QUASI-SYMMETRIC FUNCTIONS AND THE CHOW RING OF THE STACK OF EXPANDED PAIRS

JAKOB OESINGHAUS

ABSTRACT. We show that the Hopf algebra of quasi-symmetric functions arises naturally as the integral Chow ring of the algebraic stack of expanded pairs originally described by J. Li, using a more combinatorial description in terms of configurations of line bundles. In particular, we exhibit a gluing map which gives rise to the comultiplication. We then apply the result to calculate the Chow rings of certain stacks of semistable curves.

CONTENTS

1. INTRODUCTION

In order to study relative Gromov-Witten invariants of a pair (X, D) of a scheme X over $\mathbb{C}$ and a Cartier divisor $D \subset \mathbb{C}$, J. Li introduced the notion of a family of *expansions* of the pair (X, D) (cf. [Li01, Li02]). An expansion of the pair (X, D) of length ℓ is constructed by gluing ℓ copies of the projectivized normal bundle of D to X:

$$X(\ell) := X \sqcup_D \underbrace{P \sqcup_D \cdots \sqcup_D P}_{\ell \text{ times}},$$

1

where $P = \mathbb{P}(\mathcal{O}_X(D)|_D \oplus \mathcal{O}_D)$. Since P carries a $\mathbb{G}_m$-action by scaling, an expansion of length ℓ comes with an action of $\mathbb{G}_m^\ell$.

Let $\mathcal{A} := [\mathbb{A}/\mathbb{G}_m]$ be the stack quotient of the affine line by the standard action of the multiplicative group; it is the moduli space of pairs $(\mathcal{L}, s)$ of a line bundle $\mathcal{L}$ and a section s of $\mathcal{L}$. Let $\mathcal{D} := [0/\mathbb{G}_m] \cong \mathbb{G}_m$ be the vanishing locus of the universal section. The pair $(\mathcal{A}, \mathcal{D})$ forms the universal pair of an algebraic stack and a Cartier divisor on it. In [ACFW13], the authors used this fact to define a stack $\mathcal{T}$ of expansions of any expansions of the universal pair $(\mathcal{A}, \mathcal{D})$, proved that it is an algebraic stack locally of finite type. The same object has also been studied in [GV05].

The stack $\mathcal{A}$ and its powers $\mathcal{A}^n \cong [\mathbb{A}^n/\mathbb{G}_m^n]$ are connected to logarithmic geometry; in fact, they form an open substack of the stack of logarithmic structures ([Ols03]). It is possible to adopt the logarithmic point of view to identify $\mathcal{T}$ as the stack of aligned log structures (cf [ACFW13, 8] and [BV12]). This allows a more combinatorial description of $\mathcal{T}$ as a colimit of the $\mathcal{A}^n$ by étale morphisms.

The Hopf algebra QSym of quasi-symmetric functions is well-studied object that arises as a generalization of symmetric functions. As an algebra, it is commutative and graded, and it is free[1] over $\mathbb{Z}$ with finitely many generators in each degree, though writing down explicit integral generators is not straightforward (cf [Haz10]). The coalgebra structure is not cocommutative. It is straightforward to see that QSym arises as a certain projective limit of polynomial rings in the category of graded algebras.

We prove that QSym arises as the Chow ring of $\mathcal{T}$. To be more precise, we calculate the Chow ring of $\mathcal{X} \times \mathcal{T}$ for a smooth algebraic stack $\mathcal{X}$ of finite type over the base field, and show that the colimit construction of $\mathcal{T}$ gives rise to an isomorphism

$$\mathrm{CH}^\bullet(\mathcal{X} \times \mathcal{T}) \cong \mathrm{CH}^\bullet(\mathcal{X}) \otimes \mathrm{QSym}$$

in Theorem 3. We then show that there exists an étale, but non-separated morphism $\mathcal{T} \times \mathcal{T} \to \mathcal{T}$ which exhibits $\mathcal{T}$ as a monoid object and induces the comultiplication of QSym on the level of Chow cohomology. It can be shown that

In section 5, we use the fact that $\mathcal{T}$ has an interpretation as an open substack of the moduli stack of 3-pointed semistable curves $\mathfrak{M}_{0,3}^{ss}$ to calculate the intersection rings of $\mathfrak{M}_{0,3}^{ss}$. Moreover, the stack $\mathfrak{M}_{0,2}^{ss}$ can be seen as a non-rigid variant of $\mathcal{T}$, which allows us to compute its Chow ring, and the action of the natural involution on this ring, as a corollary of our computations for $\mathcal{T}$.

Notation. Throughout this note, we let k be an algebraically closed field of characteristic 0. There is no loss for the reader in assuming k to be $\mathbb{C}$ throughout. We let $\mathfrak{C}$ be the category of finite ordered sets and order-preserving injections, and we let $\mathfrak{C}_{\leq m}$ be the full subcategory consisting of sets with m or less elements.

---

[1]I.e., isomorphic to a polynomial algebra.

2. QUASISYMMETRIC FUNCTIONS

Quasisymmetric functions are a generalization of the well-known Hopf algebra of symmetric functions due to Gessel ([Ges84]). Since we will identify the intersection ring of the stack of expanded pairs as the ring of quasisymmetric functions, we first take some time to describe this ring in more detail. All of the material in this section is classical, except for possibly Proposition 5, which we did not find in the literature. A very in-depth treatment of quasisymmetric functions can be found in [LMvW13, GR14].

2.1. **Definitions.** Given any totally ordered set $\mathfrak{I}$, we can consider the commutative graded ring

$$\mathbb{Z}[\![\alpha_i]\!]_{i \in \mathfrak{I}} \tag{1}$$

of formal power series in the variables $\{\alpha_i\}_{i \in \mathfrak{I}}$ and with $\mathbb{Z}$-coefficients.

Definition 1. The algebra of *quasisymmetric functions on the index set* $\mathfrak{I}$ is the subring $\mathrm{QSym}_{\mathfrak{I}} \subset \mathbb{Z}[\![\alpha_i]\!]_{i \in \mathfrak{I}}$ consisting of power series f of bounded degree satisfying the following condition:

For every two increasing sequences $i_1 < \cdots < i_\ell$ and $j_1 < \cdots < j_\ell$ of elements of $\mathfrak{I}$ of length ℓ, and for every $(I_1, \ldots, I_\ell) \in \mathbb{Z}_{>0}^\ell$, the coefficients of f for the monomials $\alpha_{i_1}^{I_1} \cdots \alpha_{i_\ell}^{I_\ell}$ and $\alpha_{j_1}^{I_1} \cdots \alpha_{j_\ell}^{I_\ell}$ are equal.

It is also possible to define quasisymmetric functions with R-coefficients for any commutative ring R as $\mathrm{QSym}_{\mathfrak{I}} \otimes R$.

Notation. We will usually consider the index set $\mathfrak{I} = \mathbb{Z}_{>0}$, and write $\mathrm{QSym} := \mathrm{QSym}_{\mathbb{Z}_{>0}}$. We also define $\mathrm{QSym}_n := \mathrm{QSym}_{[n]}$, where $[n] := \{1 < \cdots < n\}$.

Definition 2. Let $n \in \mathbb{N}$. A *composition* I of n (of length $\ell(I) := \ell$) is an ordered ℓ-tuple $(I_1, \ldots, I_\ell) \in \mathbb{Z}_{>0}$ of positive integers such that $I_1 + \cdots + I_\ell = n$. Let Comp be the set of all compositions, and let Comp_n be the set of compositions of n. Given two compositions $I = (I_1, \ldots, I_k)$ and $J = (J_1, \ldots, J_\ell)$, we write

$$I \cdot J := (I_1, \ldots, I_k, J_1, \ldots, J_\ell).$$

There is a natural basis of $\mathrm{QSym}_{\mathfrak{I}}$ called the *monomial basis*, indexed by compositions. For a composition I, let M_I be the monomial

$$M_I := \sum_{i_1 < \cdots < i_\ell} \alpha_{i_1}^{I_1} \cdots \alpha_{i_\ell}^{I_\ell}. \tag{2}$$

Then the monomials M_I form a basis of $\mathrm{QSym}_{\mathfrak{I}}$ if $\mathfrak{I}$ is infinite, and the monomials indexed by compositions I with $\ell(I) \leq |\mathfrak{I}|$ form a basis of $\mathrm{QSym}_{\mathfrak{I}}$ if $\mathfrak{I}$ is finite[2]. In particular, this implies that $\mathrm{QSym}_{\mathfrak{I}}$ does not depend on the index set as long as $\mathfrak{I}$ is infinite. Nonetheless, keeping track of indices can often be useful.

