Stanley symmetric functions for signed involutions

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Abstract

An involution in a Coxeter group has an associated set of *involution words*, a variation on reduced words. These words are saturated chains in a partial order first considered by Richardson and Springer in their study of symmetric varieties. In the symmetric group, involution words can be enumerated in terms of tableaux using appropriate analogues of the symmetric functions introduced by Stanley to accomplish the same task for reduced words. We adapt this approach to the group of signed permutations. We show that involution words for the longest element in the Coxeter group C_n are in bijection with reduced words for the longest element in $A_n = S_{n+1}$, which are known to be in bijection with standard tableaux of shape $(n, n-1, \ldots, 2, 1)$.

Contents

1 Introduction

Let W be a Coxeter group with simple generating set S. A *reduced word* for $w \in W$ is a minimal-length sequence $(r_1, r_2, \ldots, r_\ell)$ of simple generators $r_i \in S$ with $w = r_1 r_2 \cdots r_\ell$. Let $\mathcal{R}(w)$ be the set of reduced words for w.

Of primary interest are the finite Coxeter groups of classical types A and C, given as follows. Fix a positive integer n and let $[n] = \{1, 2, \dots, n\}$ and $[\pm n] = \{\pm 1, \pm 2, \dots, \pm n\}$. Let $A_n = S_{n+1}$ be the group of permutations of $[n+1]$. Let C_n be the group of permutations w of $[\pm n]$ with $w(-i) = -w(i)$ for all i. Define $s_1, s_2, \dots, s_n \in A_n$ and $t_0, t_1, \dots, t_{n-1} \in C_n$ by

$$s_i = (i, i+1), \qquad t_0 = (-1, 1), \qquad \text{and} \qquad t_i = (-i-1, -i)(i, i+1) \text{ for } i \neq 0. \tag{1.1}$$

Then A_n is a Coxeter group relative to the generating set $S = \{s_1, s_2, \dots, s_n\}$ while C_n is a Coxeter group relative to the generating set $S = \{t_0, t_1, \dots, t_{n-1}\}$. We refer to elements of C_n as *signed permutations*.

Each finite Coxeter group contains a unique element of maximal length, where the *length* of an element w refers to the common length of any word in $\mathcal{R}(w)$. Let w_n^A and w_n^C denote the longest elements of A_n and C_n. Then w_n^A is the permutation given in one-line notation by $(n+1)n \cdots 321$ while w_n^C is the signed permutation given by the negation map $i \mapsto -i$. There are attractive product formulas for the number of reduced words for both of these permutations:

$$\left|\mathcal{R}\left(w_n^A\right)\right| = \frac{\binom{n+1}{2}!}{\prod_{i=1}^{n}(2i-1)^i} \qquad \text{and} \qquad \left|\mathcal{R}\left(w_n^C\right)\right| = \frac{(n^2)!}{n^n \prod_{i=1}^{n-1}[i(2n-i)]^i}. \tag{1.2}$$

Stanley proved the first of these identities [30, Corollary 4.3] and conjectured the second, which was later shown by Haiman [8, Theorem 5.12].

Let $\mathrm{SYT}(\lambda)$ be the set of *standard Young tableaux* of shape λ. Define $\delta_n = (n, n-1, \dots, 2, 1)$ and write (n^n) for the partition with n parts of size n. The identities (1.2) are equivalent to $\left|\mathcal{R}\left(w_n^A\right)\right| = |\mathrm{SYT}(\delta_n)|$ and $\left|\mathcal{R}\left(w_n^C\right)\right| = |\mathrm{SYT}((n^n))|$ via the *hook-length formula* [29, Corollary 7.21.6]. As one would expect from this formulation, there are natural bijective proofs of the identities (1.2), due to Edelman and Greene [6] in type A and to Haiman [8] and Kraśkiewicz [20] in type C.

The main result of this paper is a product formula similar to (1.2) for the cardinality of a set of reduced-word-like objects associated to w_n^C. Write $\ell : W \to \mathbb{N}$ for the length function of the Coxeter system (W, S). There exists a unique associative product $\circ : W \times W \to W$ with $s \circ s = s$ for any $s \in S$ and $u \circ v = uv$ for any $u, v \in W$ such that $\ell(uv) = \ell(u) + \ell(v)$ [19, Theorem 7.1]. This is sometimes called the *Demazure product* or *Hecke product* of (W, S). The pair $(W, \circ)$ is sometimes called the *0-Hecke monoid* of (W, S).

Let $\mathcal{I}(W) \overset{\text{def}}{=} \{y \in W : y = y^{-1}\}$ be the set of involutions in W. This set is preserved by the conjugation action $w : y \mapsto w^{-1} \circ y \circ w$ of the 0-Hecke monoid $(W, \circ)$. Indeed, it is a straightforward exercise from the exchange principle for Coxeter systems (see [4, §1.5]) to check the identity

$$s \circ y \circ s = \begin{cases} sys & \text{if } \ell(ys) > \ell(y) \text{ and } ys \neq sy \\ ys & \text{if } \ell(ys) > \ell(y) \text{ and } ys = sy \\ y & \text{if } \ell(ys) < \ell(y) \end{cases} \qquad \text{for } y \in \mathcal{I}(W) \text{ and } s \in S, \tag{1.3}$$

which is equivalent to [18, Lemma 3.4]. An *involution word* for $y \in \mathcal{I}(W)$ is a minimal-length sequence $(r_1, r_2, \dots, r_\ell)$ of simple generators $r_i \in S$ such that

$$y = r_\ell \circ (\cdots \circ (r_2 \circ (r_1 \circ 1 \circ r_1) \circ r_2) \circ \cdots) \circ r_\ell.$$

The parentheses make clear how to evaluate the right hand expression using (1.3), but are actually superfluous since $\circ$ is associative. Let $\hat{\mathcal{R}}(y)$ be the set of involution words for $y \in \mathcal{I}(W)$. This set is always nonempty, with $\hat{\mathcal{R}}(1) = \{\emptyset\}$ where $\emptyset$ is the empty word. Define $\hat{\ell}(y)$ for $y \in \mathcal{I}(W)$ to be the common length of any word in $\hat{\mathcal{R}}(y)$.

Example 1.1. Let $s_i \in A_n = S_{n+1}$ and $t_i \in C_n$ be as in (1.1). In A_2, we have

$$s_2 \circ (s_1 \circ 1 \circ s_1) \circ s_2 = s_2 \circ s_1 \circ s_2 = s_2 s_1 s_2 \quad \text{and} \quad s_1 \circ (s_2 \circ 1 \circ s_2) \circ s_1 = s_1 \circ s_2 \circ s_1 = s_1 s_2 s_1$$

and it holds that $w_2^A = s_1 s_2 s_1 = s_2 s_1 s_2$ and $\hat{\mathcal{R}}(w_2^A) = \{(s_1, s_2), (s_1, s_2)\}$. In C_2, we have

$$t_0 \circ (t_1 \circ (t_0 \circ 1 \circ t_0) \circ t_1) \circ t_0 = t_0 \circ (t_1 \circ t_0 \circ t_1) \circ t_0 = t_0 \circ t_1 t_0 t_1 \circ t_0 = t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0 = w_2^C$$

and $t_1 \circ (t_0 \circ (t_1 \circ 1 \circ t_1) \circ t_0) \circ t_1 = w_2^C$ and it holds that $\hat{\mathcal{R}}(w_2^C) = \{(t_0, t_1, t_0), (t_1, t_0, t_1)\}$.

Involution words first appeared in work of Richardson and Springer [27, 28], and have since been studied by various authors: Can, Joyce and Wyser [5], the authors and Hamaker [9, 10, 11, 12, 13], Hu and Zhang [16, 17], Hultman [18], and Hansson and Hultman [15]. In [9], the authors and Hamaker showed that

$$|\hat{\mathcal{R}}(w_n^A)| = \binom{\binom{p+1}{2} + \binom{q+1}{2}}{\binom{p+1}{2}} |\operatorname{SYT}(\delta_p)||\operatorname{SYT}(\delta_q)| \tag{1.4}$$

where $p = \lfloor \frac{n}{2} \rfloor$ and $q = \lceil \frac{n}{2} \rceil$, and conjectured the following theorem, which is our main result.

Theorem 1.2. For any positive integer n, it holds that $|\hat{\mathcal{R}}(w_n^C)| = |\operatorname{SYT}(\delta_n)| = |\mathcal{R}(w_n^A)|$.

There is an algebraic approach to enumerating $\mathcal{R}(w_n^A)$, $\mathcal{R}(w_n^C)$, $\hat{\mathcal{R}}(w_n^A)$, and $\hat{\mathcal{R}}(w_n^C)$ by means of certain generating functions called *Stanley symmetric functions*. We write $[x_1 x_2 \cdots]f$ for the coefficient of a square-free monomial in a homogeneous symmetric function f. The Stanley symmetric functions of interest, which will be defined in Section 2.2. have the following properties:

- The (type A) Stanley symmetric function F_w of $w \in A_n$ has $[x_1 x_2 \cdots]F_w = |\mathcal{R}(w)|$.
- The (type C) Stanley symmetric function G_w of $w \in C_n$ has $[x_1 x_2 \cdots]G_w = 2^{\ell(w)}|\mathcal{R}(w)|$.
- The (type A) involution Stanley symmetric function $\hat{F}_y$ of $y \in \mathcal{I}(A_n)$ is a multiplicity-free sum of certain instances of F_w, and has $[x_1 x_2 \cdots]\hat{F}_y = |\hat{\mathcal{R}}(y)|$.
- The (type C) involution Stanley symmetric function $\hat{G}_y$ of $y \in \mathcal{I}(C_n)$ is a multiplicity-free sum of certain instances of G_w, and has $[x_1 x_2 \cdots]\hat{G}_y = 2^{\hat{\ell}(y)}|\hat{\mathcal{R}}(y)|$.

There are expressions for $F_{w_n^A}$, $G_{w_n^C}$, and $\hat{F}_{w_n^A}$ as *Schur functions* s_λ, *Schur Q-functions* Q_λ, and *Schur S-functions* S_λ. For the definitions of these symmetric functions, see Section 2.1. The identities (1.2) and (1.4) are corollaries of these formulas:

Theorem 1.3 (Stanley [30, Corollary 4.2])**.** It holds that $F_{w_n^A} = s_{\delta_n}$.

Theorem 1.4 (Worley [31, Eq. (7.19)]; Billey and Haiman [2, Proposition 3.14])**.** It holds that

$$G_{w_n^C} = Q_{(2n-1, 2n-3, \ldots, 3, 1)} = S_{(n^n)}.$$

Theorem 1.5 (Hamaker, Marberg, and Pawlowski [12, Corollary 1.14])**.** It holds that

$$\hat{F}_{w_n^A} = 2^{-q} Q_{(n, n-2, n-4, \ldots)} = s_{\delta_p} s_{\delta_q}$$

where $p = \lfloor \frac{n}{2} \rfloor$ and $q = \lceil \frac{n}{2} \rceil$.

We prove Theorem 1.2 enumerating $\hat{\mathcal{R}}(w_n^C)$ by adding an entry for $\hat{G}_{w_n^C}$ to this list.

Theorem 1.6. It holds that $\hat{G}_{w_n^C} = G_{w_n^A} = S_{\delta_n}$.

Our proof in Section 5 of this result proceeds as follows. One can define $\hat{G}_{w_n^C}$ as a sum $\sum_{v\in\mathcal{A}_n} G_v$ indexed by a certain set $\mathcal{A}_n$ of signed permutations $v \in C_n$, the *atoms* of w_n^C. The transition formula of Lascoux-Schützenberger [21] as adapted by Billey [1] generates various identities between sums of type C Stanley symmetric functions. Work of Lam implies that $G_{w_n^A} = S_{\delta_n}$ [23], and we apply Billey's transition formula iteratively to rewrite $G_{w_n^A}$ as the sum $\sum_{v\in\mathcal{A}_n} G_v$. The fact that this is possible is somewhat miraculous. Our arguments rely heavily on a recent characterization of the atoms of w_n^C by the first author and Hamaker [14].

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2 Preliminaries

2.1 Symmetric functions

Fix a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$. The *Young diagram* of λ is the set of pairs $D_\lambda \overset{\text{def}}{=} \{(i,j) : i \in [k] \text{ and } j \in [\lambda_i]\}$, which we envision as a collection of left-justified boxes oriented as in a matrix. A *semistandard tableau* of shape λ is a filling of the boxes of the Young diagram D_λ by positive integers, such that each row is weakly increasing from left to right and each column is (strictly) increasing from top to bottom. Such a tableau is *standard* if its boxes contain exactly the numbers $1, 2, \dots, |\lambda|$.

Similarly, a *marked semistandard tableau* of shape λ is a filling of the Young diagram of λ by numbers from the alphabet of primed and unprimed positive integers $\{1,2,3,\dots\} \sqcup \{1',2',3',\dots\}$ such that (i) the rows and columns are weakly increasing under the order $1' < 1 < 2' < 2 < \cdots$, (ii) no unprimed letter i appears twice in the same column, and (iii) no primed letter i' appears twice in the same row.

Assume λ is a strict partition, i.e., has all distinct parts. A *marked semistandard shifted tableau* of shape λ is a filling of the *shifted Young diagram* $\{(i, i+j-1) : (i,j) \in D_\lambda\}$ with primed and unprimed positive integers satisfying properties (i)-(iii) from the previous paragraph. A semistandard marked (shifted) tableau T of shape λ is *standard* if exactly one of i or i' appears in T for each $i = 1, 2, \dots, |\lambda|$.

Given a (marked) semistandard (shifted) tableau T, write x^T for the monomial formed by replacing the boxes in T containing i or i' by x_i and then multiplying the resulting variables.

Example 2.1. If T, U, and V are the tableaux of shape $\lambda = (4,3,1)$ given by

$$T = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 3 \\ \hline 3 & 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-3} 5 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \qquad \text{and} \qquad U = \begin{array}{|c|c|c|c|}\hline 1' & 1 & 1 & 3 \\ \hline 1' & 3 & 4' & \multicolumn{1}{c}{} \\ \cline{1-3} 5 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \qquad \text{and} \qquad V = \begin{array}{|c|c|c|c|}\hline 1 & 2' & 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 2' & 4 & 6 \\ \cline{2-4} \multicolumn{2}{c|}{} & 5 & \multicolumn{1}{c}{} \\ \cline{3-3}\end{array}$$

then T is semistandard, U is marked and semistandard, and V is marked, semistandard, and shifted. We have $x^T = x_2^3 x_3^3 x_4 x_5$ and $x^U = x_1^4 x_3^2 x_4 x_5$ and $x^V = x_1 x_2^2 x_3^2 x_4 x_5 x_6$.

