

h^* -POLYNOMIALS WITH ROOTS ON THE UNIT CIRCLE

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ABSTRACT. For an n -dimensional lattice simplex $\Delta_{(1,\mathbf{q})}$ with vertices given by the standard basis vectors and $-\mathbf{q}$ where \mathbf{q} has positive entries, we investigate when the Ehrhart h^* -polynomial for $\Delta_{(1,\mathbf{q})}$ factors as a product of geometric series in powers of z . Our motivation is a theorem of Rodriguez-Villegas implying that when the h^* -polynomial of a lattice polytope P has all roots on the unit circle, then the Ehrhart polynomial of P has positive coefficients. We focus on those $\Delta_{(1,\mathbf{q})}$ for which \mathbf{q} has only two or three distinct entries, providing both theoretical results and conjectures/questions motivated by experimental evidence.

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Date: 19 July 2018.

2010 *Mathematics Subject Classification.* Primary: 52B20, 05A15, 26C10.

The first author was partially supported by grant H98230-16-1-0045 from the U.S. National Security Agency. The second author was partially supported by a grant from the Simons Foundation #426756. This material is also based in part upon work supported by the National Science Foundation under Grant No. DMS-1440140 while both authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2017 semester.

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1. INTRODUCTION

1.1. Background and Motivation. Assume for this paper that P is a full-dimensional lattice polytope in \mathbb{R}^n , i.e. P is given by the convex hull of a finite subset of \mathbb{Z}^n and the affine hull of P has dimension n . Letting tP denote the dilation of P by t , the *Ehrhart polynomial* $L_P(t)$ is defined to be the degree n polynomial satisfying

$$L_P(t) := |tP \cap \mathbb{Z}^n|$$

for $t \in \mathbb{Z}_{\geq 1}$, which is known to exist due to work of Ehrhart [8]. Much is known about the roots and coefficients of Ehrhart polynomials, but major open questions remain. One area of active investigation [13] is to identify criteria that imply $L_P(t) \in \mathbb{Q}_{>0}[t]$, in which case we say that P is *Ehrhart positive*.

Given a polynomial $f(t) \in \mathbb{R}[t]$ of degree n , if all the roots of $f(t)$ have negative real parts, then expanding $f(t)$ as a product of terms of the form $(t+r)$ and $(t+r+bi)(t+r-bi)$ implies that $f(t) \in \mathbb{R}_{>0}[t]$. Thus, Ehrhart positivity is a consequence when $L_P(t)$ has roots with only negative real parts. One approach to investigating those P such that $L_P(t)$ has roots with only non-negative real parts is to consider the generating function for $L_P(t)$. For any polynomial $f(t) \in \mathbb{R}[t]$ of degree n , there exist values $h_j^* \in \mathbb{R}$ with $\sum_{j=0}^n h_j^* \neq 0$ such that

$$\sum_{t=0}^{\infty} f(t)z^t = \frac{\sum_{j=0}^n h_j^* z^j}{(1-z)^{n+1}}.$$

When $f(t) = L_P(t)$, it is known due to work of Stanley [17] that $h_j^* \in \mathbb{Z}_{\geq 0}$ for all j , and we refer to the polynomial $h^*(P; z) := \sum_{j=0}^n h_j^* z^j$ as the *h^* -polynomial of P* . Further, $h_0^* = 1$ and $h_n^* = |\text{int}(P) \cap \mathbb{Z}^n|$ where $\text{int}(P)$ denotes the topological interior of P . Our connection to Ehrhart positivity is provided by the following theorem, which is a special case of a more general result proved by Rodriguez-Villegas.

Theorem 1.1 (Rodriguez-Villegas [15]). If $f(t) \in \mathbb{R}[t]$ is of degree n and the associated polynomial $\sum_{j=0}^n h_j^* z^j$ is also of degree n with all roots on the unit circle, then the roots of $f(t)$ all have real part equal to $-1/2$.

As a consequence of Ehrhart-MacDonald Reciprocity, those lattice polytopes P whose Ehrhart polynomials have roots with real parts equal to $-1/2$ form a subfamily of the class of reflexive polytopes, where P is *reflexive* if some translate P' of P by an integer vector contains the origin in its interior and satisfies that the polar dual of P' is also a lattice polytope. By a result due to Hibi [10], it is known that P is reflexive if and only if $h_i^* = h_{n-i}^*$ for all i . Since $h_0^* = 1$ for all lattice polytopes, it follows that reflexive P have $h_n^* = 1$.

Lattice polytopes satisfying $h_n^* = |\text{int}(P) \cap \mathbb{Z}^n| = 1$ are called *canonical Fano* polytopes, and thus reflexive polytopes are contained within this broader class.

To summarize, if one can apply Theorem 1.1 to $L_P(t)$, then we must have that $h^*(P; z)$ is monic of degree n with all of its roots on the unit circle. The h^* -polynomials with these properties fall within a large and well-studied family.

Definition 1.2. A *Kronecker polynomial* is a monic integer polynomial with all roots inside the complex unit disk.

It is known as a consequence of results due to Hensley [9] and Lagarias and Ziegler [12] that for each dimension n , there are only a finite number of canonical Fano polytopes (up to unimodular equivalence). The following classical theorem complements this fact.

Theorem 1.3 (Kronecker [11], Damianou [7]). For each fixed n , there are only finitely many Kronecker polynomials of degree n . Further, if $h(z) \in \mathbb{Z}[z]$ is a Kronecker polynomial, then all the roots of $h(z)$ are roots of unity, and $h(z)$ factors as a product of cyclotomic polynomials.

Combining Theorem 1.1 and Theorem 1.3 in the setting of Ehrhart h^* -polynomials, we obtain the following corollary.

Corollary 1.4 (see Corollary 2.2.4 in [13]). If the h^* -polynomial of a canonical Fano polytope is a Kronecker polynomial, then P is reflexive and $L_P(t)$ is Ehrhart positive.

1.2. Our Contributions. One way for an h^* -polynomial to be Kronecker is to factor as a product of geometric series in powers of z , which we refer to as a *geometric factorization*. Motivated by Corollary 1.4, we explore geometric factorizations for lattice simplices of the following form: let $\Delta_{(1, \mathbf{q})}$ be the simplex with vertices given by the standard basis vectors and $-\mathbf{q}$ where \mathbf{q} has positive entries. These simplices are related to fans defining weighted projective spaces, and their Ehrhart-theoretic properties have recently been studied by Payne [14], Braun, Davis, and Solus [4], Solus [16], and Balletti, Hibi, Meyer, and Tsuchiya [2].

In Section 2, we establish basic facts about the h^* -polynomials of these simplices and review some of their properties related to $h^*(\Delta_{(1, \mathbf{q})}; z)$ being Kronecker. In Section 3, we prove that when $\Delta_{(1, \mathbf{q})}$ is reflexive there is always a geometric series that can be factored from $h^*(\Delta_{(1, \mathbf{q})}; z)$, leading us to define a polynomial $g_{\mathbf{r}}^{\mathbf{x}}(z)$ that is our primary object of study.

Sections 4 and 5 contain our main theoretical results, focused on \mathbf{q} -vectors with two distinct entries a and $ka - 1$. In Section 4, we identify four families of \mathbf{q} -vectors for which $h^*(\Delta_{(1, \mathbf{q})}; z)$ factors as a product of geometric series. In Section 5, we prove that when \mathbf{q} has distinct entries 2 and $2k - 1$, these families essentially classify those simplices with Kronecker h^* -polynomials.

In Section 6, we provide various conjectures and questions informed by experiments using SageMath [18]. These include conjectured extensions of our result in Section 5, a conjectured Kronecker family related to Fibonacci numbers, and an exploration of the case where \mathbf{q} has three distinct entries, among other topics.

2. THE SIMPLICES $\Delta_{(1,\mathbf{q})}$

2.1. **Definition and Reflexivity.** Given a vector of positive integers $\mathbf{q} \in \mathbb{Z}_{>0}^n$, we define

$$\Delta_{(1,\mathbf{q})} := \text{conv} \left\{ \mathbf{e}_1, \dots, \mathbf{e}_n, - \sum_{i=1}^n q_i \mathbf{e}_i \right\}$$

where \mathbf{e}_i denotes the i -th standard basis vector in \mathbb{R}^n . There is a natural stratification of the family of simplices of the form $\Delta_{(1,\mathbf{q})}$ based on the distinct entries in the vector \mathbf{q} . Given a vector of distinct positive integers $\mathbf{r} = (r_1, \dots, r_d)$, write

$$(r_1^{x_1}, r_2^{x_2}, \dots, r_d^{x_d}) := \underbrace{(r_1, r_1, \dots, r_1)}_{x_1 \text{ times}}, \underbrace{(r_2, r_2, \dots, r_2)}_{x_2 \text{ times}}, \dots, \underbrace{(r_d, r_d, \dots, r_d)}_{x_d \text{ times}}.$$

Definition 2.1. We say that both \mathbf{q} and $\Delta_{(1,\mathbf{q})}$ are *supported* by the vector $\mathbf{r} = (r_1, \dots, r_d)$ with *multiplicity* $\mathbf{x} = (x_1, \dots, x_d)$ if $\mathbf{q} = (q_1, \dots, q_n) = (r_1^{x_1}, r_2^{x_2}, \dots, r_d^{x_d})$.

Since our goal is to determine when $h^*(\Delta_{(1,\mathbf{q})}; z)$ is a Kronecker polynomial, Corollary 1.4 implies that we are only interested in the case where $\Delta_{(1,\mathbf{q})}$ is reflexive. It is straightforward to show [5] that $\Delta_{(1,\mathbf{q})}$ is reflexive if and only if

$$(2.1) \quad q_i \text{ divides } 1 + \sum_{j=1}^n q_j, \quad \text{for all } 1 \leq i \leq n .$$

Equivalently, if \mathbf{q} is supported by \mathbf{r} with multiplicity \mathbf{x} , then $\Delta_{(1,\mathbf{q})}$ is reflexive if and only if $\text{lcm}(r_1, \dots, r_d)$ divides $1 + \sum_{i=1}^d x_i r_i$, which leads us to the following definition.

Definition 2.2. Say \mathbf{x} is an R -multiplicity of \mathbf{r} if $\text{lcm}(r_1, \dots, r_d)$ divides $1 + \sum_{i=1}^d x_i r_i$.

Throughout the rest of this paper, we will frequently use the following setup.

Setup 2.3. Let \mathbf{q} be supported by the vector $\mathbf{r} = (r_1, \dots, r_d) \in (\mathbb{Z}_{>0})^d$ with an R -multiplicity $\mathbf{x} = (x_1, \dots, x_d) \in (\mathbb{Z}_{>0})^d$. Let $\ell = \ell(\mathbf{q})$ be the integer defined by

$$(2.2) \quad 1 + \sum_{i=1}^d x_i r_i = \ell \cdot \text{lcm}(r_1, r_2, \dots, r_d) .$$

Finally, we define

$$(2.3) \quad \mathbf{s} := (s_1, \dots, s_d), \quad \text{where } s_i := \text{lcm}(r_1, \dots, r_d) / r_i \text{ for each } 1 \leq i \leq d .$$

Lemma 2.4. Using Setup 2.3, we have that $\text{gcd}(r_1, \dots, r_d) = 1$ and thus

$$(2.4) \quad \text{lcm}(s_1, \dots, s_d) = \text{lcm}(r_1, \dots, r_d) .$$

Proof. It follows from (2.2) that $\text{gcd}(r_1, \dots, r_d)$ has to be 1. By the definition of s_i , we can verify that

$$\text{lcm}(s_1, \dots, s_d) = \frac{\text{lcm}(r_1, \dots, r_d)}{\text{gcd}(r_1, \dots, r_d)} = \text{lcm}(r_1, \dots, r_d) . \quad \square$$

Our analysis of families of \mathbf{q} -vectors will require a precise language for studying R -multiplicities of vectors, which we introduce next.

Definition 2.5. We define $\langle n \rangle := \{0, 1, \dots, n-1\}$ and $[-n] := \{-n, -(n-1), \dots, -1\}$.

Definition 2.6. Suppose $\mathbf{s} = (s_1, \dots, s_d)$ is a vector of positive integers and $\mathbf{x} = (x_1, \dots, x_d)$ is a vector of integers. Let $\mathbf{c} = (c_1, \dots, c_d)$ and $\boldsymbol{\rho} = (\rho_1, \dots, \rho_d)$ be two vectors of integers such that for each i ,

$$(2.5) \quad x_i = c_i s_i + \rho_i.$$

We say such a pair $(\mathbf{c}, \boldsymbol{\rho})$ is an \mathbf{s} -division of \mathbf{x} , and $\boldsymbol{\rho}$ is an \mathbf{s} -remainder and \mathbf{c} is an \mathbf{s} -quotient. It is clear that any valid \mathbf{s} -quotient or \mathbf{s} -remainder determines a unique \mathbf{s} -division. However, \mathbf{s} -divisions exist nonuniquely.