It is clear that $\deg(M_I) = n$ if I is a composition of n, hence compositions of n give rise to a basis for the degree n part of QSym. The assignment $\mathfrak{I} \mapsto \mathrm{QSym}_{\mathfrak{I}}$

---

[2]All M_I for $\ell(I) > |\mathfrak{I}|$ are identically 0.

defines a contravariant functor $\mathfrak{QSym}$ from the category of totally ordered sets and order-preserving injections to the category of commutative graded rings; a morphism $g : \mathfrak{I} \hookrightarrow \mathfrak{J}$ maps to the "evaluation" homomorphism setting all α_j for $j \notin g(\mathfrak{I})$ to 0. This functor behaves nicely in the monomial basis:

Proposition 1. *For any order-preserving injection $g : \mathfrak{I} \hookrightarrow \mathfrak{J}$ and any composition I, we have*

$$\mathfrak{QSym}(g)(M_I) = M_I.$$

In particular, if $\mathfrak{J}$ is finite, then the kernel of $\mathfrak{QSym}(g)$ is generated by monomials M_I for compositions I satisfying $\ell(I) > |\mathfrak{J}|$, and the restriction of $\mathfrak{QSym}(g)$ to the subalgebra generated by monomials M_I for $\ell(I) \leq |\mathfrak{J}|$ is an isomorphism. □

Note that there also exists a covariant functor from the category of totally ordered sets and order-preserving injections to the category of graded $\mathbb{Z}$-modules, sending $M_I \to M_I$; however, these inclusions of $\mathbb{Z}$-modules are not algebra homomorphisms. For example, for the inclusion $\{1\} \hookrightarrow \{1 < 2\}$, the product $M_{(1)} \cdot M_{(1)}$ is equal to $M_{(2)}$ in the source, but it is $M_{(2)} + 2M_{(1,1)}$ in the target.

Example. Let us list all monomials of degree up to 3, with index set $\mathbb{Z}_{>0}$.

- $M_\emptyset = 1$.
- $M_{(1)} = \sum_i x_i = x_1 + x_2 + \dots$.
- $M_{(2)} = \sum_i x_i^2 = x_1^2 + x_2^2 + \dots$.
- $M_{(1,1)} = \sum_{i<j} x_i x_j = x_1 x_2 + x_1 x_3 + x_2 x_3 + \dots$.
- $M_{(3)} = \sum_i x_i^3 = x_1^3 + x_2^3 + \dots$.
- $M_{(2,1)} = \sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \dots$.
- $M_{(1,2)} = \sum_{i<j} x_i x_j^2 = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \dots$.
- $M_{(1,1,1)} = \sum_{i<j<k} x_i x_j x_k = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 + \dots$.

2.2. **Free generators for the multiplication.** There is a description of the multiplication of monomials in purely combinatorial terms.

Proposition 2 ([GR14, Prop 5.3])**.** *Fix an infinite index set $\mathfrak{I}$, and $\ell, m \in \mathbb{Z}_{>0}$. Fix two pairwise disjoint chain posets $\{i_1 < \cdots < i_\ell\}$ and $\{j_1 < \cdots < j_m\}$.*

For compositions $I = (I_1, \dots, I_\ell)$ and $J = (J_1, \dots, J_m)$, we have

$$M_I \cdot M_J = \sum_f M_{\mathrm{wt}\, f},$$

where the sum runs over all surjective, strictly order-preserving maps

$$\{i_1, \dots, i_\ell\} \sqcup \{j_1, \dots, j_m\} \to \{1, \dots, n\}$$

for some $n \in \mathbb{N}$, and the composition $\mathrm{wt}\, f$ is defined by

$$(\mathrm{wt}\, f)_x := \sum_{i_u \in f^{-1}(x)} I_u + \sum_{j_u \in f^{-1}(x)} J_u.$$

As a consequence of Proposition 1 and Proposition 2, we can perform additive computations involving only monomials of length $\leq \ell$ using any index set with ℓ elements, and we can compute the product of two monomials of degree ℓ and m using any index set with $\ell + m$ elements.

Example. We can compute the product $M_{(1,2)} \cdot M_{(1,1)}$ as follows, using Proposition 2.

$$
\begin{aligned}
M_{(1,2)} \cdot M_{(1,1)} =& M_{(1,2,1,1)} + 2M_{(1,1,2,1)} + 3M_{(1,1,1,2)} \\
&+ M_{(2,2,1)} + M_{(1,3,1)} + M_{(2,1,2)} + 2M_{(1,1,3)} + M_{(1,2,2)} \\
&+ M_{(2,3)}.
\end{aligned}
$$

It can be shown that quasisymmetric functions form a polynomial algebra, i.e., there is a subset generating QSym as an algebra, without any relations. To write down a more precise statement, we need to introduce some notation.

Definition 3. Let $I, J \in \text{Comp}$. We say that $I \leq J$ if

- either there exists an $i \leq \min\{\ell(I), \ell(J)\}$ such that $I_i < J_i$ and for every $j < i$, we have $I_j = J_j$,
- or I is a prefix of J, i.e. we can write $J = I \cdot K$ for another composition K.

This defines a total order on the set of compositions, sometimes called the *lexicographic order*, because it is equal to the lexicographic order when two compositions have the same length.

Definition 4. We say that a composition I is a *Lyndon composition* if every nonempty proper suffix of I is greater than I. Concretely, this means that whenever we can write $I = J \cdot K$ for J nonempty compositions J and K, we have $K > I$. Let $\mathfrak{L}$ be the set of Lyndon compositions.

Example. The empty composition is not Lyndon. Every composition of the form (a) is Lyndon. A composition of the form (a, b) is Lyndon if and only if $b > a$. A composition of the form (a, b, c) is Lyndon if and only if $c > a$ and $b \geq a$.

Let μ be the number-theoretic Möbius function, i.e. $\mu(d)$ is the sum of the d-th primitive roots of unity.

Proposition 3 ([Wit37]). *The number b_n of Lyndon compositions of length n equals*

$$
b_n := \frac{1}{n} \sum_{d|n} \mu(d) \left(2^{n/d} - 1 \right).
$$

Hence, the numbers b_n satisfy $\sum_{d|n} d b_d = 2^n - 1$.

Remark. The sequence b_n starts as follows (sequence A059966 in OEIS):

$$
(b_1, b_2, \dots) = (1, 1, 2, 3, 6, 9, 18, \dots).
$$

Theorem 1 ([Haz01, Haz10]). QSym *is isomorphic to a graded polynomial ring with b_n generators in degree n.*

We can write down a set of free rational generators quite explicitly. This is also possible over the integers, but requires a larger notational effort[3]. In fact, the morphism

$$
\begin{aligned}
\mathbb{Q}[x_I]_{I \in \mathfrak{L}} &\to \text{QSym} \otimes \mathbb{Q} \\
x_I &\mapsto M_I
\end{aligned}
$$

---

[3]To see how an integral basis can be constructed with the same index set, see [Haz10], or [GR14, 6.5] for a more detailed explanation.

is an isomorphism of graded algebras, for $\deg x_I = |I|$.

Example. The generators corresponding to Lyndon compositions up to degree 4 are as follows:

$$M_{(1)}, M_{(2)}, M_{(3)}, M_{(1,2)}, M_{(4)}, M_{(1,3)}, M_{(1,1,2)}.$$

2.3. **Hopf algebra structure.** The comultiplication for quasisymmetric functions extend the one for symmetric functions. In the monomial basis, it can be described as follows. Let

$$\Delta \colon \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym} \qquad\qquad \varepsilon \colon \mathrm{QSym} \longrightarrow \mathbb{Z}$$
$$M_I \mapsto \sum_{I = J \cdot K} M_J \otimes M_K \qquad\qquad n + \sum_{|I| \geq 1} M_I \mapsto n.$$

and

$$S \colon \mathrm{QSym} \to \mathrm{QSym}$$
$$M_I \mapsto (-1)^{\ell(I)} \sum_{\substack{J \text{ coarser than} \\ \mathrm{rev}(I)}} M_J,$$

where the reverse $\mathrm{rev}(I)$ of a composition $I = (I_1, \ldots, I_\ell)$ is $\mathrm{rev}(I) = (I_\ell, \ldots, I_1)$ and J is said to be *coarser* than K if J can be obtained by successively summing some of the adjacent entries of K.