Definition 2.2. Let λ be a partition and let μ be a strict partition. The *Schur function* of λ, the *Schur S-function* of λ, and the *Schur Q-function* of μ are then the respective sums

$$s_\lambda \overset{\text{def}}{=} \sum_T x^T, \qquad S_\lambda \overset{\text{def}}{=} \sum_U x^U, \qquad \text{and} \qquad Q_\mu \overset{\text{def}}{=} \sum_V x^V$$

where T runs over all semistandard tableaux of shape λ, U runs over all semistandard marked tableaux of shape λ, and V runs over all marked semistandard shifted tableaux of shape μ.

The power series s_λ, S_λ, and Q_μ are all symmetric functions. For example, we have

$$\begin{aligned} 4s_{(2,1)} = Q_{(2,1)} &= \sum_{i<j<k} \left(x^{\begin{smallmatrix} i & j \\ & k \end{smallmatrix}} + x^{\begin{smallmatrix} i' & j \\ & k \end{smallmatrix}} + x^{\begin{smallmatrix} i & j' \\ & k \end{smallmatrix}} + x^{\begin{smallmatrix} i' & j' \\ & k \end{smallmatrix}} + x^{\begin{smallmatrix} i & j \\ & k' \end{smallmatrix}} + x^{\begin{smallmatrix} i' & j \\ & k' \end{smallmatrix}} + x^{\begin{smallmatrix} i & j' \\ & k' \end{smallmatrix}} + x^{\begin{smallmatrix} i' & j' \\ & k' \end{smallmatrix}} \right) \\ &\quad + \sum_{i<j} \left(x^{\begin{smallmatrix} i & i \\ & j \end{smallmatrix}} + x^{\begin{smallmatrix} i' & i \\ & j \end{smallmatrix}} + x^{\begin{smallmatrix} i & i \\ & j' \end{smallmatrix}} + x^{\begin{smallmatrix} i' & i \\ & j' \end{smallmatrix}} + x^{\begin{smallmatrix} i & j' \\ & j \end{smallmatrix}} + x^{\begin{smallmatrix} i' & j' \\ & j \end{smallmatrix}} + x^{\begin{smallmatrix} i & j' \\ & j' \end{smallmatrix}} + x^{\begin{smallmatrix} i' & j' \\ & j' \end{smallmatrix}} \right) \\ &= \sum_{i<j<k} 8x_ix_jx_k + \sum_{i<j} 4x_i^2x_j + \sum_{i<j} 4x_ix_j^2 \end{aligned}$$

and $S_{(2,1)} = Q_{(2,1)} + Q_{(3)}$.

The Schur functions s_λ, with λ ranging over all partitions, form a basis for the algebra Λ of symmetric functions. Similarly, the Schur Q-functions Q_μ, with μ ranging over all strict partitions, form a basis for the subalgebra $\Gamma \subset \Lambda$ generated by the odd-indexed power sum symmetric functions. Each Schur Q-function is itself Schur-positive, i.e., a linear combination of Schur functions with positive integer coefficients.

The set of Schur S-functions, with λ ranging over all partitions, is not linearly independent, but also spans the subalgebra Γ. The set $\{S_\lambda : \lambda \text{ is a strict partition}\}$ is a second basis for Γ. For more properties of these functions, see [25, Chaper I, §3] (for s_λ), [25, Chapter III, §8] (for Q_λ), and [25, Chapter III, §8, Ex. 7] (for S_λ).

2.2 Stanley symmetric functions

We review the definitions of the Stanley symmetric functions (see [1, 2, 7, 30]) and involution Stanley symmetric functions (see [9, 12]) mentioned in the introduction.

Definition 2.3. The *type A Stanley symmetric function* associated to $w \in A_n = S_{n+1}$ is

$$F_w \overset{\text{def}}{=} \sum_{\mathbf{a} \in \mathcal{R}(w)} \sum_{(i_1 \le i_2 \le \cdots \le i_l) \in \mathcal{C}(\mathbf{a})} x_{i_1} x_{i_2} \cdots x_{i_l}$$

where for a reduced word $\mathbf{a} = (s_{a_1}, s_{a_2}, \cdots, s_{a_l})$, the set $\mathcal{C}(\mathbf{a})$ consists of all weakly increasing sequences of positive integers $i_1 \le i_2 \le \cdots \le i_l$ such that if $a_j > a_{j+1}$ then $i_j < i_{j+1}$.

Each F_w is a linear combination of Schur functions with positive integer coefficients [6]. For example, $F_{w_2^A} = \sum_{i \le j < k} x_ix_jx_k + \sum_{i<j\le k} x_ix_jx_k = s_{(2,1)}$ as $\mathcal{R}(w_2^A) = \{(s_1, s_2, s_1), (s_2, s_1, s_2)\}$.

Definition 2.4. The *type C Stanley symmetric function* associated to $w \in C_n$ is

$$G_w \overset{\text{def}}{=} \sum_{\mathbf{a} \in \mathcal{R}(w)} \sum_{(i_1 \le i_2 \le \cdots \le i_l) \in \mathcal{D}(\mathbf{a})} 2^{|\{i_1, i_2, \ldots, i_l\}|} x_{i_1} x_{i_2} \cdots x_{i_l}$$

where for a reduced word $\mathbf{a} = (t_{a_1}, t_{a_2}, \cdots, t_{a_l})$, the set $\mathcal{D}(\mathbf{a})$ consists of all weakly increasing sequences of positive integers $i_1 \leq i_2 \leq \cdots \leq i_l$ such that if $a_{j-1} < a_j > a_{j+1}$ for some $j \in [l-1]$ then either $i_{j-1} < i_j \leq i_{j+1}$ or $i_{j-1} \leq i_j < i_{j+1}$.

Each G_w is a linear combination of Schur Q-functions with positive integer coefficients [23, Theorem 3.12]. It is an instructive exercise to check that $G_{w_2^C} = Q_{(3,1)} = S_{(2,2)}$ as predicted by Theorem 1.4; the details are more involved than in our calculation of $F_{w_2^A}$, however.

Remark. The finite Coxeter groups of classical type B are the same as the groups C_n, but there is a distinct notion of type B Stanley symmetric functions. These only differ from G_w by a scalar factor, however: the type B Stanley symmetric function of $w \in C_n$ is $2^{-\ell_0(w)} G_w$ where $\ell_0(w)$ is the number of indices $i \in [n]$ with $w(i) < 0$; see [1, 2, 7].

Notation. The symbols for Stanley symmetric functions are somewhat inconsistent across the literature. The use of F_w for type A Stanley symmetric functions, following [30], is fairly widespread. Nevertheless, these functions are denoted G_w in [2, 3], while in [1, 3] the type C Stanley symmetric functions are denoted F_w. Some authors have also used F_w [2] and G_w [1, 22, 23] for the type B Stanley symmetric functions mentioned in the previous remark.

There is a unique injective group homomorphism $\iota : A_{n-1} \hookrightarrow C_n$ with $\iota(s_i) = t_i$ for $i \in [n-1]$. If $w \in A_{n-1} = S_n$ then $\iota(w)$ is the signed permutation with $\pm i \mapsto \pm w(i)$ for each $i \in [n]$. We define

$$G_w \overset{\text{def}}{=} G_{\iota(w)} \qquad \text{for } w \in A_{n-1}.$$

Although $s_i \mapsto t_i$ induces a bijection $\mathcal{R}(w) \to \mathcal{R}(\iota(w))$, it is not obvious from the definitions how to relate F_w and G_w for $w \in A_{n-1}$. There is a simple connection, however. Define

$$\Lambda \overset{\text{def}}{=} \mathbb{Q}\text{-span}\{s_\lambda\} \qquad \text{and} \qquad \Gamma \overset{\text{def}}{=} \mathbb{Q}\text{-span}\{Q_\mu\} = \mathbb{Q}\text{-span}\{S_\mu\}$$

where the first span is over all partitions λ and the second two are over all strict partitions μ. The *superfication map* $\phi : \Lambda \to \Gamma$ is the linear map with $\phi(s_\lambda) = S_\lambda$ for all partitions λ. This is well-defined since each S_λ is a linear combination of S_μ's with μ strict.

Theorem 2.5 (Lam [22, Theorem 3.10])**.** If $w \in A_{n-1}$ then $\phi(F_w) = G_w$.

We turn to involution Stanley symmetric functions. Let (W, S) be a Coxeter system with length function ℓ. Recall the definition of the Demazure product $\circ : W \times W \to W$ from the introduction.

Definition 2.6. For each $y \in \mathcal{I}(W) = \{z \in W : z = z^{-1}\}$ let $\mathcal{A}(y)$ be the set of elements $w \in W$ with minimal length such that $w^{-1} \circ w = y$. The elements of this set are the *atoms* of y.

The associativity of $\circ$ implies that the set of involution words $\hat{\mathcal{R}}(y)$ for $y \in \mathcal{I}(W)$ is the disjoint union $\bigsqcup_{w \in \mathcal{A}(y)} \mathcal{R}(w)$. The involution length of y is $\hat{\ell}(y) = \ell(w)$ for any $w \in \mathcal{A}(y)$.

Definition 2.7. The *type A* and *type C involution Stanley symmetric functions* associated to $y \in \mathcal{I}(A_n)$ and $z \in \mathcal{I}(C_n)$ are $\hat{F}_y \overset{\text{def}}{=} \sum_{w \in \mathcal{A}(y)} F_w$ and $\hat{G}_z \overset{\text{def}}{=} \sum_{w \in \mathcal{A}(z)} G_w$, respectively.

Since F_w is Schur-positive and G_w is Schur-Q-positive, it holds by construction that $\hat{F}_y$ and $\hat{G}_z$ are respectively Schur-positive and Schur-Q-positive. For $\hat{F}_y$, a stronger statement holds: if $\kappa(y)$ is the number of 2-cycles in $y \in \mathcal{I}(A_n)$, then $2^{\kappa(y)} \hat{F}_y$ is also Schur-Q-positive [12, Corollary 4.62]. We do not know if $\hat{G}_z$ has any stronger positivity property along these lines; see Section 6.2.

Example 2.8. From Example 1.1, we see that $\mathcal{A}(w_2^C) = \{t_0t_1t_0, t_1t_0t_1\}$. Therefore

$$\hat{G}_{w_2^C} = \sum_{\substack{i\leq j\leq k\\ i<j \text{ or } j<k}} 2^{|\{i,j,k\}|} x_ix_jx_k + \sum_{i\leq j\leq k} 2^{|\{i,j,k\}|} x_ix_jx_k = Q_{(2,1)} + Q_{(3)} = S_{(2,1)}.$$

Define $\hat{G}_y \overset{\text{def}}{=} \hat{G}_{\iota(y)}$ for $y \in \mathcal{I}(A_{n-1})$. Since $\mathcal{A}(\iota(y)) = \iota(\mathcal{A}(y))$, the following holds:

Corollary 2.9. If $y \in \mathcal{I}(A_{n-1})$ then $\hat{G}_y = \phi(\hat{F}_y)$.

2.3 Transition formulas

We use the term *word* to refer to a finite sequence of nonzero integers. The *one-line representation* of a signed permutation $w \in C_n$ is the word $w_1w_2\cdots w_n$ where we set $w_i = w(i)$. We usually write $\overline{m}$ in place of $-m$ so that, for example, the eight elements of C_2 are 12, $\overline{1}2$, $1\overline{2}$, $\overline{1}\,\overline{2}$, 21, $\overline{2}1$, $2\overline{1}$, and $\overline{2}\,\overline{1}$. In this notation, the longest element of C_n is

$$w_n^C = \overline{1}\,\overline{2}\,\overline{3}\cdots\overline{n}.$$

The map $w_1w_2\cdots w_n \mapsto w_1w_2\cdots w_n(n{+}1)$ is an inclusion $C_n \hookrightarrow C_{n+1}$. We do not distinguish between w and its image under this map. If $w \in C_n$ then the words $w_1w_2\cdots w_n$ and $w_1w_2\cdots w_n(n+1)(n+2)\cdots(n+m)$ represent the same signed permutation for all $m \in \mathbb{N}$.

Let $w \in C_n$. Define $\mathrm{inv}_\pm(w)$ as the number of pairs $(i,j) \in [\pm n]\times[\pm n]$ with $i<j$ and $w_i > w_j$. Define $\ell_0(w)$ as the number of integers $i \in [n]$ with $w_i < 0$.

Lemma 2.10 (See [1, §3])**.** The length function of C_n has the formula $\ell(w) = \frac{1}{2}\left(\mathrm{inv}_\pm(w) + \ell_0(w)\right)$.

A *reflection* in a Coxeter group is an element conjugate to a simple generator. With our notation as in [1, §3], the reflections in C_n are the following elements:

(1) $s_{ii} \overset{\text{def}}{=} 1\cdots\overline{i}\cdots n = (i,\overline{i})$ for $i \in [n]$.

(2) $s_{ij} = s_{ji} \overset{\text{def}}{=} 1\cdots\overline{j}\cdots\overline{i}\cdots n = (i,\overline{j})(\overline{i},j)$ for $i,j \in [n]$ with $i<j$.

(3) $t_{ij} = t_{ji} \overset{\text{def}}{=} 1\cdots j\cdots i\cdots n = (i,j)(\overline{i},\overline{j})$ for $i,j \in [n]$ with $i<j$.

Observe that $t_0 = s_{11}$ and $t_i = t_{i,i+1}$ and $s_{ij} = s_{ii}t_{ij}s_{ii} = s_{jj}t_{ij}s_{jj}$ for $i,j \in [n]$ with $i<j$. If $u,v \in C_n$ are any elements and $t \in C_n$ is a reflection such that $v = ut$ and $\ell(v) = \ell(u)+1$, then we write $u \lessdot v$, so that $\lessdot$ is the covering relation of the *Bruhat order* of C_n.

Lemma 2.11 ([1, Lemmas 1 and 2])**.** Let $w = w_1w_2\cdots w_n \in C_n$ and $i,j \in [n]$.

(a) One has $w \lessdot ws_{ii}$ if and only if $w_i > 0$ and $-w_i < e < w_i \Rightarrow e \notin \{w_1, w_2, \ldots, w_{i-1}\}$.

