Generalized Cullen Numbers in Linear Recurrence Sequences

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Abstract

A Cullen number is a number of the form $m2^m + 1$, where m is a positive integer. In 2004, Luca and Stănică proved, among other things, that the largest Fibonacci number in the Cullen sequence is $F_4 = 3$. Actually, they searched for generalized Cullen numbers among some binary recurrence sequences. In this paper, we will work on higher order recurrence sequences. For a given linear recurrence $(G_n)_n$, under weak assumptions, and a given polynomial $T(x) \in \mathbb{Z}[x]$, we shall prove that if $G_n = mx^m + T(x)$, then

 $m \ll \log \log |x| \log^2(\log \log |x|)$ and $n \ll \log |x| \log \log |x| \log^2(\log \log |x|)$,

where the implied constant depends only on $(G_n)_n$ and T(x).

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1. Introduction

A Cullen number is a number of the form $m2^m + 1$ (denoted by C_m), where m is a nonnegative integer. A few terms of this sequence are

$$1, 3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, \ldots$$

which is the OEIS [31] sequence A002064 (this sequence was introduced in 1905 by the Father J. Cullen [6] and it was mentioned in the well-known Guy's book [11, Section **B20**]). These numbers gained great interest in 1976, when C. Hooley [13] showed that almost all Cullen numbers are composite. However, despite being very scarce, it is still conjectured the existence of infinitely many *Cullen primes*. For instance, $C_{6679881}$ is a prime number with more than 2 millions of digits (PrimeGrid, August 2009).

These numbers can be generalized to the *generalized Cullen numbers* which are numbers of the form

$$C_{m,s} = ms^m + 1,$$

where $m \ge 1$ and $s \ge 2$. Clearly, one has that $C_{m,2} = C_m$, for all $m \ge 1$. For simplicity, we call $C_{m,s}$ of *s*-Cullen number. This family was introduced by H. Dubner [7] and is one of the main sources for prime number "hunters". A big prime of the form $C_{m,s}$ is $C_{139948,151}$ an integer with 304949 digits.

Many authors have searched for special properties of Cullen numbers and their generalizations. Concerning these numbers, we refer to [10, 12, 15] for primality results and [20] for their greatest common divisor. The problem of finding Cullen numbers belonging to others known sequences has attracted much attention in the last two decades. We cite [21] for pseudoprime Cullen numbers, and [1] for Cullen numbers which are both Riesel and Sierpiński numbers.

A sequence $(G_n)_{n\geq 0}$ is a linear recurrence sequence with coefficients c_0 , c_1, \ldots, c_{k-1} , with $c_0 \neq 0$, if

$$G_{n+k} = c_{k-1}G_{n+k-1} + \dots + c_1G_{n+1} + c_0G_n, \tag{1}$$

for all positive integer n. A recurrence sequence is therefore completely determined by the *initial values* G_0, \ldots, G_{k-1} , and by the coefficients $c_0, c_1, \ldots, c_{k-1}$. The integer k is called the *order* of the linear recurrence. The *characteristic polynomial* of the sequence $(G_n)_{n>0}$ is given by

$$G(x) = x^{k} - c_{k-1}x^{k-1} - \dots - c_{1}x - c_{0}.$$

It is well-known that for all n

$$G_n = g_1(n)r_1^n + \dots + g_\ell(n)r_\ell^n, \tag{2}$$

where r_j is a root of G(x) and $g_j(x)$ is a polynomial over a certain number field, for $j = 1, ..., \ell$. In this paper, we consider only integer recurrence sequences, i.e., recurrence sequences whose coefficients and initial values are integers. Hence, $g_j(n)$ is an algebraic number, for all $j = 1, ..., \ell$, and $n \in \mathbb{Z}$.

A general Lucas sequence $(U_n)_{n\geq 0}$ given by $U_{n+2} = aU_{n+1} + bU_n$, for $n \geq 0$, where the values a, b, U_0 and U_1 are previously fixed, is an example of a linear recurrence of order 2 (also called *binary*). For instance, if $U_0 = 0$ and $U_1 = a = b = 1$, then $(U_n)_{n\geq 0} = (F_n)_{n\geq 0}$ is the well-known Fibonacci sequence and for $U_0 = 2$ and $U_1 = a = b = 1$, then $(U_n)_{n\geq 0} = (L_n)_{n\geq 0}$ is the sequence of the Lucas numbers:

In 2003, Luca and Stănică [19] showed, in particular, that Cullen numbers occur only finitely many times in a binary recurrent sequence satisfying some additional conditions. As application, they proved that the largest Fibonacci number in the Cullen sequence is $F_4 = 3 = 1 \cdot 2^1 + 1$. Very recently, Marques [25] searched for Fibonacci numbers in s-Cullen sequences. In particular, he proved that there is no Fibonacci number that is also a nontrivial s-Cullen number when all divisors of s are not Wall-Sun-Sun primes (i.e., $p^2 \nmid F_{p-(5/p)}$). See also [26]. We remark that no Wall-Sun-Sun prime is known as of July 2017, moreover if any exist, they must be greater than $2.6 \cdot 10^{17}$.