Proposition 4. *The triple* $(\mathrm{QSym}, \Delta, \varepsilon)$ *defines a coassociative coalgebra structure on* QSym *which is compatible with the algebra structure. Moreover, the bialgebra* QSym *is a graded Hopf algebra with antipode* S. $\qquad\qquad\square$

Example. Consider the monomial $M := M_{(3,1,4)}$. Then

$$\Delta(M) = M_{(3,1,4)} \otimes 1 + M_{(3,1)} \otimes M_{(4)} + M_{(3)} \otimes M_{(1,4)} + 1 \otimes M_{(3,1,4)}$$

and

$$S(M) = - \left(M_{(4,1,3)} + M_{(5,3)} + M_{(4,4)} + M_{(8)} \right).$$

2.4. **Quasisymmetric functions as a limit.** There is a construction of quasisymmetric functions as a categorical limit, which is useful for our purposes. Recall that $\mathfrak{C}$ denotes the category of finite ordered sets and order-preserving injections. The assignment

$$S \mapsto \mathbb{Z}[\alpha_s]_{s \in S} \qquad (\phi : S \to T) \mapsto g_\phi,$$

where

$$g_\phi(x_t) = \begin{cases} \alpha_s, & \text{if there exist some } s \in S \text{ such that } \phi(s) = t, \\ 0, & \text{otherwise.} \end{cases}$$

defines a contravariant functor from $\mathfrak{C}$ to the category of graded rings.

Proposition 5. *For every finite ordered set* S, *consider the restriction morphism* $\mathrm{QSym} \to \mathrm{QSym}_S$ *from Proposition 1 for any order-preserving injection* $S \hookrightarrow \mathbb{Z}$. *Then* QSym, *together with this family of restriction morphisms, satisfies the universal property of the colimit*

$$\varprojlim_{\mathfrak{C}} \mathbb{Z}[\alpha_s]_{s \in S}.$$

Proof. For simplicity, we replace $\mathfrak{C}$ by the equivalent full subcategory consisting only of the objects $[n] := \{1 < \cdots < n\}$ for $n \in \mathbb{N}$. Note that the morphisms are generated by the family of morphisms $j_i^n : [n] \to [n+1]$ for $i \in \{1, \ldots, n+1\}$, where j_i^n is the unique map whose image does not contain i. Denote by

$$\pi_n : \mathrm{QSym} \to \mathrm{QSym}_{[n]} \hookrightarrow \mathbb{Z}[\alpha_i]_{i=1,\ldots,n}$$

the projections, and let $g_i^n := g_{j_i^n}$. It is clear from the definition that the projections π_n satisfy the necessary compatibility conditions, since they preserve monomials.

To prove that QSym satisfies the universal property, let now $R^\bullet$ be any graded ring, and assume we are given morphisms of graded rings

$$f_n : R^\bullet \to \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$$

such that $g_i^n \circ f_{n+1} = f_n$, then for any $r \in R^d$ and every $1 \leq i \leq n+1$, we have

$$f_{n+1}(r)(\alpha_1, \ldots, \alpha_{i-1}, 0, \alpha_i, \ldots, \alpha_n) = f_n(r)(\alpha_1, \ldots, \alpha_n). \tag{3}$$

This implies that for indices $i_1 < \cdots < i_s$ and $j_1 < \cdots < j_s$ and exponents $k_1 + \cdots + k_s = d$, the coefficient of $\alpha_{i_1}^{k_1} \cdots \alpha_{i_s}^{k_s}$ and the coefficient of $\alpha_{j_1}^{k_1} \cdots \alpha_{j_s}^{k_s}$ in the homogeneous degree d polynomial $f_{n+1}(r)$ agree. This is proved by iterating the above equalities (3), inserting zero everywhere except for indices in $\{i_1, \ldots, i_s\}$, respectively $\{j_1, \ldots, j_s\}$. We conclude that $f_n(r)$ is a quasisymmetric function of degree d in n variables, for every n. Moreover, the value $f_e(r)$ for $e \geq 0$ is uniquely determined by $f_d(r)$, because a quasisymmetric function of degree d in any number of variables is determined by its coefficients for monomials in the first d variables, and because $f_0(r), \ldots, f_{d-1}(r)$ are determined by (3). Hence, we can define $\psi(r) \in A_d$ by extending the quasisymmetric function $f_d(r)$ to a countable number of variables. By construction, we have $\pi_n \circ \psi = f_n$. The lifting ψ is unique, because the map $\pi_d|_{A_d}$ is injective. $\qquad\square$

Remark. With the same proof, we can also conclude that

$$\varprojlim_{\mathfrak{C}_{\leq m}} \mathbb{Z}[\alpha_s]_{s \in S} \cong \mathrm{QSym}_m.$$

3. Definitions and setup

We denote the Chow groups of an algebraic stack $\mathcal{X}$ by $\mathrm{CH}_\bullet(\mathcal{X})$, and the Chow ring of a smooth, equidimensional algebraic stack by

$$\mathrm{CH}^\bullet(\mathcal{X}) = \mathrm{CH}_{\dim(\mathcal{X})-\bullet}(\mathcal{X}).$$

with cohomological grading, assuming the latter is defined (for example, if it has a stratification by quotient stacks). All algebraic stacks are assumed to be defined over the field k.

3.1. **Chow groups for algebraic stacks admitting a good filtration.** In [Kre99], Chow groups, and the intersection ring for smooth algebraic stacks, are only defined for an algebraic stack of finite type over k. We will need to extend the definition to stacks which are close enough to being of finite type for the purpose of intersection theory.

Definition 5. A *good filtration by finite-type substacks* on an algebraic stack $\mathcal{X}$, which is assumed to be locally of finite type and of finite dimension, is a collection $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ such that

- $\mathcal{X}_n \subset \mathcal{X}$ is an open substack of finite type;
- $\mathcal{X}_n \subset \mathcal{X}_m$ for $n < m$;
- $\dim(\mathcal{X} \setminus \mathcal{X}_n) < \dim X - n$.

In this situation, we will abbreviate to say that $\mathcal{X}$ *admits a good filtration*. A morphism of algebraic stacks admitting a good filtration *respects the filtration* if it factors through the filtrations in the natural way, up to a degree shift.

Remark. The last condition could be weakened to only requiring that the $\mathcal{X}_n$ jointly cover $\mathcal{X}$, and that $\lim_{n \to \infty} \operatorname{codim}(\cup_{i=0}^n \mathcal{X}_i) = \infty$; however, all the stacks in our example already come with a natural good filtration, and it is possible to pass from any filtration by open substacks of finite type to a good filtration.

Proposition 6. *There is a Chow group functor* $\operatorname{CH}_d(\mathcal{X}) := \varprojlim_n \operatorname{CH}_d(\mathcal{X}_n)$ *defined for stacks admitting a good filtration, satisfying the usual properties as in* [Kre99], *with functoriality for morphisms of algebraic stacks which respect the filtration. In fact, we have* $\operatorname{CH}_d(\mathcal{X}) = \operatorname{CH}_d(\mathcal{X}_{\dim(\mathcal{X})+d})$. *If* $\mathcal{X}$ *is smooth and admits a stratification by quotients stacks, there is also a ring structure defined on the Chow groups, and we can compute the product* $\operatorname{CH}^{d'}(\mathcal{X}) \otimes \operatorname{CH}^{d''}(\mathcal{X}) \to \operatorname{CH}^{d'+d''}(\mathcal{X})$ *on* $\mathcal{X}_{d'+d''}$.

Proof. This is a straightforward consequence of excision. $\qquad\qquad\square$

All of the algebraic stacks appearing in this note admit a good filtration, and we will use the colimit above implicitly when discussing their Chow groups.

3.2. **Étale-local models.** We collect some facts about the stack quotients $\mathcal{A}^n := [\mathbb{A}^n/\mathbb{G}_m^n] = (\mathcal{A}^1)^{\times n}$ for $n \in \mathbb{Z}_{\geq 0}$ and their Chow rings. Recall that $\mathcal{A}^1$ is the moduli stack of pairs $(\mathcal{L}, s)$ of a line bundle $\mathcal{L}$ and a section s of $\mathcal{L}$, and hence $\mathcal{A}^n$ is the moduli stack of n-tuples of such pairs.

Since $\mathcal{A}^n$ is a vector bundle of rank n over $B\mathbb{G}_m^n \cong (B\mathbb{G}_m)^{\times n}$, the pullback of cycle classes from $B\mathbb{G}_m^n$ is an isomorphism. Let $[\mathbb{A}^1/\mathbb{G}_m^n]_i$ correspond to the one-dimensional representation of $B\mathbb{G}_m^n$ given by $x \mapsto t_i x$.

