SPECIAL IDENTITIES FOR COMTRANS ALGEBRAS

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ABSTRACT. Comtrans algebras, arising in web geometry, have two trilinear operations, commutator and translator. We determine a Gröbner basis for the comtrans operad, and state a conjecture on its dimension formula. We study multilinear polynomial identities for the special commutator $[x,y,z] = xyz - yxz$ and special translator $\langle x,y,z\rangle = xyz - yzx$ in associative triple systems. In degree 3, the defining identities for comtrans algebras generate all identities. In degree 5, we simplify known identities for each operation and determine new identities relating the operations. In degree 7, we use representation theory of the symmetric group to show that each operation satisfies identities which do not follow from those of lower degree but there are no new identities relating the operations. We use noncommutative Gröbner bases to construct the universal associative envelope for the special comtrans algebra on 2×2 matrices.

1. INTRODUCTION

Comtrans algebras were introduced by Smith [27, §3] to answer a problem in web geometry [2, 12] posed by Goldberg [11, Problem X.3.9]: to find the algebraic structure on the tangent bundle of the coordinate ternary loop of a 4-web [18, §3.7]. Comtrans algebras are a common ternary generalization of Lie algebras, Malcev algebras [21] and Akivis algebras [1]: every such algebra can be given the structure of a comtrans algebra [24, §5]. A generalization of Lie's Third Fundamental Theorem connects formal ternary loops with comtrans algebras [27, §5]. For physical applications of comtrans algebras, see [28, p. 321]: "...the Lorentz metric on 4-dimensional real space-time provides a simple comtrans algebra that extends the 3-dimensional vector triple product comtrans algebra."

Definition 1.1. Smith [27, §3]. A *comtrans algebra* is a vector space A with two trilinear operations $A \times A \times A \to A$, the *commutator* $[x,y,z]$ and the *translator* $\langle x,y,z\rangle$, satisfying the following polynomial identities for all $x,y,z \in A$:

$$[x,y,z] + [y,x,z] = 0, \tag{1}$$

$$\langle x,y,z\rangle + \langle y,z,x\rangle + \langle z,x,y\rangle = 0, \tag{2}$$

$$[x,y,z] + [z,y,x] = \langle x,y,z\rangle + \langle z,y,x\rangle. \tag{3}$$

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The commutator alternates in the first two arguments (1), the translator satisfies the Jacobi identity (2), and the operations are related by the comtrans identity (3).

Example 1.2. If T is a Lie triple system with bracket $[x, y, z]$ then letting both commutator and translator equal $[x, y, z]$ gives T the structure of a comtrans algebra T^{CT}. If T is obtained from a Lie algebra L by $[x, y, z] = [[x, y], z]$ then Shen & Smith [26, Theorem 3.2] have shown that L is simple if and only if T^{CT} is simple.

Example 1.3. Let $A_{m,n}$ denote the vector space of $m \times n$ matrices over $\mathbb{F}$. Fix matrices p $(n \times n)$ and q $(m \times m)$ over $\mathbb{F}$. Define a trilinear operation on $A_{m,n}$ by $(x, y, z) = xpy^tqz$. Setting $[x, y, z] = (x, y, z) - (y, x, z)$ and $\langle x, y, z\rangle = (x, y, z) - (y, z, x)$ gives $A_{m,n}$ the structure of a comtrans algebra [26, Example 2.1].

Definition 1.4. An *associative triple system* [22] is a vector space A with a trilinear operation xyz satisfying $(vw(xyz)) = (v(wxy)z) = (vw(xyz))$. The *special commutator* and *special translator* in A are $[x, y, z] = xyz - yxz$ and $\langle x, y, z\rangle = xyz - yzx$.

Remark 1.5. In the terminology of [8], two trilinear operations in $\mathbb{Q}S_3$ are equivalent if they generate the same left ideal. The special commutator is equivalent to the $q = 2$ case of the q-deformed anti-Jordan triple product. The special translator is the same as the cyclic commutator, and is equivalent to the operation $2xyz - yzx - zxy$ which represents to the identity matrix I_2 in the simple 2-sided ideal corresponding to partition 2+1.

In §3 we recall basic definitions from the theory of algebraic operads, a new approach to multilinear polynomial identities for nonassociative structures. We calculate a Gröbner basis for the comtrans operad and explain how Gröbner bases for operads may be used to compute polynomial identities. In §4 we show that every identity in degree 3 for the commutator and translator follows from (1)–(3). In §5 we use computer algebra to find explicit generators for the S_5-module of identities which do not follow from those of degree 3. We consider three cases: the commutator by itself, the translator by itself, and identities in which each term contains both operations. In §6 we use a constructive version of the representation theory of the symmetric group to demonstrate that there are new identities in degree 7 for each operation separately but no new identities relating the operations. In §7 we construct the universal associative enveloping algebra of the special comtrans algebra M^{CT} obtained from the associative triple system M of 2×2 matrices.

Our results are valid over any field of characteristic 0. Our computations were performed using Maple worksheets written by the authors.

2. Preliminaries

Lemma 2.1. *In an associative triple system A, the special commutator and special translator satisfy the relations* (1)–(3).

Definition 2.2. Let M be an associative triple system and let M^{CT} be the comtrans algebra defined on the underlying vector space by the special commutator and special translator. We say that a comtrans algebra A is *special* if there exists an associative triple system M and an injective morphism $A \to M^{CT}$ of comtrans algebras; otherwise, A is *exceptional*. (This terminology is motivated by the definitions of special and exceptional Jordan algebras [23].)

Remark 2.3. Tercom (ternary commutator) algebras were introduced by Rossmanith & Smith [24]. They have trilinear operations $\lambda(x,y,z)$ and $\rho(x,y,z)$ which satisfy the following polynomial identities:

$$\lambda(x,y,z)+\lambda(y,x,z)=0, \qquad \rho(x,y,z)+\rho(x,z,y)=0,$$
$$\lambda(x,y,z)+\lambda(y,z,x)+\lambda(z,x,y)=\rho(x,y,z)+\rho(y,z,x)+\rho(z,x,y).$$

These identities hold for the left and right commutators $\lambda(x,y,z)=xyz-yxz$ and $\rho(x,y,z)=xyz-xzy$ in every associative triple system. Tercom algebras are equivalent to comtrans algebras in the sense that

$$[x,y,z]=\lambda(x,y,z),\ \langle x,y,z\rangle=\lambda(x,y,z)+\rho(y,x,z),\ \rho(x,y,z)=\langle y,x,z\rangle-[y,x,z].$$

Remark 2.4. The weakly anticommutative operation [8], which is equivalent to the operation $\{x,y,z\}=xyz+xzy-2zyx$, satisfies the symmetric sum identity in every associative triple system:

$$\sum_{\sigma\in S_3}\{x^\sigma,y^\sigma,z^\sigma\}=0.$$

This allows us to define special comtrans algebras in terms of a single operation:

$$[x,y,z]=\tfrac{1}{2}(\{z,x,y\}-\{z,y,x\}),$$
$$\langle x,y,z\rangle=\tfrac{1}{6}(4\{x,z,y\}+2\{y,z,x\}+\{z,x,y\}-\{z,y,x\}),$$
$$\{x,y,z\}=[y,z,x]+\langle x,y,z\rangle+\langle x,z,y\rangle.$$

Remark 2.5. Every special comtrans algebra becomes a special anti-Jordan triple system by means of the trilinear operation $(x,y,z)=[x,y,z]-\langle z,y,x\rangle=xyz-zyx$.