Suppose further $\mathbf{r} = (r_1, \dots, r_d)$ is a vector of positive integers such that \mathbf{r} and \mathbf{s} are related as in (2.3). We say $\boldsymbol{\rho}$ (or \mathbf{c} or $(\mathbf{c}, \boldsymbol{\rho})$) is *desirable* if

$$\sum_{i=1}^d \rho_i r_i = -1.$$

Example 2.7. Assume Setup 2.3 with $\mathbf{r} = (a, ka-1)$ for some positive integers a and k . Then $r_1 = s_2 = a$ and $r_2 = s_1 = ka-1$. Suppose $\mathbf{x} = (c_1(ka-1) - k, c_2 a + 1)$. (In fact, one can show that any R-multiplicity of \mathbf{x} is in the form. See Subsection 4.1 and Example 4.2.) Then there is a desirable \mathbf{s} -division of \mathbf{x} with

$$\rho_1 = -k, \quad \rho_2 = 1,$$

which follows from observing that $(-k)a + 1 \cdot (ka-1) = -1$.

Lemma 2.8. Suppose two vectors of positive integers $\mathbf{s} = (s_1, \dots, s_d)$ and $\mathbf{r} = (r_1, \dots, r_d)$ are related as in (2.3). Then a vector of integers $\boldsymbol{\rho}$ satisfies

$$(2.6) \quad \sum_{i=1}^d \rho_i r_i \equiv -1 \pmod{\text{lcm}(r_1, \dots, r_d)}$$

if and only if $\boldsymbol{\rho}$ is an \mathbf{s} -remainder of some R-multiplicity \mathbf{x} of \mathbf{r} . Moreover, if \mathbf{x} is an R-multiplicity of \mathbf{r} , there exists a desirable \mathbf{s} -remainder $\boldsymbol{\rho}$ of \mathbf{x} such that for each i ,

$$\rho_i \in \langle s_i \rangle \text{ or } [-s_i].$$

Proof. Suppose $\boldsymbol{\rho}$ is an \mathbf{s} -remainder of some R-multiplicity \mathbf{x} of \mathbf{r} . Plugging in $x_i = c_i s_i + \rho_i$ and using the fact that $s_i r_i = \text{lcm}(r_1, \dots, r_m)$, we obtain

$$1 + \sum_{i=1}^d x_i r_i = 1 + \sum_{i=1}^d (c_i s_i r_i + \rho_i r_i) \equiv 1 + \sum_{i=1}^d \rho_i r_i \pmod{\text{lcm}(r_1, \dots, r_d)}.$$

Thus, (2.6) follows from the fact that \mathbf{x} is an R-multiplicity. Conversely, if (2.6) holds, one sees that $\boldsymbol{\rho}$ is an \mathbf{s} -remainder of $\mathbf{x} = \boldsymbol{\rho}$ which is an R-multiplicity of \mathbf{r} .

We next show the existence of our specified desirable remainder. Let $(\mathbf{c}, \boldsymbol{\rho})$ be the (unique) \mathbf{s} -division of \mathbf{x} such that $\rho_i \in \langle s_i \rangle$ for each i . As $0 \leq \rho_i < s_i$, we have $0 \leq \rho_i r_i < s_i r_i = \text{lcm}(r_1, \dots, r_d)$. Hence,

$$0 \leq \sum_{i=1}^d \rho_i r_i \leq d \cdot \text{lcm}(r_1, \dots, r_d) - d.$$

Thus, Equation (2.6) implies that $\sum_{i=1}^d \rho_i r_i = m \cdot \text{lcm}(r_1, \dots, r_d) - 1$ for some $1 \leq m \leq \max(1, d-1)$. Note that for each i ,

$$x_i = c_i s_i + \rho_i = (c_i + 1)s_i + (\rho_i - s_i),$$

where $\rho_i - s_i \in [-s_i]$. It is straightforward to verify that if we let $(\mathbf{c}', \boldsymbol{\rho}')$ be the \mathbf{s} -division of \mathbf{x} obtained from $(\mathbf{c}, \boldsymbol{\rho})$ by choosing m indices j_1, \dots, j_m and replacing each (c_{j_p}, ρ_{j_p}) with $(c_{j_p} + 1, \rho_{j_p} - s_{j_p})$, then $(\mathbf{c}', \boldsymbol{\rho}')$ is desirable and satisfies that $\rho'_i \in \langle s_i \rangle$ or $[-s_i]$ for each i . \square

Lemma 2.9. Assume Setup 2.3. Suppose $(\mathbf{c}, \boldsymbol{\rho})$ is a desirable \mathbf{s} -division of \mathbf{x} . Then

$$\ell = \ell(\mathbf{q}) = \sum_{i=1}^d c_i.$$

Proof. $1 + \sum_{i=1}^d x_i r_i = 1 + \sum_{i=1}^d (c_i s_i + \rho_i) r_i = \left(\sum_{i=1}^d c_i \right) \cdot \text{lcm}(r_1, \dots, r_d) + \left(1 + \sum_{i=1}^d \rho_i r_i \right)$. \square

Example 2.10. Building on Example 2.7 where $\mathbf{r} = (a, ka - 1)$ and $\mathbf{x} = (c_1(ka - 1) - k, c_2a + 1)$, it is elementary to verify that

$$1 + (c_1(ka - 1) - k)a + (c_2a + 1)(ka - 1) = (c_1 + c_2)a(ka - 1).$$

2.2. h^* -Polynomials and Geometric Factorizations. The following theorem shows that the h^* -polynomial for any $\Delta_{(1, \mathbf{q})}$ can be expressed purely in terms of the vector \mathbf{q} .

Theorem 2.11 (Braun, Davis, and Solus [4]). The h^* -polynomial of $\Delta_{(1, \mathbf{q})}$ is given by

$$\sum_{b=0}^{q_1+q_2+\dots+q_n} z^{w(b)}$$

where

$$(2.7) \quad w(b) = b - \sum_{i=1}^n \left\lfloor \frac{b q_i}{1 + \sum_{j=1}^n q_j} \right\rfloor.$$

Example 2.12. For integers $w \geq 0$, $a \geq 3$, and $t \geq w + 2$, Payne [14] introduced the reflexive simplex $\Delta_{(1, \mathbf{q})}$ with

$$(2.8) \quad \mathbf{q} = (\underbrace{1, 1, \dots, 1}_{at-1 \text{ times}}, \underbrace{a, a, \dots, a}_{w+1 \text{ times}}),$$

in other words we have $\mathbf{r} = (1, a)$ with R-multiplicity $\mathbf{x} = (at - 1, w + 1)$. It follows from Theorems 3.2 and 4.5 below that

$$h^*(\Delta_{(1, \mathbf{q})}; z) = (1 + z^t + z^{2t} + \dots + z^{(a-1)t})(1 + z + z^2 + \dots + z^{t+w}).$$

In this work we are primarily interested in studying when $h^*(\Delta_{(1, \mathbf{p})}; z)$ factors as a product of geometric series, similarly to Payne's simplices in Example 2.12. We next define language and notation for working with products of geometric series in varying powers of z .

Definition 2.13. For any $e \in \mathbb{Z}_{>0}$ and $\gamma \in \mathbb{Z}_{\geq 2}$, we call

$$\sum_{i=0}^{\gamma-1} z^{ie} = 1 + z^e + z^{2e} + \dots + z^{(\gamma-1)e}$$

a *geometric series* (in powers of z) of length γ and with *exponent* e . We say a polynomial $f(z)$ in z is a *product of geometric series* (in powers of z) if there exists $p \in \mathbb{Z}_{>0}$, $e_1, e_2, \dots, e_p \in \mathbb{Z}_{>0}$ and $\gamma_1, \dots, \gamma_p \in \mathbb{Z}_{\geq 2}$ such that

$$(2.9) \quad f(z) = \prod_{j=1}^p \sum_{i=0}^{\gamma_j-1} z^{ie_j} = \prod_{j=1}^p (1 + z^{e_j} + z^{2e_j} + \dots + z^{(\gamma_j-1)e_j}).$$

We also call the right hand side of the above equation a *geometric factorization* of $f(z)$.

We remark that geometric factorizations of a polynomial f are not necessarily unique, e.g, $f(z) = 1 + z + z^2 + z^3$ is a geometric series itself, but can also be expressed as $(1 + z)(1 + z^2)$. As our first observation regarding geometric factorizations, we show that ordinary geometric series are h^* -polynomials for only one family of $\Delta_{(1, \mathbf{q})}$ simplices.

Proposition 2.14. Assume Setup 2.3. Then $h^*(\Delta_{(1, \mathbf{q})}; z)$ is a geometric series if and only if \mathbf{q} is supported on one integer.

Proof. Suppose \mathbf{q} is supported on one integer r , i.e. $q = (r^x)$ for some positive integers r and x . Since r divides $1 + xr$, we have that $r = 1$ and x can be any positive integer. Applying Theorem 2.11, we immediately obtain that

$$h^*(\Delta_{(1, \mathbf{q})}; z) = \sum_{b=0}^x z^{w(b)} = \sum_{b=0}^{xr} z^b,$$

which is a geometric series of length $1 + xr$ and with exponent 1.

Conversely, assume $h^*(\Delta_{(1, \mathbf{q})}; z)$ is a geometric series. Note that

$$w(1) = 1 - \sum_{i=1}^d x_i \left[\frac{r_i}{1 + \sum_{j=1}^d x_j r_j} \right] = 1.$$

Hence, z^1 appears in $h^*(\Delta_{(1, \mathbf{q})}; z)$. This implies that $h^*(\Delta_{(1, \mathbf{q})}; z)$ is a geometric series with exponent 1. Thus, we must have that for each b with $0 \leq b \leq \sum_{j=1}^d x_j r_j$,

$$w(b) = b - \sum_{i=1}^d x_i \left[\frac{br_i}{1 + \sum_{j=1}^d x_j r_j} \right] = b.$$

Thus, $br_i < 1 + \sum_{j=1}^d x_j r_j$ for all such b . Considering the case where $b = \sum_{j=1}^d x_j r_j$, we must have $r_i = 1$ for all i . Hence $\Delta_{(1, \mathbf{q})}$ is supported on one integer $r = 1$. \square

2.3. Free Sums Create New Kronecker $h^*(\Delta_{(1,q)}; z)$. For two reflexive simplices $\Delta_{(1,q)}$ and $\Delta_{(1,p)}$ with Kronecker h^* -polynomials, there exists an operation that produces a new simplex $\Delta_{(1,y)}$ that is reflexive with a Kronecker h^* -polynomial. We say that $P \oplus Q := \text{conv}\{P \cup Q\}$ is an *affine free sum* if, up to unimodular equivalence, $P \cap Q = \{0\}$ and the affine span of P and Q are orthogonal coordinate subspaces of \mathbb{R}^n . Suppose further that $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ are reflexive polytopes with $0 \in P$ and the vertices of Q labeled as v_0, v_1, \dots, v_m . For every $i = 0, 1, \dots, m$, we define the polytope

$$P *_i Q := \text{conv}\{(P \times 0^m) \cup (0^n \times Q - v_i)\} \subset \mathbb{R}^{n+m}.$$

The following theorem indicates that affine free sum decompositions can be detected from the \mathbf{q} -vector defining $\Delta_{(1,q)}$ and induce a product structure for h^* -polynomials.

Theorem 2.15 (Braun, Davis [3]). If $\Delta_{(1,p)}$ and $\Delta_{(1,q)}$ are full-dimensional reflexive simplices with $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_m)$, respectively, then $\Delta_{(1,p)} *_0 \Delta_{(1,q)}$ is a reflexive simplex $\Delta_{(1,y)}$ with $\mathbf{y} = (p_1, \dots, p_n, sq_1, \dots, sq_m)$ where $s = 1 + \sum_{j=1}^n p_j$. Moreover, if $\Delta_{(1,y)}$ arises in this form, then it decomposes as a free sum. Further, if $\Delta_{(1,p)}$ and $\Delta_{(1,q)}$ are reflexive, then $h^*(\Delta_{(1,p)} *_0 \Delta_{(1,q)}; z) = h^*(\Delta_{(1,p)}; z)h^*(\Delta_{(1,q)}; z)$.

Corollary 2.16. If $h^*(\Delta_{(1,p)}; z)$ and $h^*(\Delta_{(1,q)}; z)$ are Kronecker polynomials, then we also have that $h^*(\Delta_{(1,p)} *_0 \Delta_{(1,q)}; z)$ is a Kronecker polynomial.

Remark 2.17. More generally, if P and Q are reflexive polytopes, then free sums of P and Q have h^* -polynomials obtained as products of the h^* -polynomials of their free summands. Thus, the resulting h^* -polynomials are also Kronecker when the summands have Kronecker h^* -polynomials.

3. FACTORING $h^*(\Delta_{(1,q)}; z)$ FOR REFLEXIVE $\Delta_{(1,q)}$

3.1. Reflexive $\Delta_{(1,q)}$ Always Have a Geometric Series Factor in $h^*(\Delta_{(1,q)}; z)$. In this subsection, we show that for a reflexive $\Delta_{(1,q)}$, it is always possible to factor a geometric series from $h^*(\Delta_{(1,q)}; z)$. The following polynomial plays a fundamental role in this factorization.