In this paper, we are interested in much more general Cullen numbers among terms of linear recurrences. More precisely, our goal is to work on the Diophantine equation

$$G_n = mx^m + T(x), (3)$$

for a given polynomial $T(x) \in \mathbb{Z}[x]$. Observe that when x is fixed and T(x) = 1 the right-hand side of (3) is an x-Cullen number.

We remark that several authors investigated the related equation

$$G_n = ax^m + T(x),$$

where a is fixed. Among the results for a general recurrence (under some technical hypotheses), it was proved finiteness of solutions for T(x) = 0 by Shorey and Stewart [30], for T(x) = c by Stewart [32] and for any T(x) by Nemes and Pethő [27]. Moreover, these results are effective.

Here, our main result is the following

Theorem 1. Let $(G_n)_n$ be an integer linear recurrence with roots r_1, \ldots, r_k satisfying either

- (i) $|r_1| > 1 > |r_j| > 0$, for j > 1; or
- (*ii*) $|r_1| > |r_2| > |r_j| > 0$, for j > 2.

Moreover, we suppose that r_1 is a simple root. Let $T(x) \in \mathbb{Z}[x]$ be a polynomial. There exist effectively computable constants C_1, C_2 , depending only on $(G_n)_n$ and T(x), such that if (m, n, x) is a solution of the Diophantine equation (3), then

 $m \leq C_1 \log \log |x| \log^2 (\log \log |x|)$ and $n \leq C_2 \log |x| \log \log |x| \log^2 (\log \log |x|)$.

Observe that we cannot ensure here finitely many values for |x|. For example, for $G_n = 2L_n$ and T(x) = -4, one has that $(n, m, x) = (4t, 2, L_{2t})$ is solution for Eq. (3) for all $t \ge 0$.

Let $k \geq 2$ and denote $F^{(k)} := (F_n^{(k)})_{n \geq -(k-2)}$, the *k*-generalized Fibonacci sequence whose terms satisfy the recurrence relation

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + F_{n+k-2}^{(k)} + \dots + F_n^{(k)},$$
(4)

with initial conditions 0, 0, ..., 0, 1 (k terms) and such that the first nonzero term is $F_1^{(k)} = 1$.

The above sequence is one among the several generalizations of Fibonacci numbers. Such a sequence is also called *k*-step Fibonacci sequence, the Fibonacci k-sequence, or k-bonacci sequence. Clearly for k = 2, we obtain the classical Fibonacci numbers, for k = 3, the Tribonacci numbers, for k = 4, the Tetranacci numbers, etc.

Recently, these sequences have been the main subject of many papers. We refer to [4] for results on the largest prime factor of $F_n^{(k)}$ and we refer to [2] for the solution of the problem of finding powers of two belonging to these sequences. In 2013, two conjectures concerning these numbers were proved. The first one, proved by Bravo and Luca [5] is related to *repdigits* (i.e., numbers with only one distinct digit in its decimal expansion) among k-Fibonacci numbers (proposed by Marques [24]) and the second one, a conjecture (proposed by Noe and Post [28]) about coincidences between terms of these sequences, proved independently by Bravo-Luca [3] and Marques [23].

If we use Theorem 1 to sequence $(G_n)_n = (F_n^{(k)})_n$, we get finitely many solutions for Eq. (3), for each $k \ge 2$. However, we shall improve the method

and we find an upper bound for the number of Cullen numbers (case x = 2 and T(x) = 1) in $\bigcup_{k \ge 2} F^{(k)}$. More precisely,

Theorem 2. If (m, n, k) is a solution of the Diophantine equation

$$F_n^{(k)} = m2^m + 1 \tag{5}$$

in positive integers m, n and $k \geq 2$, then

$$m < 9.5 \cdot 10^{23}, n < 2.4 \cdot 10^{24} \text{ and } k \le 158.$$

Let us give a brief overview of our strategy for proving Theorem 2. First, we use a Dresden and Du formula [9, Formula (2)] to get an upper bound for a linear form in three logarithms related to equation (5). After, we use a lower bound due to Matveev to obtain an upper bound for m and n in terms of k. Very recently, Bravo and Luca solved the equation $F_n^{(k)} = 2^m$ and for that they used a nice argument combining some estimates together with the Mean Value Theorem (this can be seen in pages 77 and 78 of [2]). In our case, we use Bravo-Luca's approach to get an inequality involving a linear form in two logarithms. In the other case, we use a lower bound due to Laurent to get substantially upper bounds for m, n and k. The computations in the paper were performed using *Mathematica*[®]