3. Algebraic operads

We consider operads in the symmetric monoidal category of vector spaces over a field of characteristic 0; equivalently, $\mathbb{Z}$-graded vector spaces concentrated in degree 0. We say *degree* instead of *arity* since our motivation comes from nonassociative algebra and we never refer to the homological degree (that is, all operations have homological degree 0). For background on algebraic operads, see [6].

3.1. **Basic definitions.** A monomial of weight w and degree $d(w)=2w{+}1$ in trilinear operations γ, δ can be represented as a complete ternary tree with w internal nodes each labelled by one of the operations and $d(w)$ leaves labelled by a permutation of the arguments $x_1,\dots,x_{d(w)}$.

Definition 3.1. Let $\mathcal{T}$ be the free weight-graded operad generated by ternary operations γ, δ with no symmetry, and let $\mathcal{T}(w)$ be the subspace of weight w. Let $\mathcal{A}$ be the weight-graded operad generated by a ternary operation α with no symmetry satisfying ternary associativity (Definition 1.4), and let $\mathcal{T}(w)$ be the subspace of weight w. We have (using the formula for ternary Catalan numbers)

$$\dim\mathcal{T}(w)=\frac{2^w}{2w{+}1}\binom{3w}{w}(2w{+}1)!\,, \qquad \dim\mathcal{A}(w)=(2w{+}1)!\,.$$

Since we usee the weight grading, $\mathcal{T}(w)$ and $\mathcal{A}(w)$ are modules over S_{2w+1}.

Definition 3.2. Since $\mathcal{T}$ is free, a morphism with domain $\mathcal{T}$ is determined by its values on γ, δ. We define the *expansion map* $X\colon \mathcal{T} \to \mathcal{A}$ by

$$X(\gamma) = \alpha - (12)\cdot\alpha, \quad X(\delta) = \alpha - (132)\cdot\alpha, \quad p \cdot x_1x_2x_3 = x_{p^{-1}(1)}x_{p^{-1}(2)}x_{p^{-1}(3)}.$$

Permutations act on positions of the arguments (not subscripts). Thus

$$X(\gamma)(x,y,z) = \alpha(x,y,z) - \alpha(y,x,z), \qquad X(\delta)(x,y,z) = \alpha(x,y,z) - \alpha(y,z,x).$$

We use the more convenient notation

$$\gamma(x,y,z) = [x,y,z], \qquad \delta(x,y,z) = \langle x,y,z\rangle, \qquad \alpha(x,y,z) = xyz.$$

Thus X may be written in terms of the special ternary commutator and translator:

$$[x,y,z] \xrightarrow{\;X\;} xyz - yxz, \qquad\qquad \langle x,y,z\rangle \xrightarrow{\;X\;} xyz - yzx.$$

We write X_w for the restriction of X to $\mathcal{T}(w)$.

Lemma 3.3. *We have $X_w(\mathcal{T}(w)) \subseteq \mathcal{A}(w)$ for $w \ge 0$. The kernel of X_w is the S_{2w+1}-submodule of $\mathcal{T}(w)$ of multilinear polynomial identities of degree $2w{+}1$ satisfied by the special commutator and translator in every associative triple system.*

$$\left[\begin{array}{cccccc|cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\hline
1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
-1 & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & -1 & \cdot & -1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & -1 & \cdot & -1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & -1 & \cdot & -1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & -1 & \cdot & -1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
-1 & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right]
\xrightarrow{\text{RCF}}
\left[\begin{array}{cccccc|cccccc}
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & -1 \\
\cdot & 1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & 1 & -1 & \cdot & 1 \\
\cdot & \cdot & 1 & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 & -1 & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & -1 & 1 & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1
\end{array}\right]$$

FIGURE 1. Computation of a Gröbner basis for the comtrans operad

3.2. **A Gröbner basis for the comtrans operad.** Relations (1)–(3) are linear (weight 1) relations between ternary operations[1]. For relations of weight 1, the computation of a Gröbner basis for the operad [6, 14] reduces to Gaussian elimination in the S_n-module of relations. Thus to find a Gröbner basis for the comtrans operad, we form the 18×12 matrix (Figure 1) whose 6×12 blocks contain all permutations of (1)–(3) with respect to the ordered basis $\langle x,y,z\rangle, \dots, \langle z,y,x\rangle$, $[x,y,z], \dots, [z,y,x]$. The nonzero rows of the row canonical form (RCF) represent the relations forming the Gröbner basis:

$$\begin{aligned}
&\langle x,y,z\rangle + \langle z,y,x\rangle + [y,x,z] - [z,y,x],\\
&\langle x,z,y\rangle - \langle z,x,y\rangle + \langle z,y,x\rangle + [y,x,z] + [z,x,y],\\
&\langle y,x,z\rangle + \langle z,x,y\rangle - [y,x,z] - [z,x,y],\\
&\langle y,z,x\rangle + \langle z,x,y\rangle - \langle z,y,x\rangle - [y,x,z] + [z,y,x],
\end{aligned}$$

[1]We thank Vladimir Dotsenko for clarifying some points in this subsection and the next.

$$[x,y,z]+[y,x,z], \quad [x,z,y]+[z,x,y], \quad [y,z,x]+[z,y,x].$$

Equivalently, in terms of rewrite rules, we have:

$$\begin{array}{lcl}
\langle x,y,z\rangle & \mapsto & -\langle z,y,x\rangle-[y,x,z]+[z,y,x], \\
\langle x,z,y\rangle & \mapsto & \langle z,x,y\rangle-\langle z,y,x\rangle-[y,x,z]-[z,x,y], \\
\langle y,x,z\rangle & \mapsto & -\langle z,x,y\rangle+[y,x,z]+[z,x,y], \\
\langle y,z,x\rangle & \mapsto & -\langle z,x,y\rangle+\langle z,y,x\rangle+[y,x,z]-[z,y,x], \\
[x,y,z] & \mapsto & -[y,x,z], \quad [x,z,y] \mapsto -[z,x,y], \quad [y,z,x] \mapsto -[z,y,x].
\end{array}$$

If x, y, z are multilinear monomials of arbitrary degree, and $x \prec y \prec z$ in the deglex extension of the operation order, then the rewrite rules show how to move greater factors to the left, to the extent allowed by the comtrans relations.

Conjecture 3.4. If $\mathcal{CT}$ is the weight-graded comtrans operad then

$$\dim \mathcal{CT}(w) = \frac{(3w)!}{w!\,(3!\,)^w} \cdot 5^w \qquad (w \geq 0).$$

Using terminology from nonassociative algebra, this is the dimension of the multilinear subspace of degree $2w{+}1$ in the free comtrans algebra on $2w{+}1$ generators.

We implemented the rewrite rules in Maple to verify the first four terms 1, 5, 250, 35000 by computing all multilinear normal forms in degrees $1,3,5,7$. The factor 5^w comes from the number of normal forms in weight 1 (degree 3). The other factor is OEIS sequence A025035 which counts (i) set partitions of $\{1,2,\dots,3w\}$ into parts of size 3, (ii) rooted phylogenetic (non-planar) complete ternary trees with w internal vertices, (iii) distinct multilinear monomials in degree $2w{+}1$ for a symmetric ternary operation: $(x^\sigma, y^\sigma, z^\sigma) = (x,y,z)$ for $\sigma \in S_3$.