Definition 3.1. Suppose $\mathbf{r}, \mathbf{x}, \ell$ and \mathbf{s} are as given in Setup 2.3. We define

$$g_{\mathbf{r}}^{\mathbf{x}}(z) := \sum_{0 \leq \alpha < \text{lcm}(r_1, \dots, r_d)} z^{u(\alpha)}$$

where

$$u(\alpha) = u_{\mathbf{r}}^{\mathbf{x}}(\alpha) := \alpha\ell - \sum_{i=1}^d x_i \left\lfloor \frac{\alpha}{s_i} \right\rfloor.$$

Theorem 3.2. Assuming Setup 2.3, we have that

$$h^*(\Delta_{(1,q)}; z) = \left(\sum_{t=0}^{\ell-1} z^t \right) \cdot g_{\mathbf{r}}^{\mathbf{x}}(z).$$

Proof. Let $M := \text{lcm}(r_1, \dots, r_d)$. Let $0 \leq b < \ell M$ and write $b = \alpha\ell + \beta$ for $0 \leq \alpha < M$ and $0 \leq \beta < \ell$. Then using (2.7) we have:

$$\begin{aligned} w(b) &= w(\alpha\ell + \beta) = \alpha\ell + \beta - \sum_{i=1}^d x_i \left\lfloor \frac{(\alpha\ell + \beta)r_i}{\ell M} \right\rfloor \\ &= \beta + \alpha\ell - \sum_{i=1}^d x_i \left\lfloor \frac{\alpha\ell + \beta}{\ell s_i} \right\rfloor = \beta + \alpha\ell - \sum_{i=1}^d x_i \left\lfloor \frac{\alpha}{s_i} + \frac{\beta}{\ell s_i} \right\rfloor. \end{aligned}$$

Since $0 \leq \beta < \ell$, we have

$$0 \leq \frac{\alpha}{s_i} + \frac{\beta}{\ell s_i} < \frac{\alpha + 1}{s_i}$$

and thus

$$w(b) = w(\alpha\ell + \beta) = \beta + \alpha\ell - \sum_{i=1}^d x_i \left\lfloor \frac{\alpha}{s_i} \right\rfloor = \beta + u(\alpha).$$

Hence, it follows from Theorem 2.11 that

$$h^*(\Delta_{(1,\mathbf{q})}; z) = \sum_{b=0}^{\ell s} z^{w(b)} = \sum_{\substack{0 \leq \alpha < M \\ 0 \leq \beta < \ell}} z^{\beta + u(\alpha)} = \left(\sum_{0 \leq \beta < \ell} z^\beta \right) g_{\mathbf{r}}^{\mathbf{x}}(z). \quad \square$$

The following is an immediate consequence of Theorem 3.2.

Corollary 3.3. For $\mathbf{q} = (r_1^{x_1}, \dots, r_d^{x_d})$, we have $h^*(\Delta_{(1,\mathbf{q})}; z)$ is a Kronecker polynomial if and only if $g_{\mathbf{r}}^{\mathbf{x}}(z)$ is a Kronecker polynomial.

Remark 3.4. If $h^*(\Delta_{(1,\mathbf{q})}; z)$ has a geometric factorization, then $g_{\mathbf{r}}^{\mathbf{x}}(z)$ does not necessarily have a geometric factorization, although the converse is clearly true. The smallest counterexample is when $\mathbf{r} = (2, 5)$ and $\mathbf{x} = (7, 5)$. In this case,

$$h^*(\Delta_{(1,\mathbf{q})}; z) = 1 + z + 2z^2 + 4z^3 + 4z^4 + 5z^5 + 6z^6 + 5z^7 + 4z^8 + 4z^9 + 2z^{10} + z^{11} + z^{12},$$

which can be factored as $(1 + z^2)(1 + z^3)^2(1 + z + z^2 + z^3 + z^4)$, and

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = 1 + z^2 + 2z^3 + z^4 + z^5 + 2z^6 + z^7 + z^9,$$

which cannot be written as a product of geometric series.

Remark 3.5. Another area of interest is identifying lattice polytopes where $h^*(P; z)$ has only real roots; see recent work by Solus [16] for an investigation of $\Delta_{(1,\mathbf{q})}$ with this property. Theorem 3.2 implies that if $\Delta_{(1,\mathbf{q})}$ is reflexive with $\ell \geq 3$, then $h^*(\Delta_{(1,\mathbf{q})}; z)$ is not real-rooted. Further, while our primary focus in this paper is on factoring h^* -polynomials as products of geometric series, there are techniques related to real-rootedness that count the number of unit circle roots of a given polynomial. For example, if $f(z)$ is degree n and does not have 1 as a root, then the transformation $g(z) = (z + i)^n f\left(\frac{z - i}{z + i}\right)$ sends unit circle roots of f to real roots of g [6, Page 7]. Thus, in this setting f has all unit circle roots if and only if g has only real roots. It would be of interest to determine if these techniques can be applied productively in the setting of h^* -polynomials.

The next result shows that extending $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ by $\text{lcm}(r_1, \dots, r_d)$ does not alter the structure of $g_{\mathbf{r}}^{\mathbf{x}}(z)$.

Theorem 3.6. Let $\mathbf{q} = (r_1^{x_1}, \dots, r_d^{x_d})$ where \mathbf{x} is an R-multiplicity of \mathbf{r} and $\ell = \ell(\mathbf{q})$. Then $\mathbf{q}' = (r_1^{x_1}, \dots, r_d^{x_d}, \text{lcm}(r_1, \dots, r_d)^y)$ satisfies

$$h^*(\Delta_{(1, \mathbf{q}')}; z) = \left(\sum_{t=0}^{\ell+y-1} z^t \right) \cdot g_{\mathbf{r}}^{\mathbf{x}}(z).$$

Proof. Let $M := \text{lcm}(r_1, \dots, r_d)$. First observe that if \mathbf{x} is an R-multiplicity of \mathbf{r} , then

$$1 + \sum_{i=1}^d x_i r_i + yM = (\ell + y)M$$

and thus (\mathbf{x}, y) is clearly an R-multiplicity of (\mathbf{r}, M) . Further,

$$\text{lcm}(r_1, \dots, r_d, M) = \text{lcm}(r_1, \dots, r_d)$$

and thus

$$g_{(\mathbf{r}, M)}^{(\mathbf{x}, y)}(z) := \sum_{0 \leq \alpha < \text{lcm}(r_1, \dots, r_d)} z^{u(\alpha)}$$

where

$$u_{(\mathbf{r}, M)}^{(\mathbf{x}, y)}(\alpha) = \alpha(\ell + y) - \sum_{i=1}^d x_i \left\lfloor \frac{\alpha}{s_i} \right\rfloor - y \left\lfloor \frac{\alpha}{1} \right\rfloor = \alpha(\ell) - \sum_{i=1}^d x_i \left\lfloor \frac{\alpha}{s_i} \right\rfloor = u_{\mathbf{r}}^{\mathbf{x}}(\alpha).$$

Hence,

$$g_{(\mathbf{r}, M)}^{(\mathbf{x}, y)}(z) = g_{\mathbf{r}}^{\mathbf{x}}(z)$$

and the result follows. \square

3.2. A Useful Form for $g_{\mathbf{r}}^{\mathbf{x}}(z)$. Our goal in this subsection is to prove Theorem 3.10 below, providing a reformulation of $g_{\mathbf{r}}^{\mathbf{x}}(z)$ that is helpful for establishing factorizations. We will require the following theorem from elementary number theory.

Theorem 3.7 (Generalized Chinese Remainder Theorem). Suppose m_1, m_2, \dots, m_d are positive integers and $i_1, i_2, \dots, i_d \in \mathbb{Z}$. Then the system of congruences

$$(3.1) \quad \begin{cases} x \equiv i_1 & \text{mod } m_1 \\ x \equiv i_2 & \text{mod } m_2 \\ \vdots & \vdots \\ x \equiv i_d & \text{mod } m_d \end{cases}$$

has a solution if and only if $\text{gcd}(m_j, m_{j'}) \mid (i_j - i_{j'})$ for any pair of indices (j, j') , where $1 \leq j < j' \leq d$. Moreover, when there is a solution, it is unique modulo $\text{lcm}(m_1, m_2, \dots, m_d)$.

Motivated by the above theorem, for two vectors \mathbf{r} and \mathbf{s} related by (2.3) we define

$$I = I(\mathbf{r}) := \{\mathbf{i} = (i_1, \dots, i_d) \in \langle s_1 \rangle \times \dots \times \langle s_d \rangle : \text{gcd}(s_j, s_{j'}) \mid (i_j - i_{j'}) \text{ for all } 1 \leq j < j' \leq d\}.$$

The following result is a direct consequence of Theorem 3.7 and (2.4).

Corollary 3.8. For each $\mathbf{i} \in I(\mathbf{r})$, there exists a unique $\alpha \in \langle \text{lcm}(r_1, \dots, r_d) \rangle$ such that $\alpha \equiv i_j \pmod{s_j}$ for each $1 \leq j \leq d$.

Definition 3.9. We denote by $\alpha(\mathbf{i})$ the unique α assumed by the above corollary, and let

$$(3.2) \quad \omega_j = \omega_j(\mathbf{i}) := \left\lfloor \frac{\alpha(\mathbf{i})}{s_j} \right\rfloor.$$

Thus,

$$(3.3) \quad \alpha(\mathbf{i}) = \omega_j(\mathbf{i}) \cdot s_j + i_j.$$

The following theorem provides an expression for $g_{\mathbf{r}}^{\mathbf{x}}(z)$ that we will rely on throughout the remainder of this work.

Theorem 3.10. Assume Setup 2.3. Suppose $(\mathbf{c}, \boldsymbol{\rho})$ is a desirable \mathbf{s} -division of \mathbf{x} . Then

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = \sum_{\mathbf{i} \in I(\mathbf{r})} z^{\sum_{j=1}^d (c_j i_j - \rho_j \omega_j(\mathbf{i}))}.$$

Proof. By Definition 3.1 and Corollary 3.8, it is enough to verify that for each $\mathbf{i} \in I(\mathbf{r})$, we have

$$(3.4) \quad u(\alpha(\mathbf{i})) = \alpha(\mathbf{i})\ell - \sum_{j=1}^d x_j \left\lfloor \frac{\alpha(\mathbf{i})}{s_j} \right\rfloor = \sum_{j=1}^d (c_j i_j - \rho_j \omega_j(\mathbf{i})).$$

However, it is straightforward to show this by using (2.5), (3.2), (3.3), and Lemma 2.9. \square

In the case where \mathbf{r} and \mathbf{s} are related by (2.3) with the entries of \mathbf{s} pairwise coprime, the following proposition provides an alternative description of ω_j , and hence of $g_{\mathbf{r}}^{\mathbf{x}}(z)$. Recall that $(a \bmod b)$ is the unique integer $a' \in \langle b \rangle$ satisfying $a \equiv a' \pmod{b}$.

Proposition 3.11. Assume Setup 2.3 where s_1, \dots, s_d are pairwise coprime. Then

$$\text{lcm}(r_1, \dots, r_d) = \text{lcm}(s_1, \dots, s_d) = s_1 s_2 \dots s_d$$

and thus for each $1 \leq j \leq d$, we have $r_j = \prod_{j' \neq j} s_{j'}$.

Suppose $(\mathbf{c}, \boldsymbol{\rho})$ is an \mathbf{s} -division of \mathbf{x} . Then for each $\mathbf{i} \in I(\mathbf{r})$,

$$(3.5) \quad \alpha(\mathbf{i}) = \left(- \sum_{t=1}^d \rho_t r_t i_t \bmod s_1 s_2 \dots s_d \right).$$

Furthermore, if $\boldsymbol{\rho}$ is desirable, then for each $1 \leq j \leq d$,

$$(3.6) \quad \omega_j(\mathbf{i}) = \left(\sum_{t \neq j} \rho_t \frac{r_t}{s_j} (i_j - i_t) \bmod r_j \right).$$

Proof. It is straightforward to verify the conclusions in the first paragraph.

By the definition of $\alpha(\mathbf{i})$ and because the s_j 's are pairwise coprime, in order to show (3.5) it is enough to prove that for each $1 \leq j \leq d$,

$$(3.7) \quad - \sum_{t=1}^d \rho_t r_t i_t \equiv i_j \pmod{s_j}.$$

However, since $r_t = \prod_{j' \neq t} s_{j'}$, clearly s_j divides r_t for each $t \neq j$. Hence, $-\sum_{t=1}^d \rho_t r_t i_t \equiv -\rho_j r_j i_j$

(mod s_j). Next, it follows from Lemma 2.8 that $\sum_{j=1}^d \rho_j r_j \equiv -1 \pmod{s_j}$. Again, as s_j divides r_t whenever $t \neq j$, we conclude that $\rho_j r_j \equiv -1 \pmod{s_j}$. Thus, (3.7) follows.