2. Auxiliary results

In this section, we recall some results that will be very useful for the proof of the above theorems. Let G(x) be the characteristic polynomial of a linear recurrence G_n . One can factor G(x) over the set of complex numbers as

$$G(x) = (x - r_1)^{m_1} (x - r_2)^{m_2} \cdots (x - r_\ell)^{m_\ell},$$

where r_1, \ldots, r_ℓ are distinct non-zero complex numbers (called the *roots* of the recurrence) and m_1, \ldots, m_ℓ are positive integers. A root r_j of the recurrence is called a *dominant root* if $|r_j| > |r_i|$, for all $j \neq i \in \{1, \ldots, \ell\}$. The corresponding polynomial $g_j(n)$ is named the *dominant polynomial* of the recurrence. A fundamental result in the theory of recurrence sequences asserts that there exist uniquely determined non-zero polynomials $g_1, \ldots, g_\ell \in \mathbb{Q}(\{r_j\}_{j=1}^\ell)[x]$, with deg $g_j \leq m_j - 1$, for $j = 1, \ldots, \ell$, such that

$$G_n = g_1(n)r_1^n + \dots + g_\ell(n)r_\ell^n, \text{ for all } n.$$
(6)

For more details, see [29, Theorem C.1].

In the case of the Fibonacci sequence, the above formula is known as *Binet's formula*:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\tag{7}$$

where $\alpha = (1 + \sqrt{5})/2$ (the golden number) and $\beta = (1 - \sqrt{5})/2 = -1/\alpha$. Equation (6) and some tricks will allow us to obtain linear forms in three logarithms and then determine lower bounds à *la Baker* for these linear forms. From the main result of Matveev [22], we deduce the following lemma.

Lemma 1. Let $\gamma_1, \ldots, \gamma_t$ be real algebraic numbers and let b_1, \ldots, b_t be nonzero rational integer numbers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_t)$ over \mathbb{Q} and let A_j be a positive real number satisfying

$$A_j \ge \max\{Dh(\gamma_j), |\log \gamma_j|, 0.16\}, \text{ for } j = 1, \dots, t.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_t|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_t^{b_t} \neq 1$, then $|\gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1| \ge \exp(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t).$

As usual, in the previous statement, the *logarithmic height* of an *n*-degree algebraic number α is defined as

$$h(\alpha) = \frac{1}{n} (\log|a| + \sum_{j=1}^{n} \log \max\{1, |\alpha^{(j)}|\}),$$

where a is the leading coefficient of the minimal polynomial of α (over \mathbb{Z}) and $(\alpha^{(j)})_{1 \le j \le n}$ are the conjugates of α .

Now, we are ready to deal with the proofs of our results.

3. The proof of Theorem 1

Throughout the proof, the numerical constants implied by \ll depend only on $(G_n)_n$ and T(x). Also, without loss of generality, we may suppose $|x| \ge 2$ (i.e., $x \ne -1, 0, 1$).

First, since r_1 is a simple dominant root then $g_1(n)$ in formula (6) is a constant, say g (because the degree of $g_1(n)$ would be at most $m_1 - 1 = 1 - 1 = 0$). Now, we rewrite Eq. (3) as

$$mx^m - gr_1^n = B(x, n),$$

where $B(x, n) := \sum_{j=2}^{k} g_j(n) r_j^n - T(x)$. Then

$$\frac{mx^m r_1^{-n}}{g} - 1 = \frac{B(x,n)}{gr_1^n}$$

If B(x,n) = 0 and (ii) holds, we use the same argument than Nemes and Pethő to get $m \ll 1$ and the proof is complete (see lines 25-35 in page 231 of [27]). However, if B(x,n) = 0 and (i) holds, we get the relation $mx^m = gr_1^n$. So, we can take the conjugates of this relation in $\mathbb{Q}(r_1)$ to get $mx^m = g^{(t)}r_t^n$, where the $g_1^{(i)}$'s are the conjugates of g_1 over $\mathbb{Q}(r_1)$. Thus, by taking absolute values and using that $|r_t| < 1$ we obtain $m2^m \ll 1$ yielding $m \ll 1$.