3.3. **Polynomial identities and operadic Gröbner bases.** In principle, we can use Gröbner bases to determine polynomial identities satisfied by the special commutator and translator in every associative triple system. We describe the method briefly in this special case to illustrate the connection between polynomial identities and Gröbner bases for operads. (We will not use this method; instead, we apply computational techniques based on the LLL algorithm for lattice basis reduction [9] and the representation theory of the symmetric group [7].)

We express the expansion map (Definition 3.2) in terms of a single operad. Let $\mathcal{P}$ be the free symmetric operad generated by the ternary operations α, δ, γ with no symmetries. Let $\mathcal{O} = \mathcal{P}/\mathcal{I}$ where $\mathcal{I}$ is generated by these relations:

$$\gamma(x,y,z)+\gamma(y,x,z), \tag{4}$$
$$\delta(x,y,z)+\delta(y,z,x)+\delta(z,x,y), \tag{5}$$
$$\gamma(x,y,z)+\gamma(z,y,x)-\delta(x,y,z)-\delta(z,y,x), \tag{6}$$
$$\alpha(x,y,z)-\alpha(y,x,z)-\gamma(x,y,z), \tag{7}$$
$$\alpha(x,y,z)-\alpha(y,z,x)-\delta(x,y,z), \tag{8}$$
$$\alpha(v,\alpha(w,x,y),z)-\alpha(\alpha(v,w,x),y,z), \tag{9}$$
$$\alpha(v,w,\alpha(x,y,z))-\alpha(\alpha(v,w,x),y,z). \tag{10}$$

Relations (4)-(6) are the defining identities for comtrans algebras; (7)-(8) express the comtrans operations γ, δ in terms of the associative operation α; and (9)-(10) are the defining identities for associative triple systems.

The operation order $\alpha \succ \delta \succ \gamma$ extends to the revdeglex (reverse degree lexicographical) order on the tree monomials forming a basis of $\mathcal{P}$. If we compute the

Gröbner basis for $\mathcal{O}$ using this monomial order then we obtain tree polynomials of the form $f + g$ where f consists of the terms containing α (and possibly also δ, γ) and g consists of the terms not containing α. If the algorithm produces a polynomial $f + g$ with $f = 0$ then $\mathcal{I}$ contains a polynomial g involving only δ, γ which is identically 0 in $\mathcal{O}$. Hence g is a relation between δ, γ which belongs to the kernel of the expansion map, and so g is a polynomial identity relating δ, γ.

4. Identities of degree 3

Lemma 4.1. *Every multilinear identity satisfied by the special commutator and special translator in every associative triple system follows from* (1)–(3).

Proof. We follow [9]. For the expansion map $X_1 \colon \mathcal{T}(1) \to \mathcal{A}(1)$ in arity 3, the permutations $xyz, \dots, zyx$ in lex order form a basis of $\mathcal{A}(1)$, and the monomials $[x, y, z], \dots, [z, y, x], \langle x, y, z\rangle, \dots, \langle z, y, x\rangle$ form an ordered basis of $\mathcal{T}(1)$. The matrix representing X_1 has rank 5 and nullity 7:

$$[X_1] = \left[\begin{array}{cccccccccccc} 1 & \cdot & -1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot & -1 & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & 1 & \cdot & -1 & -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & 1 & \cdot & -1 & \cdot & \cdot & \cdot & 1 \end{array}\right]$$

We compute the Hermite normal form H of the transpose $[X_1]^t$ and an integer matrix U such that $\det(U) = \pm 1$ and $U[X_1]^t = H$. Rows 6-12 of H are zero and so rows 6-12 of U form a lattice basis N_1 for the integer nullspace of $[X_1]$:

$$N_1 = \left[\begin{array}{cccccccccccc} \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & -1 & 1 & \cdot & -1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & -1 & \cdot & 1 & -1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right]$$

The rows of N_1 have squared Euclidean lengths $2, 2, 2, 4, 4, 7, 7$ (sorted) with product 6272. We apply the LLL algorithm to the lattice L spanned by the rows of N_1, obtain a reduced basis of L, sort the vectors by increasing length, and multiply each row by ± 1 to make its leading entry positive:

$$N_2 = \left[\begin{array}{cccccccccccc} \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & -1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & -1 & \cdot & -1 & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & -1 & 1 & -1 & -1 & \cdot & \cdot & \cdot \end{array}\right]$$

The rows of N_2 have squared lengths $2, 2, 2, 3, 4, 4, 5$ with product 1920. The lattice L is a module over $\mathbb{Z}S_3$, where S_3 acts by permuting x, y, z in the ordered basis of $\mathcal{T}(1)$. Rows $1, 4, 7$ of N_2 are the lex-minimal subset which generates L as a $\mathbb{Z}S_3$-module. These rows represent relations (1)–(3). □

Remark 4.2. Consider the 40 pairs of trilinear operations $xyz \pm x^p y^p z^p$ and $xyz \pm x^q y^q z^q$ in an associative triple system where p, q are distinct non-identity permutations of x, y, z. In every case, a single operation generates the same S_3-module as the original pair. Up to equivalence there are four possibilities for this operation:

the translator $xyz - yzx$, the weakly commutative operation $xyz - xzy + 2zyx$, the weakly anticommutative operation $xyz + xzy - 2zyx$, and the original associative operation xyz.

5. Identities of degree 5

We first consider each operation separately and then the two operations together.

Lemma 5.1. *Every multilinear identity of degree 5 satisfied by the special commutator in every associative triple system follows from relation* (1) *and*

$$T_{x,z}(v,w,y) + T_{x,z}(w,y,v) + T_{x,z}(y,v,w) = 0, \tag{11}$$

where $T_{x,z}(v,w,y) = [[v,w,x],y,z] - [v,w,[x,y,z]]$. *(This simplifies the* $q = 2$ *case of the deformed anti-Jordan triple product in [8].)*

Proof. It is easy to check by hand that (11) is satisfied by the special commutator. We must verify that every identity in degree 5 satisfied by the special commutator follows from (1) and (11). We use computer algebra [9].

Let $\mathcal{T}$ be the free weight-graded operad generated by one ternary operation $[-,-,-]$ with no symmetry. The homogeneous space $\mathcal{T}(2)$ is isomorphic to the direct sum of three copies of the regular representation $\mathbb{F}S_5$ corresponding to the three association types $[[-,-,-],-,-]$, $[-,[-,-,-],-]$, $[-,-,[-,-,-]]$ in that order. Each association type has 120 permutations of v, w, x, y, z in lex order.

Each of these nonassociative monomials expands using the special commutator to a linear combination of monomials in the operad $\mathcal{A}$ of Definition 3.1: for example,

$$[[v,w,x],y,z] \;\mapsto\; vwxyz - wvxyz - yvwxz + ywvxz.$$

We construct the 120×360 matrix M whose (i,j) entry is the coefficient of the i-th associative monomial in the expansion of the j-th nonassociative monomial.

We compute the Hermite normal form H of the transpose M^t and an integer matrix U with $\det(U) = \pm 1$ and $UM^t = H$. We find that H has rank 70 so the bottom 290 rows of H are 0. Hence the bottom 290 rows of U, denoted N, form an integer basis for the nullspace of M; that is, the kernel of the expansion map X_5. The largest squared Euclidean length of the rows of N is 49352.

We apply the LLL algorithm to the rows of N and obtain a matrix N' whose rows generate the same lattice. The largest squared Euclidean length of the rows of N' is only 6. Let N'' consist of the rows of N' sorted by increasing length.