(b) If $i<j$ and $w_i > 0$, then $w \lessdot ws_{ij}$ if and only if

$$0 < -w_j < w_i \qquad \text{and} \qquad \begin{cases} -w_j < e < w_i \;\Rightarrow\; e \notin \{w_1, w_2, \ldots, w_{i-1}\}, \\ -w_i < e < w_j \;\Rightarrow\; e \notin \{w_{i+1}, w_{i+2}, \ldots, w_{j-1}\}. \end{cases}$$

(c) If $i<j$ then $w \lessdot wt_{ij}$ if and only if

$$w_i < w_j \qquad \text{and} \qquad w_i < e < w_j \ \Rightarrow\ e \notin \{w_{i+1}, w_{i+2}, \dots w_{j-1}\}.$$

For example, it holds that $1\overline{2}43 \lessdot 1\overline{2}43 \cdot t_{14} = 3\overline{2}41$ while $1243 \not\lessdot 1243 \cdot t_{14} = 3241$, and it holds that $324\overline{1} \lessdot 324\overline{1} \cdot s_{14} = 124\overline{3}$ while $3\overline{2}4\overline{1} \not\lessdot 3\overline{2}4\overline{1} \cdot s_{14} = 1\overline{2}4\overline{3}$.

Lemma 2.11(c) says that $w \lessdot wt_{ij}$ if and only if $w_i < w_j$ and no entry in $w_1 w_2 \cdots w_n$ between positions i and j is between w_i and w_j in value. Lemma 2.11(a) can be described in the same way using "symmetric" one-line notation: one has $w \lessdot ws_{ii}$ if and only if $\overline{w_i} < w_i$ and no number between $\overline{w_i}$ and w_i appears in the word $\overline{w_{i-1}} \cdots \overline{w_2}\,\overline{w_1} w_1 w_2 \cdots w_{i-1}$. One can express Lemma 2.11(b) similarly. We frequently only need the following special cases of these conditions:

Lemma 2.12. Let $w = w_1 w_2 \cdots w_n \in C_n$ and $j \in \{2, 3, \dots, n\}$. Assume $w_1 > 0$.

(a) $wt_0 \lessdot w$.

(b) $ws_{1j} \lessdot w$ if and only if $w_1 < -w_j$ and no $e \in \{w_2, w_3, \dots, w_{j-1}\}$ has $w_j < e < -w_1$.

(c) $wt_{1j} \lessdot w$ if and only if $w_1 > w_j$ and no $e \in \{w_2, w_3, \dots, w_{j-1}\}$ has $w_1 > e > w_j$.

Let $[m, n] = \{i \in \mathbb{Z} : m \leq i \leq n\}$. For $w \in C_n$ and $j \in [n]$, we define three sets:

$$\begin{aligned} \mathcal{T}^+_j(w) &\overset{\text{def}}{=} \{wt_{jk} : k \in [j+1, n+1], w \lessdot wt_{jk}\} \subseteq C_{n+1}, \\ \mathcal{T}^-_j(w) &\overset{\text{def}}{=} \{wt_{ij} : i \in [j-1], w \lessdot wt_{ij}\} \subseteq C_n, \\ \mathcal{S}_j(w) &\overset{\text{def}}{=} \{ws_{ij} : i \in [n], w \lessdot ws_{ij}\} \subseteq C_n. \end{aligned} \tag{2.1}$$

The next theorem, which is analogous to the transition formulas of Lascoux and Schützenberger [21], is the main technical tool we require to work with type C Stanley symmetric functions.

Theorem 2.13 (Billey [1, Lemma 8])**.** If $w \in C_n$ and $j \in [n]$ then

$$\sum_{u \in \mathcal{T}^+_j(w)} G_u = \sum_{v \in \mathcal{S}_j(w)} G_v + \sum_{v \in \mathcal{T}^-_j(w)} G_v.$$

This result leads to an effective algorithm for computing the Schur Q-expansion of G_w.

Theorem 2.14 (Billey [1, Corollary 9])**.** Suppose $w \in C_n$.

(a) If $w_1 < \cdots < w_r < 0 < w_{r+1} < \cdots < w_n$ for some $r \in [n]$, then $G_w = Q_{(-w_1, -w_2, \dots, -w_r)}$.

(b) Suppose $(r, s) \in [n] \times [n]$ is lexicographically maximal such that $r < s$ and $w_r > w_s$. Let $v = wt_{rs}$. Then $G_w = \sum_{i \in [n], v \lessdot vs_{ir}} G_{vs_{ir}} + \sum_{i \in [r-1], v \lessdot vt_{ir}} G_{vt_{ir}}$.

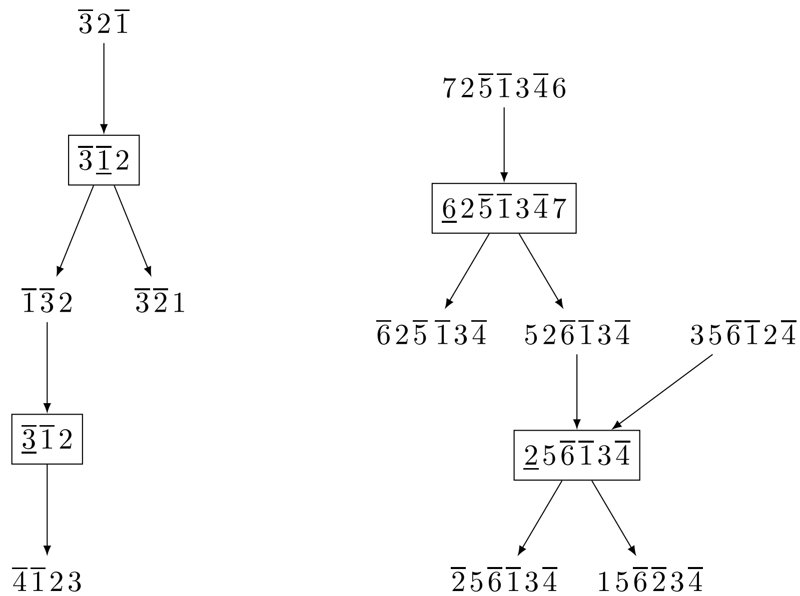
The theorem gives a recursion for G_w which terminates when w is strictly increasing. Billey shows that this recursion always terminates in a finite number of steps [1, Theorem 4].

Example 2.15. The results of [14] (see Section 3) imply that $\mathcal{A}(\overline{8}\,\overline{7}\,\overline{6}\,\overline{5}\,\overline{4}\,\overline{3}\,\overline{2}\,\overline{1}) = \{\overline{8}\,\overline{6}\,\overline{4}\,\overline{2}1357\}$ so

$$\hat{G}_{\overline{8}\,\overline{7}\,\overline{6}\,\overline{5}\,\overline{4}\,\overline{3}\,\overline{2}\,\overline{1}} = G_{\overline{8}\,\overline{6}\,\overline{4}\,\overline{2}1357} = Q_{(8,6,4,2)}$$

by Theorem 2.14(a). It follows that the number of involution words for $\overline{8}\,\overline{7}\,\overline{6}\,\overline{5}\,\overline{4}\,\overline{3}\,\overline{2}\,\overline{1}$ times $2^{8+6+4+2}$ is equal to the number of marked standard shifted tableaux of shape $(8, 6, 4, 2)$. This example generalizes in a straightforward way from C_8 to any C_n.

Example 2.16. For each boxed vertex w in the directed bipartite graphs $\overrightarrow{\mathcal{G}}$ below, the identity $\sum_{\{u\to w\}\in\overrightarrow{\mathcal{G}}} G_u = \sum_{\{w\to u\}\in\overrightarrow{\mathcal{G}}} G_u$ holds, and is an instance of Theorem 2.13 with j as the index of the underlined letter of w.



The graph on the left, illustrating Theorem 2.14, is constructed as follows: from each unboxed vertex w which is not increasing (starting with $\overline{3}2\overline{1}$), draw an arrow to the boxed vertex $v = wt_{rs}$ and underline the letter of v in position r. In reading these graphs one should keep in mind that we identify $w_1w_2\cdots w_n \in C_n$ with $w_1w_2\cdots w_n(n{+}1) \in C_{n+1}$. For both graphs $\overrightarrow{\mathcal{G}}$ it holds that

$$\sum_{u\in\mathsf{Sink}(\overrightarrow{\mathcal{G}})} G_u = \sum_{u\in\mathsf{Source}(\overrightarrow{\mathcal{G}})} G_u$$

(cf. Lemma 2.17 below), i.e.

$$G_{\overline{3}2\overline{1}} = G_{\overline{3}\overline{2}1} + G_{\overline{4}\overline{1}23} = Q_{(3,2)} + Q_{(4,1)}$$

(using Theorem 2.14(a)) and

$$G_{72\overline{5}\overline{1}3\overline{4}6} + G_{35\overline{6}\overline{1}2\overline{4}} = G_{\overline{6}2\overline{5}\overline{1}3\overline{4}} + G_{\overline{2}5\overline{6}\overline{1}3\overline{4}} + G_{15\overline{6}\overline{2}3\overline{4}}.$$

Suppose $\overrightarrow{\mathcal{G}}$ is a directed graph with x a vertex. Write $\mathrm{sdeg}(x)$ for the indegree of x minus its outdegree, and $\deg(x)$ for the indegree of x plus its outdegree.

Lemma 2.17. Let $\overrightarrow{\mathcal{G}}$ be a bipartite directed graph with vertex set $\mathcal{V}$ and bipartition $\mathcal{V} = \mathcal{V}^- \sqcup \mathcal{V}^+$ such that $\mathrm{sdeg}(u) \in \{-1,0,1\}$ for all $u \in W$. Suppose $f : \mathcal{V} \to A$ is a function to an abelian group A such if $w \in \mathcal{V}^-$ then $f(w) = 0$ and $\sum_{\{u\to w\}\in\overrightarrow{\mathcal{G}}} f(u) = \sum_{\{w\to u\}\in\overrightarrow{\mathcal{G}}} f(u)$. Then

$$\sum_{u\in\mathsf{Sink}(\overrightarrow{\mathcal{G}})} \deg(u)f(u) = \sum_{u\in\mathsf{Source}(\overrightarrow{\mathcal{G}})} \deg(u)f(u).$$

Proof. By assumption $\sum_{\{x\to y\}\in\overrightarrow{\mathcal{G}}}(f(y)-f(x)) = \sum_{w\in\mathcal{V}^-}\left(\sum_{w\to u} f(u) - \sum_{u\to w} f(u)\right) = 0$, while $\sum_{\{x\to y\}\in\overrightarrow{\mathcal{G}}}(f(y)-f(x)) = \sum_{u\in\mathcal{V}^+}\mathrm{sdeg}(u)f(u) = \sum_{u\in\mathsf{Sink}(\overrightarrow{\mathcal{G}})}\deg(u)f(u) - \sum_{u\in\mathsf{Source}(\overrightarrow{\mathcal{G}})}\deg(u)f(u)$. $\square$

3 Atoms

The atoms of the longest element $w_n^C \in \mathcal{I}(C_n)$ have a number of special properties, which we review in this section. Let $S \subset \mathbb{Z}$ be a set of integers. A *perfect matching* on a set S is a set M of pairwise disjoint 2-element subsets $\{i,j\}$, referred to as *blocks*, whose union is S. A perfect matching M is *symmetric* if $\{i,j\} \in M$ implies $-\{i,j\} = \{-i,-j\} \in M$, and *noncrossing* if it does not occur that $i < a < j < b$ for any $\{i,j\},\{a,b\} \in M$. Let $\mathsf{NCSP}(n)$ denote the set of noncrossing, symmetric, perfect matchings on the set $[\pm n]$. The three elements of $\mathsf{NCSP}(3)$ are

$$\{\{\pm 1\},\{\pm 2\},\{\pm 3\}\}, \qquad \{\pm\{1,2\},\{\pm 3\}\}, \qquad \text{and} \qquad \{\{\pm 1\},\pm\{2,3\}\}.$$

In general, $|\mathsf{NCSP}(n)| = \binom{n}{\lfloor n/2 \rfloor}$; see [26, A001405]. We emphasize the following basic observation:

Fact. If $M \in \mathsf{NCSP}(n)$ and $\{i,j\} \in M$, then i and j have the same sign or $i = -j$.

If $w = w_1 w_2 \cdots w_n$ is a word then we write $[[w]]$ for the subword formed by omitting each repeated letter after its first appearance. For example, $[[31231124]] = 3124$. Suppose M is a symmetric noncrossing perfect matching of a subset of $[\pm n]$. Define

$$\mathrm{Pair}(M) \overset{\text{def}}{=} \{(a,-b) : \{a,b\} \in M \text{ and } 0 < a < b\} \sqcup \{(-a,-a) : \{-a,a\} \in M \text{ and } 0 < a\}.$$

Let $(a_1,b_1),(a_2,b_2),\dots,(a_l,b_l)$ (respectively, $(c_1,d_1),(c_2,d_2),\dots,(c_l,d_l)$) be the elements $\mathrm{Pair}(M)$ listed in order such that $b_1 < b_2 < \cdots < b_l$ (respectively, $c_1 < c_2 < \cdots < c_l$). Define the words

$$\alpha_{\min}(M) \overset{\text{def}}{=} [[a_1 b_1 a_2 b_2 \cdots a_l b_l]] \qquad \text{and} \qquad \alpha_{\max}(M) \overset{\text{def}}{=} [[c_1 d_1 c_2 d_2 \cdots c_l d_l]].$$

If $M \in \mathsf{NCSP}(n)$, then $\alpha_{\min}(M)$ and $\alpha_{\max}(M)$ contain exactly one letter from $\{\pm i\}$ for each $i \in [n]$, so may be interpreted as elements of C_n.

Example 3.1. If $M = \{\pm\{1,3\},\pm\{4,7\},\pm\{5,6\},\{\pm 8\}\}$ then

$$\mathrm{Pair}(M) = \{(-8,-8),(4,-7),(5,-6),(1,-3)\} = \{(-8,-8),(1,-3),(4,-7),(5,-6)\}\}$$

and $\alpha_{\min}(M) = \overline{8}4\overline{7}5\overline{6}1\overline{3}$ and $\alpha_{\max}(M) = \overline{8}1\overline{3}4\overline{7}5\overline{6}$.