Thus, we may suppose $B(x, n) \neq 0$ and in this case Nemes and Pethő [27, p. 232] proved that $|B(x, n)| \leq r_1^{n(1-\delta)}$, for some $\delta \ll 1$. Therefore

$$\left|\frac{mx^m r_1^{-n}}{g} - 1\right| \ll \frac{1}{r_1^{n\delta}}.$$
(8)

Let $\Lambda = \log(m/g) - n \log r_1 + m \log x$. Since $x < e^x - 1$ and for x < 0, $|e^x - 1| = 1 - e^{-|x|}$, then the previous inequality yields $|\Lambda| \ll 1/r_1^{n\delta+O(1)}$ yielding

$$\log|\Lambda| \ll -(n\delta + O(1))\log r_1. \tag{9}$$

Now, we will apply Lemma 1. To this end, take

$$t := 3, \ \gamma_1 := m/g, \ \gamma_2 := x, \ \gamma_3 := r_1,$$

and

 $b_1 := 1, \ b_2 := m, \ b_3 := -n.$

For this choice, we have $D = [\mathbb{Q}(g, r_1) : \mathbb{Q}] \leq k!$. Also $h(\gamma_1) \leq \log m + h(g) \ll \log m$, $h(\gamma_2) = \log |x|$ and $h(\gamma_3) \ll \log r_1$, where we used the well-known facts that $h(xy) \leq h(x) + h(y)$ and $h(x) = h(x^{-1})$.

Note that Eq. (3) implies that $m \log |x| \approx n$. In fact, one has that

$$r_1^{n-O(1)} \ll |G_n| = |mx^m + T(x)| \ll mx^m,$$

where we used that $|T(x)| \ll |x|^{\deg T}$. Thus, we obtain $n \ll m \log |x|$. On the other hand, $m|x|^m \leq |g|r_1^n + |B(x,n)| \ll r_1^{O(n)}$ (here we used that $|B(x,n)| \leq r_1^{n(1-\delta)}$). By applying the log function we arrive at

$$1 + m \log |x| \ll \log m + m \log |x| \ll n$$

and thus $m \log |x| \ll n$. Therefore, we have that $B \ll m \log |x|$.

Since $B(x, n) \neq 0$, the left-hand side of (9) is nonzero and so the conditions to apply Lemma 1 are fulfilled yielding

$$\log |\Lambda| > -\log^2 m \log |x| \log \log |x|.$$
(10)

Combining estimates (9) and (10) we have

$$n \ll \log^2 m \log |x| \log \log |x|. \tag{11}$$

Combining this estimate with (11) we get

$$\frac{m}{\log^2 m} \ll \log \log |x|.$$

As we shall prove in (18), the inequality above implies

 $m \ll \log \log |x| \log^2(\log \log |x|).$

Now, we use the estimate $n \ll m \log |x|$ to get the desired inequality on n, i.e.,

 $n \ll \log |x| \log \log |x| \log^2(\log \log |x|).$

The proof is then complete.

4. The proof of Theorem 2

4.1. Auxiliary results

Before proceeding further, we will recall some facts and properties of these sequences which will be used after.

We know that the characteristic polynomial of $(F_n^{(k)})_n$ is

$$\psi_k(x) := x^k - x^{k-1} - \dots - x - 1$$

and it is irreducible over $\mathbb{Q}[x]$ with just one zero outside the unit circle. That single zero is located between $2(1-2^{-k})$ and 2 (as it can be seen in [16]). Also,

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in a recent paper, G. Dresden and Z. Du [9, Theorem 1] gave a simplified "Binet-like" formula for $F_n^{(k)}$:

$$F_n^{(k)} = \sum_{i=1}^k \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1},$$
(12)

for $\alpha = \alpha_1, \ldots, \alpha_k$ being the roots of $\psi_k(x)$. Also, it was proved in [5, Lemma 1] that

$$\alpha^{n-2} \le F_n^{(k)} \le \alpha^{n-1}, \text{ for all } n \ge 1,$$
(13)

where α is the dominant root of $\psi_k(x)$. Also, the contribution of the roots inside the unit circle in formula (12) is almost trivial. More precisely, it was proved in [9] that

$$|F_n^{(k)} - g(\alpha, k)\alpha^{n-1}| < \frac{1}{2},\tag{14}$$

where we adopt throughout the notation g(x, y) := (x-1)/(2+(y+1)(x-2)).

Very recently, Bravo and Luca [2] found all powers of two in k-generalized Fibonacci sequences. Their nice method can be slightly changed to show that (n, k, m) = (1, 4, 2) and (5, 2, 2) are the only solutions of the equation $F_n^{(k)} = 2^m + 1$, with $k \ge 2$. Thus, the only solution of Eq. (5) such that mis a power of two is (n, k, m) = (1, 4, 2). So, throughout the paper, we shall suppose that m is not a power of two and that $m \ge 10$ (the case m < 10 can be easily solved). Note also that, by definition, $F_n^{(k)}$ is a power of two for all $1 \le n \le k + 1$ and hence these values cannot be Cullen numbers. Thus, it is enough to consider n > k + 1. Finally, due to [19, Theorem 3], we can suppose that $k \ge 3$.

