts of two kinds. For example, $A_3 = 10$ since we have

$$\begin{aligned} &3, \quad \bar{3}, \quad 2 + 1, \quad \bar{2} + \bar{1}, \quad \bar{2} + 1, \quad 2 + \bar{1}, \\ &1 + 1 + 1, \quad \bar{1} + \bar{1} + \bar{1}, \quad \bar{1} + 1 + 1, \quad 1 + \bar{1} + \bar{1}. \end{aligned}$$

Let $q(n)$ denote the number of partitions of n into distinct parts. Then by Merca [5, Theorem 5.2],

$$\sum_{k=0}^n p\left(n - \binom{k+1}{2}\right) = \sum_{m=0}^n q(m) q(n - m),$$

a result much like Theorem 1. Also, from Merca [3, equation 12], we have

$$q(k, n) = p_k\left(n - \binom{k+1}{2}\right), \quad (3)$$

where $p_k(n)$ denotes the number of partitions of n with no part greater than k . Clearly

$$q\left(k, n + \binom{k+1}{2}\right) = p_k(n) = p(n)$$

for $k \geq n$. Again, this identity is similar to Theorem 1.

2. Proof of Theorem 1

Let $k, n \geq 0$ be integers. Applying a change of variables to (1),

$$p\left(k, n + \binom{k+1}{2}\right) = \sum_{m=\binom{k+1}{2}}^{n+\binom{k+1}{2}} a(k, m) p\left(n + \binom{k+1}{2} - m\right). \quad (4)$$

Since $q(k+1, m) = 0$ for $m < \binom{k+2}{2} = \binom{k+1}{2} + k + 1$, then by (2) and (3),

$$a(k, m) = q(k, m) = p_k\left(m - \binom{k+1}{2}\right) = p\left(m - \binom{k+1}{2}\right)$$

for $m \leq \binom{k+1}{2} + k$. Thus by (4),

$$\begin{aligned} p\left(k, n + \binom{k+1}{2}\right) &= \sum_{m=\binom{k+1}{2}}^{n+\binom{k+1}{2}} p\left(m - \binom{k+1}{2}\right) p\left(n + \binom{k+1}{2} - m\right) \\ &= \sum_{m=0}^n p(m) p(n - m) \end{aligned}$$

for $k \geq n$ and the proof is complete.

3. Combinatorial proof of Theorem 1

Let $k, n \geq 0$ be integers with $n \leq k$. For $m \in [0, n]$, take a partition $a_1 + \cdots + a_r$ of m with $1 \leq a_1 \leq \cdots \leq a_r$ and a partition $b_1 + \cdots + b_s$ of $n - m$ with $1 \leq b_1 \leq \cdots \leq b_s$. Then

$$r + s \leq m + s \leq m + (n - m) = n \leq k,$$

so $m \leq k - s$ and

$$\begin{aligned} n + \binom{k+1}{2} &= m + (n - m) + 1 + 2 + \cdots + k \\ &= a_1 + \cdots + a_r + b_1 + \cdots + b_s + 1 + 2 + \cdots + k \\ &= a_1 + \cdots + a_r + c_1 + \cdots + c_k \end{aligned}$$

where

$$c_j = \begin{cases} j, & 1 \leq j \leq k - s \\ j + b_{j-(k-s)}, & k - s + 1 \leq j \leq k \end{cases}.$$

Clearly, the parts a_1, \dots, a_r are contained in $\{c_1, c_2, \dots, c_{k-s}\}$ and $c_1 < c_2 < \cdots < c_k$, so each pair of partitions of m and $n - m$ uniquely defines a partition

$$a_1 + \cdots + a_r + c_1 + \cdots + c_k$$

of $n + \binom{k+1}{2}$ having exactly k different values for the parts. The product $p(m)p(n-m)$ then counts the number of partitions that may be constructed in this fashion for each $m \in [0, n]$, thus completing the proof.

The idea of this proof was suggested by a referee reading a previous version of this paper. We also see that setting $c_j = \overline{b_j}$ for $1 \leq j \leq s$ would instead give us the partition

$$a_1 + \cdots + a_r + \overline{b_1} + \cdots + \overline{b_s}$$

of n into parts of two kinds. This justifies the earlier remark on sequence $(A_n)_{n \geq 0}$ and provides some motivation for the given combinatorial proof.

The combinatorial proof gives us a simple way of writing down the partitions of $n + \binom{k+1}{2}$ having exactly k different values for the parts for any $k \geq n$. All we require are the partitions of m for each $m \in [0, n]$. For example, Theorem 1 implies there are ten partitions of $3 + \binom{k+1}{2}$ having exactly k different values for the parts for any $k \geq 3$. The case $k = 3$ is shown below.

m	r	s	$a_1 + \cdots + a_r$	$b_1 + \cdots + b_s$	$c_1 + c_2 + c_3$	partitions
0	0	1	0	3	1 + 2 + 6	1 + 2 + 6
	0	2	0	1 + 2	1 + 3 + 5	1 + 3 + 5
	0	3	0	1 + 1 + 1	2 + 3 + 4	2 + 3 + 4
1	1	1	1	2	1 + 2 + 5	1 + 1 + 2 + 5
	1	2	1	1 + 1	1 + 3 + 4	1 + 1 + 3 + 4
2	1	1	2	1	1 + 2 + 4	1 + 2 + 2 + 4
	2	1	1 + 1	1	1 + 2 + 4	1 + 1 + 1 + 2 + 4
3	1	0	3	0	1 + 2 + 3	1 + 2 + 3 + 3
	2	0	1 + 2	0	1 + 2 + 3	1 + 1 + 2 + 2 + 3
	3	0	1 + 1 + 1	0	1 + 2 + 3	1 + 1 + 1 + 1 + 2 + 3

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