Proposition 7. *The graded ring* $\operatorname{CH}^\bullet(B\mathbb{G}_m^n)$ *is isomorphic to* $\mathbb{Z}[\alpha_1, \ldots, \alpha_n]$, *where* α_i *corresponds to the class of the zero section of* $[\mathbb{A}^1/\mathbb{G}_m^n]_i$ *under the Gysin homomorphism* $\operatorname{CH}^\bullet([\mathbb{A}^1/\mathbb{G}_m^n]_i) \to \operatorname{CH}^\bullet(B\mathbb{G}_m^n)$.

Corollary 8. *We have*
$$\operatorname{CH}^\bullet(\mathcal{A}^n) \cong \mathbb{Z}[\alpha_1, \ldots, \alpha_n].$$

Lemma 9. *The normal bundle of* $B\mathbb{G}_m^n$ *in* $\mathcal{A}^n$ *has top Chern class* $\alpha_1 \cdots \alpha_n$.

Proof. We can compute the normal bundle on any smooth atlas. Choose the atlas $\mathbb{A}^n$ to find that the normal bundle is $\mathcal{A}^n$ itself, seen as a vector bundle over $B\mathbb{G}_m$. This a sum of line bundles
$$N_{B\mathbb{G}_m^n} \mathcal{A}^n = \bigoplus_{i=1}^n [\mathbb{A}^1/\mathbb{G}_m^n]_i.$$

Hence we have

$$c_n(N_{B\mathbb{G}_m^n}\mathcal{A}^n) = \prod_{i=1}^n \alpha_i.$$

$\square$

Definition 6. We say that an algebraic stack $\mathfrak{Y}$ over k has the *Chow Künneth property* if for all algebraic stacks $\mathcal{X}$ of finite type over k, the natural morphism

$$\mathrm{CH}^\bullet(\mathcal{X}) \otimes \mathrm{CH}^\bullet(\mathfrak{Y}) \to \mathrm{CH}^\bullet(\mathcal{X} \times_k \mathfrak{Y}) \tag{4}$$

induced by functoriality is an isomorphism.

Remark. If $\mathfrak{Y}$ has the Chow Künneth property, then (4) will also be true if $\mathcal{X}$ only admits a good filtration.

Lemma 10. *The classifying stack $B\mathbb{G}_m$ has the Chow Künneth property.*

Proof. First note that for any $n \geq 1$, projective space $\mathbb{P}^{n-1}$ has the Chow Künneth property, by the formula for the Chow rings of projective bundles, specialized to the case of a trivial bundle (cf. [Ful98, 3.3]). To prove the lemma, note that $\mathcal{X} \times [\mathbb{A}^n/\mathbb{G}_m]$ is a vector bundle over $\mathcal{X} \times B\mathbb{G}_m$, which is $\mathcal{X} \times \mathbb{P}^{n-1}$ in codimension smaller than n.[4] By choosing n high enough and applying the Chow Künneth property of projective space, we obtain the statement in any fixed degree. $\square$

As a consequence, also $\mathcal{A}^n$, for any n, has the Chow Künneth property. Occasionally, we will use coordinates indexed by a finite set J instead of $\{1, \ldots, n\}$. In this situation, we will use the symbols $\mathbb{A}^J$ and $\mathcal{A}^J$.

3.3. **The stack of configurations of line bundles.** We continue to use the notation $\mathcal{A}^n$ from the last section. In this section, we introduce the stack of expanded pairs $\mathcal{T}$. The following treatment has been adapted from [ACFW13]. There are many equivalent moduli definitions (and universal families) for $\mathcal{T}$; we will describe one of them for the convenience of the reader, where $\mathcal{T}$ appears in the form of the moduli stack of *configurations of line bundles*. This point of view has been inspired from the approach to logarithmic structures in terms of line bundles in [BV12].

Definition 7. A *sheaf of totally ordered finite sets* is a constructible sheaf E of partially ordered nonempty[5] finite sets on the étale site of a scheme S such that any two sections are locally comparable. We identify E with the stack on the étale site of S whose objects are sections of E, and where a unique morphism $x \to y$ exists if and only if $x \geq y$.

We let $\mathfrak{Pic}$ be the category of line bundles. The objects of $\mathfrak{Pic}$ over S are line bundles on S, and its morphisms are morphisms of line bundles. We should remark that morphisms are *not* required to be isomorphisms of line bundles, so $\mathfrak{Pic}$ is not a category fibered in groupoids, as opposed to $B\mathbb{G}_m$.

Definition 8. Let $\mathcal{T}$ be the stack whose objects are pairs (E, L) of a sheaf of totally ordered sets E and a morphism of stacks $L : E \to \mathfrak{Pic}$ such that

---

[4]In the topological setting, we have $BU(1) = \mathbb{CP}^\infty$. In this spirit, one could regard $B\mathbb{G}_m$ as an algebraic version of $\mathbb{P}^\infty$.

[5]except over the empty set.

(1) If $x \geq y$ are sections of E and $L(x) \to L(y)$ is an isomorphism, then $x = y$.

(2) $L(0) = \mathcal{O}$, where 0 is the unique section of E that is minimal in all fibers.

We call such an L a *diagram of line bundles* indexed by E. We require that morphisms in $\mathcal{T}$ are the identity on $L(0)$. We call $\mathcal{T}$ the stack of totally ordered configurations of line bundles.

We recall the definition of the moduli stack of genus g semistable curves with n marked points $\mathfrak{M}_{g,n}^{ss} \subset \mathfrak{M}_{g,n}$, whose objects are prestable curves of genus g endowed with n sections such that the relative dualizing sheaf, twisted by the sections, has non-negative multidegree. It is classical that $\mathfrak{M}_{g,n}^{ss}$ is an algebraic stack locally of finite presentation over k.

We will only need the genus 0 case; for $n > 0$, its geometric points are nodal curves such that each component is isomorphic to projective space and such that each component has at least two special points (either nodes or marked points).

Theorem 2 ([ACFW13]). *There is an isomorphism of $\mathcal{T}$ with the open substack of $\mathfrak{M}_{0,3}^{ss}$ where the last two points lie on the same component.*

There is also a useful description of $\mathcal{T}$ as a colimit. Given a finite, possibly empty, ordered set J, there is a natural augmentation $\tilde{J}$, which is the union $J \sqcup \{0\}$ of I and a smallest element 0. An order-preserving injection of finite sets $J \to K$ induces an open embedding $\mathcal{A}^J \to \mathcal{A}^K$. Assigning $J \mapsto \mathcal{A}^J$, together with the above morphisms, give rise to diagrams of algebraic stacks indexed by $\mathcal{C}_{\leq k}$, respectively $\mathcal{C}$.

There are natural diagrams of line bundles on $\mathcal{A}^J$ as follows. Consider the universal family on $\mathcal{A}^J$, that is, a collection $((\mathcal{L}_i, s_i))_{i \in J}$ of line bundles and sections. Let E be the quotient of the constant sheaf $\tilde{J}$ on $\mathcal{A}^J$ by the relation $i \sim i+1$ on the locus where s_{i+1} is nonzero. Denote the elements of J by $\{1 < \cdots < n\}$ for $|J| = n$. Then the following sequence of morphisms of line bundles defines a morphism $\tilde{J} \to \mathfrak{Pic}$ which descends to a diagram of line bundles on $\mathcal{A}^J$ indexed by E.

$$L_1^{\vee} \otimes \cdots \otimes L_n^{\vee} \xrightarrow{s_n} L_1^{\vee} \otimes \cdots \otimes L_{n-1}^{\vee} \xrightarrow{s_{n-1}} \ldots \xrightarrow{s_2} L_1^{\vee} \xrightarrow{s_1} \mathcal{O}$$

This gives rise to morphisms $\mathcal{A}^J \to \mathcal{T}$ for every J, compatible with the embeddings $\mathcal{A}^J \hookrightarrow \mathcal{A}^K$.