4.2. The proof

First, we use Eq. (5) together with the formula (12) to obtain

$$g(\alpha, k)\alpha^{n-1} - m2^m = 1 - \sum_{i=2}^k g(\alpha_i, k)\alpha_i^{n-1} \in (1/2, 3/2),$$
(15)

where we used (14). Thus, equation (15) implies that

$$0 < g(\alpha, k)\alpha^{n-1} - m2^m < 3/2.$$

So, dividing by $m2^m$, we get

$$\left|\frac{g(\alpha,k)\alpha^{n-1}}{m2^m} - 1\right| < 1/2^{m+1},\tag{16}$$

for $m \geq 3$.

In order to use Lemma 1, we take

$$t := 3, \ \gamma_1 := g(\alpha, k)/m, \ \gamma_2 := 2, \ \gamma_3 := \alpha$$

and

$$b_1 := 1, \ b_2 := -m, \ b_3 := n - 1.$$

For this choice, we have $D = [\mathbb{Q}(\alpha) : \mathbb{Q}] = k$. Also $h(\gamma_1) \leq \log((4k+4)m)$, $h(\gamma_2) = \log 2$ and $h(\gamma_3) < 0.7/k$. Thus, we can take $A_1 := k \log((4k+4)m)$, $A_2 := k \log 2$ and $A_3 := 0.7$.

Moreover, using the inequalities (13), we get

$$(7/4)^{n-2} < \alpha^{n-2} < F_n^{(k)} = m2^m + 1 < 2^{2m-1}$$

and so n < 2.5m + 0.8. Note that $\max\{|b_1|, |b_2|, |b_3|\} = \max\{m, n-1\} \le 2.5m + 0.8 =: B$. Since $g(\alpha, k)\alpha^{n-1}2^{-m}/m > 1$ (by (15)), we are in position to apply Lemma 1. This lemma together with a straightforward calculation gives

$$\left|\frac{g(\alpha,k)\alpha^{n-1}}{m2^m} - 1\right| > \exp(-6.7 \cdot 10^{11}k^4 \log^2 m),\tag{17}$$

where we used that $1 + \log k < 2 \log k$, for $k \ge 2$, $1 + \log(2.5m + 0.8) < 1.9 \log m$, for $m \ge 10$, and $\log((4k + 4)m) < 2.5 \log m$ (to prove this last inequality, we used that 2.5m + 0.8 > n > k + 1).

By combining (16) and (17), we obtain

$$\frac{m}{\log^2 m} < 9.7 \cdot 10^{11} k^4 \log k.$$

Since the function $x \mapsto x/\log^2 x$ is increasing for x > e, then it is a simple matter to prove that

$$\frac{x}{\log^2 x} < A \text{ implies that } x < 2A \log^2 A \text{ (for } A \ge 10^7\text{)}.$$
(18)

In fact, suppose the contrary, i.e. $x \ge 2A \log^2 A$. Then

$$\frac{x}{\log^2 x} \ge \frac{2A\log^2 A}{\log^2(2A\log^2 A)} > A,$$

which contradicts our inequality. Here we used that $\log^2(2A\log^2 A) < 2\log^2 A$, for $A \ge 10^7$.

Thus, using (18) for x := m and $A := 9.7 \cdot 10^{11} k^4 \log k$, we have that

$$m < 2(9.7 \cdot 10^{11}k^4 \log k) \log^2(9.7 \cdot 10^{11}k^4 \log k).$$

A straightforward calculation gives

$$m < 5.9 \cdot 10^{13} k^4 \log^2 k. \tag{19}$$

Now, we shall prove that there is no solution when $k \ge 159$. In this case, (19) implies

$$n < 2.5m + 0.8 < 11.8 \cdot 10^{13} k^4 \log^2 k + 0.8 < 2^{k/2}.$$

Now, we use a key argument due to Bravo and Luca [2, p. 77-78].