By the definition of $\omega_j(\mathbf{i})$, we see that $\omega_j(\mathbf{i}) \in \langle r_j \rangle$. Hence, (3.6) is equivalent to

$$(3.8) \quad \omega_j(\mathbf{i}) \equiv \sum_{t \neq j} \rho_t \frac{r_t}{s_j} (i_j - i_t) \pmod{r_j},$$

By (3.5), we have that $\alpha(\mathbf{i}) = -\sum_{t=1}^d \rho_t r_t i_t + M s_1 s_2 \dots s_d = -\sum_{t=1}^d \rho_t r_t i_t + M s_j r_j$ for some integer M . Hence,

$$\omega_j(\mathbf{i}) = \frac{\alpha(\mathbf{i}) - i_j}{s_j} \equiv \frac{-\sum_{t=1}^d \rho_t r_t i_t - i_j}{s_j} \pmod{r_j}.$$

Since $\boldsymbol{\rho}$ is desirable, $\sum_{t=1}^d \rho_t r_t = -1$. Hence, we can replace $-i_j$ with $\sum_{t=1}^d \rho_t r_t i_j$ in the above equation, from which (3.8) follows. \square

4. SOME KRONECKER h^* -POLYNOMIALS WHEN $\mathbf{r} = (a, ka - 1)$

We have seen in Proposition 2.14 that any reflexive $\Delta_{(1, \mathbf{q})}$ supported on one integer has $\mathbf{r} = (1)$. The next level of complexity of \mathbf{q} -vectors are those for which \mathbf{q} has two distinct entries. Payne's simplices from Example 2.12 are an important example of this type in Ehrhart theory, as they are reflexive polytopes whose h^* -polynomials are not unimodal; further, their h^* -polynomials factor as a product of geometric series. In this section we prove four theorems establishing Kronecker h^* -polynomials, each theorem corresponding to a family of \mathbf{q} -vectors supported on two integers. We use the following setup throughout this section.

4.1. Setup. Recall from elementary number theory that for $\mathbf{r} = (r_1, r_2) \in (\mathbb{Z}_{>0})^2$ such that $\gcd(r_1, r_2) = 1$, there exists an integer solution $\boldsymbol{\rho} = (\rho_1, \rho_2)$ to $\rho_1 r_1 + \rho_2 r_2 = -1$. Furthermore, if $\boldsymbol{\rho}^* = (\rho_1^*, \rho_2^*)$ is a special integer solution to $\rho_1 r_1 + \rho_2 r_2 = -1$, then all integer solutions are in the form of

$$\rho_1 = \rho_1^* - r_2 k, \quad \rho_2 = \rho_2^* + r_1 k, \quad \text{for some integer } k.$$

It then follows that there exists a unique integer solution $\boldsymbol{\rho} = (\rho_1, \rho_2)$ to $\rho_1 r_1 + \rho_2 r_2 = -1$ where $\rho_1 \in [-r_2]$ and $\rho_2 \in \langle r_1 \rangle$. This implies that desirable \mathbf{s} -remainders are unique in this context.

Setup 4.1. Let $\mathbf{r} = (r_1, r_2) \in (\mathbb{Z}_{>0})^2$ satisfy $\gcd(r_1, r_2) = 1$, and let $\mathbf{s} = (s_1, s_2) = (r_2, r_1)$. Let $\boldsymbol{\rho} = (\rho_1, \rho_2)$ be the unique solution to $\rho_1 r_1 + \rho_2 r_2 = -1$ such that $\rho_1 \in [-s_1]$ and $\rho_2 \in \langle s_2 \rangle$. Let \mathbf{q} be the vector supported by \mathbf{r} with the R-multiplicity $\mathbf{x} = (x_1, x_2) \in (\mathbb{Z}_{>0})^2$ having the property that $\boldsymbol{\rho}$ is an \mathbf{s} -remainder of \mathbf{x} ; that is, for some integers c_1, c_2 ,

$$x_1 = c_1 s_1 + \rho_1 \text{ and } x_2 = c_2 s_2 + \rho_2.$$

Thus, $\ell = \ell(\mathbf{q}) = c_1 + c_2$.

Example 4.2. Suppose $\mathbf{r} = (a, ka - 1)$ for some integers $a \geq 2$ and $k \geq 1$. Then $\mathbf{s} = (ka - 1, a)$, $\boldsymbol{\rho} = (-k, 1)$, and $\mathbf{x} = (c_1(ka - 1) - k, c_2a + 1)$ for some integers $c_1 > k/(ka - 1)$ and $c_2 \geq 0$.

Since $\gcd(s_1, s_2) = \gcd(r_1, r_2) = 1$ for this setup, we can always apply Proposition 3.11, yielding the following corollary.

Corollary 4.3. Assume Setup 4.1. Then for each $\mathbf{i} = (i_1, i_2) \in I(\mathbf{r}) = \langle r_2 \rangle \times \langle r_1 \rangle$, we have:

$$\begin{aligned}\alpha(\mathbf{i}) &= (-\rho_1 r_1 i_1 - \rho_2 r_2 i_2 \bmod r_1 r_2) \\ \omega_1(\mathbf{i}) &= (\rho_2(i_1 - i_2) \bmod r_1) \\ \omega_2(\mathbf{i}) &= (\rho_1(i_2 - i_1) \bmod r_2)\end{aligned}$$

The following lemma will be used in the proofs of the theorems in the next subsection.

Lemma 4.4. Let $\mathbf{r} = (a, ka - 1)$ for some $k \geq 1$ and $a \geq 2$. Let $(\mathbf{c}, \boldsymbol{\rho})$ be the desirable \mathbf{s} -division of \mathbf{x} with $\boldsymbol{\rho} = (-k, 1)$. Then

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = \sum_{\mathbf{i} \in \langle ka-1 \rangle \times \langle a \rangle} z^{c_1 i_1 + c_2 i_2 - \lfloor \frac{i_1 - i_2}{a} \rfloor}.$$

Proof. Let $M = \lfloor \frac{i_1 - i_2}{a} \rfloor$. Note that $I(\mathbf{r}) = \langle r_2 \rangle \times \langle r_1 \rangle = \langle ka - 1 \rangle \times \langle a \rangle$. Hence, we only need to show that, using the notation from Definition 3.9,

$$-k\omega_1(\mathbf{i}) + \omega_2(\mathbf{i}) = \left\lfloor \frac{i_1 - i_2}{a} \right\rfloor = M,$$

and the result follows from Theorem 3.10. Applying Corollary 4.3, we get

$$\omega_1(\mathbf{i}) = ((i_1 - i_2) \bmod a) \quad \text{and} \quad \omega_2(\mathbf{i}) = (k(i_1 - i_2) \bmod (ka - 1)).$$

Thus, $i_1 - i_2 = aM + \omega_1(\mathbf{i})$ and

$$(4.1) \quad \omega_2(\mathbf{i}) = (k(i_1 - i_2) \bmod (ka - 1)) = (M + k\omega_1(\mathbf{i}) \bmod (ka - 1)).$$

Since $(i_1, i_2) \in \langle ka - 1 \rangle \times \langle a \rangle$, we have that $-(a - 1) \leq i_1 - i_2 \leq (ka - 2)$. The proof is complete after we show that the right hand side of (4.1) is equal to $M + k\omega_1(\mathbf{i})$, which is equivalent to

$$0 \leq M + k\omega_1(\mathbf{i}) \leq ka - 2.$$

It is straightforward to verify the left-hand inequality

$$0 \leq M + k\omega_1(\mathbf{i}) = \left\lfloor \frac{i_1 - i_2}{a} \right\rfloor + k((i_1 - i_2) \bmod a)$$

holds by considering the two cases $i_1 - i_2 < 0$ and $i_1 - i_2 \geq 0$, noting the assumption that $k \geq 1$. One can similarly verify the right-hand inequality holds by considering the two cases $i_1 - i_2 < (k - 1)a$ and $i_1 - i_2 \geq (k - 1)a$. \square

4.2. Four Main Theorems.

Theorem 4.5. For $\mathbf{r} = (1, a)$ or $(a, 1)$ and any R-multiplicity \mathbf{x} , the resulting $g_{\mathbf{r}}^{\mathbf{x}}(z)$ is a geometric series, which is a Kronecker polynomial.

Proof. Suppose $\mathbf{r} = (1, a)$ for some integer $a \geq 2$. Then $\mathbf{s} = (a, 1)$, $\boldsymbol{\rho} = (-1, 0)$, and $\mathbf{x} = (ac_1 - 1, c_2)$ for some positive integers c_1, c_2 . Then $\omega_1(\mathbf{i}) = (\rho_2(i_1 - i_2) \bmod r_1) = (0 \bmod r_1) = 0$. Thus,

$$\rho_1\omega_1(\mathbf{i}) + \rho_2\omega_2(\mathbf{i}) = -1 \cdot 0 + 0 \cdot \omega_2(\mathbf{i}) = 0.$$

Hence,

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = \sum_{\mathbf{i} \in \langle a \rangle \times \langle 1 \rangle} z^{c_1 i_1 + c_2 i_2} = \sum_{i_1 \in \langle a \rangle} z^{c_1 i_1}$$

is a Kronecker polynomial.

The proof in the case where $\mathbf{r} = (a, 1)$ for some integer $a \geq 2$ is identical. \square

Theorem 4.6. Let $a \geq 2$, $k \geq 1$, and $c \geq 1$. For $\mathbf{r} = (a, ka - 1)$ and $\mathbf{x} = ((ka - 1)c - k, a((ka - 1)c - k) + 1)$, we have

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = \left(\sum_{j_1 \in \langle ka-1 \rangle} z^{(ac-1)j_1} \right) \left(\sum_{j_2 \in \langle a \rangle} z^{cj_2} \right),$$

which is a Kronecker polynomial.

Proof. With the given \mathbf{x} , we have the desirable \mathbf{s} -division with

$$\mathbf{c} = (c, (ka - 1)c - k) \text{ and } \boldsymbol{\rho} = (-k, 1).$$

Observe that $r_1 = s_2 = a$ and $r_2 = s_1 = ka - 1$. Thus, by Lemma 4.4,

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = \sum_{\mathbf{i} \in \langle ka-1 \rangle \times \langle a \rangle} z^{ci_1 + ((ka-1)c-k)i_2 - \lfloor \frac{i_1-i_2}{a} \rfloor}.$$

One sees that it is enough to show that there exists a bijection φ on $\langle ka - 1 \rangle \times \langle a \rangle$ such that for any $\mathbf{i} = (i_1, i_2) \in \langle ka - 1 \rangle \times \langle a \rangle$, if $\mathbf{j} = (j_1, j_2) = \varphi(\mathbf{i})$, then

$$(4.2) \quad ci_1 + ((ka - 1)c - k)i_2 - \left\lfloor \frac{i_1 - i_2}{a} \right\rfloor = (ac - 1)j_1 + cj_2.$$

We will construct such a bijection below.

For any $\mathbf{i} = (i_1, i_2) \in \langle ka - 1 \rangle \times \langle a \rangle$, we define

$$\varphi_1(\mathbf{i}) = (ka - 1)i_2 + i_1, \text{ and } \varphi_2(\mathbf{i}) = ai_1 + i_2.$$

It is easy to see that both φ_1 and φ_2 are bijections from $\langle ka - 1 \rangle \times \langle a \rangle$ to $\langle a(ka - 1) \rangle$. Therefore, $\varphi := \varphi_2^{-1} \circ \varphi_1$ is a bijection on $\langle ka - 1 \rangle \times \langle a \rangle$. Now suppose $\mathbf{j} = (j_1, j_2) = \varphi(\mathbf{i}) = \varphi(i_1, i_2)$. By the definition of φ , we have

$$j_1 = \left\lfloor \frac{(ka - 1)i_2 + i_1}{a} \right\rfloor, \text{ and } j_2 = ((ka - 1)i_2 + i_1) - aj_1.$$

Thus,

$$j_1 = ki_2 + \left\lfloor \frac{i_1 - i_2}{a} \right\rfloor, \text{ and } j_2 = i_1 - i_2 - a \left\lfloor \frac{i_1 - i_2}{a} \right\rfloor.$$

One then can show (4.2) holds directly by plugging in the above. \square

Theorem 4.7. Let $a \geq 2$ and $c \geq 1$. For $\mathbf{r} = (a, a - 1)$ and $\mathbf{x} = ((a - 1)c - 1, ac + 1)$, we have

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = (1 + z^{c+1}) \left(\sum_{j=0}^{\lfloor \frac{a-1}{2} \rfloor} z^{cj} \right) \left(\sum_{j=0}^{\lceil \frac{a-1}{2} \rceil - 1} z^{2cj} \right),$$

which is a Kronecker polynomial.