If u and v are words, both with $n \geq i+2$ letters, then we write $u \lessdot_i v$ to mean that

$$u_i u_{i+1} u_{i+2} = cab, \quad v_i v_{i+1} v_{i+2} = bca, \quad \text{and} \quad u_j = v_j \text{ for } j \notin \{i, i+1, i+2\} \tag{3.1}$$

for some numbers $a < b < c$. Define $<_{\mathcal{A}}$ as the transitive closure of the relations $\lessdot_i$ for all $i \geq 1$. Equivalently, $<_{\mathcal{A}}$ is the transitive closure of the relation on words with

$$\cdots cab \cdots <_{\mathcal{A}} \cdots bca \cdots \tag{3.2}$$

whenever $a < b < c$ and the corresponding ellipses mask identical subsequences. This relation is a partial order since it is a sub-relation of lexicographic order. We apply $<_{\mathcal{A}}$ to signed permutations via their one-line representations. Define

$$\mathcal{A}_n \overset{\text{def}}{=} \mathcal{A}(\overline{1}\,\overline{2}\,\overline{3}\cdots\overline{n}) \subset C_n$$

and for each $M \in \mathsf{NCSP}(n)$ let $\mathcal{A}_M \overset{\text{def}}{=} \{w \in C_n : \alpha_{\min}(M) \leq_{\mathcal{A}} w \leq_{\mathcal{A}} \alpha_{\max}(M)\}$.

Example 3.2. Let $M = \{\{\pm 1\}, \pm\{2,3\}, \pm\{4,5\}\} \in \mathsf{NCSP}(5)$. The interval $\mathcal{A}_M$ is

$$\begin{tikzcd}[column sep=small]
 & \overline{1}2\overline{3}4\overline{5} \arrow[dl, dash] \arrow[dr, dash] & \\
\overline{1}24\overline{5}\,\overline{3} \arrow[d, dash] & & 2\,\overline{3}\,\overline{1}\,4\,\overline{5} \arrow[d, dash] \\
\overline{1}4\overline{5}2\overline{3} \arrow[d, dash] & & 2\overline{3}4\overline{51} \arrow[d, dash] \\
4\overline{5}\,\overline{1}2\overline{3} \arrow[dr, dash] & & 24\overline{5}\,\overline{3}\,\overline{1} \arrow[dl, dash] \\
 & 4\overline{5}2\overline{3}\,\overline{1} &
\end{tikzcd}$$

Theorem 3.3 (See [14])**.** There is a disjoint decomposition $\mathcal{A}_n = \bigsqcup_{M \in \mathsf{NCSP}(n)} \mathcal{A}_M$.

This result remains true when $n = 0$ if we take both $\mathcal{A}_0$ and C_0 to be the singleton set containing just the empty word $\emptyset$, and define $\alpha_{\min}(M) = \alpha_{\max}(M) = \emptyset$ if $M = \varnothing \in \mathsf{NCSP}(0)$.

Proof. [14, Theorem 5.6] describes the connected components of $\mathcal{A}(z)^{-1} = \{w^{-1} : w \in \mathcal{A}(z)\}$ under $\leq_{\mathcal{A}}$ for any $z \in \mathcal{I}(C_n)$ in terms of symmetric noncrossing matchings, and implies that $\mathcal{A}_n^{-1} = \bigsqcup_{M \in \mathsf{NCSP}(n)} \mathcal{A}_M$. The theorem follows since $\mathcal{A}_n = \mathcal{A}_n^{-1}$ by [14, Proposition 2.7]. $\square$

Fix an atom $w \in \mathcal{A}_n$. We define the *shape* of w to be the unique matching $M(w) \in \mathsf{NCSP}(n)$ with $w \in \mathcal{A}_{M(w)}$. It is helpful to understand how the shape of w can be extracted from the one-line representation $w_1 w_2 \cdots w_n$. This can be done as follows.

From $w \in \mathcal{A}_n$, we produce a sequence of words $w^0, w^1, \dots, w^l$. Start by letting $w^0 = w_1 w_2 \cdots w_n$. For each $i > 0$, form w^i by removing an arbitrary descent from w^{i-1}, where a *descent* in a word $a_1 a_2 \cdots a_n$ is a consecutive subword $a_i a_{i+1}$ with $a_i > a_{i+1}$. The sequence terminates when we obtain an increasing word w^l. Let $\{c_1, c_2, \dots, c_k\}$ be the set of letters in w^l and suppose $q_i p_i$ is the descent removed from w^{i-1} to form w^i. We then define

$$\mathrm{NNeg}(w) \overset{\text{def}}{=} \{-c_1, -c_2, \dots, -c_k\} \qquad \text{and} \qquad \mathrm{NDes}(w) \overset{\text{def}}{=} \{(q_1, p_1), (q_2, p_2), \dots, (q_l, p_l)\}.$$

We refer to $\mathrm{NNeg}(w)$ and $\mathrm{NDes}(w)$ as the *nested negated set* and *nested descent set* of w.

Theorem 3.4 (See [14])**.** No matter how the words $w^0, w^1, \dots, w^l$ are constructed, we have:

(a) $\mathrm{NNeg}(w) = \{a : \{-a, a\} \in M(w) \text{ and } 0 < a\}$.

(b) $\mathrm{NDes}(w) = \{(a, -b) : \{a, b\} \in M(w) \text{ and } 0 < a < b\}$.

Proof. This is equivalent to [14, Theorem-Definition 3.10] since $\mathcal{A}_n = \mathcal{A}_n^{-1}$ [14, Proposition 2.7]. $\square$

Example 3.5. If $M = \{\pm\{1,3\}, \pm\{4,7\}, \pm\{5,6\}, \{\pm 8\}\}$ as in Example 3.1, then

$$w = \overline{8}14\overline{7}5\overline{6}\,\overline{3} \lessdot_5 \overline{8}14\overline{7}\,\overline{3}5\overline{6} \lessdot_3 \overline{8}1\overline{3}4\overline{7}5\overline{6} = \alpha_{\max}(M),$$

so $w \in \mathcal{A}_M$. There are two ways to progressively remove descents from w as described above:

$$w = w^0 = \overline{8}14\overline{7}5\overline{6}\,\overline{3}, \quad w^1 = \overline{8}15\overline{6}\,\overline{3}, \quad w^2 = \overline{8}1\overline{3}, \quad w^3 = \overline{8}$$
$$w = w^0 = \overline{8}14\overline{7}5\overline{6}\,\overline{3}, \quad w^1 = \overline{8}14\overline{7}\,\overline{3}, \quad w^2 = \overline{8}1\overline{3}, \quad w^3 = \overline{8}$$

Both give $\mathrm{NNeg}(w) = \{8\}$ and $\mathrm{NDes}(w) = \{(1,-3), (4,-7), (5,-6)\}$ as claimed by Theorem 3.4.

The preceding theorem has several implications, starting with the following observation.

Corollary 3.6. Let $M \in \mathsf{NCSP}(n)$ and $w \in \mathcal{A}_M$. If $w_i > w_{i+1}$ for some $i \in [n-1]$ then $0 < w_i < -w_{i+1}$ and $\{w_i, -w_{i+1}\} \in M$. The word $w_1 w_2 \cdots w_n$ therefore contains no consecutive subwords of the form ba where $0 < a < b$ or $a < b < 0$, or of the form cba where $a < b < c$.

Subwords in the following lemma need not be consecutive.

Lemma 3.7. Let $M \in \mathsf{NCSP}(n)$ and $w \in \mathcal{A}_M$. Suppose $0 < a < b < c < d$.

(a) If $\{a, d\}, \{b, c\} \in M$ then $a\overline{d}b\overline{c}$ is a subword of $w_1 w_2 \cdots w_n$.

(b) If $\{a, b\}, \{\pm c\} \in M$ then $\overline{c}a\overline{b}$ is a subword of $w_1 w_2 \cdots w_n$.

(c) If $\{\pm a\}, \{\pm b\} \in M$ then $\overline{b}\overline{a}$ is a subword of $w_1 w_2 \cdots w_n$.

Proof. Suppose $S = -S \subset [\pm n]$ is a union of blocks in M. Given $v \in \mathcal{A}_n$, let v_S be the subword of $v_1 v_2 \cdots v_n$ with all letters not in S removed. It follows from Corollary 3.6 that if $u, v \in \mathcal{A}_n$ and $u \leq_{\mathcal{A}} v$ then $u_S \leq_{\mathcal{A}} v_S$. Let $u = \alpha_{\min}(M)$ and $v = \alpha_{\max}(M)$. If $\{a, d\}, \{b, c\} \in M$ and $S = \{\pm a, \pm b, \pm c, \pm d\}$ then it follows that $u_S \leq_{\mathcal{A}} w_S \leq_{\mathcal{A}} v_S$. Since in this case $u_S = v_S = a\overline{d}b\overline{c}$, we deduce that $w_S = a\overline{d}b\overline{c}$ is a subword of $w_1 w_2 \cdots w_n$. Parts (b) and (c) follow similarly. $\square$

Let $w \in \mathcal{A}_n$. If $1 \leq i < j \leq n$ are indices such that $0 < w_i < -w_j$ and $\{w_i, -w_j\} \in M(w)$, then we say that i and j are *complementary indices* in w. If $i \in [n]$ is such that $w_i < 0$ and $\{\pm w_i\} \in M(w)$, then we say that i is a *symmetric index* in w.

Corollary 3.8. If $w \in \mathcal{A}_n$ then each $i \in [n]$ is symmetric or belongs to complementary pair.

Therefore, if $w \in \mathcal{A}_n$ and $i \in [n]$ is such that $w_i > 0$, then there exists a complementary index $j \in [n]$ with $i < j$ and $w_i < -w_j$ and $\{w_i, -w_j\} \in M(w)$. In turn, since 1 cannot be the second index in a complementary pair, if $w \in \mathcal{A}_n$ and $w_1 < 0$ then we must have $\{\pm w_1\} \in M(w)$.

Lemma 3.9. Suppose $1 \leq i < j \leq n$ are complementary indices for $w \in \mathcal{A}_n$ and $e \in [\pm n]$.

(a) If $w_j < e < -w_i < 0$, then $e \notin \{w_1, w_2, \ldots, w_j, w_{j+1}\}$.

(b) If $0 < w_i < e < -w_j$, then $e \notin \{w_1, w_2, \ldots, w_j\}$.

Proof. We have $\{w_i, -w_j\} \in M(w)$, so if $e \in \{w_1, w_2, \ldots, w_n\}$ and $w_i < |e| < -w_j$, then the noncrossing matching $M(w)$ must contain a block $\{a, b\}$ with $|e| \in \{a, b\}$ and $w_i < a < b < -w_j$, and in this case $w_i w_j a\overline{b}$ must be a subword of $w_1 w_2 \ldots w_n$ by Lemma 3.7(a). $\square$

Lemma 3.10. If $1 \leq i < j \leq n$ are complementary indices for $w \in \mathcal{A}_n$ then $ws_{ij} \lessdot w$.

Proof. This is immediate from Lemmas 2.11(b) and 3.9. $\square$

4 Quasi-atoms

Given a word $w = w_1 w_2 \cdots w_n$ such that $|w_1|$, $|w_2|$, $\ldots$, $|w_n|$ are distinct and nonzero, define $\mathrm{fl}_\pm(w) \in C_n$ to be the signed permutation whose one-line representation is formed by replacing each letter of w by its image under the order-preserving bijection $\{\pm w_1, \pm w_2, \ldots, \pm w_n\} \to [\pm n]$. For example, we have $\mathrm{fl}_\pm(3\overline{2}5\overline{7}) = 2\overline{1}3\overline{4} \in C_4$. If M is a partition of a symmetric $2n$-element subset $X = -X \subset [\pm m]$, then define $\mathrm{fl}_\pm(M)$ to be the partition of $[\pm n]$ formed by replacing each element of each block of M by its image under the order-preserving bijection $X \to [\pm n]$.

Suppose $w \in C_n$ and $v = \mathrm{fl}_\pm(w_2 w_3 \cdots w_n) \in \mathcal{A}_{n-1}$. Define $M'(w)$ to be the unique perfect matching on $[n] \setminus \{\pm w_1\}$ with $\mathrm{fl}_\pm(M'(w)) = M(v)$. Since $M(v)$ is symmetric and noncrossing, $M'(w)$ is symmetric and noncrossing.

The matching $M'(w)$ may be read off directly from the one-line representation of w by the following procedure. Let $w^0, w^1, w^2, \ldots, w^l$ be any sequence of words whose first term is $w^0 = w_2 w_3 \cdots w_n$ (note the deliberate omission of w_1) and whose final term is strictly increasing, in which w^i for $i > 0$ is formed from w^{i-1} by removing a single descent $q_i p_i$. Let $\{c_1, c_2, \ldots, c_k\}$ be the set of letters in w^l. Then $M'(w)$ is the matching whose blocks consist of $\{p, -q\}$, $\{-p, q\}$, and $\{\pm c\}$ for each descent $(q, p) \in \{(q_1, p_1), (q_2, p_2), \ldots, (q_l, p_l)\}$ and each $c \in \{c_1, c_2, \ldots, c_k\}$. This construction is independent of the choices of descents by Theorem 3.4.

Example 4.1. Let $w = 316\overline{7}4\overline{5}\,\overline{2}$. One sequence of words $w^0, w^1, \ldots, w^l$ as described above is

$$w^0 = 1\,6\,\overline{7}\,4\,\overline{5}\,\overline{2}, \quad w^1 = 1\,4\,\overline{5}\,\overline{2}, \quad w^2 = 1\,\overline{2}, \quad w^3 = \emptyset,$$

so $M'(w) = \{\pm\{1,2\}, \pm\{4,5\}, \pm\{6,7\}\}$. Setting $v = \mathrm{fl}_\pm(w_2 w_3 \cdots w_n)$, we have

$$\alpha_{\min}(M) = 5\overline{6}3\overline{4}1\overline{2} \lhd_3 5\overline{6}13\overline{4}\,\overline{2} \lhd_1 15\overline{6}3\overline{4}\,\overline{2} = v$$

where $M = \{\pm\{1,2\}, \pm\{3,4\}, \pm\{5,6\}\} = \mathrm{fl}_\pm(M'(w))$, so $v \in \mathcal{A}_M$.

We define $\mathcal{A}_0$ to be the singleton set containing just the empty word $\emptyset$.

Definition 4.2. An element $w \in C_n$ is a *quasi-atom* if the following conditions hold:

(a) One has $w_1 > 0$ and $\mathrm{fl}_\pm(w_2 w_3 \cdots w_n) \in \mathcal{A}_{n-1}$, so $M'(w)$ is defined.

(b) At most one block $\{a, b\} \in M'(w)$ has $0 < a < w_1 < b$.