Setting $\lambda = 2 - \alpha$, we deduce that $0 < \lambda < 1/2^{k-1}$ (because $2(1 - 2^{-k}) < \alpha < 2$). So

$$\alpha^{n-1} = (2-\lambda)^{n-1} = 2^{n-1} \left(1 - \frac{\lambda}{2}\right)^{n-1} > 2^{n-1} (1 - (n-1)\lambda),$$

since that the inequality $(1-x)^n > 1-2nx$ holds for all $n \ge 1$ and 0 < x < 1. Moreover, $(n-1)\lambda < 2^{k/2}/2^{k-1} = 2/2^{k/2}$ and hence

$$2^{n-1} - \frac{2^n}{2^{k/2}} < \alpha^{n-1} < 2^{n-1} + \frac{2^n}{2^{k/2}},$$

yielding

$$|\alpha^{n-1} - 2^{n-1}| < \frac{2^n}{2^{k/2}}.$$
(20)

Now, we define for $x > 2(1 - 2^{-k})$ the function f(x) := g(x, k) which is differentiable in the interval $[\alpha, 2]$. So, by the Mean Value Theorem, there exists $\xi \in (\alpha, 2)$, such that $f(\alpha) - f(2) = f'(\xi)(\alpha - 2)$. Thus

$$|f(\alpha) - f(2)| < \frac{2k}{2^k},$$
(21)

where we used the bounds $|\alpha - 2| < 1/2^{k-1}$ and $|f'(\xi)| < k$. For simplicity, we denote $\delta = \alpha^{n-1} - 2^{n-1}$ and $\eta = f(\alpha) - f(2) = f(\alpha) - 1/2$. After some calculations, we arrive at

$$2^{n-2} = f(\alpha)\alpha^{n-1} - 2^{n-1}\eta - \frac{\delta}{2} - \delta\eta.$$

Therefore

$$\begin{aligned} |2^{n-2} - m2^{m}| &\leq \frac{3}{2} + 2^{n-1}|\eta| + \left|\frac{\delta}{2}\right| + |\delta\eta| \\ &\leq \frac{3}{2} + \frac{2^{n}k}{2^{k}} + \frac{2^{n-1}}{2^{k/2}} + \frac{2^{n+1}k}{2^{3k/2}}, \end{aligned}$$

where we used (20) and (21). Since n > k + 1, one has that $2^{n-2}/2^{k/2} \ge 2^{k/2} > 3/2$ (for $k \ge 2$) and we rewrite the above inequality as

$$|2^{n-2} - m2^{m}| < \frac{2^{n-2}}{2^{k/2}} + \left(\frac{4k}{2^{k/2}}\right)\frac{2^{n-2}}{2^{k/2}} + 2 \cdot \frac{2^{n-2}}{2^{k/2}} + \left(\frac{8k}{2^{k}}\right)\frac{2^{n-2}}{2^{k/2}}.$$

Since the inequality $\max_{k \ge 159} \{4k/2^{k-1}, 8k/2^{k/2}\} < 5.4 \cdot 10^{-22}$ holds, then

$$|2^{n-2} - m2^m| < \frac{3 \cdot 2 \cdot 2^{n-2}}{2^{k/2}},\tag{22}$$

or equivalently

$$|1 - m2^{-(n-m-2)}| < \frac{3.2}{2^{k/2}}.$$
(23)

Since $m \ge 10$, we have

- If $\log m / \log 2 + m + 3 \le n$, then $1 m/2^{n-m-2} \ge 1/2$ yielding $k \le 5$;
- If $\log m / \log 2 + m + 1 \ge n$, then $m 2^{n-m-2} 1 \ge 1$ leading to $k \le 3$

which is not possible. Since $\log m / \log 2 \notin \mathbb{Q}$ when *m* is not a power of 2, we may suppose that $n = \lfloor \log m / \log 2 \rfloor + m + \delta$, for $\delta \in \{2, 3\}$.

Note that (23) is equivalent to

$$|1 - e^{\Lambda}| < \frac{3.2}{2^{k/5}},\tag{24}$$

where $\Lambda := \log m - (\lfloor \log m / \log 2 \rfloor + \delta - 2) \log 2$.

Since *m* is not a power of 2, then *m* and 2 are multiplicatively independent. In particular, $\Lambda \neq 0$. If $\Lambda > 0$, then $\Lambda < e^{\Lambda} - 1 < 3.2/2^{k/2}$. In the case of $\Lambda < 0$, we use $1 - e^{-|\Lambda|} = |e^{\Lambda} - 1| < 3.2/2^{k/2}$ to get $e^{|\Lambda|} < 1/(1 - 3.2 \cdot 2^{-k/2})$. Thus

$$|\Lambda| < e^{|\Lambda|} - 1 < \frac{3 \cdot 2 \cdot 2^{-k/2}}{1 - 3 \cdot 2 \cdot 2^{-k/2}} < 3 \cdot 6 \cdot 2^{-k/2},$$

where we used that $1/(1 - 3.2 \cdot 2^{-k/2}) < 1.1$, for $k \ge 159$. In any case, we have

$$|\Lambda| < 3.6 \cdot 2^{-k/2} \tag{25}$$

and so

$$\log|\Lambda| < \log(3.6) - \frac{k}{2}\log 2.$$
(26)

