A SIMPLE BIJECTION FOR CLASSICAL AND ENHANCED k-NONCROSSING PARTITIONS

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Abstract. In this short note, we give a simple bijection from partitions of subsets of $[n]$ to partitions of $[n+1]$, in which enhanced k-crossings correspond to classical k-crossings. This resolves a recent conjecture of Zhicong Lin involving the binomial transform of the sequence that enumerates enhanced k-noncrossing partitions.

Given a set partition π of a totally ordered set, an *enhanced k-crossing* is a sequence

$$a_1 < a_2 < \cdots < a_k \leq b_1 < b_2 < \cdots < b_k$$

such that each pair $a_i < b_i$ appears consecutively in a block (that is, there are no elements x in the block with $a_i < x < b_i$). A *classical k-crossing* additionally requires that $a_k < b_1$.

Let $C_k(n)$ (resp. $E_k(n)$) denote the number of set partitions of $[n] = \{1, \ldots, n\}$ which avoid classical (resp. enhanced) k-crossings. For $k = 2$, $C_2(n)$ are the Catalan numbers (A000108 in the OEIS [2]) and $E_2(n)$ are the Motzkin numbers A001006. It is known that $C_2(n+1) = \sum_{i=0}^{n} \binom{n}{i} E_2(i)$. In [1], Zhicong Lin showed that $C_3(n+1) = \sum_{i=0}^{n} \binom{n}{i} E_3(i)$ (see sequences A108304 and A108307) and conjectured the following.

Theorem 1. *For integers $n \geq 0$ and $k \geq 1$,*

$$C_k(n+1) = \sum_{i=0}^{n} \binom{n}{i} E_k(i).$$

Proof. For fixed $n \geq 0$ and any partition π of a subset of $[n]$, we define a partition $\hat{\pi}$ of $[n+1]$ as follows. For each pair of vertices $v < w$ appearing consecutively in the same block of π, place v and $w+1$ in the same block of $\hat{\pi}$. For each singleton u of π, place u and $u+1$ in the same block of $\hat{\pi}$. Any remaining vertices in $[n+1]$ appear as singletons of $\hat{\pi}$.

It is now straightforward to confirm this is a bijection and that enhanced k-crossings of π correspond to classical k-crossings of $\hat{\pi}$. To reverse this map, let $\hat{\pi}$ be a partition of $[n+1]$. For any consecutive vertices $x < y$ in the same block of $\hat{\pi}$, if $y = x+1$, include the singleton x in π, otherwise place the pair $x < y-1$ in the same block of π. With the map $\pi \to \hat{\pi}$, an enhanced k-crossing $a_1 < \cdots < a_k \leq b_1 < \cdots < b_k$ of π gives a classical k-crossing $a_1 < \cdots < a_k < b_1 + 1 < \cdots < b_k + 1$ of $\hat{\pi}$. Similarly, with the reverse map, a classical k-crossing $a_1 < \cdots < a_k < b_1 < \cdots < b_k$ of $\hat{\pi}$ gives an enhanced k-crossing $a_1 < \cdots < a_k \leq b_1 - 1 < \cdots < b_k - 1$ of π.

Finally, interpreting the right-hand side as the number of partitions of subsets of $[n]$ with no enhanced k-crossing, these are in bijection with partitions of $[n+1]$ with no classical k-crossing. ☐

2010 *Mathematics Subject Classification.* 05A19 (Primary); 05A18 (Secondary).

Key words and phrases. k-noncrossing partition, enhanced k-noncrossing partition.

For example, given the partition $\pi = \{1479, 25, 3, 6\}$ of the set $[9] \setminus \{8\}$, our bijection returns the partition $\hat{\pi} = \{15, 267\mathrm{T}, 348, 9\}$. A beautiful geometric description of this bijection can be obtained by extending the lines in the arc diagram of π to arrive at $\hat{\pi}$. Here the singletons are interpreted as having trivial arcs, see Fig. 1.

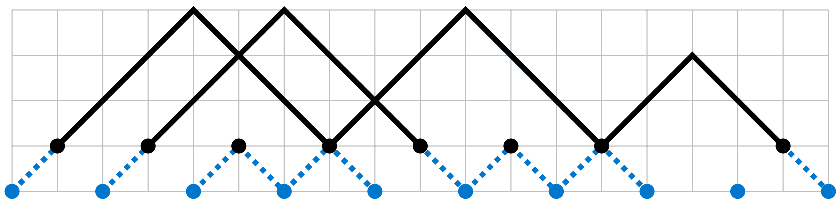

FIGURE 1. The diagram for $\pi = \{1479, 25, 3, 6\}$ and $\hat{\pi} = \{15, 267\mathrm{T}, 348, 9\}$.

This bijection allows us to prove a significant refinement of Theorem 1. The distances of the arcs in π (including trivial arcs) are one less than those of $\hat{\pi}$. The enhanced k-crossings of π correspond precisely to classical k-crossings of $\hat{\pi}$. This includes the case $k = 1$, where a classical 1-crossing is a (nontrivial) arc, and an enhanced 1-crossing is a (possibly trivial) arc. It is not hard to see that it also gives a correspondence between *enhanced k-nestings* in π, which are sequences

$$a_1 < a_2 < \cdots < a_k \leq b_k < \cdots < b_2 < b_1$$

such that each pair $a_i < b_i$ appears consecutively in a block, and *classical k-nestings* in $\hat{\pi}$, which additionally require that $a_k < b_k$.

Remark 2. This bijection provides an alternative combinatorial proof that the sequence (B_n) of Bell numbers A000110, which counts the number of partitions of $[n]$, is an eigensequence of the binomial transform. That is, $B_{n+1} = \sum_{i=0}^{n} \binom{n}{i} B_i$. The usual combinatorial proof of this fact simply removes the part containing $n+1$ from a partition of $[n+1]$ to obtain a partition of a subset of $[n]$.

REFERENCES

[1] Zhicong Lin, Restricted inversion sequences and enhanced 3-noncrossing partitions, preprint arXiv:1706.07213, 2017.

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