(c) No symmetric block $\{\pm c\} \in M'(w)$ has $0 < w_1 < c$.

A quasi-atom w is *odd* if no block $\{a, b\} \in M'(w)$ exists with $0 < a < w_1 < b$; otherwise, w is *even*. We write $\mathcal{Q}_n^+$ and $\mathcal{Q}_n^-$ for the sets of even and odd quasi-atoms in C_n, and define $\mathcal{Q}_n \overset{\text{def}}{=} \mathcal{Q}_n^+ \sqcup \mathcal{Q}_n^-$.

Example 4.3. By convention we have

$\mathcal{A}_0 = \{\emptyset\}$,

$\mathcal{Q}_1^+ = \varnothing$ and $\mathcal{Q}_1^- = \{1\}$.

In rank two we have:

$\mathcal{A}_1 = \{\overline{1}\}$,

$$\mathcal{Q}_2^+ = \varnothing \text{ and } \mathcal{Q}_2^- = \{2\overline{1}\}.$$

In rank three we have:

$$\mathcal{A}_2 = \{\overline{2}\,\overline{1}, 1\overline{2}\},$$

$$\mathcal{Q}_3^+ = \{21\overline{3}\} \text{ and } \mathcal{Q}_3^- = \{3\overline{2}\,\overline{1}, 31\overline{2}, 12\overline{3}\}.$$

In rank four we have:

$$\mathcal{A}_3 = \{\overline{3}\,\overline{2}\,\overline{1}, \overline{3}1\overline{2}, 2\overline{3}\,\overline{1}, \overline{1}2\overline{3}\},$$

$$\mathcal{Q}_4^+ = \{32\overline{4}\,\overline{1}, 3\overline{1}2\overline{4}\} \text{ and } \mathcal{Q}_4^- = \{4\overline{3}\,\overline{2}\,\overline{1}, 4\overline{3}1\overline{2}, 42\overline{3}\,\overline{1}, 4\overline{1}2\overline{3}, 23\overline{4}\overline{1}, 2\overline{1}3\overline{4}\}.$$

The sequences of cardinalities

$$\begin{aligned}
(|\mathcal{A}_n| : n = 1,2,3,\dots) &= (1,\ 2,\ 4,\ 11,\ 30,\ 101,\ 336,\ 1310,\ 5039,\ \dots) \\
\big(|\mathcal{Q}_n^+| : n = 1,2,3,\dots\big) &= (0,\ 0,\ 1,\ 2,\ 11,\ 30,\ 151,\ 501,\ 2592,\ \dots) \\
\big(|\mathcal{Q}_n^-| : n = 1,2,3,\dots\big) &= (1,\ 1,\ 3,\ 6,\ 21,\ 57,\ 228,\ 753,\ 3359,\ \dots) \\
(|\mathcal{Q}_n| : n = 1,2,3,\dots) &= (1,\ 1,\ 4,\ 8,\ 32,\ 87,\ 379,\ 1254,\ 5951,\ \dots)
\end{aligned}$$

do not match any existing entries in [26].

It can happen that $w \in \mathcal{A}_n$ and $\mathrm{fl}_\pm(w_2w_3\cdots w_n) \in \mathcal{A}_{n-1}$, in which case $M(w)$ and $M'(w)$ are both defined but unequal. For quasi-atoms, however, this ambiguity does not arise:

Proposition 4.4. The sets $\mathcal{A}_n$ and $\mathcal{Q}_n$ are disjoint.

Proof. Suppose $w \in \mathcal{A}_n \cap \mathcal{Q}_n$. Since $w_1 > 0$, the index 1 must be complementary to some $j \in [2, n]$, but then $0 < w_1 < -w_j$ and necessarily $\{\pm w_j\} \in M'(w)$, contradicting the definition of $\mathcal{Q}_n$. $\square$

Define $<_\mathcal{Q}$ as the transitive closure of the relations $\lhd_i$ from (3.1) for $i \geq 2$. This is the partial order with $v <_\mathcal{Q} w$ if and only if $v_1 = w_1$ and $v_2v_3\cdots v_n <_\mathcal{A} w_2w_3\cdots w_n$, or equivalently the transitive closure of the relation on words with

$$x \cdots cab \cdots <_\mathcal{Q} x \cdots bca \cdots$$

whenever a, b, c, x are integers with $a < b < c$ and the corresponding ellipses mask identical subsequences. Each interval $\mathcal{A}_M \subset \mathcal{A}_n$ is preserved by $<_\mathcal{A}$, so $\mathcal{Q}_n^+$ and $\mathcal{Q}_n^-$ are preserved by $<_\mathcal{Q}$.

Let $e \in [n]$ and suppose M is a perfect matching on $[\pm n] \setminus \{\pm e\}$ which is symmetric and noncrossing. Assume M has no blocks $\{\pm c\}$ with $0 < e < c$. Define $\mathsf{NCSQ}^+(n, e)$ as the set of such matchings with exactly one block $\{a, b\}$ such that $0 < a < e < b$; define $\mathsf{NCSQ}^-(n, e)$ as the set of such matchings with no blocks $\{a, b\}$ such that $0 < a < e < b$. Let

$$\mathsf{NCSQ}^+(n) \overset{\text{def}}{=} \bigsqcup_{e \in [n]} \mathsf{NCSQ}^+(n, e) \qquad \text{and} \qquad \mathsf{NCSQ}^-(n) \overset{\text{def}}{=} \bigsqcup_{e \in [n]} \mathsf{NCSQ}^-(n, e).$$

Given $M \in \mathsf{NCSQ}^\pm(n, e)$, define $\alpha'_{\min}(M) \overset{\text{def}}{=} u_1u_2u_3\cdots u_n$ and $\alpha'_{\max}(M) \overset{\text{def}}{=} v_1v_2v_3\cdots v_n$ where

$$u_1 = v_1 = e \qquad \text{and} \qquad u_2u_3\cdots u_n = \alpha_{\min}(M) \qquad \text{and} \qquad v_2v_3\cdots v_n = \alpha_{\max}(M).$$

Finally let $\mathcal{Q}_M \overset{\text{def}}{=} \{w \in C_n : \alpha'_{\min}(M) \leq_\mathcal{Q} w \leq_\mathcal{Q} \alpha'_{\max}(M)\}$.

Example 4.5. We have

$$\{\pm\{1,7\},\pm\{2,3\},\pm\{5,6\}\} \in \mathsf{NCSQ}^+(7,4) \quad \text{and} \quad \{\pm\{2,7\},\pm\{3,4\},\pm\{5,6\}\} \in \mathsf{NCSQ}^-(7,1).$$

If $M = \{\{\pm 3\},\pm\{1,2\},\pm\{4,8\},\pm\{6,7\}\} \in \mathsf{NCSQ}^+(8,5)$, then

$$\alpha_{\min}(M) = 54\overline{8}6\overline{7}\overline{3}1\overline{2} \qquad \text{and} \qquad \alpha_{\max}(M) = 5\overline{3}1\overline{2}4\overline{8}6\overline{7}.$$

Proposition 4.6. It holds that $\mathcal{Q}_n^+ = \bigsqcup_{M\in\mathsf{NCSQ}^+(n)} \mathcal{Q}_M$ and $\mathcal{Q}_n^- = \bigsqcup_{M\in\mathsf{NCSQ}^-(n)} \mathcal{Q}_M$.

Proof. This is clear since if $M \in \mathsf{NCSQ}^\pm(n)$ then $\mathcal{Q}_M = \{w \in \mathcal{Q}_n : M'(w) = M\}$. $\square$

Let $w \in \mathcal{Q}_n$ be a quasi-atom. Mimicking our terminology in the previous section, define indices $2 \leq i < j \leq n$ to be *complementary* in w if $0 < w_i < -w_j$ and $\{w_i, -w_j\} \in M'(w)$, and define an index $2 \leq i \leq n$ to be *symmetric* for w if $w_i < 0$ and $\{\pm w_i\} \in M'(w)$. In view of Proposition 4.4, there is no risk of these notions conflicting with our earlier definitions for atoms.

With minor changes, the technical properties of atoms in the previous section remain true for quasi-atoms. The following summarizes the main facts we will need.

Lemma 4.7. Consider a quasi-atom $w \in \mathcal{Q}_n$.

(a) Each index $i \in [2,n]$ is symmetric or part of a complementary pair for w.

(b) If $w_i > w_{i+1}$ for some $i \in [2, n-1]$ then $0 < w_i < -w_{i+1}$.

(c) Suppose $0 < a < b < c < d$.

  i. If $\{a, d\}, \{b, c\} \in M'(w)$ then $a\overline{d}b\overline{c}$ is a subword of $w_2w_3\cdots w_n$.

  ii. If $\{a, b\}, \{\pm c\} \in M'(w)$ then $\overline{c}a\overline{b}$ is a subword of $w_2w_3\cdots w_n$.

  iii. If $\{\pm a\}, \{\pm b\} \in M'(w)$ then $\overline{b}\overline{a}$ is a subword of $w_2w_3\cdots w_n$.

(d) Suppose $2 \leq i < j \leq n$ are complementary indices for w and $e \in [\pm n]$.

  i. If $w_j < e < -w_i < 0$ then $e \notin \{w_2, w_3, \ldots, w_j, w_{j+1}\}$.

  ii. If $0 < w_i < e < -w_j$ then $e \notin \{w_2, w_3, \ldots, w_j\}$.

Proof. Since $\mathrm{fl}_\pm(w_2w_3\cdots w_n)$ is required to belong to $\mathcal{A}_M$ for some matching $M \in \mathsf{NCSP}(n-1)$, and since $M'(w)$ is defined to be the matching on $[\pm n] \setminus \{\pm w_1\}$ with $\mathrm{fl}_\pm(M'(w)) = M$, these properties just restate Corollary 3.6, Lemma 3.7, Corollary 3.8, and Lemma 3.9. $\square$

If $w \in \mathcal{Q}_n^+$ is an even quasi-atom then there exists a unique pair of complementary indices $2 \leq i < j \leq n$ with $0 < w_i < w_1 < -w_j$. We call these the *distinguished indices* of w.

Corollary 4.8. Suppose $2 \leq i < j \leq n$ are the distinguished indices of an even quasi-atom $w \in \mathcal{Q}_n$. Then $wt_{1i} \lessdot w$ and $ws_{1j} \lessdot w$.

Proof. This is immediate from Lemmas 2.12 and 4.7(d). $\square$

5 Transition graphs

We define a directed bipartite graph $\overrightarrow{\mathcal{L}_n}$ with vertex set $\mathcal{A}_n \sqcup \mathcal{Q}_n$. We use the letter $\mathcal{L}$ to denote this graph since it will later serve as one "layer" in a larger graph of interest. Each edge $\overrightarrow{\mathcal{L}_n}$ will pass either from an even quasi-atom to an odd quasi-atom, from an odd quasi-atom to an even quasi-atom, or from an odd quasi-atom to an atom. The atoms of $w_n^C = \overline{1}\overline{2}\overline{3}\cdots\overline{n}$ will each have a unique incoming edge, and all even quasi-atoms will have one incoming and one outgoing edge. These properties will not be immediately clear from the following definition.

First suppose $v \in \mathcal{Q}_n$ is an even quasi-atom. Let $b = v_1 > 0$ and suppose $\{a, c\} \in M'(v)$ is the unique block with $0 < a < b < c$. Let $2 \leq i < j \leq n$ be the distinguished indices with $a = v_i$ and $b = -v_j$. In $\overrightarrow{\mathcal{L}_n}$, we define v to have a unique incoming edge $u \to v$ where

$$u = vs_{1j} = t_{bc}v = \overline{v_j}v_2 \cdots v_{j-1}\overline{v_1}v_j \cdots v_n \tag{5.1}$$

and a unique outgoing edge $v \to w$ where

$$w = vt_{1i} = t_{ab}v = v_iv_2 \cdots v_{i-1}v_1v_{i+1} \cdots v_n. \tag{5.2}$$

Next suppose $v \in \mathcal{A}_n$. If $v_1 < 0$ then let

$$u = vt_0 = \overline{v_1}v_2 \cdots v_n. \tag{5.3}$$

If $v_1 > 0$ and $j \in [2, n]$ is the unique index with $\{v_1, -v_j\} \in M(v)$, then let

$$u = vs_{1j} = t_{bc}v = \overline{v_j}v_2 \cdots v_{j-1}\overline{v_1}v_{j+1} \cdots v_n \tag{5.4}$$

where $b = v_1 < -v_j = c$. We define v to have a single incoming edge $u \to v$ in $\overrightarrow{\mathcal{L}_n}$. Figure 1 shows $\overrightarrow{\mathcal{L}_n}$ for $n = 1, 2, 3, 4$ and Figure 2 shows a part of $\overrightarrow{\mathcal{L}_5}$.

Lemma 5.1. Let $v, v' \in \mathcal{A}_n \sqcup \mathcal{Q}_n^+$ and $i \in [2, n-2]$. Define $\lhd_i$ as in (3.1).

(a) If $u \to v$ is an edge in $\overrightarrow{\mathcal{L}_n}$ and $v \lhd_i v'$, then there is an edge $u' \to v'$ in $\overrightarrow{\mathcal{L}_n}$ with $u \lhd_i u'$.

(b) If $v \to w$ is an edge in $\overrightarrow{\mathcal{L}_n}$ and $v' \lhd_i v$, then there is an edge $v' \to w'$ in $\overrightarrow{\mathcal{L}_n}$ with $w' \lhd_i w$.

Proof. First suppose $v, v' \in \mathcal{A}_n$ and $v \lhd_i v'$. Let $u \to v$ and $u' \to v'$ be the unique edges incident to v and v' in $\overrightarrow{\mathcal{L}_n}$. If $v_1 < 0$ then $u = vt_0$ and $u' = v't_0$ and clearly $u \lhd_i u'$. Assume $b = v_1 > 0$ and suppose $\{b, c\} \in M(v)$ is the unique block with $0 < b < c$, so that $u = t_{bc}v$ and $u' = t_{bc}v'$. We can write $v_iv_{i+1}v_{i+2} = zxy$ where $x < y < z$. By Corollary 3.6, it must hold that $0 < z < -x$. Since $u_2u_3 \cdots u_n$ is given by replacing $\overline{c}$ by $\overline{b}$ in $v_2v_3 \cdots v_n$, the only way we can fail to have $u \lhd_i u'$ is if $v_i = -c$ and $-c < v_{i+1} < -b$. But this is impossible by Lemma 3.9(a).