Now, we will determine a lower bound for Λ . We remark that the bounds available for linear forms in two logarithms are substantially better than those available for linear forms in three logarithms. Here we choose to use a result due to Laurent [17, Corollary 2] with m = 24 and $C_2 = 18.8$. First let us introduce some notations. Let α_1, α_2 be real algebraic numbers, with $|\alpha_j| \geq 1, b_1, b_2$ be positive integer numbers and

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1.$$

Let A_j be real numbers such that

$$\log A_j \ge \max\{h(\alpha_j), |\log \alpha_j|/D, 1/D\}, j \in \{1, 2\},\$$

where D is the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2)$ over \mathbb{Q} . Define

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}$$

Laurent's result asserts that if α_1, α_2 are multiplicatively independent, then

$$\log |\Lambda| \ge -18.8 \cdot D^4 \left(\max\{\log b' + 0.38, m/D, 1\} \right)^2 \cdot \log A_1 \log A_2.$$

We then take

$$D = 1, \ b_1 = \lfloor \log m / \log 2 \rfloor + \delta - 2, \ b_2 = 1, \ \alpha_1 = 2, \ \alpha_2 = m.$$

We choose $\log A_1 = 1$ and $\log A_2 = \log m$. So we get

$$b' = \frac{\lfloor \log m / \log 2 \rfloor + \delta - 2}{\log m} + 1 < \frac{1}{\log m} + \frac{1}{\log 2} + 1.$$

Thus, by Corollary 2 of [17] we get

$$\log|\Lambda| \ge -13.1 \cdot 24^2 \log m. \tag{27}$$

Now, we combine the estimates (26) and (27) to obtain $k < 21761 \log m$. On the other hand, inequality (19) gives $k < 2.1 \cdot 10^6$. Therefore, $m < 2.5 \cdot 10^{41}$. Now, we come back to (25) and by using Mathematica, we arrive at

$$\frac{2^{-k/2}}{\log 2} > \min_{\substack{\theta \in \{0,1\}, 3 \le m \le 2.5 \cdot 10^{41}, m \ne 2^{s} \\ 3 \le m \le 2.5 \cdot 10^{41}, m \ne 2^{s}}} \left\{ \left| \frac{\log m}{\log 2} - \left(\left\lfloor \frac{\log m}{\log 2} \right\rfloor + \theta \right) \right| \right\} \\
\geq \min_{\substack{3 \le m \le 2.5 \cdot 10^{41}, m \ne 2^{s} \\ > 8.2 \cdot 10^{-42},}} \min \left\{ \left\{ \frac{\log m}{\log 2} \right\}, 1 - \left\{ \frac{\log m}{\log 2} \right\} \right\}$$

where this minimum occurs when $m = 2^{137} \pm 1$ (here, as usual $\{x\}$ denotes the fractional part of a real number x). This yields $k \leq 274$ implying $m < 1.1 \cdot 10^{25}$. Now, we repeat the above process two times (with the minimum occuring in $m = 2^{83} \pm 1$ and $m = 2^{79} \pm 1$) to obtain $k \leq 158$. This contradicts the assumption of $k \geq 159$.

Remark 1. We remark to the reader that it must be possible to improve the upper bound for m, n and k in Theorem 2. Unfortunately, it is not possible to decrease them to fulfill all remaining cases. On the other hand, the usual approach to finish the finite many cases is by using the Baker-Davenport reduction method (mainly, results related to a Dujella-Pethö theorem). However, for this problem, we have a form like

$$(n-1)\gamma_k - m + \mu_{k,m},$$

where $\gamma_k := \log \alpha / \log 2$ and $\mu_{m,k} := \log(g(\alpha, k)/m) / \log 2$. To use the reduction method, we should get a positive lower bound for a quantity (called ϵ) depending on, in this case, k and m. The problem here is the dependence on m which by its size ($\approx 10^{23}$) becomes the calculation "impossible", by our computational tools.