Proof. With the given \mathbf{x} , we have the desirable \mathbf{s} -division with

$$\mathbf{c} = (c, c) \text{ and } \boldsymbol{\rho} = (-k, 1).$$

Observe that $r_1 = s_2 = a$ and $r_2 = s_1 = a - 1$. Thus, by Lemma 4.4,

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = \sum_{\mathbf{i} \in \langle a-1 \rangle \times \langle a \rangle} z^{c(i_1+i_2) - \lfloor \frac{i_1-i_2}{a} \rfloor}.$$

We have further that $0 \leq i_1 \leq a - 2$ and $0 \leq i_2 \leq a - 1$, and thus

$$(4.3) \quad \left\lfloor \frac{i_1 - i_2}{a} \right\rfloor = \begin{cases} 0 & \text{if } i_1 \geq i_2 \\ -1 & \text{if } i_1 < i_2 \end{cases}.$$

Define $A := \{(i_1, i_2) \in \langle a - 1 \rangle \times \langle a \rangle : i_1 \geq i_2\}$ and $B := \{(i_1, i_2) \in \langle a - 1 \rangle \times \langle a \rangle : i_1 < i_2\}$. We define a bijection $\phi : \langle a - 1 \rangle \times \langle a \rangle \rightarrow \langle a - 1 \rangle \times \langle a \rangle$ by sending $(i_1, i_2) \in A$ to the element $(i_2, i_1 + 1) \in B$ and sending the element $(i_1, i_2) \in B$ to the element $(i_2 - 1, i_1) \in A$.

Now, using (3.4) and (4.3) we see that for $(i_1, i_2) \in A$, we have

$$u(\phi(\alpha(i_1, i_2))) = c(i_2 + i_1 + 1) - (-1) = c(i_1 + i_2) + c + 1 = u(\alpha(i_1, i_2)) + c + 1.$$

Thus,

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = (1 - z^{d+1}) \sum_{(i_1, i_2) \in A} z^{c(i_1+i_2)}.$$

To complete the proof, it is enough to show that

$$\sum_{0 \leq i_2 \leq i_1 \leq a-2} z^{c(i_1+i_2)} = \left(\sum_{j=0}^{\lfloor \frac{a-1}{2} \rfloor} z^{cj} \right) \left(\sum_{j=0}^{\lceil \frac{a-1}{2} \rceil - 1} z^{2cj} \right),$$

which is a straightforward exercise using induction on a . \square

Theorem 4.8. Let $a \geq 2$ and $c \geq 1$. For $\mathbf{r} = (a, a^2 - 1)$ and $\mathbf{x} = ((a^2 - 1)c - a, a(ac - 1) + 1)$, we have

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = \left(\sum_{j_1 \in \langle a \rangle} z^{(ac-1)j_1} \right) \left(\sum_{j_2 \in \langle a+1 \rangle} z^{cj_2} \right) \left(\sum_{j_3 \in \langle a-1 \rangle} z^{(ac+c-1)j_3} \right),$$

which is a Kronecker polynomial.

Proof. With the given \mathbf{x} , we have the desirable \mathbf{s} -division with

$$\mathbf{c} = (c, ac - 1) \text{ and } \boldsymbol{\rho} = (-a, 1).$$

Observe that $r_1 = s_2 = a$ and $r_2 = s_1 = a^2 - 1$. For convenience, for any $\mathbf{i} \in \mathbb{Z}^2$, we let

$$u(\mathbf{i}) := ci_1 + (ac - 1)i_2 - \left\lfloor \frac{i_1 - i_2}{a} \right\rfloor.$$

It is straightforward to verify that for any $m = 1, 2, \dots, a - 1$, we have

$$(4.4) \quad u(ma - 1, a - 1) = u(a^2 - 1, m - 1).$$

Notice that

$$I' := \langle a^2 \rangle \times \langle a \rangle \setminus \{(ma - 1, a - 1) : m = 1, 2, \dots, a\}$$

is the set obtained from $I(\mathbf{r}) = \langle a^2 - 1 \rangle \times \langle a \rangle$ by replacing each $(ma - 1, a - 1)$ with $(a^2 - 1, m - 1)$ for $m = 1, 2, \dots, a - 1$. Hence, it follows from Lemma 4.4 and (4.4) that

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = \sum_{\mathbf{i} \in I(\mathbf{r})} z^{u(\mathbf{i})} = \sum_{\mathbf{i} \in I'} z^{u(\mathbf{i})}.$$

Next, one sees that if we let $I_0 := \langle a \rangle \times \langle a \rangle \setminus \{(a - 1, a - 1)\}$, then I' can be decomposed as

$$I' = \bigsqcup_{j_1 \in \langle a \rangle} \{(j_1 a + i_1, i_2) : (i_1, i_2) \in I_0\}.$$

Since $u(j_1 a + i_1, i_2) = (ac - 1)j_1 + u(i_1, i_2)$, we immediately have that

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = \left(\sum_{j_1 \in \langle a \rangle} z^{(ac-1)j_1} \right) \left(\sum_{\mathbf{i} \in I_0} z^{u(\mathbf{i})} \right).$$

Finally, one sees that for each $\mathbf{i} = (i_1, i_2) \in I_0$, there exists a unique $(j_2, j_3) \in \langle a + 1 \rangle \times \langle a - 1 \rangle$ such that

$$ai_2 + i_1 = (a + 1)j_3 + j_2.$$

This defines a bijection Ψ from I_0 to $\langle a + 1 \rangle \times \langle a - 1 \rangle$. Since

$$ai_2 + i_1 = (a + 1)i_2 + (i_1 - i_2) = (a + 1)(i_2 - 1) + (a + 1 + i_1 - i_2)$$

and $-(a - 1) \leq i_1 - i_2 \leq a - 1$, we conclude that if $(j_2, j_3) = \Psi(i_1, i_2)$, then

$$j_2 = i_1 - i_2 - (a + 1) \left\lfloor \frac{i_1 - i_2}{a} \right\rfloor, \quad j_3 = i_2 + \left\lfloor \frac{i_1 - i_2}{a} \right\rfloor.$$

Using the above, it is easy to verify that

$$cj_2 + (ac + c - 1)j_3 = u(\mathbf{i}) = ci_1 + (ac - 1)i_2 - \left\lfloor \frac{i_1 - i_2}{a} \right\rfloor.$$

Then our conclusion follows. □

5. A CLASSIFICATION WHEN $\mathbf{r} = (2, 2k - 1)$

Given the positive results in Section 4, it is natural to ask if it is possible to classify those (\mathbf{r}, \mathbf{x}) such that $g_{\mathbf{r}}^{\mathbf{x}}(z)$ admits a geometric factorization. In this section, we prove Theorem 5.2, providing a first step in response to this question. We will work in the context of the following setup.

5.1. Setup and Classification.

Setup 5.1. Let $\mathbf{r} = (2, 2k - 1)$ for some integer $k \geq 2$. Then $\boldsymbol{\rho} = (-k, 1)$ and $\mathbf{x} = ((2k - 1)c_1 - k, 2c_2 + 1)$ for some integers $c_1 \geq 1$ and $c_2 \geq 0$. Applying Lemma 4.4, we have that

$$(5.1) \quad \begin{aligned} g_{\mathbf{r}}^{\mathbf{x}}(z) &= \sum_{i \in \langle 2k-1 \rangle \times \langle 2 \rangle} z^{c_1 i_1 + c_2 i_2 - \lfloor \frac{i_1 - i_2}{2} \rfloor} \\ &= \begin{pmatrix} z^0 + z^{c_1} + z^{2c_1-1} + z^{3c_1-1} + \dots + \\ z^{(2k-3)c_1 - (k-2)} + z^{(2k-2)c_1 - (k-1)} + \\ z^{c_2+1} + z^{c_1+c_2} + z^{2c_1+c_2} + z^{3c_1+c_2-1} + \dots + \\ z^{(2k-3)c_1+c_2 - (k-2)} + z^{(2k-2)c_1+c_2 - (k-2)} \end{pmatrix} \end{aligned}$$

where the first two lines of (5.1) correspond to summands with $i_2 = 0$ and the last two lines of (5.1) correspond to summands with $i_2 = 1$. Suppose further in our setup that if $g_{\mathbf{r}}^{\mathbf{x}}(z)$ has a geometric factorization, it is given as follows for some $\gamma_1, \dots, \gamma_p \geq 2$ and $e_1 \leq e_2 \leq \dots \leq e_p$.

$$(5.2) \quad g_{\mathbf{r}}^{\mathbf{x}}(z) = \prod_{j=1}^p \sum_{i=0}^{\gamma_j-1} z^{ie_j} = \prod_{j=1}^p (1 + z^{e_j} + z^{2e_j} + \dots + z^{(\gamma_j-1)e_j})$$

Our main result of this section is the following.

Theorem 5.2. Suppose $\mathbf{r} = (2, 2k - 1)$ for some integer $k \geq 2$. Then $g_{\mathbf{r}}^{\mathbf{x}}(z)$ has a geometric factorization if and only if $(\mathbf{r}, \mathbf{x}) = ((2, 9), (4, 3))$ or (\mathbf{r}, \mathbf{x}) is one of the cases given by Theorems 4.6, 4.7, and 4.8. Specifically, assume Setup 5.1 holds and $g_{\mathbf{r}}^{\mathbf{x}}(z)$ admits a geometric factorization. Then $c_1 \neq c_2 + 1$ and two cases arise:

- (1) Suppose $c_2 + 1 < c_1$.
 - (a) If $c_1 = 2(c_2 + 1)$, then $\mathbf{r} = (2, 3)$ and c_2 can be any non-negative integer, which corresponds to applying Theorem 4.6 with $a = 3$ and $k = 1$ to obtain $\mathbf{x} = (6c - 2, 2c - 1)$ for $c \geq 1$.
 - (b) If $c_1 \neq 2(c_2 + 1)$, then $\mathbf{r} = (2, 3)$ and $c_2 = c_1 - 2$, which corresponds to applying Theorem 4.7 with $a = 3$ to obtain $\mathbf{x} = (3c + 1, 2c - 1)$ for $c \geq 2$.
- (2) Suppose $c_1 < c_2 + 1$.
 - (a) If $c_2 + 1 = 2c_1$, then either $\mathbf{r} = (2, 9)$ and $c_1 = 1$ (so $\mathbf{x} = (4, 3)$), or $\mathbf{r} = (2, 3)$ and c_1 can be any positive integer. Note that the latter situation corresponds to applying Theorem 4.8 with $a = 2$ to obtain $\mathbf{x} = (3c - 2, 4c - 1)$ for $c \geq 1$.
 - (b) If $c_2 + 1 \neq 2c_1$, then $c_2 = (2k - 1)c_1 - k$, which corresponds to cases given by Theorem 4.6 with $a = 2$.

Our proof will require the following two lemmas. Recall that $[z^t]f(z)$ denotes the coefficient of z^t in $f(z)$.

Lemma 5.3. Suppose $f(z)$ has a geometric factorization as given in (2.9). Assume $e_1 \leq e_2 \leq \dots \leq e_p$ and express $f(z)$ as

$$(5.3) \quad f(z) = 1 + z^{\mu_1} + z^{\mu_2} + \dots + z^{\mu_M} \quad \text{with } 0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_M.$$

Then the following are true.

- (i) $e_1 = \mu_1$. Furthermore, if $[z^{e_1}]f = m$, then $e_1 = e_2 = \dots = e_m \neq e_{m+1}$.
- (ii) If $\mu_2 \neq 2\mu_1$, then $e_2 = \mu_2$.
- (iii) If $z^{\mu_1 + \mu_2}$ does not appear in (5.3), then $\mu_2 = 2\mu_1$ and $\gamma_1 = 3$. So $(1 + z^{\mu_1} + z^{2\mu_1})$ is a factor in the geometric factorization (2.9) of $f(z)$.
- (iv) For any $i \in \{2, 3, \dots, M\}$, if μ_i cannot be written as a non-negative integer linear combination of μ_1, \dots, μ_{j-1} , then $\mu_i = e_j$ for some j . In particular, if μ_i is not a multiple of μ_1 , but $\mu_{i'}$ is a multiple of μ_1 for every $1 \leq i' < i$, then $\mu_i = e_j$ for some j .
- (v) For any subset $S \subseteq \{1, 2, \dots, p\}$, and any $t \in \mathbb{Z}_{\geq 0}$,

$$[z^t]f(z) \geq [z^t] \left(\prod_{j \in S} \sum_{i=0}^{\gamma_j-1} z^{ie_j} \right).$$

- (vi) For any exponent e_j of the factorization and any $e \geq e_j$, we have

$$[z^{e-e_j}]f + [z^{e+e_j}]f \geq [z^e]f.$$

Proof. We omit the proof for all but parts (iii) and (vi), as the others are straightforward exercises from the definition. For part (iii), if $z^{\mu_1 + \mu_2}$ does not appear in (5.3), then we must have $\mu_2 \neq e_2$. Hence, by the contrapositive of part (ii), $\mu_2 = 2\mu_1$. Since we assumed that $3\mu_1 = \mu_2 + \mu_1$ is not an exponent in (5.3), then $\gamma_1 = 3$, and we have our desired factor.