Now let $v, v' \in \mathcal{Q}_n^+$. Suppose $b = v_1$ and $\{a, c\} \in M'(v)$ is the unique block with $0 < a < b < c$. Set $u = t_{bc}v$ and $w = t_{ab}v$ so that $u \to v$ and $v \to w$ are the edges incident to v in $\overrightarrow{\mathcal{L}_n}$.

Suppose $v \lhd_i v'$ and $u' = t_{bc}v'$ so that $u' \to v'$ is an edge in $\overrightarrow{\mathcal{L}_n}$. Our argument is similar to the previous case. We can write $v_iv_{i+1}v_{i+2} = zxy$ where $x < y < z$, and $0 < z < -x$ by Lemma 4.7(b). Since $u_2u_3 \cdots u_n$ is given by replacing $\overline{c}$ by $\overline{b}$ in $v_2v_3 \cdots v_n$, the only way we can fail to have $u \lhd_i u'$ is if $v_{i+1} = -c$ and $-c < v_{i+2} < -b$. But this is impossible by Lemma 4.7(d).

Finally suppose $v' \lhd_i v$ and $w' = t_{ab}v'$ so that $v' \to w'$ is an edge in $\overrightarrow{\mathcal{L}_n}$. We can write $v_iv_{i+1}v_{i+2} = yzx$ where $x < y < z$, and $0 < z < -x$ by Lemma 4.7(b). Since $w_2w_3 \cdots w_n$ is given by replacing a by b in $v_2v_3 \cdots v_n$, the only way we can fail to have $w' \lhd_i w$ is if $v_i = a$ and $a < v_{i+1} < b$. But this is impossible by Lemma 4.7(d). $\square$

$$
\begin{tikzcd}
1 \arrow[d] \\
\overline{1}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& 2\overline{1} \arrow[dl] \arrow[dr] & \\
1\overline{2} & & \overline{2}\,\overline{1}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& 31\overline{2} \arrow[dl] \arrow[dr] & \\
\overline{3}1\overline{2} & & 21\overline{3} \arrow[d] \\
& & 12\overline{3} \arrow[d] \\
& & \overline{1}2\overline{3}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& 3\overline{2}\,\overline{1} \arrow[dl] \arrow[dr] & \\
\overline{3}\,\overline{2}\,\overline{1} & & 2\overline{3}\,\overline{1}
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
& 4\overline{3}1\overline{2} \arrow[dl] \arrow[dr] & \\
\overline{4}\,\overline{3}1\overline{2} & & 3\overline{4}1\overline{2}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& 4\overline{1}2\overline{3} \arrow[dl] \arrow[d] \arrow[dr] & \\
\overline{4}\,\overline{1}2\overline{3} & 3\overline{1}2\overline{4} \arrow[d] & 1\overline{4}2\overline{3} \\
& 2\overline{1}3\overline{4} \arrow[dl] \arrow[dr] & \\
\overline{2}\,\overline{1}3\overline{4} & & 1\overline{2}3\overline{4}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& 4\overline{3}\,\overline{2}\,\overline{1} \arrow[dl] \arrow[dr] & \\
\overline{4}\,\overline{3}\,\overline{2}\,\overline{1} & & 3\overline{4}\,\overline{2}\,\overline{1}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& 42\overline{3}\,\overline{1} \arrow[dl] \arrow[dr] & \\
32\overline{4}\,\overline{1} \arrow[d] & & \overline{4}2\overline{3}\,\overline{1} \\
23\overline{4}\,\overline{1} \arrow[d] \arrow[dr] & & \\
\overline{2}3\overline{4}\,\overline{1} & 13\overline{4}\,\overline{2} &
\end{tikzcd}
$$

Figure 1: The directed graphs $\overrightarrow{\mathcal{L}_n}$ for $n = 1, 2, 3, 4$.

Lemma 5.2. Let $M \in \mathsf{NCSP}(n)$ and $v \in \mathcal{A}_M$. If $v_1 < 0$ then define M' by removing $\{\pm v_1\}$ from M. Otherwise $-v_1$ and v_1 belong to distinct blocks in M, and we define M' by removing these blocks and then adding $\{\pm v_1\}$. In either case we have $M' \in \mathsf{NCSQ}^-(n)$. If v is maximal with respect to $<_{\mathcal{Q}}$ then $\alpha'_{\max}(M') \to v$ is the unique edge incident to v in $\overrightarrow{\mathcal{L}_n}$.

Proof. If $v_1 < 0$ then clearly $M' \in \mathsf{NCSQ}^-(n, v_1)$. If $v_1 > 0$ and $j \in [2, n]$ is such that $0 < v_1 < -v_j$ and $\{v_1, -v_j\} \in M$, then Lemma 3.7 implies that M has no blocks $\{a, b\}$ with $a < v_1 < -v_j < b$, from which it follows that $M' \in \mathsf{NCSQ}^-(n, v_j)$.

Assume v is maximal with respect to $<_{\mathcal{Q}}$. Using Theorem 3.4, it is not hard to show that

$$v = v_1 c_1 c_2 \cdots c_k a_1 \overline{b_1} a_2 \overline{b_2} \cdots a_l \overline{b_l}$$

for some numbers where $k, l \in \mathbb{N}$ and $c_1 < c_2 < \cdots < c_k < a_1 < a_2 < \cdots < a_l$ and $a_i < -b_i$ for $i \in [l]$. If $v_1 < 0$ then $vt_0 \to v$ is the unique edge incident to v in $\overrightarrow{\mathcal{L}_n}$, and it follows from Theorem 3.4 that either $k = 0$ or $v_1 < c_1$, and in turn that $v = \alpha_{\max}(M)$ and $vt_0 = \alpha'_{\max}(M')$.

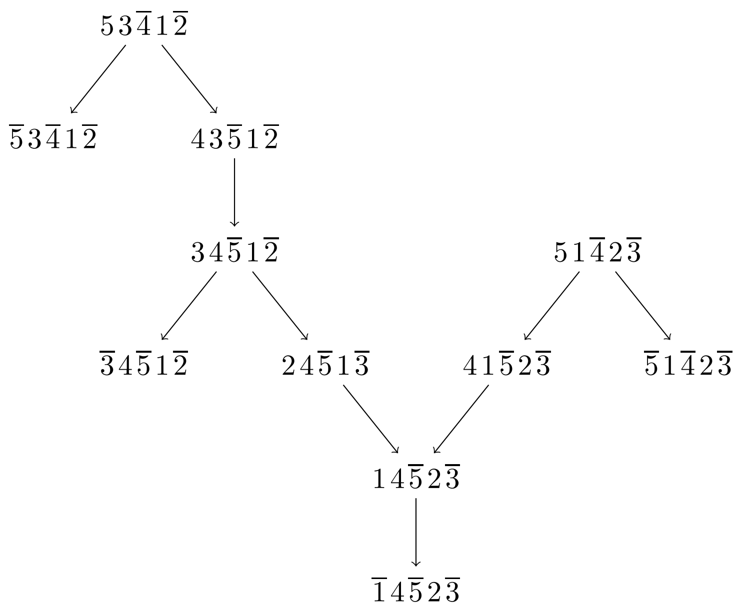


Figure 2: The most interesting connected component of $\overrightarrow{\mathcal{L}_5}$. As demonstrated by this example, the graphs $\overrightarrow{\mathcal{L}_n}$ are not always directed forests.

Instead suppose $v_1 > 0$. Theorem 3.4 then implies that $k > 0$, $v_1 < -c_1$, $\{v_1, -c_1\} \in M$, and $\{\pm c_i\} \in M$ for $i \in [2, k]$. In this case $vs_{12} \to v$ is the unique edge incident to v in $\overrightarrow{\mathcal{L}_n}$. Since M is noncrossing, we must have $c_1 < -v_1 < c_i$ for all $i \in [2, k]$, so $vs_{12} = \alpha'_{\max}(M')$. $\square$

The following theorem confirms that $\overrightarrow{\mathcal{L}_n}$ is indeed a bipartite graph on the vertex set $\mathcal{A}_n \sqcup \mathcal{Q}_n$.

Theorem 5.3. The edges in $\overrightarrow{\mathcal{L}_n}$ have the following properties:

(a) If $v \in \mathcal{A}_n \sqcup \mathcal{Q}_n^+$ and $u \to v$ is an edge in $\overrightarrow{\mathcal{L}_n}$, then $u \in \mathcal{Q}_n^-$ and $u \lessdot v$.

(b) If $v \in \mathcal{Q}_n^+$ and $v \to w$ is an edge in $\overrightarrow{\mathcal{L}_n}$, then $w \in \mathcal{Q}_n^-$ and $w \lessdot v$.

Proof. Let $v \in \mathcal{A}_n$ and suppose $u \to v$ is an edge in $\overrightarrow{\mathcal{L}_n}$. We deduce that $u \lessdot v$ either by Lemma 2.12(a) when $v_1 < 0$ or by Lemma 3.10 when $v_1 > 0$. Let $M = M(v)$ and define M' from M as in Lemma 5.2. It follows from Lemmas 5.1 and 5.2 that $u \leq_{\mathcal{Q}} \alpha'_{\max}(M') \in \mathcal{Q}_n^-$ so $u \in \mathcal{Q}_n^-$.

Now let $v \in \mathcal{Q}_n^+$ and suppose $u \to v$ and $v \to w$ are the edges incident to v in $\overrightarrow{\mathcal{L}_n}$. Then $u = vs_{1j}$ and $w = vt_{1i}$ where $i < j$ are the distinguished indices in v, so Corollary 4.8 implies that $u \lessdot v$ and $w \lessdot v$. Let $M = M'(v)$. Define $a = v_i$, $b = v_1$, and $c = -v_j$ so that $0 < a < b < c$ and $\{a, c\} \in M$. Construct P (respectively, Q) from M by replacing the blocks $\{a, c\}$ and $\{-a, -c\}$ by $\{a, b\}$ and $\{-a, -b\}$ (respectively, $\{b, c\}$ and $\{-b, -c\}$). Since $M \in \mathsf{NCSQ}^+(n, b)$ it follows that $P \in \mathsf{NCSQ}^-(n, c)$ and $Q \in \mathsf{NCSQ}^-(n, a)$. If v is maximal with respect to $<_{\mathcal{Q}}$ then $v = \alpha'_{\max}(M)$ and evidently $u = \alpha'_{\max}(P)$. If v is minimal with respect to $<_{\mathcal{Q}}$ then $v = \alpha'_{\min}(M)$ and $w = \alpha'_{\min}(Q)$. Lemma 5.1 therefore implies that $u \leq_{\mathcal{Q}} \alpha'_{\max}(P) \in \mathcal{Q}_n^-$ and $w \geq_{\mathcal{Q}} \alpha'_{\min}(Q) \in \mathcal{Q}_n^-$ so $u, w \in \mathcal{Q}_n^-$. $\square$

Let $\mathcal{S}(w) = \mathcal{S}_1(w)$ and $\mathcal{T}(w) = \mathcal{T}_1^+(w)$ for $w \in C_n$, where $\mathcal{S}_j(w)$ and $\mathcal{T}_j^\pm(w)$ are as in (2.1). If $b = w_1 > 0$ then we can also write $\mathcal{S}(w) = \{t_{ab}w : a \in [b-1] \text{ and } w \lessdot t_{ab}w\} \sqcup \{wt_0\}$ and $\mathcal{T}(w) = \{t_{bc}w : c \in [b+1, n+1] \text{ and } w \lessdot t_{bc}w\}$.

Lemma 5.4. Let $w, w' \in \mathcal{Q}_n^-$ and $i \in [2, n-2]$. Suppose $b = w_1$ and $1 \leq a < b < c \leq n$.

(a) If $w \lhd_i w'$ and $w \lessdot t_{ab}w \in \mathcal{S}(w)$, then $t_{ab}w \lhd_i t_{ab}w'$ and $w' \lessdot t_{ab}w' \in \mathcal{S}(w')$.

(b) If $w' \lhd_i w$ and $w \lessdot t_{bc}w \in \mathcal{T}(w)$ then $t_{bc}w' \lhd_i t_{bc}w$ and $w' \lessdot t_{bc}w' \in \mathcal{T}(w')$.

Proof. Suppose $w \lhd_i w'$ and $w \lessdot t_{ab}w$. Then for some $j \in [2, n]$ we have $w_j = -a$ and no $e \in \{w_2, w_3, \ldots, w_{j-1}\}$ has $a < -e < b$. We can write $w_i w_{i+1} w_{i+2} = zxy$ where $x < y < z$ and $0 < z < -x$. The only way we can fail to have $t_{ab}w \lhd_i t_{ab}w'$ is if $-b < w_{i+1} < -a = w_{i+2}$, but this would contradict $w \lessdot t_{ab}w$. Since the relation $\lhd_i$ is length-preserving, we also have $w' \lessdot t_{ab}w'$.

Suppose $w' \lhd_i w$ and $w \lessdot t_{bc}w$. Then for some $j \in [2, n]$ we have $w_j = c$ and no $e \in \{w_2, w_3, \ldots, w_{j-1}\}$ has $b < e < c$. We can write $w_i w_{i+1} w_{i+2} = yzx$ where $x < y < z$ and $0 < z < -x$. The only way we can fail to have $t_{bc}w' \lhd_i t_{bc}w$ is if $w_{i+1} = c > w_i > b$, but this would contradict $w \lessdot t_{bc}w$. Since the relation $\lhd_i$ is length-preserving, we have $w' \lessdot t_{bc}w'$. ☐

Lemma 5.5. Let $u \in \mathcal{Q}_n^-$ and $P = M'(u)$. Define M by adding the block $\{\pm u_1\}$ to P. Then $M \in \mathsf{NCSP}(n)$ and $ut_0 \in \mathcal{A}_M$, and $u \to ut_0$ is an edge in $\overrightarrow{\mathcal{L}_n}$.

Proof. Since P has no blocks $\{a, b\}$ with $a < u_1 < b$, the matching M belongs to $\mathsf{NCSP}(n)$ and $c = u_1$ is the largest value such that $\{\pm c\} \in M$. If $u = \alpha'_{\max}(P)$ then evidently $ut_0 = \alpha_{\max}(M)$. In general we have $u \leq_{\mathcal{Q}} \alpha'_{\max}(P)$, and this implies that $ut_0 \leq_{\mathcal{Q}} \alpha_{\max}(M)$, so $ut_0 \in \mathcal{A}_M$. ☐

Lemma 5.6. Let $u \in \mathcal{Q}_n^-$, $P = M'(u)$, and $b = u_1$. Suppose $a \in [b-1]$ is such that $\{\pm a\} \in P$ and $u \lessdot t_{ab}u$. Define M by removing the block $\{\pm a\}$ from P and then adding $\{a, b\}$ and $\{-a, -b\}$. Then $M \in \mathsf{NCSP}(n)$ and $t_{ab}u \in \mathcal{A}_M$, and $u \to t_{ab}u$ is an edge in $\overrightarrow{\mathcal{L}_n}$.