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References

References

- P. Berrizbeitia, J. G. Fernandes, M. González, F. Luca, V. Janitzio. On Cullen numbers which are both Riesel and Sierpiński numbers. J. Number Theory 132 (2012), no. 12, 2836-2841.
- [2] J. J. Bravo, F. Luca, Powers of two in generalized Fibonacci sequences, *Rev. Colombiana Mat.* 46 (2012), 67–79.
- [3] J. J. Bravo, F. Luca, Coincidences in generalized Fibonacci sequences, J. Number Theory. 133 (2013), 2121–2137.
- [4] J. J. Bravo, F. Luca, On the largest prime factor of the k-Fibonacci numbers. Int. J. Number Theory 9 (2013), 1351-1366.
- [5] J. J. Bravo and F. Luca, On a conjecture about repdigits in kgeneralized Fibonacci sequences, *Publ. Math. Debrecen* 82 Fasc. 3-4 (2013).
- [6] J. Cullen. Question 15897. Educ. Times, (Dec.):534, 1905.
- [7] H. Dubner. Generalized Cullen numbers. J. Recreat. Math., 21 (1989), 190-194.
- [8] A. Dujella and A. Pethö, A generalization of a theorem of Baker and Davenport, Quart. J. Math. Oxford Ser. (2) 49 (1998), 291–306.
- [9] G. P. Dresden, Z. Du, A simplified Binet formula for k-generalized Fibonacci numbers, J. Integer Seq. 17 (2014) No. 4, Article 14.4.7.
- [10] J. M. Grau, A. M. Oller-Marcén, An O(log²(N)) time primality test for generalized Cullen numbers, *Math. Comp.* 80 (2011), 2315-2323.
- [11] R. Guy, Unsolved Problems in Number Theory (2nd ed.), Springer-Verlag, New York, 1994.
- [12] F. Heppner, Über Primzahlen der Form $n2^n+1$ bzw. $p2^p+1$, Monatsh. Math. 85 (1978), 99-103.
- [13] C. Hooley, Applications of the Sieve Methods to the Theory of Numbers, Cambridge University Press, Cambridge, 1976.

- [14] D. Kalman, R. Mena, The Fibonacci numbers exposed, *Math. Mag.* 76 (2003), no. 3, 167–181.
- [15] W. Keller, New Cullen primes, Math. Comput. 64 (1995), 1733-1741.
- [16] H. L. Keng, W. Yuan, Application of Number Theory to Numerical Analysis, Springer Verlag, 1981.
- [17] M. Laurent, Linear forms in two logarithms and interpolation determinants II, Acta Arith. 133.4 (2008), 325–348.
- [18] M. Laurent, Équations exponentielles polynômes et suites récurrentes linéeaires II, J. Number Theory 31 (1989), 24-53.
- [19] F. Luca and P. Stănică, Cullen numbers in binary recurrent sequences, Applications of Fibonacci numbers, vol. 10, Kluwer Academic Publishers, 2004, 167–175.
- [20] F. Luca, On the greatest common divisor of two Cullen numbers, Abh. Math. Sem. Univ. Hamburg 73 (2003), 253-270.
- [21] F. Luca, I. Shparlinski, Pseudoprime Cullen and Woodall numbers, Colloq. Math. 107 (2007), 35-43.
- [22] E. M. Matveev, An explict lower bound for a homogeneous rational linear form in logarithms of algebraic numbers, II, *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000), 125–180. English transl. in *Izv. Math.* **64** (2000), 1217–1269.
- [23] D. Marques, The proof of a conjecture concerning the intersection of k-generalized Fibonacci sequences, Bull. Brazilian Math. Soc. 44 (3) (2013), 455-468.
- [24] D. Marques, On k-generalized Fibonacci numbers with only one distinct digit, To appear in Util. Math.
- [25] D. Marques, On generalized Cullen and Woodall numbers which are also Fibonacci numbers, J. Integer Sequences, 17 (2014), Article 14.9.4.
- [26] D. Marques, A. P. Chaves, Fibonacci s-Cullen and s-Woodall numbers, J. Integer Sequences, 18 (2015), Article 15.1.4.

- [27] I. Nemes, A. Pethő, Polynomial values in linear recurrences, *Publica*tions Math. Debrecen **31** (1984) p. 229–233.
- [28] T. D. Noe and J. V. Post, Primes in Fibonacci n-step and Lucas n-step sequences, J. Integer Seq., 8 (2005), Article 05.4.4.
- [29] T. N. Shorey and R. Tijdeman, Exponential Diophantine Equations, Cambridge Tracts in Mathematics 87, Cambridge University Press, Cambridge, 1986.
- [30] T. N. Shorey, C. L. Stewart, Pure powers in recurrence sequences and some related diophantine equations, J. Number Theory, 27 (1987), 324-352.
- [31] N. J. A. Sloane, *The On-Line Encyclopedia* of *Integer Sequences*, published electronically at http://www.research.att.com/~njas/sequences/
- [32] C. L. Stewart, On some Diophantine equations and related linear recurrence sequences, *seminare Delang-Pisot-Poitou Theorie des Nombres* (1980-1981), 317-321.
- [33] Wolfram Research, Inc., Mathematica, Version 7.0, Champaign, IL (2008).
- [34] A. Wolfram, Solving generalized Fibonacci recurrences, Fibonacci Quart. 36 (1998), 129–145.