Let $I(x,y,z) = [x,y,z] + [y,x,z]$ denote relation (1). We generate all consequences of I in degree 5 by partial compositions in the operad; that is, substituting $[-,-,-]$ into I, or substituting I into $[-,-,-]$:

$$\begin{array}{lll} I([x,v,w],y,z), & I(x,[y,v,w],z), & I(x,y,[z,v,w]), \\ {[I(x,y,z),v,w]}, & [v,I(x,y,z),w], & [v,w,I(x,y,z)]. \end{array} \tag{12}$$

These relations generate the S_5-module of identities which follow from (1).

We construct the 720×360 matrix C whose rows contain all permutations of the relations (12). We compute the RCF and find that C has rank 270. Since N has rank 290, there is a 20-dimensional quotient S_5-module of new identities.

For each row i of N'', we stack C on top of the matrix containing all permutations of the relation represented by row i. Row 194 is the first which increases the rank from 270 to 290. This row is the coefficient vector of relation (11). ∎

Remark 5.2. The special commutator is equivalent to $xyz + xzy - yxz + yzx - zxy - zyx$, case $q = 2$ of the deformed anti-Jordan triple product [8]. This follows from the equality of the RCFs of the representation matrices for the operations:

$$\left[0, \left[\begin{array}{cc} 1 & 2 \\ 0 & 0 \end{array} \right], 1 \right]$$

Relation (11) is much simpler than the relation with 24 terms in [8, equation (55)].

Lemma 5.3. *Every multilinear identity in degree 5 satisfied by the special translator in every associative triple system follows from relation* (2) *and*

$$R_{y,z}(\langle v,w,x \rangle) = \langle R_{y,z}(v), w, x \rangle + \langle v, R_{y,z}(w), x \rangle + \langle v, w, R_{y,z}(x) \rangle, \tag{13}$$

where $R_{y,z}(u) = \langle u, y, z \rangle$*. (Compare the result for the cyclic commutator in [8].)*

Proof. Similar to the proof of Lemma 5.1. □

Theorem 5.4. *Every multilinear identity in degree 5 satisfied by the special commutator and special translator in every associative triple system follows from* (1)*–*(3)*,* (11)*,* (13)*, and the following new identities involving both operations:*

$$[[vwx]yz] + [[xvy]wz] - [\langle vwx \rangle yz] + [\langle vyw \rangle xz] - [\langle xyw \rangle vz] = 0, \tag{14}$$

$$[[vwx]yz] - [[vyw]xz] + \langle [wvz]xy \rangle + [\langle vyw \rangle xz] + [vw\langle zxy \rangle] = 0, \tag{15}$$

$$\begin{aligned} &[[vwx]yz] + [[vwz]xy] - [[xwy]vz] - [[zwy]xv] + \langle [wvz]xy \rangle \\ &+ \langle [xwz]yv \rangle + \langle [zwy]xv \rangle + [\langle wvx \rangle yz] + [\langle wvz \rangle xy] - [\langle wyv \rangle xz] \\ &- \langle \langle wvz \rangle xy \rangle + \langle \langle wyv \rangle xz \rangle + \langle wx \langle zyv \rangle \rangle = 0. \end{aligned} \tag{16}$$

Proof. Let $\mathcal{T}$ be the free operad generated by two ternary operations $[-,-,-]$ and $\langle -,-,- \rangle$ with no symmetry. The homogeneous space $\mathcal{T}(2)$ is isomorphic to the direct sum of 12 copies of the regular representation $\mathbb{F}S_5$ corresponding to the association types ordered as follows:

$$\begin{array}{llll} [[-,-,-],-,-], & \langle [-,-,-],-,- \rangle, & [\langle -,-,- \rangle,-,-], & \langle \langle -,-,- \rangle,-,- \rangle, \\ [-,[-,-,-],-], & \langle -,[-,-,-],- \rangle, & [-,\langle -,-,- \rangle,-], & \langle -,\langle -,-,- \rangle,- \rangle, \\ [-,-,[-,-,-]], & \langle -,-,[-,-,- \rangle, & [-,-,\langle -,-,- \rangle], & \langle -,-,\langle -,-,- \rangle \rangle. \end{array}$$

Each association type has 120 permutations of v, w, x, y, z in lex order. We construct the 120×1440 matrix M in which the (i,j) entry is the coefficient of the i-th associative monomial in the expansion of the j-th nonassociative monomial. There are 36 consequences of identities (1)–(3) since we can substitute either of two operations. We must exclude these consequences together with all permutations of relations (11) and (13). The rest is similar to the proof of Lemma 5.1. □

Definition 5.5. A *Smith algebra* is a comtrans algebra satisfying the five identities of Lemmas 5.1, 5.3 and Theorem 5.4. (Since these identities have weights 1 and 2, the corresponding operad is quadratic and hence has a Koszul dual.)

Remark 5.6. The weakly anticommutative operation satisfies a 141-dimensional S_5-module of multilinear identities which are not consequences of the symmetric sum identity (Remark 2.4). If $T_{yz}(u) = \{u,y,z\} + \{u,z,y\}$ then the simplest new identity that we found (which however does not generate all new identities) is

$$T_{yz}(\{v,w,x\}) = \{v, T_{yz}(w), x\} + \{v, T_{yz}(w), x\} + \{v, w, T_{yz}(x)\}.$$

6. Identities of degree 7

Every special comtrans algebra is a Smith algebra, but the converse is false: we demonstrate the existence of identities in degree 7 satisfied by every special comtrans algebra but not by every Smith algebra. There are new identities for each operation separately, but none relating the operations.

For degree 7, the methods of the previous sections are impractical: for one (resp. two) operation(s), there are 12 (resp. 96) association types and 60480 (resp. 483840) multilinear monomials. We use a constructive version of the representation theory of the symmetric group to decompose the kernel of the expansion map X_7 into isotypic components corresponding to the partitions of 7. We provide only an outline; these methods have been discussed in detail in previous papers [3, 7, 10]. We order the 12 association types for one ternary operation as follows:

$$
\begin{array}{lll}
[[[-,-,-],-,-],-,-], & [[-,[-,-,-],-],-,-], & [[-,-,[-,-,-]],-,-], \\
[[-,-,-],[-,-,-],-], & [[-,-,-],-,[-,-,-]], & [-,[[-,-,-],-,-],-], \\
[-,[-,[-,-,-],-],-], & [-,[-,-,[-,-,-]],-], & [-,[-,-,-],[-,-,-]], \\
[-,-,[[-,-,-],-,-]], & [-,-,[-,[-,-,-],-]], & [-,-,[-,-,[-,-,-]]].
\end{array}
\tag{17}
$$

Recall that an identity is *new* if it does not follow from those of lower degree.

Lemma 6.1. *The S_7-module of new identities for the special commutator is nonzero, and has the following decomposition into isotypic components:*

$$[43] \oplus [421]^2 \oplus [41^3] \oplus [3^21]^3 \oplus [32^2]^2 \oplus [321^2]^5 \oplus [31^4]^2 \oplus [2^31]^2 \oplus [2^21^3]^3 \oplus [21^5].$$

We write $[\lambda]^m$ if the irreducible representation for partition λ has multiplicity m.