Proposition 11 ([ACFW13, Prop 8.3.1]). *These morphisms are étale and induce an equivalence*

$$\varinjlim_{\mathcal{C}} \mathcal{A}^J \xrightarrow{\sim} \mathcal{T}. \tag{5}$$

Setting

$$\mathcal{T}^{\leq d} := \varinjlim_{\mathcal{C}_{\leq d}} \mathcal{A}^J \hookrightarrow \mathcal{T},$$

we see that the collection $\{\mathcal{T}^{\leq d}\}_k$ forms a good filtration of $\mathcal{T}$. The following observation will be useful: $\mathcal{T}^{\leq d}$ possesses a unique closed point with stabilizer $B\mathbb{G}_m^d$, whose complement is $\mathcal{T}^{\leq d-1}$. It pulls back to the image of the origin in $\mathcal{A}^J$ for $|J| = d$. We will sometimes call a geometric point of $\mathcal{T}$ over this point an *accordion of length d*, according to the geometric picture in $\mathfrak{M}_{0,3}^{ss}$: it is a nodal curve which is a chain of $d+1$ projective spaces with one marking on one end two markings on the

other end, and the action of the automorphism group $\mathbb{G}_m^d$ looks like an accordion being played.

4. CYCLE GROUP CALCULATIONS

4.1. **The Chow ring of $\mathcal{T}$.** We will prove the following result.

Theorem 3. *Let $\mathcal{X}$ be an algebraic stack, smooth over k and admitting a good filtration, which has a stratification by quotient stacks.*

(1) The natural morphism

$$\mathrm{CH}^\bullet(\mathcal{X} \times \mathcal{T}^{\leq d}) = \mathrm{CH}^\bullet(\varinjlim_{\mathcal{C}_{\leq d}} \mathcal{X} \times \mathcal{A}^n) \to \varprojlim_{\mathcal{C}_{\leq d}} \mathrm{CH}^\bullet(\mathcal{X} \times \mathcal{A}^n) = \mathrm{CH}^\bullet(\mathcal{X}) \otimes \mathrm{QSym}_d \quad (6)$$

induced (via functoriality with regard to étale morphisms) by the colimit (5) is an isomorphism for all d.

(2) The natural morphism

$$\mathrm{CH}^\bullet(\mathcal{X} \times \mathcal{T}) = \mathrm{CH}^\bullet(\varinjlim_{\mathcal{C}} \mathcal{X} \times \mathcal{A}^n) \to \varprojlim_{\mathcal{C}} \mathrm{CH}^\bullet(\mathcal{X} \times \mathcal{A}^n) = \mathrm{CH}^\bullet(\mathcal{X}) \otimes \mathrm{QSym} \quad (7)$$

is an isomorphism.

In particular, the stack $\mathcal{T}$ has the Chow Künneth property, and $\mathrm{CH}^\bullet(\mathcal{T}) \cong \mathrm{QSym}$.

Remark 1. If $\mathcal{X}$ admits a good filtration, but is not smooth, Theorem 3 is still true, using almost the same proof, for the Chow groups without ring structure.

Proof. First we note that the second part follows from the first. To see this, note that the complement of $\mathcal{T}^{\leq d}$ has codimension $d + 1$, hence for any $d' \leq d$,

$$\mathrm{CH}^{d'}(\mathcal{T} \times \mathcal{X}) \cong \mathrm{CH}^{d'}(\mathcal{T}^{\leq d} \times \mathcal{X})$$

by excision. To prove the first part, we fix notation by setting $B := \mathrm{CH}^\bullet(\mathcal{X})$. Also, for any algebraic stack $\mathfrak{Y}$, let $\mathfrak{Y}_\mathcal{X} := \mathcal{X} \times \mathfrak{Y}$, and for morphisms $f : \mathfrak{Y} \to \mathfrak{Z}$, let $f_\mathcal{X} := \mathrm{id} \times f : \mathfrak{Y}_\mathcal{X} \to \mathfrak{Z}_\mathcal{X}$.

We prove the statement by induction on d. More precisely, we prove by induction that the morphism $\mathrm{CH}^\bullet(\mathcal{T}_{\overline{\mathcal{X}}}^{\leq d}) \to \mathrm{CH}^\bullet(\mathcal{A}_\mathcal{X}^d)$ is injective, with image equal to $B \otimes \mathrm{QSym}_d$. The case $d = 0$ is obvious. Suppose the theorem has been proven for $d - 1$. By Lemma 9, the map $\mathrm{CH}^\bullet((B\mathbb{G}_m^d)_\mathcal{X}) \to \mathrm{CH}^{\bullet+d}(\mathcal{A}_\mathcal{X}^d)$ induced by inclusion is injective. Since $(p_d)_\mathcal{X}$ is étale, the same holds for $\mathrm{CH}^\bullet((B\mathbb{G}_m^d)_\mathcal{X}) \to \mathrm{CH}^{\bullet+d}(\mathcal{T}_{\overline{\mathcal{X}}}^{\leq d})$. We will determine its image. By excision, the following commutative diagram is exact.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{CH}^\bullet((B\mathbb{G}_m^d)_\mathcal{X}) & \longrightarrow & \mathrm{CH}^{\bullet+d}(\mathcal{A}_\mathcal{X}^d) & \longrightarrow & \mathrm{CH}^{\bullet+d}((\mathcal{A}^d \setminus B\mathbb{G}_m^d)_\mathcal{X}) & \longrightarrow & 0 \\
& & (\mathrm{id}_{B\mathbb{G}_m^d})_\mathcal{X}^* \uparrow & & (p_d)_\mathcal{X}^* \uparrow & & (p_d|_{(\mathcal{A}^d \setminus B\mathbb{G}_m^d)})_\mathcal{X}^* \uparrow & & \\
0 & \longrightarrow & \mathrm{CH}^\bullet((B\mathbb{G}_m^d)_\mathcal{X}) & \longrightarrow & \mathrm{CH}^{\bullet+d}(\mathcal{T}_{\overline{\mathcal{X}}}^{\leq d}) & \longrightarrow & \mathrm{CH}^{\bullet+d}(\mathcal{T}_{\overline{\mathcal{X}}}^{\leq d-1}) & \longrightarrow & 0
\end{array}
\quad (8)
$$

Let $\psi_i : \mathcal{A}_{\mathcal{X}}^{d-1} \to \mathcal{A}_{\mathcal{X}}^d$ induced by j_i^{d-1}. Consider the following (not necessarily commutative) diagram.

$$
\begin{array}{c}
\mathrm{CH}^\bullet((\mathcal{A}^d \setminus B\mathbb{G}_m^d)_{\mathcal{X}}) \\
\end{array}
\tag{9}
$$

with maps ψ_i^*, ψ_j^*, $(p_d|_{(\mathcal{A}^d \setminus B\mathbb{G}_m^d)})^*_{\mathcal{X}}$, $\mathrm{CH}^\bullet(\mathcal{A}_{\mathcal{X}}^{d-1}) \xrightarrow{\sim} \mathrm{CH}^\bullet(\mathcal{A}_{\mathcal{X}}^{d-1})$, $(p_{d-1})^*_{\mathcal{X}}$, $\mathrm{CH}^\bullet(\mathcal{T}_{\widetilde{\mathcal{X}}}^{\leq d-1})$, $(p_{d-1})^*_{\mathcal{X}}$.

We take the horizontal map h to be induced by the order-preserving bijection $\{1, \ldots, \hat{j}, \ldots, d\} \to \{1, \ldots, \hat{i}, \ldots, d\}$. Then, because $\mathcal{T}_{\mathcal{X}}^{d-1}$ is a colimit over $\mathcal{C}_{\leq d-1}$,

$$
h \circ \psi_i^* \circ (p_d|_{(\mathcal{A}^d \setminus B\mathbb{G}_m^d)})^*_{\mathcal{X}} = (p_{d-1})^*_{\mathcal{X}} = \psi_j^* \circ (p_d|_{(\mathcal{A}^d \setminus B\mathbb{G}_m^d)})^*_{\mathcal{X}}.
\tag{10}
$$

Since $(p_{d-1})^*_{\mathcal{X}}$ is injective by induction hypothesis, so is $(p_d|_{(\mathcal{A}^d \setminus B\mathbb{G}_m^d)})^*_{\mathcal{X}}$. By the five lemma, applied to (8), we deduce that $(p_d)^*_{\mathcal{X}}$ is injective. First, fill in the known (by the induction hypothesis) terms in (8).