For part (vi), if e is written as a non-negative integer linear combination \mathcal{C} of e_1, \dots, e_p using less than $\gamma_j - 1$ e_j 's, then $\mathcal{C} + e_j$ contributes an exponent in (5.3). If e can only be written as a non-negative integer linear combination \mathcal{C} of e_1, \dots, e_p using all of the $\gamma_j - 1$ e_j 's, then $\mathcal{C} - e_j$ contributes an exponent in (5.3). Thus, for each non-negative integer linear combination \mathcal{C} giving e , we obtain at least one combination giving either $e - e_j$ or $e + e_j$. \square

Lemma 5.4. If Setup 5.1 holds and $g_{\mathbf{r}}^{\mathbf{x}}(z)$ admits a geometric factorization, then the following are true.

- (i) $2(2k - 1) = \prod_{j=1}^p \gamma_j$. Thus, exactly one of γ_j 's is even.
- (ii) $c_1 \neq c_2 + 1$.
- (iii) If $(c_1, c_2) = (1, 1)$, then $k = 2$ or 5 , that is, $\mathbf{r} = (2, 3)$ or $(2, 9)$.

Proof. (i) Comparing the number of monomials in equations (5.1) and (5.2), the result follows.

- (ii) Assume the contrary that $c_1 = c_2 + 1$. Then (5.1) becomes

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = \begin{array}{cccccccc} z^0 & +z^{c_1} & +z^{2c_1-1} & +z^{3c_1-1} & +\dots+ & z^{(2k-3)c_1-(k-2)} & +z^{(2k-2)c_1-(k-1)} & + \\ z^{c_1} & +z^{2c_1-1} & +z^{3c_1-1} & +z^{4c_1-2} & +\dots+ & z^{(2k-2)c_1-(k-1)} & +z^{(2k-1)c_1-(k-1)}. \end{array}$$

We consider two cases. If $c_1 = 1$, then by Lemma 5.3 part (i), we have $e_1 = e_2 = e_3 = e_4 = 1$. This implies that $[z^2]g_{\mathbf{r}}^{\mathbf{x}}(z) \geq \binom{4}{2} = 6$. However, one sees that the expression above contains at most 4 copies z^2 , a contradiction. If $c_1 > 1$, then by Lemma 5.3 part (i) again, we have $e_1 = e_2 = c_1$. It then follows from Lemma 5.3 part (v) that $[z^{2c_1}]g_{\mathbf{r}}^{\mathbf{x}}(z) \geq 1$, contradicting with the fact that z^{2c_1} does not appear in the expression above. Therefore, we must have that $c_1 \neq c_2 + 1$.

(iii) It is easy to verify the following:

- when $\mathbf{r} = (2, 3)$, $g_{\mathbf{r}}^{\mathbf{x}}(z)$ has a geometric factorization $(1+z)(1+z+z^2)$,
- when $\mathbf{r} = (2, 9)$, $g_{\mathbf{r}}^{\mathbf{x}}(z)$ has a geometric factorization $(1+z+z^2)(1+z+z^2)(1+z^2)$,
- when $\mathbf{r} = (2, 5)$ or $(2, 7)$, $g_{\mathbf{r}}^{\mathbf{x}}(z)$ does not have a geometric factorization.

Now assume $k \geq 6$. (We will find a contradiction.) Then using (5.1) we have

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = 1 + 2z + 4z^2 + 4z^3 + 4z^4 + 4z^5 + cz^6 + z^7 f(z),$$

where $f(z) \in \mathbb{Z}_{\geq 0}[z]$ and $c = 2$ or 4 . It follows from Lemma 5.3 part (i) that $e_1 = e_2 = 1 \neq e_3$.

It follows from part (i) that one of γ_1 and γ_2 is not 2. We next show that both γ_1 and γ_2 are not 2. Suppose one of them is 2. Without loss of generality (due to $e_1 = e_2$), assume $\gamma_1 = 2$. Then $\gamma_2 \geq 3$. Thus,

$$\prod_{j=3}^p \sum_{i=0}^{\gamma_j-1} z^{ie_j} = g_{\mathbf{r}}^{\mathbf{x}}(z) / ((1+z)(1+z+z^2+\dots+z^{\gamma_2-1})) = 1 + 2z^2 + z^3 h(z),$$

for some polynomial $h(z)$. Thus, by Lemma 5.3 part (i) again, we conclude that $e_3 = e_4 = 2$. However,

$$[z^3] \left(\prod_{j=1}^4 \sum_{i=0}^{\gamma_j-1} z^{ie_j} \right) \geq [z^3] ((1+z)(1+z+z^2)(1+z^2)(1+z^2)) = 5 > 4 = [z^3]g_{\mathbf{r}}^{\mathbf{x}}(z),$$

contradicting Lemma 5.3 part (v). Therefore, $\gamma_1 \geq 3$.

Now given $\gamma_1 \geq 3$ and $\gamma_2 \geq 3$, we can show $e_3 = 2$ using similar arguments as above. Then one checks that

$$[z^4] \left(\prod_{j=1}^3 \sum_{i=0}^{\gamma_j-1} z^{ie_j} \right) \geq [z^4] ((1+z+z^2)(1+z+z^2)(1+z^2)) = 4 = [z^4]g_{\mathbf{r}}^{\mathbf{x}}(z),$$

where the equality in “ \geq ” holds if and only if $(\gamma_1, \gamma_2, \gamma_3) = (3, 3, 2)$. Hence, by Lemma 5.3 part (v), we must have $(\gamma_1, \gamma_2, \gamma_3) = (3, 3, 2)$. Let $g_0(z) = \prod_{j=4}^p \sum_{i=0}^{\gamma_j-1} z^{ie_j} = g_{\mathbf{r}}^{\mathbf{x}}(z) / ((1+z)(1+z+z^2)(1+z+z^2)(1+z^2))$. Then

$$g_{\mathbf{r}}^{\mathbf{x}}(z) = (1+z+z^2)(1+z+z^2)(1+z^2)g_0(z).$$

By comparing the coefficients of z^5 on both sides, we must have that $[z^5]g_0(z) = 2$. But this implies that

$$[z^6] ((1+z+z^2)(1+z+z^2)(1+z^2)g_0(z)) \geq 5 > 4$$

contradicting with the assumption that $[z^6]g_{\mathbf{r}}^{\mathbf{x}}(z) = 2$ or 4 . □

5.2. Proof of Theorem 5.2. Note that Lemma 5.4 part (ii) provides the assertion that $c_1 \neq c_2 + 1$. In the proof of Lemma 5.4 part (iii), we showed that if $(\mathbf{r}, \mathbf{x}) = ((2, 9), (4, 3))$, $g_{\mathbf{r}}^{\mathbf{x}}(z)$ has a geometric factorization. This, together with, Theorems 4.6, 4.7, and 4.8, provides one direction for the if and only if condition in Theorem 5.2. We providing separate proofs of the other direction for parts (1), (2a), and (2b) of Theorem 5.2.

Proof of Part (1) of Theorem 5.2. Since $c_2 + 1 < c_1$, we have $c_1 \geq 2$. Express $g_{\mathbf{r}}^{\mathbf{x}}(z)$ as

$$(5.4) \quad g_{\mathbf{r}}^{\mathbf{x}}(z) = 1 + z^{\mu_1} + z^{\mu_2} + \cdots + z^{\mu_M} \quad \text{with } 0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_M.$$

Then by (5.1), $\mu_1 = c_2 + 1$ and $\mu_2 = c_1$. Hence, by Lemma 5.3 part (i), $e_1 = \mu_1 = c_2 + 1$.

(a) Suppose $c_1 = 2(c_2 + 1)$. Let $c = c_2 + 1$. Then

$$\mu_1 = c, \mu_2 = 2c, \mu_3 = 3c - 1, \mu_4 = 4c - 1, \mu_5 = 5c - 1,$$

and if $k \geq 3$,

$$\mu_6 = 6c - 1, \mu_7 = 7c - 2, \mu_8 = 8c - 2, \mu_9 = 9c - 2.$$

Hence $z^{\mu_1 + \mu_2} = z^{3c}$ does not appear in $g_{\mathbf{r}}^{\mathbf{x}}(z)$. Thus, it follows from part (iii) of Lemma 5.3 that $(1 + z^c + z^{2c})$ is a factor of given geometric factorization of $g_{\mathbf{r}}^{\mathbf{x}}(z)$. Next, one sees that by Lemma 5.3 part (iv) we must have that $e_2 = \mu_3 = 3c - 1$. Given $z^{2(3c-1)}$ does not appear in $g_{\mathbf{r}}^{\mathbf{x}}(z)$, we conclude that $\gamma_2 = 2$. Hence,

$$(1 + z^c + z^{2c})(1 + z^{3c-1}) = 1 + z^c + z^{2c} + z^{3c-1} + z^{4c-1} + z^{5c-1}$$

appears in the geometric factorization of $g_{\mathbf{r}}^{\mathbf{x}}(z)$. If $k = 2$, i.e., $\mathbf{r} = (2, 3)$, the above expression is exactly the geometric factorization of $g_{\mathbf{r}}^{\mathbf{x}}(z)$. If $k \geq 3$, one can show that $e_3 = 6c - 1$ which implies that $z^{c+6c-1} = z^{7c-1}$ appears in (5.4), a contradiction.

(b) Suppose $c_1 \neq 2(c_2 + 1)$, so $\mu_2 \neq 2\mu_1$. By Lemma 5.3 parts (ii) and (iii), $e_2 = \mu_2 = c_1$ and $z^{\mu_1 + \mu_2} = z^{c_1 + c_2 + 1}$ must appear in (5.4). However, by looking at Expression (5.1), we see that the only term that could be $z^{c_1 + c_2 + 1}$ is $z^{2c_1 - 1}$. Hence, $c_1 + c_2 + 1 = 2c_1 - 1$, equivalently, $c_2 = c_1 - 2$. Since $2 = 2(0 + 1)$ and $c_1 \neq 2(c_2 + 1)$, we conclude that $c_1 \geq 3$. Let $c = c_1 - 1 \geq 2$. Then

$$e_1 = \mu_1 = c, e_2 = \mu_2 = c + 1, \mu_3 = 2c, \mu_4 = 2c + 1, \mu_5 = 3c + 1.$$

Since $2c + 1 < 2c + 2 < 3c + 1$, the term z^{2c+2} does not appear in Expression (5.4) of $g_{\mathbf{r}}^{\mathbf{x}}(z)$. This implies that $\gamma_2 = 2$, that is, $(1 + z^{c+1})$ is a factor in the geometric factorization (5.2) of $g_{\mathbf{r}}^{\mathbf{x}}(z)$. Then it follows from Lemma 5.4 part (i) that γ_1 must be an odd number. In particular $\gamma_1 \geq 3$. One sees that z^{3c} does not appear in Expression (5.4) of $g_{\mathbf{r}}^{\mathbf{x}}(z)$. Hence, $\gamma_1 = 3$. Therefore,

$$(1 + z^c + z^{2c})(1 + z^{c+1}) = 1 + z^c + z^{c+1} + z^{2c} + z^{2c+1} + z^{3c+1}$$

appears in the geometric factorization of $g_{\mathbf{r}}^{\mathbf{x}}(z)$. Then similarly to part (a), we can show that \mathbf{r} has to be $(2, 3)$. \square

Proof of Part (2a) of Theorem 5.2. Express $g_{\mathbf{r}}^{\mathbf{x}}(z)$ as (5.4). Since $c_1 < c_2 + 1$, one sees that $\mu_1 = c_1$. Hence, by Lemma 5.3 part (i), $e_1 = \mu_1 = c_1$.

Suppose $c_2 + 1 = 2c_1$. If $c_1 = 1$, then $(c_1, c_2) = (1, 1)$, and Lemma 5.4 part (iii) applies. Hence, we only need to show that if $c_1 \geq 2$, then $\mathbf{r} = (2, 3)$, or equivalently $k = 2$. We prove by contradiction. Suppose $c_1 \geq 2$ and $k \geq 3$. Let $c = c_1 \geq 2$. Then

$$\mu_1 = c, \mu_2 = 2c - 1, \mu_3 = 2c, \mu_4 = \mu_5 = 3c - 1, \mu_6 = 4c - 2, \mu_7 = 4c - 1, \mu_8 = 5c - 2.$$

It follows from Lemma 5.3 part (ii), we have $e_2 = \mu_2 = 2c - 1$. Since $\mu_2 < 2c < \mu_4$, the term z^{2c} appears exactly once in $g_{\mathbf{r}}^{\mathbf{x}}(z)$. Hence, if $e_3 = 2c$, we must have that $\gamma_1 = 2$ because $(1 + z^c + z^{2c})(1 + z^{2c})$ has two copies of z^{2c} . However, in this case

$$(1 + z^c) (1 + z^{2c} + \dots + z^{2c(\gamma_3-1)}) = \sum_{i=0}^{2\gamma_3-1} z^{ic},$$

which is a geometric series with exponent c and of length $2\gamma_3$. Therefore, we may assume $\gamma_1 \geq 3$, and $e_3 \neq 2c$. Now notice that

$$\prod_{i=1}^{\gamma_1-1} z^{ic} \prod_{i=1}^{\gamma_2-1} z^{i(2c-1)} = 1 + z^c + z^{2c-1} + z^{2c} + z^{3c-1} + z^{4c-1} + z^{3c}h(z),$$

for some polynomial $h(z)$, and we have previously seen that $[z^{3c-1}]g_{\mathbf{r}}^{\mathbf{x}}(z) = 2$. Thus, we must have that $e_3 = 3c - 1$. However, this implies that z^{4c-1} appears in Expression (5.4) of $g_{\mathbf{r}}^{\mathbf{x}}(z)$ at least twice as $z^c \cdot z^{3c-1}$ and $z^{2c} \cdot z^{2c-1}$, contradicting with the observation that z^{4c-1} only appears once. \square

The proof of (2b) of Theorem 5.2 is more completed than the other parts, requiring the following lemma.