Proof. There are six consequences (12) in degree 5 of identity (1). Combining these with (11), we obtain seven identities $J(v,w,x,y,z)$ each of which produces eight consequences $K(t,\dots,z)$ in degree 7; these can be expressed using partial compositions as $J \circ_k \gamma$ $(1 \le k \le 5)$ and $\gamma \circ_k J$ $(1 \le k \le 3)$ where γ denotes the commutator. For each partition λ of 7 we write $R_\lambda \colon \mathbb{Q}S_7 \to M_{d_\lambda}(\mathbb{Q})$ for the corresponding irreducible representation; d_λ is the number of standard tableaux of shape λ. For each $p \in S_7$ we use Clifton's algorithm [7, 13] to compute $R_\lambda(p)$. For each λ we construct a $56d_\lambda \times 12d_\lambda$ matrix C_λ partitioned into $d_\lambda \times d_\lambda$ blocks. Each consequence K_i $(1 \le i \le 56)$ is a sum of 12 components $K_i^1, \dots K_i^{12}$ corresponding to the association types (17). Block (i,j) of C_λ contains $R_\lambda(K_i^j)$. We compute the RCF of C_λ; its rank c_λ is the multiplicity of representation $[\lambda]$ in the S_7-module of all consequences of the identities of lower degree (Figure 2).

We substitute the identity permutation of $t, u, \dots, z$ into each association type, and obtain monomials $\xi_1, \dots, \xi_{12}$ which we expand using the special commutator into a linear combination of eight associative monomials. For each λ we construct a $d_\lambda \times 12d_\lambda$ matrix E_λ: one row of $d_\lambda \times d_\lambda$ blocks in which block j is $R_\lambda(\xi_j)^t$. We compute the RCF of E_λ and find its rank e_λ; then $a_\lambda = 12d_\lambda - e_\lambda$ is the multiplicity of $[\lambda]$ in the kernel of the expansion map (Figure 2). We compute the $a_\lambda \times 12d_\lambda$ matrix N_λ in RCF whose row space is the nullspace of E_λ, and check that the row space of N_λ contains the row space of C_λ. The difference $n_\lambda = a_\lambda - c_\lambda$ is the multiplicity of $[\lambda]$ in the quotient module N_λ / C_λ (we identify each matrix with its row space) of new identities in degree 7 for partition λ (Figure 2). □

λ	7	61	52	51^2	43	421	41^3	3^21	32^2	321^2	31^4	2^31	2^21^3	21^5	1^7
d_λ	1	6	14	15	14	35	20	21	21	35	15	14	14	6	1
c_λ	12	71	162	173	157	394	225	231	233	385	166	153	152	66	11
a_λ	12	71	162	173	158	396	226	234	235	390	168	155	155	67	11
n_λ	0	0	0	0	1	2	1	3	2	5	2	2	3	1	0

FIGURE 2. Multiplicities in degree 7 for the ternary commutator

Lemma 6.2. *The S_7-module of new identities for the special translator is nonzero and has the following decomposition into isotypic components:*

$$[52] \oplus [51^2] \oplus [43] \oplus [421]^3 \oplus [41^3]^2 \oplus [3^21]^2 \oplus [32^2]^2 \oplus [321^2]^3 \oplus [31^4] \oplus [2^31] \oplus [2^21^3].$$

Proof. Similar to the proof of Lemma 6.1; see Figure 3. □

λ	7	61	52	51^2	43	421	41^3	3^21	32^2	321^2	31^4	2^31	2^21^3	21^5	1^7
d_λ	1	6	14	15	14	35	20	21	21	35	15	14	14	6	1
c_λ	12	68	156	168	155	388	222	232	232	388	168	155	156	68	12
a_λ	12	68	157	169	156	391	224	234	234	391	169	156	157	68	12
n_λ	0	0	1	1	1	3	2	2	2	3	1	1	1	0	0

FIGURE 3. Multiplicities in degree 7 for the ternary translator

For two ternary operations we have $2^3 \cdot 12 = 96$ association types: in each type for one operation (17), each operation may be replaced by either of two operations.

Theorem 6.3. *Every identity in degree 7 satisfied by the special commutator and translator in every associative triple system follows from* (1)–(3), (11), (13), *and* (14)–(16)*, together with the new identities for the commutator and translator separately whose existence is demonstrated by Lemmas 6.1 and 6.2. That is, special comtrans algebras satisfy no new identities in which every term has both operations.*

Proof. Each identity J in equations (1)–(3) produces 12 consequences in degree 5: the partial compositions $J \circ_k \gamma$, $J \circ_k \delta$ $(1 \le k \le 3)$, $\gamma \circ_k J$, $\gamma \circ_k J$ $(1 \le k \le 3)$, where γ, δ denote commutator and translator. Including (11), (13), (14)–(16), we obtain 41 consequences K in degree 5. Each of these produces 16 consequences in degree 7: $K \circ_k \gamma$, $K \circ_k \delta$ $(1 \le k \le 5)$, $\gamma \circ_\ell K$, $\gamma \circ_\ell K$ $(1 \le \ell \le 3)$. For each partition λ of 7, the matrix C_λ representing the consequences has size $656 d_\lambda \times 96 d_\lambda$. Let D^1_λ, D^2_λ be obtained from the matrices C_λ in the proofs of Lemmas 6.1, 6.2 by embedding the 12 association types for one operation into the 96 association types for two operations. The row space of D^1_λ (resp. D^2_λ) contains the identities satisfied by the commutator (resp. translator) in degree 7. We stack C_λ on top of D^1_λ and D^2_λ to obtain CD_λ; we denote its rank by c_λ. The expansion matrix E_λ has size $d_\lambda \times 96 d_\lambda$; we denote its rank by e_λ, so it has nullity $a_\lambda = 96 d_\lambda - e_\lambda$. Let N_λ be the matrix whose row space is the nullspace of E_λ. For each partition λ, we find that $c_\lambda = a_\lambda$ and that the RCFs of C_λ and N_λ coincide. □

7. ENVELOPING ALGEBRAS FOR 2 × 2 MATRICES

In this final section we use noncommutative Gröbner bases [5] to construct the universal associative enveloping algebras of the nonassociative triple systems A^C,

A^T, and A^{CT}, obtained by applying the special commutator, special translator, and both together, to the associative triple system $A = (a_{ij})$ of 2×2 matrices. This method has previously been used for the universal envelopes of triple systems [16] obtained by applying trilinear operations [8] to the 2×2 matrices with $a_{11} = a_{22} = 0$, and for infinite families of simple anti-Jordan triple systems [15, 17]. This approach to the representation theory of comtrans algebras is based on the special commutator and special translator in an associative triple system, and therefore differs essentially from the approach of Im, Shen & Smith [19, 20, 25].

Definition 7.1. Let A be the associative triple system of 2×2 matrices. Let B and X be the basis of matrix units and a set of symbols in bijection with B:

$$B = \{E_{ij} \mid 1 \le i, j \le 2\}, \qquad X = \{e_{ij} \mid 1 \le i, j \le 2\}, \qquad \eta(E_{ij}) = e_{ij}.$$

Extend η linearly to the injective map $\eta\colon A \to F\langle X\rangle$ where $F\langle X\rangle$ is the free associative algebra generated by X. Define three ideals in $F\langle X\rangle$ as follows:

$$\begin{aligned} I^C &= \langle G^C\rangle, & G^C &= \{e_{ij}e_{kl}e_{st} - e_{kl}e_{ij}e_{st} - \eta([E_{ij}, E_{kl}, E_{st}])\}, \\ I^T &= \langle G^T\rangle, & G^T &= \{e_{ij}e_{kl}e_{st} - e_{kl}e_{st}e_{ij} - \eta(\langle E_{ij}, E_{kl}, E_{st}\rangle)\}, \\ I^{CT} &= \langle G^{CT}\rangle, & G^{CT} &= G^T \cup G^C. \end{aligned}$$

The corresponding universal associative enveloping algebras are

$$U(A^C) = F\langle X\rangle/I^C, \qquad U(A^T) = F\langle X\rangle/I^T, \qquad U(A^{CT}) = F\langle X\rangle/I^{CT}.$$

We write $e_{ij} \prec e_{kl}$ if $i < k$, or $i = k$, $j < l$; and also $e_{11}, e_{12}, e_{21}, e_{22} = a, b, c, d$.