Proof. Since $u \lessdot t_{ab}u$, we must have $u_j = -a$ for some $j \in [2, n]$, and no numbers between $-b$ and $-a$ can appear in $u_2 u_3 \cdots u_{j-1}$. There can be no blocks $\{x, y\} \in P$ with $x < a < y$: since $P \in \mathsf{NCSQ}^-(n, b)$, such a block necessarily satisfies $x = -y < a < y < b$, and then Lemma 4.7(c)(iii) contradicts the previous sentence. This is enough to conclude that $M \in \mathsf{NCSP}(n)$, and that $e = a$ is the largest number with $\{\pm e\} \in P$. Assume $u = \alpha'_{\max}(P)$ is maximal under $<_{\mathcal{Q}}$. We must then have $j = 2$ and we can write

$$u = b\overline{a}c_1c_2 \cdots c_k a_1 \overline{b_1} a_2 \overline{b_2} \cdots a_l \overline{b_l}$$

where $\{\pm c_i\}$ for $i \in [k]$ together with $\{\pm a\}$ are the symmetric blocks in P, where $\{a_i, b_i\}$ for $i \in [l]$ are the blocks in P with $0 < a_i < b_i$, and where $-a < c_1 < \cdots < c_k < 0 < a_1 < \cdots < a_l$. Hence

$$t_{ab}u = a\overline{b}c_1c_2 \cdots c_k a_1 \overline{b_1} a_2 \overline{b_2} \cdots a_l \overline{b_l}$$

and it is easy to see that

$$t_{ab}u \leq_{\mathcal{A}} \alpha_{\max}(M) = c_1c_2 \cdots c_k a_1 \overline{b_1} \cdots a\overline{b} \cdots a_\ell \overline{b_\ell}$$

since $-b < c_i < 0$ for each $i \in [k]$ and $-b < -b_i < 0$ for each $i \in [l]$ with $a_i < a$ as P is noncrossing. Therefore $t_{ab}u \in \mathcal{A}_M$. If u is not maximal under $<_{\mathcal{Q}}$, then it follows from Lemma 5.4(a) that we still have $t_{ab}u \leq_{\mathcal{A}} \alpha_{\max}(M)$ so again $t_{ab}u \in \mathcal{A}_M$. Once we know that $t_{ab}u \in \mathcal{A}_M$, the claim that $u \to t_{ab}u$ is an edge in $\overrightarrow{\mathcal{L}_n}$ holds by definition. ☐

Theorem 5.7. Let $u, w \in \mathcal{Q}_n^-$ and $v \in C_n$ with $w_1 < n$.

(a) It holds that $v \in \mathcal{S}(u)$ if and only if $u \to v$ is an edge in $\overrightarrow{\mathcal{L}_n}$.

(b) It holds that $v \in \mathcal{T}(w)$ if and only if $v \to w$ is an edge in $\overrightarrow{\mathcal{L}_n}$.

Proof. Theorem 5.3 shows that if $u \to v$ is an edge in $\overrightarrow{\mathcal{L}_n}$ then $v \in \mathcal{S}(u)$ and that if $v \to w$ is an edge in $\overrightarrow{\mathcal{L}_n}$ then $v \in \mathcal{T}(w)$. It remains to show the converse.

Let $P = M'(u)$ and $v \in \mathcal{S}(u)$. If $v = ut_0$ or if $v = t_{ab}u$ where $0 < a < b = u_1$ and $\{\pm a\} \in P$, then $u \to v$ is an edge in $\overrightarrow{\mathcal{L}_n}$ by Lemmas 5.5 and 5.6. Assume we are not in these cases. There must be numbers $0 < a < b < c = u_1$ with $\{a, b\} \in P$ and $v = t_{bc}u$. Let $j \in [n]$ be such that $v_j = -b$. By definition $P \in \mathsf{NCSQ}^-(n, c)$ has no blocks $\{x, y\}$ with $x < c < y$. Since $\{u_2, u_3, \dots, u_{j-1}\}$ contains no numbers between $-c$ and $-b$ as $u \lessdot v$, it follows from Lemma 4.7(c) that P has no blocks $\{x, y\}$ with $x < a < b < y$. Form M from P by replacing the blocks $\{a, b\}$ and $\{-a, -b\}$ by $\{a, c\}$ and $\{-a, -c\}$. Then $M \in \mathsf{NCSQ}^+(n, b)$, and to show that $u \to v$ is an edge in $\overrightarrow{\mathcal{L}_n}$ it suffices to check that $v \in \mathcal{Q}_M$. If u is maximal with respect to $<_{\mathcal{Q}}$ then $u = \alpha'_{\max}(P)$ and evidently $v = \alpha'_{\max}(M)$. In general, Lemma 5.4(a) implies that $v \leq_{\mathcal{Q}} \alpha'_{\max}(M)$ so $v \in \mathcal{Q}_M$ as desired.

Next let $Q = M'(w)$ and $v \in \mathcal{T}(w)$. Write $a = w_1$. Since $a < n$ and $Q \in \mathsf{NCSQ}^-(n, a)$, we must have $v = wt_{1i} = t_{ab}w$ for some $i \in [2, n]$ where $0 < a < b = w_i$, and there must exist a block $\{b, c\} \in Q$ with $b < c$. By definition Q has no blocks $\{x, y\}$ with $x < a < y$. Since $\{w_2, w_3, \dots, w_{i-1}\}$ contains no numbers between a and b as $w \lessdot v$, it follows from Lemma 4.7(c) that Q has no blocks $\{x, y\}$ with $x < b < c < y$. Form M from Q by replacing $\{b, c\}$ and $\{-b, -c\}$ by $\{a, c\}$ and $\{-a, -c\}$. Then $M \in \mathsf{NCSQ}^+(n, b)$, and to show that $v \to w$ is an edge in $\overrightarrow{\mathcal{L}_n}$ it suffices to check that $v \in \mathcal{Q}_M$. If w is minimal with respect to $<_{\mathcal{Q}}$, then $w = \alpha'_{\min}(Q)$ and evidently $v = \alpha'_{\min}(M)$. In general, it follows from Lemma 5.4(b) that $v \geq_{\mathcal{Q}} \alpha'_{\min}(M)$ so $v \in \mathcal{Q}_M$ as desired. ☐

The previous theorem does not apply when $w \in \mathcal{Q}_n^-$ has $w_1 = n$, since then $\mathcal{T}(w)$ consists of the single element $(n+1)w_2w_3 \cdots w_n n \in C_{n+1}$ but there are no edges $v \to w$ in $\overrightarrow{\mathcal{L}_n}$.

Corollary 5.8. A vertex $w \in \mathcal{A}_n \sqcup \mathcal{Q}_n$ is a source in $\overrightarrow{\mathcal{L}_n}$ if and only if $w_1 = n$, in which case $w \in \mathcal{Q}_n^-$. Thus the sources in $\overrightarrow{\mathcal{L}_n}$ are the elements $nv_1v_2 \cdots v_{n-1} \in C_n$ where $v_1v_2 \cdots v_{n-1} \in \mathcal{A}_{n-1}$.

Proof. By definition no element in $\mathcal{A}_n \sqcup \mathcal{Q}_n^+$ is a source in $\overrightarrow{\mathcal{L}_n}$. Theorem 3.4 implies that no atom $v \in \mathcal{A}_n$ has $v_1 = n$, and that if $v \in \mathcal{Q}_n^+$ then $v_i = -n$ for some $i \in [n]$. It follows that an odd quasi-atom $w \in \mathcal{Q}_n^-$ with $w_1 = n$ cannot be the target of an edge $v \to w$ in $\overrightarrow{\mathcal{L}_n}$.

Suppose $w \in \mathcal{Q}_n^-$ has $w_1 \in [n-1]$. It remains to show that w is not a source in $\overrightarrow{\mathcal{L}_n}$. By Theorem 5.7, it suffices to check that $\mathcal{T}(w) \neq \varnothing$. Since $M'(w)$ has no blocks $\{a, b\}$ with $a < w_1 < b$, the interval $[w_1 + 1, n]$ must be a non-empty union of blocks in $M'(w)$. It follows that $0 < w_1 < w_i$ for some $i \in [2, n]$, and if i is minimal with these properties then $wt_{1i} \in \mathcal{T}(w)$, so $\mathcal{T}(w) \neq \varnothing$ as desired. ☐

Corollary 5.9. A vertex $w \in \mathcal{A}_n \sqcup \mathcal{Q}_n$ is a sink in $\overrightarrow{\mathcal{L}_n}$ if and only if $w \in \mathcal{A}_n$.

Proof. Since $w \to wt_0$ is always an edge in $\overrightarrow{\mathcal{L}_n}$ if $w \in \mathcal{Q}_n^-$, this follows from the definition of $\overrightarrow{\mathcal{L}_n}$. ☐

For integers $0 < m < n$, write $\uparrow_m^n : C_m \to C_n$ for the transformation

$$\uparrow_m^n(v_1v_2 \cdots v_m) \overset{\text{def}}{=} n \cdots (m+3)(m+2)v_1v_2 \cdots v_m(m+1) \in C_n.$$

Recall that, by convention, C_0 is the set consisting of just the empty word $\emptyset$. We define $\uparrow_0^n$ to be the map $\emptyset \mapsto n \cdots 321$ and view $\overrightarrow{\mathcal{L}_0}$ as the graph with no edges and a single vertex $\emptyset$.

First define $\overrightarrow{\mathcal{L}}_{m,n}$ for $0 \leq m < n$ to be the directed graph given by replacing each vertex in $\overrightarrow{\mathcal{L}_m}$ by its image under $\uparrow_m^n$. One may interpret $\uparrow_n^{n+1}$ as the identity map $C_n \to C_n \hookrightarrow C_{n+1}$ and identify $\overrightarrow{\mathcal{L}}_{n,n+1}$ with $\overrightarrow{\mathcal{L}_n}$. Next define $\overrightarrow{\mathcal{G}_n}$ to be the graph given by the disjoint union

$$\overrightarrow{\mathcal{L}}_{0,n+1} \sqcup \overrightarrow{\mathcal{L}}_{1,n+1} \sqcup \overrightarrow{\mathcal{L}}_{2,n+1} \cdots \sqcup \overrightarrow{\mathcal{L}}_{n,n+1}$$

with these additional edges: for each $m \in [n]$ and $w \in \mathcal{A}_{m-1}$, include an edge from the sink

$$\uparrow_{m-1}^{n+1}(w_1 w_2 \cdots w_{m-1}) = (n+1) \cdots (m+2)(m+1) w_1 w_2 \cdots w_{m-1} m \tag{5.5}$$

in $\overrightarrow{\mathcal{L}}_{m-1,n+1}$ to the source

$$\uparrow_m^{n+1}(m w_1 w_2 \cdots w_{m-1}) = (n+1) \cdots (m+3)(m+2) m w_1 w_2 \cdots w_m (m+1) \tag{5.6}$$

in $\overrightarrow{\mathcal{L}}_{m,n+1}$. Figure 3 shows $\overrightarrow{\mathcal{G}_n}$ for $n = 4$.

A vertex in $\overrightarrow{\mathcal{L}}_{m,n+1}$ is *odd* it is the image under $\uparrow_m^{n+1}$ of an odd quasi-atom in $\overrightarrow{\mathcal{L}_m}$. All other vertices in $\overrightarrow{\mathcal{L}}_{m,n+1}$ or $\overrightarrow{\mathcal{G}_n}$ are *even*. Since every source in $\overrightarrow{\mathcal{L}_m}$ is an odd quasi-atom and every sink is an atom, the resulting division into even and odd vertices affords a bipartition of $\overrightarrow{\mathcal{G}_n}$.

Recall that $w_n^C = \overline{1}\,\overline{2}\,\overline{3} \cdots \overline{n}$ and $w_n^A = (n+1)n \cdots 321$ and $\delta_n = (n, n-1, \dots, 3, 2, 1)$.

Theorem (Theorem 1.6)**.** It holds that $\hat{G}_{w_n^C} = G_{w_n^A} = S_{\delta_n}$.

Proof. Let $w \in C_n$. For $j \in [n]$, define $\mathcal{T}_j^{\pm}(w)$ and $\mathcal{S}_j(w)$ as in (1.1), and recall that $\mathcal{T}(w) = \mathcal{T}_1^+(w)$ and $\mathcal{S}(w) = \mathcal{S}_1(w)$. If $0 \leq m \leq n$ and $r = n - m + 1$ and $\tilde{w} = \uparrow_m^{n+1}(w) \in C_n$, then evidently

$$\mathcal{T}_r^+(\tilde{w}) = \uparrow_m^{n+1}(\mathcal{T}(w)), \qquad \mathcal{S}_r(\tilde{w}) = \uparrow_m^{n+1}(\mathcal{S}(w)), \qquad \text{and} \qquad \mathcal{T}_r^-(\tilde{w}) = \varnothing.$$

Likewise, if u and v are the elements (5.5) and (5.6) then $\mathcal{T}_{n-m}^+(v) = \{u\}$. By Theorem 5.7 if v is any odd vertex in $\overrightarrow{\mathcal{G}_n}$, then $\sum_{\{u \to v\} \in \overrightarrow{\mathcal{G}_n}} G_u = \sum_{\{v \to w\} \in \overrightarrow{\mathcal{G}_n}} G_w$, so Lemma 2.17 implies that

$$\sum_{u \in \mathsf{Source}(\overrightarrow{\mathcal{G}_n})} G_u = \sum_{v \in \mathsf{Sink}(\overrightarrow{\mathcal{G}_n})} G_v.$$

Since $\uparrow_0^{n+1}(\emptyset) = (n+1)n \cdots 321 = w_n^A$ is the unique source in $\overrightarrow{\mathcal{G}_n}$ and since the set of sinks in $\overrightarrow{\mathcal{G}_n}$ is precisely $\uparrow_n^{n+1}(\mathcal{A}_n) = \mathcal{A}_n$, we have $\hat{G}_{w_n^C} = G_{w_n^A}$. The latter is S_{δ_n} by Theorems 1.3 and 2.5. $\square$

Corollary (Theorem 1.2)**.** It holds that $|\hat{\mathcal{R}}(w_n^C)| = |\operatorname{SYT}(\delta_n)| = |\mathcal{R}(w_n^A)|$.