Lemma 5.5. Assume that Setup 5.1 holds and $g_{\mathbf{r}}^{\mathbf{x}}(z)$ admits a geometric factorization. Suppose further that $c_1 < c_2 + 1$ and $2c_1 \neq c_2 + 1$. Then $e_1 = c_1$. Furthermore, $(1 + z^{c_1})$ is a factor in the geometric factorization (5.2) of $g_{\mathbf{r}}^{\mathbf{x}}(z)$. Thus, we may assume $\gamma_1 = 2$.

Proof. Since $c_1 < c_2 + 1$ and $2c_1 \neq c_2 + 1$, it is clear that $e_1 = c_1$ and $c_2 \geq 2$. Assume the contrary that $(1 + z^{c_1})$ is not a factor in the geometric factorization (5.2) of $g_{\mathbf{r}}^{\mathbf{x}}(z)$. We consider two cases.

Suppose $c_1 = 1$. Then by Lemma 5.3 part (i), we have $e_1 = e_2 = c_1 = 1$. Since $(1 + z)$ is not a factor in the geometric factorization, we have $\gamma_1 \geq 3$ and $\gamma_2 \geq 3$. It follow from Lemma 5.3 part (v) that

$$[z^2]g_{\mathbf{r}}^{\mathbf{x}}(z) \geq [z^2] \left(\prod_{j=1}^2 \sum_{i=0}^{\gamma_j-1} z^i \right) \geq [z^2]((1 + z + z^2)(1 + z + z^2)) = 3.$$

However, since $c_2 + 1 \geq 3$, one sees that there are at most 2 copies of z^2 in Expression (5.1) of $g_{\mathbf{r}}^{\mathbf{x}}(z)$, which is a contradiction.

Suppose $c_1 \geq 2$. By assumption, we have $\gamma_1 \geq 3$. It then follows that z^{2c_1} appears at least once in $g_{\mathbf{r}}^{\mathbf{x}}(z)$. However, the only term in the Expression (5.1) that could be z^{2c_1} is $z^{c_1+c_2}$. Thus, $2c_1 = c_1 + c_2$, or equivalently, $c_2 = c_1$. Then one sees that $c_1 + 1 = c_2 + 1$ is the second lowest positive order in (5.1). Thus, by Lemma 5.3 part (ii), we have $e_2 = c_1 + 1$. It follows

that $z^{e_1+e_2} = z^{2c_1+1}$ has to appear in $g_r^x(z)$. However, the only term that could be z^{2c_1+1} is z^{3c_1-1} , which implies that $c_1 = 2$. (So $e_1 = c_2 = c_1 = 2$.) Then (5.1) becomes

$$g_r^x(z) = \begin{array}{cccccccc} z^0 & +z^2 & +z^3 & +z^5 & +\dots & +z^{3k-4} & +z^{3k-3} & + \\ z^3 & +z^4 & +z^6 & +z^7 & +\dots & +z^{3k-2} & +z^{3k} & \end{array}$$

Applying Part (i) of Lemma 5.3 to $g_r^x(z)/(\sum_{i=0}^{\gamma_1-1} z^{2i})$, we obtain that $e_2 = e_3 = 3$. It then follows from Lemma 5.3 part (v) that

$$[z^5]g_r^x(z) \geq [z^5] \left(\prod_{j=1}^3 \sum_{i=0}^{\gamma_j-1} z^i \right) \geq [z^5]((1+z^2)(1+z^3)(1+z^3)) = 2,$$

contradicting the fact that there is at most one copy of z^5 in $g_r^x(z)$. \square

Proof of Part (2b) of Theorem 5.2. By Lemma 5.5, we may assume $e_1 = c_1$ and $\gamma_1 = 2$. Let $g(z) = g_r^x(z)/(1+z^{c_1})$. Then $g(z)$ has a geometric factorization

$$(5.5) \quad g(z) = \prod_{j=2}^p \sum_{i=0}^{\gamma_j-1} z^{ie_j} = \prod_{j=1}^p (1 + z^{e_j} + z^{2e_j} + \dots + z^{(\gamma_j-1)e_j}).$$

Thus, $g(z) \in \mathbb{Z}_{\geq 0}[z]$. Dividing (5.1) by $(1+z^{c_1})$ gives

$$(5.6) \quad g(z) = 1 + z^{2c_1-1} + z^{2(2c_1-1)} + \dots + z^{(k-2)(2c_1-1)}$$

$$(5.7) \quad + z^{c_1+c_2} (1 + z^{2c_1-1} + z^{2(2c_1-1)} + \dots + z^{(k-2)(2c_1-1)})$$

$$(5.8) \quad + \frac{z^{(k-1)(2c_1-1)} + z^{c_2+1}}{z^{c_1} + 1}.$$

Since $z^{c_1} + 1$ is a factor of $z^a + z^b$ if and only if $a - b$ is an odd multiple of c_1 , we have that

$$c_2 + 1 = (k-1)(2c_1-1) + (2m+1)c_1, \quad \text{for some integer } m.$$

If $m = 0$, then we recover the situations given by Theorem 4.6 with $a = 2$. Therefore, it is left to show that it is impossible to have $m \neq 0$, which we prove by contradiction.

Suppose $m > 0$. Then the part (5.8) of $g(z)$ becomes

$$z^{(k-1)(2c_1-1)} (1 - z^{c_1} + z^{2c_1} - \dots - z^{(2m-1)c_1} + z^{2mc_1}).$$

As $m > 0$, we see that the summand $-z^{(k-1)(2c_1-1)+c_1}$ with a negative coefficient appears in the above expression. Since $g(z)$ has non-negative coefficients, at least one summand in either (5.6) or (5.7) should have power $(k-1)(2c_1-1) + c_1$. However, every exponent in (5.6) is less than $(k-1)(2c_1-1)$ and every exponent appearing in (5.7) is no less than $c_1 + c_2$. However, we have

$$(k-1)(2c_1-1) < (k-1)(2c_1-1) + c_1 \leq (c_2+1) - 3c_1 + c_1 < c_1 + c_2,$$

a contradiction.

Suppose $m < 0$. For convenience, let $m' = -(m+1) \geq 0$. Then $2m+1 = 2(m+1) - 1 = -(2m'+1)$, and thus the part (5.8) of $g(z)$ becomes

$$(5.9) \quad z^{c_2+1} (1 - z^{c_1} + z^{2c_1} - \dots - z^{(2m'-1)c_1} + z^{2m'c_1}).$$

We consider two cases.

Suppose $c_2 + 1$ is not a multiple of $2c_1 - 1$. Note that this implies that $c_1 > 1$. One can show, using Lemma 5.3 part (i), that e_2 , the smallest exponent in the geometric factorization (5.5) of $g(z)$, is $\min(2c_1 - 1, c_2 + 1)$. Then it follows from Lemma 5.3 part (iv) that $\max(2c_1 - 1, c_2 + 1) = e_j$ for some $j \geq 3$. Thus, $z^{(2c_1 - 1) + (c_2 + 1)} = z^{2c_1 + c_2}$ has to be a term appearing in $g(z)$. However, $z^{2c_1 + c_2}$ is neither a term in (5.6) since $c_2 + 1$ is not a multiple of $2c_1 - 1$, nor a term in (5.9) as $c_2 + 1 < 2c_1 + c_2 < 2c_1 + c_2 + 1$. Hence, it must appear in (5.7). Thus, $2c_1 + c_2 = c_1 + c_2 + n(2c_1 - 1)$ for some non-negative integer n . Then $c_1 = n(2c_1 - 1)$. Since $c_1 > 1$, we deduce that $n = 0$ and then $c_1 = 0$, which is a contradiction.

Suppose $c_2 + 1$ is a multiple of $2c_1 - 1$. We first show that c_1 has to be 1. If $m' > 0$, then the summand $-z^{c_1 + c_2 + 1}$ with a negative coefficient appearing in (5.9). Similarly to our prior argument, at least one summand in either (5.6) or (5.7) should have power $c_1 + c_2 + 1$. The only possible term in (5.7) that could have the desired power is $z^{c_1 + c_2 + (2c_1 - 1)}$, which would imply $c_1 = 1$. If a term in (5.6) has the desired power, then we get that $c_1 + c_2 + 1$ is a multiple of $2c_1 - 1$ as well, which implies that $c_1 = (c_1 + c_2 + 1) - (c_2 + 1)$ is a multiple of $2c_1 - 1$. It then follows that $c_1 = 1$. Now we assume $m' = 0$. Then $2m + 1 = -(2m' + 1) = -1$, and we have $c_2 + 1 = (k - 1)(2c_1 - 1) - c_1$. Thus, $c_1 + c_2 + 1 = (k - 1)(2c_1 - 1)$ is a multiple of $2c_1 - 1$ again. Then similar to above, we have $c_1 = 1$. Therefore, in all cases, we have shown that $c_1 = 1$. Plugging $c_1 = 1$ into the expressions we have for $g(z)$, we can show (in two cases $m' > 0$ and $m' = 0$) that

$$[z^{c_2}]g = 1, \quad [z^{c_2 + 1}]g = 3, \quad [z^{c_2 + 2}]g = 1.$$

Noting that $1 + 1 < 3$ and observing that 1 has to be an exponent in any factorization of $g(z)$, we find a contradiction to Lemma 5.3 part (vi). This completes our proof. \square

6. CONJECTURES AND QUESTIONS

In this concluding section, we present a variety of conjectures and questions based on experimental evidence.

6.1. Classifying Kronecker h^* -Polynomials When $\mathbf{r} = (a, ka - 1)$. In an exhaustive search of all \mathbf{q} supported on $\mathbf{r} = (r_1, r_2)$ with R-multiplicity $\mathbf{x} = (x_1, x_2)$ where $1 \leq r_i \leq 40$ and $1 \leq x_i \leq 100$, the only $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ corresponding to Kronecker $h^*(\Delta_{(1, \mathbf{q})}; z)$ that are not covered by our results in Section 4 are given in Table 1. Based on these experiments, we offer the following conjecture and question.

Conjecture 6.1. For the family of \mathbf{q} -vectors supported on two integers:

- (1) Section 4 describes all of the \mathbf{q} -vectors supported on \mathbf{r} of the form $(a, ka - 1)$ or $(a - 1, a)$ such that $h^*(\Delta_{(1, \mathbf{q})}; z)$ factors as a product of geometric series in powers of z , with the exception of the twelve (\mathbf{r}, \mathbf{x}) -pairs of this form listed in Table 1.
- (2) For each vector $\mathbf{r} = (r_1, r_2)$ that is not of the form $(a, ka - 1)$, there are only finitely many \mathbf{x} such that $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ has a Kronecker $h^*(\Delta_{(1, \mathbf{q})}; z)$.

Question 6.2. Is it true that when $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ is supported on two integers, $g_{\mathbf{r}}^{\mathbf{x}}(z)$ is a geometric series in powers of z if and only if $\mathbf{r} = (1, a)$ or $(a, 1)$?

\mathbf{r}	\mathbf{x}	\mathbf{r}	\mathbf{x}
(3, 7)	(9, 14)	(2, 5)	(7, 5)
(3, 10)	(3, 5)	(2, 7)	(10, 7)
(5, 7)	(25, 7)	(2, 9)	(4, 3)
(5, 8)	(35, 13)	(3, 4)	(9, 11)
(5, 13)	(5, 13)	(3, 5)	(13, 10)
(5, 17)	(10, 17)	(3, 8)	(5, 4)
(5, 18)	(25, 18)	(3, 8)	(21, 13)
(7, 9)	(14, 3)	(3, 14)	(9, 7)
(7, 11)	(14, 33)	(4, 5)	(6, 7)
(7, 33)	(14, 11)	(4, 5)	(11, 15)
(10, 17)	(5, 17)	(5, 6)	(7, 9)
(11, 14)	(33, 7)	(5, 9)	(7, 6)
(11, 26)	(33, 52)		
(13, 18)	(65, 18)		
(13, 34)	(13, 34)		
(17, 29)	(17, 58)		
(26, 33)	(52, 11)		

TABLE 1. Pairs \mathbf{r} and \mathbf{x} where $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ has Kronecker h^* -polynomial, but \mathbf{q} is not covered by a theorem in Section 4. These are aggregated by whether or not \mathbf{r} is of one of the forms $(a, ka - 1)$ or $(a - 1, a)$.