Lemma 7.2. *The universal enveloping algebra $U(A^C)$ has basis*

$$\mathfrak{B}^C = \{\, 1,\, a,\, b,\, c,\, d,\, a^2,\, ab,\, ca,\, cb \,\}.$$

Proof. The set G^T contains 64 elements; after putting each generator in standard form (reverse deglex order and monic), only 24 distinct generators remain:

$$\begin{array}{llll} G_1 = ba^2-aba, & G_2 = bab-ab^2, & G_3 = bac-abc+a, & G_4 = bad-abd+b, \\ G_5 = ca^2-aca-c, & G_6 = cab-acb-d, & G_7 = cac-ac^2, & G_8 = cad-acd, \\ G_9 = cba-bca+a, & G_{10} = cb^2-bcb+b, & G_{11} = cbc-bc^2-c, & G_{12} = cbd-bcd-d, \\ G_{13} = da^2-ada, & G_{14} = dab-adb, & G_{15} = dac-adc, & G_{16} = dad-ad^2, \\ G_{17} = dba-bda, & G_{18} = db^2-bdb, & G_{19} = dbc-bdc+a, & G_{20} = dbd-bd^2+b, \\ G_{21} = dca-cda-c, & G_{22} = dcb-cdb-d, & G_{23} = dc^2-cdc, & G_{24} = dcd-cd^2. \end{array}$$

There are only 56 distinct normal forms of the compositions of these generators; the corresponding compositions appear in Figure 4.

To compute normal forms we eliminate all occurrences of leading monomials of the generators. We write $\equiv$ for congruence modulo $G_1, \ldots, G_{24}$. For example, to find the normal form of $\mathcal{S}_1$, we eliminate the leading monomials of G_5, G_1, G_3:

$$\begin{aligned} \mathcal{S}_1 &= (bac - abc + a)a^2 - ba(ca^2 - aca - c) = (-abc + a)a^2 - ba(-aca - c) \\ &\equiv -ab(aca + c) + a^3 + (aba)ca + bac \\ &\equiv -a((abc - a)a + bc) + a^3 + a(abc - a)a + (abc - a) = a^3 - a. \end{aligned}$$

$$
\begin{array}{llll}
\mathcal{S}_1 = G_3a^2 - baG_5, & \mathcal{S}_2 = G_3ab - baG_6, & \mathcal{S}_3 = G_3ac - baG_7, & \mathcal{S}_4 = G_3ad - baG_8, \\
\mathcal{S}_5 = G_3ba - baG_9, & \mathcal{S}_6 = G_3b^2 - baG_{10}, & \mathcal{S}_7 = G_3bc - baG_{11}, & \mathcal{S}_8 = G_3bd - baG_{12}, \\
\mathcal{S}_9 = G_4ba - baG_{17}, & \mathcal{S}_{10} = G_4b^2 - baG_{18}, & \mathcal{S}_{11} = G_4bc - baG_{19}, & \mathcal{S}_{12} = G_4bd - baG_{20}, \\
\mathcal{S}_{13} = G_4ca - baG_{21}, & \mathcal{S}_{14} = G_4cb - baG_{22}, & \mathcal{S}_{15} = G_4c^2 - baG_{23}, & \mathcal{S}_{16} = G_4cd - bsG_{24}, \\
\mathcal{S}_{17} = G_7a^2 - caG_5, & \mathcal{S}_{18} = G_7ab - caG_6, & \mathcal{S}_{19} = G_7ac - caG_7, & \mathcal{S}_{20} = G_7ad - caG_8, \\
\mathcal{S}_{21} = G_7ba - caG_9, & \mathcal{S}_{22} = G_7b^2 - caG_{10}, & \mathcal{S}_{23} = G_7bc - caG_{11}, & \mathcal{S}_{24} = G_7bd - caG_{12}, \\
\mathcal{S}_{25} = G_8ba - caG_{17}, & \mathcal{S}_{26} = G_8b^2 - caG_{18}, & \mathcal{S}_{27} = G_8bc - caG_{19}, & \mathcal{S}_{28} = G_8bd - caG_{20}, \\
\mathcal{S}_{29} = G_9a - cG_1, & \mathcal{S}_{30} = G_9b - cG_2, & \mathcal{S}_{31} = G_9c - cG_3, & \mathcal{S}_{32} = G_9d - cG_4, \\
\mathcal{S}_{33} = G_{11}ba - cbG_9, & \mathcal{S}_{34} = G_{11}b^2 - cbG_{10}, & \mathcal{S}_{35} = G_{11}bc - cbG_{11}, & \mathcal{S}_{36} = G_{11}bd - cbG_{12}, \\
\mathcal{S}_{37} = G_{12}a^2 - cbG_{13}, & \mathcal{S}_{38} = G_{12}ab - cbG_{14}, & \mathcal{S}_{39} = G_{12}ac - cbG_{15}, & \mathcal{S}_{40} = G_{12}ad - cbG_{16}, \\
\mathcal{S}_{41} = G_{12}ba - cbG_{17}, & \mathcal{S}_{42} = G_{12}b^2 - cbG_{18}, & \mathcal{S}_{43} = G_{12}bc - cbG_{19}, & \mathcal{S}_{44} = G_{12}bd - cbG_{20}, \\
\mathcal{S}_{45} = G_{17}a - dG_1, & \mathcal{S}_{46} = G_{17}b - dG_2, & \mathcal{S}_{47} = G_{21}a - dG_5, & \mathcal{S}_{48} = G_{21}b - dG_6, \\
\mathcal{S}_{59} = G_{21}c - dG_7, & \mathcal{S}_{50} = G_{21}d - dG_8, & \mathcal{S}_{51} = G_{22}a^2 - dcG_1, & \mathcal{S}_{52} = G_{22}ab - dcG_2, \\
\mathcal{S}_{53} = G_{22}ac - dcG_3, & \mathcal{S}_{54} = G_{22}ad - dcG_4, & \mathcal{S}_{55} = G_{22}a - dG_9, & \mathcal{S}_{56} = G_{22}b - dG_{10}.
\end{array}
$$

FIGURE 4. Compositions of the generators for the ideal $I^C \subset F\langle X\rangle$