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d & \xrightarrow{\cdot \alpha_1 \cdots \alpha_d} & B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d & \longrightarrow & B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d/(\alpha_1 \cdots \alpha_d) & \longrightarrow & 0 \\
& & \uparrow^{id} & & \uparrow^{(p_d)^*_{\mathcal{X}}} & & \uparrow^{(p_d|_{(\mathcal{A}^d \setminus B\mathbb{G}_m^d)})^*_{\mathcal{X}}} & & \\
0 & \longrightarrow & B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d & \longrightarrow & \mathrm{CH}^\bullet(\mathcal{T}_{\widetilde{\mathcal{X}}}^{\leq d}) & \longrightarrow & B \otimes \mathrm{QSym}_{d-1} & \longrightarrow & 0
\end{array}
\tag{11}
$$

Likewise, we fill in the known groups in (9).

$$
\begin{array}{c}
B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d/(\alpha_1 \cdots \alpha_d) \\
\end{array}
\tag{12}
$$

with maps $\alpha_i \mapsto 0$, $\alpha_j \mapsto 0$, $(p_d|_{(\mathcal{A}^d \setminus B\mathbb{G}_m^d)})^*_{\mathcal{X}}$, $B \otimes \mathbb{Z}[\alpha_\ell]_{\ell=1, \ell \neq i}^n \xrightarrow{\sim} B \otimes \mathbb{Z}[\alpha_\ell]_{\ell=1, \ell \neq j}^n$, $(p_{d-1})^*_{\mathcal{X}}$, $B \otimes \mathrm{QSym}_{d-1}$, $(p_{d-1})^*_{\mathcal{X}}$.

To conclude, we apply the induction hypothesis and Lemma 12. The latter implies that the image of $\mathrm{CH}^\bullet(\mathcal{T}_{\widetilde{\mathcal{X}}}^{\leq d})$ in $B \otimes \mathbb{Z}[\alpha_1, \ldots, \alpha_d]$ under $(p_d)^*_{\mathcal{X}}$ can be identified with

$$
B \otimes \left(\mathbb{Z}[\alpha_1, \ldots, \alpha_d] \times_{\mathbb{Z}[\alpha_1, \ldots, \alpha_d]/(\alpha_1 \cdots \alpha_d)} \mathrm{QSym}_{d-1} \right).
$$

By the induction hypothesis, the morphism $(p_{d-1})^*_{\mathcal{X}}$ is the inclusion of $B \otimes \mathrm{QSym}_{d-1}$ into a polynomial ring over B with $d-1$ ordered variables. Hence, by (10), the image of $B \otimes \mathrm{QSym}_{d-1}$ in $B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d/(\alpha_1 \cdots \alpha_d)$ is exactly equal to (the image in $B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d/(\alpha_1 \cdots \alpha_d)$ of) those $f \in B \otimes \mathrm{QSym}_d$ such that $f \notin (\alpha_1 \cdots \alpha_d)$. By the commutativity of (8), this implies that the image of $(p_d)^*_{\mathcal{X}}$ contains a representative mod $(\alpha_1 \cdots \alpha_d)$ of every $f \in B \otimes \mathrm{QSym}_d$ such that $f \notin (\alpha_1 \cdots \alpha_d)$, or to say it differently, it contains a representative mod $(\alpha_1 \cdots \alpha_d)$ of every quasi-symmetric polynomial with B-coefficients that has weight less than d. Furthermore, it also contains $(\alpha_1, \ldots, \alpha_d) \subset \mathrm{QSym}_d$ (from the left-hand side of the fiber product). Via

a direct computation using the colimit description and functoriality, it is immediate that any polynomial in the image must be quasisymmetric, and we have shown that the image contains all quasisymmetric polynomials. This concludes the proof. □

Lemma 12 ([VV03, Lemma 4.4], graded variant)**.** *Let A, B, and C be graded rings, and let $f : B \to A$ and let $g : B \to C$ be morphisms. Suppose that there exists a homomorphism of abelian groups $\phi : A \to B$ such that:*

(1) *The sequence*

$$0 \to A \xrightarrow{\phi} B \xrightarrow{g} C \to 0$$

is exact;

(2) *the composition $f \circ \phi : A \to A$ is the multiplication by an element $a \in A$ of pure degree which is not a zero-divisor.*

Then f and g induce an isomorphism of graded rings

$$(f, g) : B \to A \times_{A/(a)} C,$$

where $A \to A/(a)$ is the projection and $C \to A/(a)$ is induced by $C \cong B/\operatorname{im}(\phi) \to A/\operatorname{im}(f \circ \phi) = A/(a)$. □

Remark 2. The geometric meaning of most of the classes in $\mathrm{CH}^\bullet(\mathfrak{T})$ is somewhat mysterious to the author. The subring of *symmetric* functions is the ring generated by closed substacks: it follows from the previous calculation that the class of $\mathfrak{T}^{\geq d}$ is M_I, where

$$I = \underbrace{(1, \ldots, 1)}_{d \text{ times}}.$$

In other words, it is the d-th elementary symmetric function. On the other hand, consider a generator L of the Picard group of $\mathcal{A}^1$. The unique (up to isomorphism) line bundle on $\mathfrak{T}$ whose restriction to the open substack $\mathcal{A}^1 \subset \mathfrak{T}$ is isomorphic to L has first Chern class $\pm M_{(1)}$. Hence, we readily obtain powers of $M_{(1)}$ as Chern classes of vector bundles, for example

$$M_{(1)}^2 = M_{(2)} + 2M_{(1,1)}.$$

However, it is not clear how we can naturally produce non-symmetric classes such as $M_{(1,2)}$.

4.2. **Hopf algebra structure.** We construct a morphism $\mu : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ that gives rise to the comultiplication of the Hopf algebra. Given two pairs (E_1, L_1) of (E_2, L_2) of a sheaf of totally ordered sets and a diagram of line bundles indexed by that sheaf, we define E as the sheaf of partially ordered sets that arises as the sheafification of the presheaf that identifies the unique minimal element of E_2 with the highest element of E_1. Then

$$L : E \to \mathfrak{Pic}$$

is the morphism which takes a section x to $L_1(x)$ if x is a section of E_1, and to $L_2(x) \otimes \tilde{L}$ if x is a section of E_2, where $\tilde{L}$ is the image under L_1 of the local maximum of E_1.

Informally, we associate to two sequences of line bundles and morphisms

$$L_{1,n_1} \to \cdots \to L_{1,1} \to \mathcal{O} \text{ and } L_{2,n_2} \to \cdots \to L_{2,1} \to \mathcal{O}$$

the sequence

$$L_{2,n_2} \otimes L_{1,n_1} \to L_{2,n_2-1} \otimes L_{1,n_1} \to \cdots \to L_{2,1} \otimes L_{1,n_1}$$
$$\to L_{1,n_1} \to L_{1,n_1-1} \to \cdots \to L_{1,1} \to \mathcal{O}.$$

For $n = n_1 + n_2$, we can identify $\Psi_{n_1,n_2} : \mathcal{A}^{n_1} \times \mathcal{A}^{n_2} \xrightarrow{\simeq} \mathcal{A}^n$ under the obvious isomorphism preserving the order of the coordinates. Since the former form an étale cover of $\mathcal{T} \times \mathcal{T}$ and the latter form an étale cover of $\mathcal{T}$, it will be important so understand how they interact.

Proposition 13. *Let $n, n_1, n_2, \Psi_{n_1,n_2}$ as above. There is an isomorphism*

$$\mu \circ (p_{n_1} \times p_{n_2}) \simeq p_n \circ \Psi_{n_1,n_2}.$$

Proof. This is clear by the definition of μ. □

Lemma 14. *The morphism μ defined by the previous construction is representable étale.*

Proof. Since the stabilizer groups of an accordion of length d is $\mathbb{G}_m^d$ and the morphism maps a pair of accordions of length d_1 and d_2 to an accordion of length d_1+d_2, the morphism is stabilizer-preserving, hence representable. For representable morphisms, we can check the property of being étale on an atlas for the target. Since the property of being étale is étale local on the source, it is enough to show that

$$\mathcal{A}^{n_1} \times \mathcal{A}^{n_2} \to \mathcal{T}^{\leq n}$$

is étale whenever $n_1 + n_2 \leq n$, and for that it is enough to show that it is étale in the case $n_1 + n_2 = n$. After base change by $\mathcal{A}^n \to \mathcal{T}^{\leq n}$, we have to show that the 2-fiber product

$$(\mathcal{A}^{n_1} \times \mathcal{A}^{n_2}) \times_{\mathcal{T}} \mathcal{A}^n$$

is étale over $\mathcal{A}^n$, but this is just the diagonal of $\mathcal{A}^n \to \mathcal{T}$. □

Remark 3. One should note that while μ is étale and quasi-finite, it is not separated, since it admits sections. For example, we have the embeddings $\mathcal{T} \cong \mathcal{T} \times \mathcal{T}^{\leq 0} \hookrightarrow \mathcal{T} \times \mathcal{T}$ and $\mathcal{T} \cong \mathcal{T}^{\leq 0} \times \mathcal{T} \hookrightarrow \mathcal{T} \times \mathcal{T}$, both of which are sections of μ.