6.2. Do Geometric Factorizations Classify Most Kronecker h^* -Polynomials? The \mathbf{q} -vector given by $(\mathbf{r}, \mathbf{x}) = ((5, 7), (25, 7))$ has a Kronecker h^* -polynomial that does not factor into geometric series in powers of z , but it is the only known \mathbf{q} -vector with this property. Given Theorem 5.2 and this experimental evidence, we make the following conjecture.

Conjecture 6.3. For all but finitely many \mathbf{q} -vectors supported on two integers, the polynomial $h^*(\Delta_{(1,\mathbf{q})}; z)$ is Kronecker if and only if it factors as a product of geometric series in powers of z .

It seems feasible that the proof technique for Theorem 5.2 might be extended to handle this general setting. However, it has proven a challenge to find a universal way to handle all \mathbf{r} -vectors, either simultaneously or partitioned as a reasonable collection of sub-families.

6.3. A Fibonacci Phenomenon. The appearance of $((5, 13), (5, 13))$ and $((13, 34), (13, 34))$ in Table 1 suggests a more general phenomenon involving Fibonacci numbers. Let $a_0 = 1$, $a_1 = 2$, and define $a_n = 3a_{n-1} - a_{n-2}$. Thus, the values a_n correspond to “every other” Fibonacci number. The following conjecture has been verified for $n \leq 7$.

Conjecture 6.4. Let \mathbf{q} be defined by $\mathbf{r} = \mathbf{x} = (a_{n+1}, a_n)$. Then

$$g_{(a_{n+1}, a_n)}^{(a_{n+1}, a_n)}(z) = \left(\sum_{i=0}^{a_n-1} z^i \right) \left(\sum_{i=0}^{a_{n+1}-1} z^i \right).$$

There are several unique aspects of Conjecture 6.4 that distinguish it from the theorems where $\mathbf{r} = (a, ka - 1)$. First, in the factorizations found in the $\mathbf{r} = (a, ka - 1)$ setting, the \mathbf{r} -vector was fixed and the \mathbf{x} -vector was varying. For this conjecture, both \mathbf{r} and \mathbf{x} are varying simultaneously. Second, the arithmetical structure of the \mathbf{r} - and \mathbf{x} -vectors in the $(a, ka - 1)$ setting are considerably simpler than in this context. For example, consider the following lemma.

Lemma 6.5. The following properties hold for the sequence (a_n) .

- (1) For $n \geq 2$, $1 + a_{n-1}^2 = a_n a_{n-2}$.
- (2) For $n \geq 0$, $1 + a_n^2 + a_{n+1}^2 = 3a_n a_{n+1}$, and thus $\mathbf{x} = (a_n, a_{n+1})$ is an R-multiplicity for $\mathbf{r} = (a_n, a_{n+1})$ with $\ell = 3$ and the corresponding $\Delta_{(1, \mathbf{q})}$ is reflexive.
- (3) $\gcd(a_n, a_{n+1}) = 1$.
- (4) For $\mathbf{r} = \mathbf{x} = (a_{n+1}, a_n)$ and $\mathbf{i} = (i_1, i_2) \in \langle a_n \rangle \times \langle a_{n+1} \rangle$, we have

$$\begin{aligned} u(\alpha(\mathbf{i})) &= 3i_1 + a_{n-1}w_1(\mathbf{i}) - a_n w_2(\mathbf{i}) \\ &= 3i_1 + a_{n-1}(a_n(i_1 - i_2) \bmod a_{n+1}) - a_n(a_{n-1}(i_1 - i_2) \bmod a_n). \end{aligned}$$

Proof. The first three claims follow from straightforward arguments using induction and application of the defining identity for a_n . For the fourth item, since $-1 = a_n a_n - a_{n-1} a_{n+1}$, we have that $\rho_1 = -a_{n-1}$ and $\rho_2 = a_n$. Thus, since $a_{n+1} = 3a_n - a_{n-1}$, we have that $c_1 = 3$, and since $a_n < a_{n+1}$ we have $c_2 = 0$. The result follows from Theorem 3.10. \square

The fact that $\ell = 3$ for all n establishes that $(1 + z + z^2)$ is a factor of the h^* -polynomial in this case, and thus one expects that $g_{(a_{n+1}, a_n)}^{(a_{n+1}, a_n)}(z)$ factors as a product of two geometric series. However, the behavior of $u(\alpha(\mathbf{i}))$ is quite subtle, in the following sense. For $\mathbf{i} = (i_1, i_2) \in \langle a_n \rangle \times \langle a_{n+1} \rangle$, define

$$v(\mathbf{i}) := a_{n-1}(a_n(i_1 - i_2) \bmod a_{n+1}) - a_n(a_{n-1}(i_1 - i_2) \bmod a_n),$$

so that $u(\alpha(\mathbf{i})) = 3i_1 + v(\mathbf{i})$. Thus, for all (i_1, i_2) , we have

$$v(i_1, i_2) = v(i_1 + 1, i_2 + 1),$$

and hence

$$u(\alpha(i_1 + 1, i_2 + 1)) = 3 + u(\alpha(i_1, i_2)).$$

This implies that the values of $u(\alpha(\mathbf{i}))$ are essentially determined by the boundary values $u(\alpha(i_1, 0))$ and $u(\alpha(0, i_2))$. Experimental data combined with an OEIS [1] search leads us to the following conjecture.

- Conjecture 6.6.**
- (1) The value of $u(\alpha(\mathbf{i}))$ is independent of n .
 - (2) For all $i_1 \geq 0$, we have

$$u(\alpha(i_1, 0)) = \left\lceil i_1 \left(\frac{1 + \sqrt{5}}{2} \right)^2 \right\rceil.$$

0	1	1	2	2	2	3	3	4	4	4	5	5
3	3	4	4	5	5	5	6	6	7	7	7	8
6	6	6	7	7	8	8	8	9	9	10	10	10
8	9	9	9	10	10	11	11	11	12	12	13	13
11	11	12	12	12	13	13	14	14	14	15	15	16

TABLE 2. Some values of $u(\alpha(i_1, i_2))$ with $i_1 \geq 0$ indexing rows and $i_2 \geq 0$ indexing columns.

(3) For all $i_2 \geq 0$, we have

$$u(\alpha(0, i_2)) = 2i_2 - \left\lfloor i_2 \left(\frac{1 + \sqrt{5}}{2} \right) \right\rfloor.$$

Some values of $u(\mathbf{i}) := u(\alpha(\mathbf{i}))$ are given in Table 2. It seems that obtaining a more precise understanding of Conjecture 6.6 and the values of $u(\mathbf{i})$ is needed to resolve Conjecture 6.4.

6.4. On Ehrhart Positivity. We conjecture that independent of the reflexivity condition, all $\Delta_{(1, \mathbf{q})}$ with \mathbf{q} supported by two integers are Ehrhart positive.

Conjecture 6.7. All $\Delta_{(1, \mathbf{q})}$ with \mathbf{q} supported on two integers are Ehrhart positive.

Conjecture 6.7 has been verified for all $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ with $1 \leq r_i \leq 15$ and $1 \leq x_i \leq 24$. Note that this general Ehrhart positivity is not a result of only Theorem 1.1 and Kronecker polynomial techniques, as most $\Delta_{(1, \mathbf{q})}$ are not reflexive.

6.5. \mathbf{q} -Vectors Supported on Three Integers. A natural next step is to consider \mathbf{q} that are supported by more than two integers. Experimental computation and Proposition 3.11 suggest that a starting point for such an exploration are 3-supported \mathbf{q} 's with \mathbf{s} entries coprime. When $\gcd(a, b) = 1$ and $\mathbf{s} = (b, a, 1)$, so that $\mathbf{r} = (a, b, ab)$, Theorem 3.6 implies that this reduces to the case where $\mathbf{r} = (a, b)$. Thus, we can consider only those \mathbf{s} such that the s_i are pairwise coprime and each $s_i \geq 2$. The first such example is $\mathbf{s} = (5, 3, 2)$, for which we have the following result.

Theorem 6.8. Let $\mathbf{s} = (5, 3, 2)$, $\mathbf{r} = (6, 10, 15)$, and $\mathbf{x} = (5c_1 - 1, 3c_2 - 1, 2c_3 + 1)$ for $c_1, c_2 \geq 1$ and $c_3 \geq 0$. For $\mathbf{q} = (\mathbf{r}, \mathbf{x})$, the following three cases imply that $h^*(\Delta_{(1, \mathbf{q})}; z)$ is Kronecker.

(1) $(c_1, c_2, c_3) = (1, 3, 1)$, where

$$g_{(6, 10, 15)}^{(4, 8, 3)}(z) = (1 + z^3)(1 + z^2 + z^4)(1 + z + z^2 + z^3 + z^4)^2$$

(2) $(c_1, c_2, c_3) = (c, c, 4c - 1)$ for $c \geq 1$, where

$$g_{(6, 10, 15)}^{(5c-1, 3c-1, 2(4c-1)+1)}(z) = (1 + z^{4c-1})(1 + z^c + z^{2c})(1 + z + z^c + z^{2c} + z^{3c} + z^{4c})$$

(3) $(c_1, c_2, c_3) = (c, 3c, 7c - 1)$ for $c \geq 1$, where

$$g_{(6, 10, 15)}^{(5c-1, 3(3c)-1, 2(7c-1)+1)}(z) = (1 + z^{7c-1})(1 + z^{3c} + z^{6c})(1 + z + z^c + z^{2c} + z^{3c} + z^{4c})$$

Proof. We sketch the proof. The case (1) is straightforward to verify directly. For case (2), we use a similar technique to those used for the proofs in Section 4 where we identify a bijection of $\langle 5 \rangle \times \langle 3 \rangle \times \langle 2 \rangle$ that yields the factorization. In this case, if we fix all elements except for the following pairs which are exchanged by the bijection, then the factorization follows:

$$\begin{aligned} (2, 2, 0) &\longleftrightarrow (0, 0, 1), & (4, 1, 0) &\longleftrightarrow (1, 0, 1) \\ (3, 2, 0) &\longleftrightarrow (0, 1, 1), & (4, 2, 0) &\longleftrightarrow (2, 0, 1) \end{aligned}$$

Similarly for case (3), if we fix all elements except for the following pairs which are exchanged by the bijection, then the factorization follows:

$$\begin{aligned} (2, 2, 0) &\longleftrightarrow (1, 0, 1), & (4, 1, 0) &\longleftrightarrow (0, 0, 1) \\ (3, 2, 0) &\longleftrightarrow (2, 0, 1), & (4, 2, 0) &\longleftrightarrow (0, 1, 1) \end{aligned} \quad \square$$

Experimental evidence suggests that these are the only \mathbf{q} supported on $(6, 10, 15)$ with Kronecker h^* -polynomials. A search over (pairwise coprime) \mathbf{s} and \mathbf{x} with $2 \leq s_i \leq 11$ and $1 \leq x_i \leq 50$ has produced only two further examples of 3-supported \mathbf{q} 's with Kronecker h^* -polynomials, specifically:

$$\mathbf{s} = (11, 4, 3), \mathbf{r} = (12, 33, 44), \mathbf{x} = (21, 11, 22)$$

and

$$\mathbf{s} = (10, 7, 3), \mathbf{r} = (21, 30, 70), \mathbf{x} = (9, 10, 5)$$

For both $\mathbf{s} = (11, 4, 3)$ and $\mathbf{s} = (10, 7, 3)$, there are no other associated $\Delta_{(1, \mathbf{q})}$ with Kronecker h^* -polynomials for any \mathbf{x} with each $1 \leq x_i \leq 75$. Hence, we present the following question.

Question 6.9. Are there other general families of \mathbf{q} -vectors supported on more than two integers such that $h^*(\Delta_{(1, \mathbf{q})}; z)$ is Kronecker? In particular, are there other 3-supported \mathbf{q} 's with \mathbf{s} entries coprime that have Kronecker $h^*(\Delta_{(1, \mathbf{q})}; z)$?

6.6. Properties of Factorizations. Our main approach in this paper has been to study factorizations of $g_{\mathbf{r}}^{\mathbf{x}}(z)$ into geometric series in powers of z . However, as Remark 3.4 shows, it is possible for $h^*(\Delta_{(1, \mathbf{q})}; z)$ to have a geometric factorization for $\mathbf{q} = (\mathbf{r}, \mathbf{x})$, yet for $g_{\mathbf{r}}^{\mathbf{x}}(z)$ to not have such a factorization, leading to the following question.

Question 6.10. Are there only finitely many $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ with Kronecker $h^*(\Delta_{(1, \mathbf{q})}; z)$ where $h^*(\Delta_{(1, \mathbf{q})}; z)$ admits a geometric factorization, but $g_{\mathbf{r}}^{\mathbf{x}}(z)$ does not?

If a polynomial is Kronecker, then it factors into cyclotomic factors. It would be interesting to determine how these factors are related to \mathbf{q} in the case of h^* -polynomials, hence the following question.

Question 6.11. How, if at all, is the factorization of a Kronecker $h^*(\Delta_{(1, \mathbf{q})}; z)$ into cyclotomic factors related to arithmetic properties of \mathbf{q} ?

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