Proof. Let $N = \binom{n+1}{2} = \deg(S_{\delta_n}) = \hat{\ell}(w_n^C)$. Then $2^N |\hat{\mathcal{R}}(w_n^C)| = [x_1 x_2 \cdots] \hat{G}_{w_n^C} = [x_1 x_2 \cdots] S_{\delta_n}$, which is the number of marked standard tableaux of shape δ_n. Since this number is evidently $2^N |\operatorname{SYT}(\delta_n)|$, and since we have already seen that $|\operatorname{SYT}(\delta_n)| = |\mathcal{R}(w_n^A)|$, the result follows. $\square$

6 Future directions

6.1 Geometry

There are geometric connections in type A for which we do not know type C analogues. The type A involution Stanley symmetric function $\hat{F}_w$ is a limit of *involution Schubert polynomials*, which are known to represent the cohomology classes of the orbit closures of $\mathrm{O}_n(\mathbb{C})$ acting on the type A complete flag variety. One can also define type C involution Schubert polynomials, which represent cohomology classes on the type C isotropic flag variety insofar as they are positive integer combinations of type C Schubert polynomials, but we do not know a more interesting description of these classes.

6.2 Positivity

As mentioned in Section 2.2, $\hat{F}_y$ is not only Schur-positive but Schur-Q-positive, with integral coefficients up to a predictable scalar [12]. It would be interesting to find a similar expression for $\hat{G}_y$ as a positive combination of some Schur-Q-positive symmetric functions in a nontrivial way. Theorem 1.6 accomplishes this for $\hat{G}_{w_n^C}$, because the Schur S-functions are Schur-Q-positive (they are in fact *skew Schur Q-functions*), but usually $\hat{G}_y$ is not Schur-S-positive.

6.3 Type D analogues

Let D_n be the subgroup of signed permutations in C_n whose one-line representations have an even number of negative letters. This is a finite Coxeter group of classical type D relative to the generating set $S = \{t_1', t_1, t_2, \dots, t_{n-1}\}$ where $t_1' \overset{\text{def}}{=} t_0 t_1 t_0$. For $w \in D_n$ and $a \in \mathcal{R}(w)$, let $\underline{a}$ be the word obtained from a by replacing each t_1' with t_1, and define $\underline{\mathcal{R}}(w) = \{\underline{a} : a \in \mathcal{R}(w)\}$. For instance, $\mathcal{R}(\overline{1}3\overline{2}) = \{(t_1, t_1', t_2), (t_1', t_1, t_2)\}$ while $\underline{\mathcal{R}}(\overline{1}3\overline{2}) = \{(t_1, t_1, t_2)\}$.

In type D it is the sets $\underline{\mathcal{R}}(w)$ that have simple tableau enumerations. Let w_n^D be the longest element of D_n. One has $w_n^D = w_n^C = \overline{1}\,\overline{2}\cdots\overline{n}$ if n is even and $w_n^D = 1\overline{2}\,\overline{3}\cdots\overline{n}$ if n is odd.

Theorem 6.1 (Billey and Haiman [2], Proposition 3.9)**.** If $n \geq 3$ then $|\underline{\mathcal{R}}(w_n^D)| = |\operatorname{SYT}((n-1)^n)|$, which is also the number of (unmarked) shifted standard tableaux of shape $(2n-2, 2n-4, \dots, 2)$.

Let (W, S) be a Coxeter system with a group automorphism $\theta : W \to W$ such that $\theta(S) = S$ and $\theta = \theta^{-1}$. The set of *twisted involutions* with respect to θ is $\mathcal{I}_\theta(W) = \{w \in W : \theta(w) = w^{-1}\}$. The set of *(twisted) atoms* $\mathcal{A}_\theta(y)$ of $y \in \mathcal{I}_\theta(W)$ consists of the minimal-length elements $w \in W$ with $\theta(w)^{-1} \circ w = y$, and the set of *(twisted) involution words* is $\hat{\mathcal{R}}_\theta(y) = \bigsqcup_{w \in \mathcal{A}_\theta(y)} \mathcal{R}(w)$.

Assume W is finite with longest element w_0. If W is A_n, C_n, or D_{2n+1} for $n > 1$, then the only possibilities for θ are the identity map and $w \mapsto w_0 w w_0$, and it holds that $|\hat{\mathcal{R}}_\theta(w_0)| = |\hat{\mathcal{R}}(w_0)|$ and (in type D) $|\hat{\underline{\mathcal{R}}}_\theta(w_0)| = |\hat{\underline{\mathcal{R}}}(w_0)|$ by [10, Corollary 3.9]. Define $*$ as the automorphism of D_n which interchanges t_1 and t_1' and fixes t_i for $i \in [2, n-1]$. When n is odd, $*$ is the inner automorphism $w \mapsto w_0 w w_0$. There appear to be involution word analogues of Theorem 6.1:

Conjecture 6.2. If $n \geq 3$ then $\left|\hat{\underline{\mathcal{R}}}(w_n^D)\right| = |\operatorname{SYT}(\lambda)|$ and $\left|\hat{\underline{\mathcal{R}}}_*(w_n^D)\right| = |\operatorname{SYT}(\mu)|$ for the partitions $\lambda = (n-1, n-2, \dots, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, \dots, 2, 1)$ and $\mu = (n-1, n-2, \dots, \lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil - 1, \dots, 2, 1)$.

For $n = 3, 4, 5, 6$, we have checked by computer that $\left|\hat{\underline{\mathcal{R}}}(w_n^D)\right| = 3,\ 70,\ 5775,\ 10720710$ and $\left|\hat{\underline{\mathcal{R}}}_*(w_n^D)\right| = 3,\ 35,\ 5775,\ 3573570$ as predicted by this conjecture.

References

[1] S. Billey, Transition equations for isotropic flag manifolds, *Discrete Math.* **193** (1998), 69–84.

[2] S. Billey and M. Haiman, Schubert polynomials for the classical groups, *Journal of AMS* **8** (1995), 443–482.

[3] S. Billey and T. K. Lam, Vexillary elements in the hyperoctahedral group, *J. Alg. Combin.* **8** (1998), no. 2, 139–152.

[4] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Maths. 231. Springer, New York, 2005.

[5] M. B. Can, M. Joyce, and B. Wyser, Chains in weak order posets associated to involutions, *J. Combin. Theory Ser. A* **137** (2016), 207–225.

[6] P. Edelman and C. Greene, Balanced tableaux, *Adv. Math.* **63** (1987), 42–99.

[7] S. Fomin and A. N. Kirillov, Combinatorial B_n-analogues of Schubert polynomials, *Trans. Amer. Math. Soc.* **348** (1996), 3591–3620.

[8] M. D. Haiman, Dual equivalence with applications, including a conjecture of Proctor, *Discrete Math.* **99** (1992), 79–113.

[9] Z. Hamaker, E. Marberg, and B. Pawlowski, Involution words: counting problems and connections to Schubert calculus for symmetric orbit closures, *J. Combin. Theory Ser. A*, to appear.

[10] Z. Hamaker, E. Marberg, and B. Pawlowski, Involution words II: braid relations and atomic structures, *J. Algebr. Comb.* **45** (2017), 701-743.

[11] Z. Hamaker, E. Marberg, and B. Pawlowski, Transition formulas for involution Schubert polynomials, *Sel. Math. New Ser.*, to appear.

[12] Z. Hamaker, E. Marberg, and B. Pawlowski, Schur P-positivity and involution Stanley symmetric functions, *IMRN* (2017), rnx274.

[13] Z. Hamaker, E. Marberg, and B. Pawlowski, Fixed-point-free involutions and Schur P-positivity, preprint (2017), `arXiv:1706.06665`.

[14] Z. Hamaker and E. Marberg, Atoms for signed permutations, preprint (2017), `arXiv:1802.09805`.

[15] M. Hansson and A. Hultman, A word property for twisted involutions in Coxeter groups, preprint (2017), `arXiv:1704.08329`.

[16] J. Hu and J. Zhang, On involutions in symmetric groups and a conjecture of Lusztig, *Adv. Math.* **287** (2016), 1–30.

[17] J. Hu and J. Zhang, On involutions in Weyl groups, *J. Lie Theory* **27** (2017), 617–706.

[18] A. Hultman, The combinatorics of twisted involutions in Coxeter groups, *Trans. Amer. Math. Soc.* **359** (2007), 2787–2798.

[19] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, Cambridge, 1990.

[20] W. Kraśkiewicz, Reduced decompositions in hyperoctahedral groups, *C. R. Acad. Sci. Paris*, Serie I, **309** (1989) 903-907.

[21] A. Lascoux and M-P. Schützenberger, Schubert polynomials and the Littlewood-Richardson rule, *Lett. Math. Phys.* **10** (1985), 111-124.

[22] T. K. Lam, B and D analogues of stable Schubert polynomials and related insertion algorithms. PhD thesis, MIT, 1995.

[23] T. K. Lam, B_n Stanley symmetric functions, *Discrete Math.* **157** (1996), 241–270.

[24] D. P. Little, Combinatorial aspects of the Lascoux-Schützenberger tree, *Adv. Math.*, **174** (2003), no. 2, 236–253.

[25] I. G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., Oxford University Press, New York, 1999.

[26] N. J. A. Sloane, editor (2003), The On-Line Encyclopedia of Integer Sequences, published electronically at `http://oeis.org/`.

[27] R. W. Richardson and T. A. Springer, The Bruhat order on symmetric varieties, *Geom. Dedicata* **35** (1990), 389–436.

[28] R. W. Richardson and T. A. Springer, Complements to: The Bruhat order on symmetric varieties, *Geom. Dedicata* **49** (1994), 231–238.

[29] R. P. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, 1999.

[30] R. P. Stanley, On the number of reduced decompositions of elements of Coxeter groups, *European J. Combin.* **5** (1984), 359–372.

[31] D. R. Worley, A theory of shifted Young tableaux, PhD Thesis, Department of Mathematics, Massachusetts Institute of Technology, 1984.

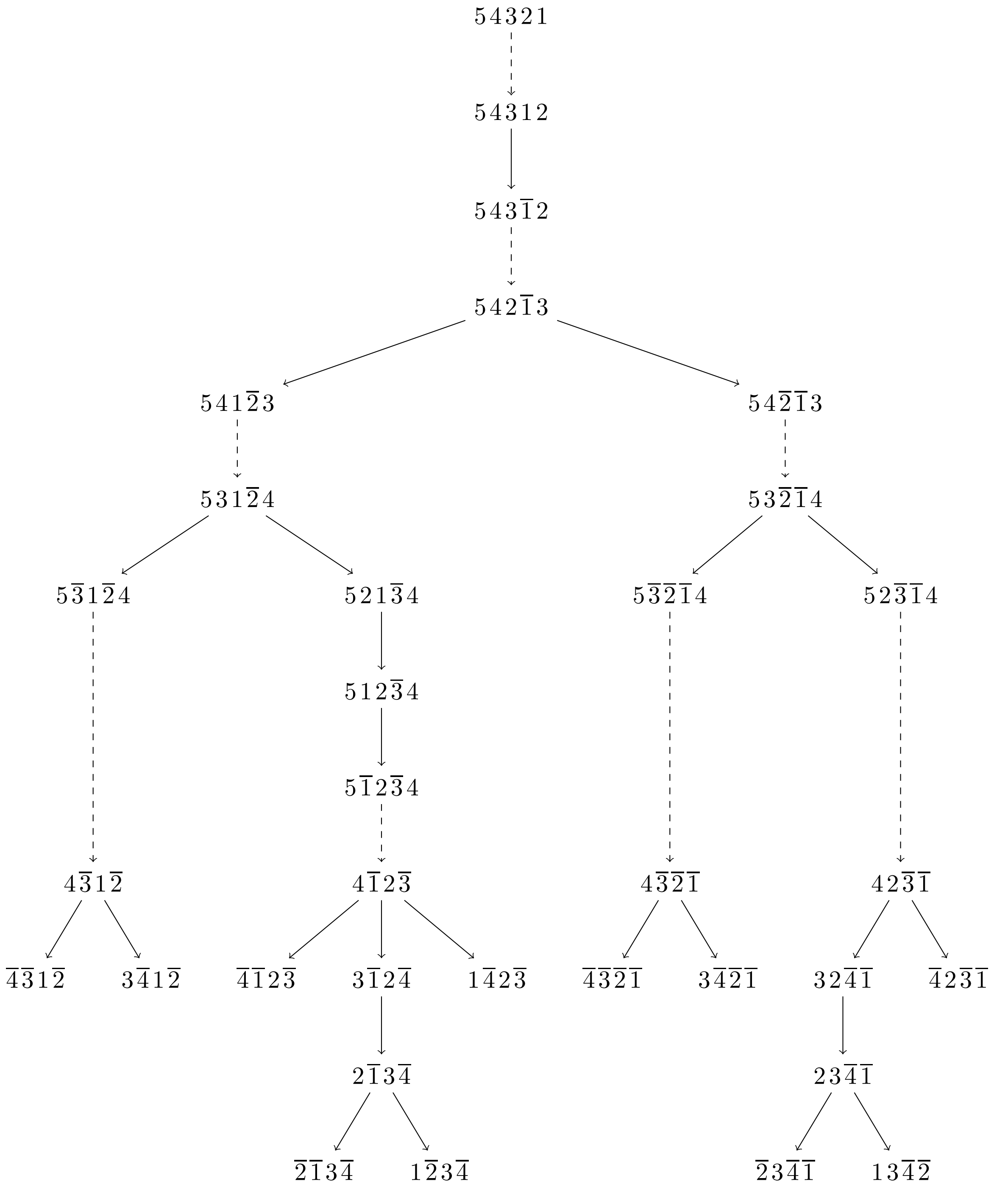


Figure 3: The directed graph $\overrightarrow{\mathcal{G}_4}$. The dashed arrows correspond to edges between vertices of the form (5.5) and (5.6). We have omitted the terminal 5 from all vertices in the final layer $\overrightarrow{\mathcal{L}}_{4,5} \subset \overrightarrow{\mathcal{G}_4}$. In contrast to what we see in this example, the graph $\overrightarrow{\mathcal{G}_n}$ is not a directed tree for $n \geq 5$.