Remark 4. As suggested by the notation, the morphism μ is the multiplication morphism exhibiting $(\mathcal{T}, \mu, \mathrm{Spec}(k) \hookrightarrow \mathcal{T})$ as a monoid object in the 2-category of algebraic stacks.

Theorem 4. *The comultiplication $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$ is equal to the pullback of cycle classes μ^*.*

Proof. It is enough to show that $\mu^*(M_I) = \Delta(M_I)$ for every composition I, where M_I is the monomial basis element indexed by I. Fix I and let $n = |I|$ be the size of I, which is also the degree of M_I. We have established in the proof of Theorem 3 that the pullback of cycle classes on $\mathcal{T}$ of cohomological degree at most n to $\mathcal{A}^n$ is injective. The Chow ring of $\mathcal{T} \times \mathcal{T}$ is naturally bigraded as a tensor product of graded algebras. Using the same technique twice, we can see that for cycle classes of bidegree at most (n_1, n_2), the pullback to $\mathcal{A}^{n_1} \times \mathcal{A}^{n_2}$ is injective.

Note that the complement of the union of the images of $\mathcal{A}^{n_1} \times \mathcal{A}^{n_2}$ as we range over pairs (n_1, n_2) with $n_1 + n_2 = n$ has codimension $n + 1$. This implies that for any $d \leq n$, a class in $\mathrm{CH}^d(\mathcal{T} \times \mathcal{T})$ can be uniquely identified by its image in

$$\prod_{n_1+n_2=n} \mathrm{CH}^d(\mathcal{A}^{n_1} \times \mathcal{A}^{n_2})$$

by the product of the various pullbacks, by excision([Kre99, Thm. 2.1.12(iv-v)]). It remains to be shown that the pullback of $\mu^* M_I$ to $\mathrm{CH}^n(\mathcal{A}^{n_1} \times \mathcal{A}^{n_2})$ is equal to the pullback of

$$\sum_{s=0}^{\ell} M_{(I_1,\ldots,I_s)} \otimes M_{(I_{s+1},\ldots,I_\ell)}. \tag{13}$$

Let $\Psi := \Psi_{n_1,n_2}$. We use the fact that $\mu \circ (p_{n_1} \times p_{n_2})$ is isomorphic to $p_n \circ \Psi$.

Under this isomorphism, the monomial M_I corresponding to a composition $I = (I_1, \ldots, I_\ell)$ pulls back to the same monomial in $\mathrm{QSym} \otimes \mathrm{QSym}$, except that we regard the first n_1 variables as coming from the left-hand side of the tensor product, and the remaining n_2 variables as coming from the right-hand side.

To make this precise, let us denote the variables in $\mathrm{CH}^\bullet(\mathcal{A}^n), \mathrm{CH}^\bullet(\mathcal{A}^{n_1})$, and $\mathrm{CH}^\bullet(\mathcal{A}^{n_2})$ by γ_i, α_i, and β_i respectively, such that $\Psi^* \gamma_i = \alpha_i$ for $i \in \{1, \ldots, n_1\}$ and $\Psi^* \gamma_{n_1+i} = \beta_i$ for $i \in \{1, \ldots, n_2\}$. Then we see that

$$
\begin{aligned}
\Psi^* M_I &= \Psi^* \left(\sum_{i_1 < \cdots < i_\ell} \gamma_{i_1}^{I_1} \cdots \gamma_{i_\ell}^{I_\ell} \right) \\
&= \sum_{i_1 < \cdots < i_\ell} \Psi^*(\gamma_{i_1})^{I_1} \cdots \Psi^*(\gamma_{i_\ell})^{I_\ell} \\
&= \sum_{\substack{i_1 < \cdots < i_k \leq n_1 \\ n_1 < i_{k+1} < \cdots < i_\ell}} \Psi^*(\gamma_{i_1})^{I_1} \cdots \Psi^*(\gamma_{i_k})^{I_k} \Psi^*(\gamma_{i_{k+1}})^{I_{k+1}} \cdots \Psi^*(\gamma_{i_\ell})^{I_\ell} \\
&= \sum_{\substack{i_1 < \cdots < i_k \leq n_1 \\ n_1 < i_{k+1} < \cdots < i_\ell}} (\alpha_{i_1} \otimes 1)^{I_1} \cdots (\alpha_{i_k} \otimes 1)^{I_k} (1 \otimes \beta_{(i_{k+1}-n_1)})^{I_{k+1}} \cdots (1 \otimes \beta_{(i_\ell-n_1)})^{I_\ell} \\
&= \sum_{\substack{I=J \cdot K \\ J \in \mathrm{Comp}_{n_1} \\ K \in \mathrm{Comp}_{n_2}}} M_J \otimes M_K = \sum_{\substack{I=J \cdot K \\ \ell(J) \leq n_1 \\ \ell(K) \leq n_2}} M_J \otimes M_K. \tag{14}
\end{aligned}
$$

On the other hand, the pullback of $p(\alpha_i) \otimes q(\beta_i) \in \mathrm{QSym} \otimes \mathrm{QSym}$ by $(p_{n_1} \times p_{n_2})$ is the evaluation homomorphism that sets all α_i to 0 for $i > n_1$ and all β_i to 0 for $i > n_2$, and is the identity on the remaining variables. Hence

$$(p_{n_1} \times p_{n_2})^* M_J \otimes M_K = \begin{cases} M_J \otimes M_K, & \text{if } \ell(J) \leq n_1 \text{ and } \ell(K) \leq n_2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by applying the previous equation to (13), we see that

$$(p_{n_1} \times p_{n_2})^*(\Delta(M_I))$$

is equal to the sum (14). $\qquad \square$

Remark. Since the structure morphism $\mathcal{T} \to \operatorname{Spec} k$ and the inclusion of the generic point $\operatorname{Spec} k \to \mathcal{T}$ induce the unit and counit of QSym, respectively, one could conjecture that there could also be a morphism $\widehat{S} : \mathcal{T} \to \mathcal{T}$ inducing the antipode. However, we have not been able to find a good candidate; among the reasons for that is that if $\widehat{S}$ exists, it cannot be representable and in particular it will not be an isomorphism, contrary to the situation for Hopf algebras. Since $\mathcal{T}$ is only a monoid, not a group, the existence of a geometric antipode seems unlikely.

4.3. **A natural involution.** The stack $\mathcal{T}$ comes equipped with an involution σ, which sends a diagram of line bundles

$$L_n \to L_{n-1} \to \cdots \to L_1 \to \mathcal{O}$$

to the "dualized" diagram

$$L_n \xrightarrow{s_1^\vee \otimes \operatorname{id}} L_1^\vee \otimes L_n \to \ldots \to L_{n-1}^\vee \otimes L_n \xrightarrow{s_n^\vee \otimes \operatorname{id}} L_n^\vee \otimes L_n \cong \mathcal{O}.$$

This corresponds to the involution $\operatorname{rev} : \mathfrak{C} \to \mathfrak{C}$ of the index categeory $\mathfrak{C}$ which sends a finite ordered set to the same set with the opposite order, and similarly for morphisms. There is an explicit description of σ^* as follows.

Proposition 15. *The homomorphism of graded algebras $\sigma^* : \operatorname{QSym} \to \operatorname{QSym}$ is characterized in the monomial basis by*

$$\sigma^* M_J = M_{\operatorname{rev}(J)}.$$

This is the unique nontrivial graded algebra automorphism of QSym *which preserves the monomial basis.*

Proof. It is enough to show that the restriction $\sigma|_{\mathcal{T}^{\leq d}}$ induces the claimed automorphism for each d. Let $J = [d]$, and consider the étale covers $p : \mathcal{A}^J \to \mathcal{T}^{\leq d}$ and $\tilde{p} : \mathcal{A}^{\operatorname{rev}(J)} \to \mathcal{T}^{\leq d}$. The involution $\sigma|_{\mathcal{T}^{\leq d}}$ does not lift to a morphism $\mathcal{A}^J \to \mathcal{A}^J$ over $\mathcal{T}^{\leq d}$, but it lifts to the identity $\operatorname{id} : \mathcal{A}^J \to \mathcal{A}^{\operatorname{rev}(J)}$, which is not induced by any morphism in $\mathfrak{C}$. Let $J = (J_1, \ldots, J_\ell)$ be a composition of length at most d. The monomial $M_J \in \operatorname{CH}(\mathcal{T}^{\leq d})$ pulls back to

$$\sum_{i_1 < \cdots < i_\ell} \alpha_{i_1}^{J_1} \cdots \alpha_{i_\ell}^{J_\ell}$$

in $\operatorname{CH}(\mathcal{A}^J)$ respectively $\operatorname{CH}(\mathcal{A}^{\operatorname{rev}(J)})$, where the indices i_j are taken from the same set $[n]$, but the order relation is inverted. Hence we see that $\tilde{p}^* \sigma^* M_J = p^* M_{\operatorname{rev}(J)}$. For the second part, see [JWY17]. $\square$

Note that there does not exist any nontrivial algebra automorphism of QSym which preserves the monomial basis *and* respects the comultiplication.