Similar calculations produce the normal forms of $\mathcal{S}_2, \dots \mathcal{S}_{56}$:

$$
\begin{array}{l}
a^2b - b,\ a^2c,\ a^2d,\ aba,\ ab^2,\ abc - a,\ abd - b,\ b^2a,\ b^3,\ b^2c,\ b^2d,\ bca - ada - a, \\
bcb - adb - b,\ bc^2 - adc,\ bcd - ad^2,\ c^2a,\ c^2b,\ c^3,\ c^2d,\ cda,\ cdb,\ cdc,\ cd^2,\ d^2a,\ d^2b, \\
d^2c - c,\ d^3 - d,\ da - bc + aa,\ db - bd + ab,\ dc - ca + 3ac,\ dd - cb + 3ad,\ bca - a,\ bcb - b, \\
bc^2,\ bcd,\ d^2a - bca + ada + a,\ d^2b - bcb + adb + b,\ d^2c - bc^2 + adc - c,\ d^3 - bcd + ad^2 - d, \\
bda,\ bdb,\ bdc - a,\ bd^2 - b,\ ba,\ b^2,\ dc - ca + ac,\ d^2 - cb + ad,\ c^2,\ cd, \\
d^2a - 3ada + abc - a^3,\ d^2b - 3adb + abd - a^2b,\ d^2c - adc + aca - a^2c - c, \\
d^3 - ad^2 + acb - a^2d - d,\ da - \frac{1}{3}bc + \frac{1}{3}a^2,\ db - \frac{1}{3}bd + \frac{1}{3}ab.
\end{array}
$$

Including these normal forms with the original 24 generators, and then self-reducing, produces the following 16 generators:

$$
\begin{array}{llllllll}
ac, & ad, & ba, & b^2, & bc - a^2, & bd - ab, & c^2, & cd, \\
da, & db, & dc - ca, & d^2 - cb, & a^3 - a, & a^2b - b, & ca^2 - c, & cab - d.
\end{array}
\tag{18}
$$

All compositions of these 16 generators reduce to 0, and so we have a Gröbner basis for I^C. A basis for $U(A^C)$ consists of the cosets of those monomials which are not divisible by the leading monomial of any element of the Gröbner basis. □

Lemma 7.3. *In $U(A^C)$ we have the relations*

$$
\begin{array}{l}
ac = ad = ba = b^2 = cd = da = db = c^2 = 0, \\
bc = a^2,\ bd = ab,\ dc = ca,\ d^2 = cb,\ a^3 = a,\ a^2b = b,\ ca^2 = c,\ cab = d.
\end{array}
$$

Proof. This follows immediately from the Gröbner basis (18). □

Lemma 7.4. *The nonzero structure constants of $U(A^C)$ are*

$$
\begin{array}{llllll}
a \cdot a = a^2, & a \cdot b = ab, & a \cdot a^2 = a, & a \cdot ab = b, & b \cdot c = a^2, & b \cdot d = ac, \\
b \cdot ca = a, & b \cdot cb = b, & c \cdot a = ca, & c \cdot b = cb, & c \cdot a^2 = c, & c \cdot ab = d, \\
d \cdot c = ca, & d \cdot d = cb, & d \cdot ca = c, & d \cdot cb = d, & a^2 \cdot a = a, & a^2 \cdot b = b, \\
a^2 \cdot a^2 = a^2, & a^2 \cdot ab = ab, & ab \cdot c = a, & ab \cdot d = b, & ab \cdot ca = a^2, & ab \cdot cb = ab, \\
ca \cdot a = c, & ca \cdot b = d, & ca \cdot a^2 = ca, & ca \cdot ab = cb, & cb \cdot c = c, & ca \cdot d = d, \\
cb \cdot ca = ca, & cb \cdot cb = cb. & & & &
\end{array}
$$

Proof. This follows immediately from Lemma 7.3. □

Theorem 7.5. *The Wedderburn decomposition of* $U(A^C)$ *is*

$$U(A^C) = \mathbb{Q} \oplus M_{2\times 2}(\mathbb{Q}) \oplus M_{2\times 2}(\mathbb{Q}),$$

where $M_{2\times 2}(\mathbb{Q})$ *is the ordinary associative algebra of* 2×2 *matrices.*

Proof. Following [4], we first verify that the radical is zero and hence $U(A^C)$ is semisimple. The center $Z(U(A^C))$ has this basis and structure constants:

$$z_1 = 1, \qquad z_2 = a + d, \qquad z_3 = a^2 + cb,$$
$$z_1 \cdot z_1 = z_1, \;\; z_1 \cdot z_2 = z_1, \;\; z_1 \cdot z_3 = z_3, \;\; z_2 \cdot z_2 = z_3, \;\; z_2 \cdot z_3 = z_2, \;\; z_3 \cdot z_3 = z_3.$$

The minimal polynomial of z_3 is $t^2 - t$ and hence $Z(U(A^C)) = J \oplus K$ where $J = \langle z_3 - z_1 \rangle$ and $K = \langle z_3 \rangle$. We have $\dim J = 1$ with basis $z_3 - z_1$, and $\dim K = 2$ with basis z_2, z_3. In K the identity element is z_3, and the minimal polynomial of z_2 is $t^2 - z_3$. Hence K splits into 1-dimensional ideals with bases $z_2 - z_3$ and $z_2 + z_3$. After scaling, we obtain a basis of orthogonal idempotents for $Z(U(A^C))$:

$$e_1 = z_1 - z_3, \qquad e_2 = \tfrac{1}{2}(z_2 - z_3), \qquad e_3 = \tfrac{1}{2}(z_2 + z_3).$$

These elements of the center correspond to the following elements of $U(A^C)$:

$$e_1 = 1 - a^2 + cb, \qquad e_2 = \tfrac{1}{2}(a + d - a^2 + cb), \qquad e_3 = \tfrac{1}{2}(a + d + a^2 + cb).$$

The ideals in $U(A^C)$ generated by e_1, e_2, e_3 have dimensions 1, 4, 4 respectively, which completes the proof. (We omit the construction of an isomorphism between each simple two-sided ideal in $U(A^C)$ and the corresponding matrix algebra.) □

Lemma 7.6. *The universal enveloping algebra* $U(A^T)$ *has basis*

$$\mathfrak{B}^T = \{\, a^m,\, b,\, c,\, ab,\, ca,\, cb,\, a^n d \mid m \ge 0,\, 0 \le n \le 2 \,\}.$$

Proof. The set G^T contains 64 elements; we put each generator in standard form (reverse deglex order and monic) and self-reduce the set, obtaining 40 generators:

$$\begin{array}{lllll}
aba{-}a^2b{+}b, & aca{-}a^2c, & ada{-}a^2d, & ba^2{-}a^2b{+}b, & bab{-}ab^2, \\
bac{-}acb, & bad{-}adb, & b^2a{-}ab^2, & bca{-}abc, & bcb{-}b^2c{-}b, \\
bda{-}abd{+}b, & bdb{-}bbd, & ca^2{-}a^2c{-}c, & cab{-}abc{-}d{+}a, & cac{-}ac^2, \\
cad{-}adc, & cba{-}acb, & cb^2{-}b^2c, & cbc{-}bc^2{-}c, & cbd{-}bdc{-}d{+}a, \\
c^2a{-}ac^2, & c^2b{-}bcc, & cda{-}acd, & cdb{-}bcd, & cdc{-}c^2d, \\
da^2{-}a^2d, & dab{-}abd{+}b, & dac{-}acd, & dad{-}ad^2, & dba{-}adb, \\
db^2{-}b^2d, & dbc{-}bcd, & dbd{-}bd^2{+}b, & dca{-}adc{-}c, & dcb{-}bdc{-}d{+}a, \\
dc^2{-}c^2d, & dcd{-}cd^2, & d^2a{-}ad^2, & d^2b{-}bd^2{+}b, & d^2c{-}cd^2{-}c.
\end{array}$$

There are 143 nontrivial compositions of these generators; including their normal forms and self-reducing produces 16 generators:

$$\begin{array}{l}
ac, \quad ba, \quad b^2, \quad bc{+}ad{-}a^2, \quad bd{-}ab, \quad c^2, \quad cd, \\
da{-}ad, \quad db, \quad dc{-}ca, \quad d^2{-}cb{-}ad, \quad a^2b{-}b, \quad ca^2{-}c, \\
cab{+}a^2d{-}a^3{-}d{+}a, \quad cad, \quad a^3d{-}a^4{-}ad{+}a^2.
\end{array} \tag{19}$$

All compositions of these generators reduce to 0, so we have a Gröbner basis of I^T. A basis for $U(A^T)$ consists of the cosets of those monomials which are not divisible by the leading monomial of any element of the Gröbner basis. □

Lemma 7.7. *In* $U(A^T)$, *we have the relations*

$$\begin{array}{l}
ac = ba = b^2 = c^2 = cd = db = cad = 0, \\
bc = -ad + a^2, \quad bd = ab, \quad da = ad, \quad dc = ca, \quad d^2 = cb + ad, \\
a^2b = b, \quad ca^2 = c, \quad cab = -a^2d + a^3 + d - a, \quad a^3d = a^4 + ad - a^2.
\end{array}$$

Proof. This follows immediately from the Gröbner basis (19). □

Theorem 7.8. *For $m, l \geq 0$ and $0 \leq n \leq 2$, the structure constants of $U(A^T)$ are*

$$a^m \cdot a^l = a^{m+n}, \qquad a^m \cdot b = \begin{cases} ab & (m \text{ odd}) \\ b & (m \text{ even}), \end{cases} \tag{20}$$

$$a^m \cdot a^n d = \begin{cases} a^{m+n} d & (m{+}n < 3) \\ a^{m+n+1} + a^{m+n-2} d - a^{m+n-1} & (m{+}n \geq 3), \end{cases} \tag{21}$$

$$b \cdot c = -ad + a^2, \qquad b \cdot ca = -a^2 d + a^3, \qquad b \cdot cb = b, \qquad b \cdot d = ab,$$

$$c \cdot a^m = \begin{cases} ca & (m \text{ odd}) \\ c & (m \text{ even}), \end{cases} \qquad c \cdot b = cb, \qquad c \cdot (ab) = -a^2 d + a^3 + d - a,$$

$$ab \cdot c = -a^2 d + a^3, \qquad ab \cdot ca = -ad + a^2, \qquad ab \cdot cb = ab,$$

$$ca \cdot b = -a^2 d + a^3 + d - a, \qquad ca \cdot ab = cb,$$

$$cb \cdot c = c, \qquad cb \cdot ca = ca, \qquad cb \cdot cb = cb, \qquad cb \cdot d = -a^2 d + a^3 + d - a,$$

$$a^n d \cdot a^m = \begin{cases} a^{m+n} d & (m{+}n < 3) \\ a^{m+n+1} + a^{m+n-2} d - a^{m+n-1} & (m{+}n \geq 3), \end{cases}$$

$$d \cdot c = ca, \qquad d^2 = cb + ad, \qquad d \cdot ca = c, \qquad d \cdot cb = -a^2 d + a^3 + d - a,$$

$$ad \cdot ca = ac, \qquad ad \cdot d = ad^2, \qquad ad \cdot ad = a^4 + ad - a^2,$$

$$ad \cdot a^2 d = a^5 - a^3 + a^2 d, \qquad a^2 d \cdot a^2 d = a^6 + ad - a^2. \tag{22}$$

Proof. The first relation of (20) is clear. For the second we use induction on m. The claim is clear for $0 \leq m \leq 1$. For $m = 2$ we have $a^2 b = b$ by Lemma 7.7. If m is odd (resp. even) then $m{-}1$ is even (resp. odd) and hence $a^m \cdot b = a(a^{m-1} b) = ab$ and $a(a^{m-1} b) = a(ab) = a^2 b = b$ by induction. Relation (21) is clear for $m{+}n < 3$. We assume $m{+}n \geq 3$ and use induction on m. For $m = 1$, Lemma 7.7 gives

$$a \cdot a^2 d = a^3 d = a^4 + ad - a^2.$$

For $m{+}n > 3$, the inductive hypothesis implies

$$\begin{aligned} a^m \cdot a^n d &= a \cdot a^{m-1} a^n d = a \cdot (a^{m+n} + a^{m+n-3} d - a^{m+n-2}) \\ &= a^{m+n+1} + a^{m+n-2} d - a^{m+n-1}. \end{aligned}$$

The proofs of the unnumbered relations are similar. For relations (22) we have

$$\begin{aligned} ad \cdot a^2 d &= adaad = aadad = aaadd = a^3 d^2 = (a^4 + ad - a^2) d \\ &= a^4 d + a(cb + ad) - a^2 d = a(a^4 + ad - a^2) + a(cb + ad) - a^2 d \\ &= a^5 + a^2 d - a^3 + acb = a^5 + a^2 d - a^3, \\ a^2 d \cdot a^2 d &= aadaad = aaadad = aaaadd = a^4 d^2 = a(a^4 + ad - a^2) d \\ &= a^5 d + a^2 d^2 - a^3 d = a^5 d + a^2 (cb + ad) - (a^4 + ad - a^2) \\ &= a^5 d + a^3 d - a^4 - ad + a^2 = (a^2 + 1) a^3 d - a^4 - ad + a^2 \\ &= (a^2 + 1)(a^4 + ad - a^2) - a^4 - ad + a^2 = a^6 + a^3 d - a^4 \\ &= a^6 + a^4 + ad - a^2 - a^4 = a^6 + ad - a^2, \end{aligned}$$

again using Lemma 7.7. □

Theorem 7.9. *We have the isomorphism $U(A^{CT}) \cong U(A^C)$.*

Proof. The set G^{CT} contains 128 elements; we put each generator in standard form and self-reduce the set, obtaining the following 44 generators:

$$
\begin{array}{lllll}
aba-a^2b+b, & aca-a^2c, & acb-abc+a, & ada-a^2d, & adb-abd+b, \\
adc-acd, & ba^2-a^2b+b, & bab-ab^2, & bac-abc+a, & bad-abd+b, \\
b^2a-ab^2, & bca-abc, & bcb-b^2c-b, & bda-abd+b, & bdb-bbd, \\
bdc-bcd-a, & ca^2-a^2c-c, & cab-abc-d+a, & cac-ac^2, & cad-acd, \\
cba-abc+a, & cb^2-b^2c, & cbc-bc^2-c, & cbd-bcd-d, & c^2a-ac^2, \\
c^2b-bc^2, & cda-acd, & cdb-bcd, & cdc-c^2d, & da^2-a^2d, \\
dab-abd+b, & dac-acd, & dad-ad^2, & dba-abd+b, & db^2-b^2d, \\
dbc-bcd, & dbd-bd^2+b, & dca-acd-c, & dcb-bcd-d, & dc^2-c^2d, \\
dcd-cd^2, & d^2a-ad^2, & d^2b-bd^2+b, & d^2c-cd^2-c. &
\end{array}
$$

There are 133 nontrivial compositions (omitted); we include their normal forms, self-reduce, and obtain 16 generators, coinciding with the Gröbner basis (18). □

Conjecture 7.10. If C is a finite dimensional comtrans algebra (special or not) then $U(C)$ is also finite dimensional.

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