5. APPLICATION TO $\mathfrak{M}_{0,2}^{ss}$ AND $\mathfrak{M}_{0,3}^{ss}$

We can now leverage our calculations and the interpretation of $\mathcal{T}$ as a substack of $\mathfrak{M}_{0,3}^{ss}$ to obtain some immediate corollaries for the Chow groups of $\mathfrak{M}_{0,n}^{ss}$ for $n \in \{2, 3\}$.

Consider the stack $\mathfrak{M}_{0,3}^{ss}$ of semistable curves of genus 0 with 3 marked points. Note that any such curve over k has precisely one stable component. We can define

a morphism $\Psi : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathfrak{M}_{0,3}^{ss}$ as follows: Given three curves C_1, C_2, C_3 with markings $(\sigma_1, \ldots, \sigma_9)$ such that 2 markings on each curve lie on the same component, glue the stable components along the three special points (in such a way that one marked points in C_1 is glued to the node in C_2, and the other marked points to the node in C_3, and similarly for marked points on the other curves). The result is a nodal semistable curve with 3 marked points and precisely one stable component. A triple of morphisms in $\mathcal{T}$, i.e. Cartesian diagrams, is taken to the glued Cartesian diagram. This is an isomorphism, because it is a representable étale morphism whose geometric fibers consist of single points. Theorem 3 then implies the following.

Corollary 16. *We have* $\mathrm{CH}^\bullet(\mathfrak{M}_{0,3}^{ss}) \cong \mathrm{QSym} \otimes \mathrm{QSym} \otimes \mathrm{QSym}$.

There is a variant of the stack of expansions, called the stack of non-rigid expansions, which can be shown to be isomorphic to $\mathfrak{M}_{0,2}^{ss}$. The two variants are connected by a rather simple relationship: moving one of the marked points on the stable component of $\mathcal{T}$, viewed as a stack of semistable curves, defines a $\mathbb{G}_m$-action, and $\mathfrak{M}_{0,2}^{ss}$ is the quotient by this action. More precisely, there is the following result:

Proposition 17 ([ACFW13, Prop. 3.3.4])**.** *There is a canonical (up to a twist by inversion) isomorphism* $\mathfrak{M}_{0,2}^{ss} \cong B\mathbb{G}_m \times \mathcal{T}$, *exhibiting* $\mathcal{T}$ *as the rigidification of* $\mathfrak{M}_{0,2}^{ss}$ *by the normal subgroup* $\mathbb{G}_m$ *of its inertia stack.*

Corollary 18. *We have* $\mathrm{CH}^\bullet(\mathfrak{M}_{0,2}^{ss}) \cong \mathrm{QSym}[\beta]$, *where* β *is the pullback of a generator under the morphism* $\mathfrak{M}_{0,2}^{ss} \to B\mathbb{G}_m$ *corresponding to the* $\mathbb{G}_m$*-action on* $\mathcal{T}$.

In fact, we can find a very explicit description of the isomorphism in Proposition 17 as follows: the stack $\mathfrak{M}_{0,2}^{ss}$ is isomorphic to the stack $\mathcal{T}^{\mathrm{nr}}$, whose definition is the same as Definition 8, except that we do not require condition (ii).[6]

Then the isomorphism $\Phi : \mathcal{T}^{\mathrm{nr}} \to \mathbb{G}_m \times \mathcal{T}$ takes a diagram

$$L_n \to \cdots \to L_1 \to L_0$$

to the pair

$$(L_0, L_n \otimes L_0^\vee \to \cdots \to L_1 \otimes L_0^\vee \to \mathcal{O}).$$

There is a natural involution $\tau : \mathfrak{M}_{0,2}^{ss} \to \mathfrak{M}_{0,2}^{ss}$ given by exchanging the two marked points. In the interpretation $\mathfrak{M}_{0,2}^{ss} \cong \mathcal{T}^{\mathrm{nr}}$, this corresponds to dualizing:

$$(L_n \to \cdots \to L_0) \mapsto (L_0^\vee \to \cdots \to L_n^\vee).$$

Proposition 19. *The action of* τ *on the Chow ring* $\mathrm{QSym}[\beta]$ *is given by mapping a quasisymmetric function* g *to* $\sigma^*(g)$ *for* σ *as in 4.3, and by mapping*

$$\beta \mapsto -\beta + M_{(1)} = -\beta + \sum_i \alpha_i.$$

Proof. First observe that τ restricts to σ on the second factor, so we only need to compute $\tau^*\beta$. Since τ^* is an automorphism of graded rings, it preserves the grading of τ, so it is enough to compute it on $\mathbb{G}_m \times \mathcal{T}^{\leq 1} \cong \mathbb{G}_m \times \mathcal{A}^1$, where we can see that the dual diagram $(L_0^\vee \to L_1^\vee)$ is sent by Φ to the pair

$$(L_1^\vee, L_0^\vee \otimes L_1 \to \mathcal{O}) = (L_0^\vee \otimes (L_0^\vee \otimes L_1)^\vee, L_0^\vee \otimes L_1 \to \mathcal{O}).$$

We conclude by remarking that $c_1((L_0^\vee \otimes L_1)^\vee) = M_{(1)}$. $\square$

---

[6]The symbol "nr" stands for "non-rigid".

References

[ACFW13] Dan Abramovich, Charles Cadman, Barbara Fantechi, and Jonathan Wise. Expanded degenerations and pairs. *Comm. Algebra*, 41(6):2346–2386, 2013.

[BV12]   Niels Borne and Angelo Vistoli. Parabolic sheaves on logarithmic schemes. *Adv. Math.*, 231(3-4):1327–1363, 2012.

[Ful98]   William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[Ges84]   Ira M. Gessel. Multipartite P-partitions and inner products of skew Schur functions. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 1984.

[GR14]   Darij Grinberg and Victor Reiner. Hopf algebras in combinatorics, 2014, arXiv:1409.8356.

[GV05]   Tom Graber and Ravi Vakil. Relative virtual localization and vanishing of tautological classes on moduli spaces of curves. *Duke Math. J.*, 130(1):1–37, 2005.

[Haz01]   Michiel Hazewinkel. The algebra of quasi-symmetric functions is free over the integers. *Adv. Math.*, 164(2):283–300, 2001.

[Haz10]   Michiel Hazewinkel. Explicit polynomial generators for the ring of quasisymmetric functions over the integers. *Acta Appl. Math.*, 109(1):39–44, 2010.

[JWY17]   Wanwan Jia, Zhengpan Wang, and Houyi Yu. Rigidity for the hopf algebra of quasisymmetric functions, 2017, arXiv:1712.06499.

[Kre99]   Andrew Kresch. Cycle groups for Artin stacks. *Invent. Math.*, 138(3):495–536, 1999.

[Li01]   Jun Li. Stable morphisms to singular schemes and relative stable morphisms. *J. Differential Geom.*, 57(3):509–578, 2001.

[Li02]   Jun Li. A degeneration formula of GW-invariants. *J. Differential Geom.*, 60(2):199–293, 2002.

[LMvW13] Kurt Luoto, Stefan Mykytiuk, and Stephanie van Willigenburg. *An introduction to quasisymmetric Schur functions*. SpringerBriefs in Mathematics. Springer, New York, 2013. Hopf algebras, quasisymmetric functions, and Young composition tableaux.

[Ols03]   Martin C. Olsson. Logarithmic geometry and algebraic stacks. *Ann. Sci. École Norm. Sup. (4)*, 36(5):747–791, 2003.

[VV03]   Gabriele Vezzosi and Angelo Vistoli. Higher algebraic K-theory for actions of diagonalizable groups. *Invent. Math.*, 153(1):1–44, 2003.

[Wit37]   Ernst Witt. Treue Darstellung Liescher Ringe. *J. Reine Angew. Math.*, 177:152–160, 1937.