P-PARTITIONS AND p-POSITIVITY

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Abstract. Using the combinatorics of α-unimodal sets, we establish two new results in the theory of quasisymmetric functions. First, we obtain the expansion of the fundamental basis into quasisymmetric power sums. Secondly, we prove that generating functions of reverse P-partitions expand positively into quasisymmetric power sums. Consequently any nonnegative linear combination of such functions is p-positive whenever it is symmetric. As an application we derive positivity results for chromatic quasisymmetric functions, unicellular and vertical strip LLT polynomials, multivariate Tutte polynomials and the more general B-polynomials, matroid quasisymmetric functions, and certain Eulerian quasisymmetric functions, thus reproving and improving on numerous results in the literature.

1. Introduction

Whenever a new family of symmetric functions is discovered, one of the most logical first steps to take is to expand them in one of the many interesting bases of the space of symmetric functions. This paradigm can be traced from Newton's identities to modern textbooks such as [Mac79]. Of special interest are expansions in which all coefficients are nonnegative integers. Such coefficients frequently encode highly nontrivial combinatorial or algebraic information.

One of the most well-studied bases is formed by the power sum symmetric functions. Symmetric functions that expand into power sum symmetric functions with nonnegative coefficients are called *p-positive*. Recent works in which p-positivity is discussed include [SW10, SSW11, Ath15, SW16, Ell16, AP17]. The expansion of a symmetric function into power sum symmetric functions can be useful, for instance, when one is working with plethystic substitution [LR10], or evaluating certain polynomials at roots of unities [Dé83, SSW11].

Suppose X is a symmetric function for which we would like to know the expaonsion into power sum symmetric functions. In some of the papers mentioned above the following pattern recurs. First, expand X into fundamental quasisymmetric functions using R. Stanley's theory of P-partitions. Secondly, conduct some analysis specific to the function X at hand to obtain the p-expansion of X. Ideally, one would ask for a more uniform approach.

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Question 1. Is there a uniform method for deriving the expansion of a given symmetric function into power sum symmetric functions whenever the theory of P-partitions is applicable?

In practice the functions of interest often belong to the larger space of quasisymmetric functions. Clearly if a quasisymmetric function expands into power sum symmetric functions, positively or not, then it has to be symmetric. This leads to results of the following type: "Suppose X belongs to some special family of quasisymmetric functions $\mathcal{F}$. Then X is p-positive if and only if X is symmetric." In this case it is very natural to ask the following.

Question 2. Is there a more general positivity phenomenon hiding in the background, which encompasses all quasisymmetric functions that belong to $\mathcal{F}$?

In this paper we answer both Question 1 and Question 2 in the affirmative. Key to these answers are the quasisymmetric power sums Ψ_α. Quasisymmetric power sums originate in the work of I. Gelfand et al. [GKL+95] on noncommutative symmetric functions, and were recently investigated by C. Ballantine et al. [BDH+17].[1] The family Ψ_α, where α ranges over all compositions of n, forms a basis of the space of homogeneous quasisymmetric functions of degree n, and refines the power sum symmetric functions.

This paper has two main results that easily fit into the existing theory of quasisymmetric functions. The first result is a formula for the expansion of the fundamental quasisymmetric functions into quasisymmetric power sums.

Theorem 1 (Theorem 3.1)**.** *Let $n \in \mathbb{N}$ and $S \subseteq [n-1]$. Then the fundamental quasisymmetric function $\mathrm{F}_{n,S}$ expands into quasisymmetric power sums as*

$$\mathrm{F}_{n,S}(\mathbf{x}) = \sum_{\alpha} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} (-1)^{|S \setminus S_\alpha|}, \tag{1}$$

where the sum ranges over all compositions α of n such that S is α-unimodal.

Here we use the standard notation $[n] \coloneqq \{1, 2, \ldots, n\}$. The definitions of α-unimodal sets and the set S_α are found in Sections 2. The quasisymmetric functions $\mathrm{F}_{n,S}$ and Ψ_α and the factor z_α are defined in Section 3. The proof of Theorem 1 relies on a new result on α-unimodal sets, and on the hook-length formula for forests.

Theorem 1 yields a new proof of results due to Y. Roichman [Roi97, Thm. 4] and C. Athanasiadis [Ath15, Prop. 3.2], both of which feature α-unimodal sets as well.

The second main result concerns reverse P-partitions, which were introduced by R. Stanley [Sta72]. In the simplest case reverse P-partitions are order-preserving maps from a finite poset P to the positive integers. The generating function of reverse P-partitions is defined as

$$K_P(\mathbf{x}) \coloneqq \sum_{\substack{f:P\to\mathbb{N}^+ \\ x<_P y \Rightarrow f(x)\leq f(y)}} \prod_{x\in P} \mathbf{x}_{f(x)}.$$

[1] There are several different quasisymmetric power sum bases. We use one denoted by Ψ_α in [BDH+17].

The function K_P is a homogeneous quasisymmetric function of degree $n = |P|$. We prove that K_P expands positively into quasisymmetric power sums Ψ_α and provide two combinatorial interpretations for the involved coefficients.

Theorem 2 (Theorems 4.2 and 5.4)**.** *Let (P, w) be a naturally labeled poset with n elements. Then*

$$K_P(\mathbf{x}) = \sum_{\alpha \vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} |\mathcal{L}^*_\alpha(P, w)| = \sum_{\alpha \vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} |\mathcal{O}^*_\alpha(P)|\,, \tag{2}$$

were both sums range over all compositions α of n. In particular, the quasisymmetric function K_P is Ψ-positive.

The set $\mathcal{L}^*_\alpha(P, w)$ consists of certain α-unimodal linear extensions[2] of P. The definition is given in Section 2. The set $\mathcal{O}^*_\alpha(P)$ consists of certain order-preserving surjections from P onto a chain. The definition is given in Section 5. The proof of Theorem 2 uses the well-known expansion of K_P into the fundamental basis, Theorem 1, and a sign-reversing involution closely related to an involution constructed by B. Ellzey in [Ell16, Thm. 4.1].

It follows from Theorem 2 that any *symmetric* function which is a positive linear combination of functions K_P for posets P is p-positive. This affirms Question 1.

It is a manifestation of the ubiquity of reverse P-partitions in algebraic combinatorics that many interesting families of symmetric and quasisymmetric functions can be expressed as nonnegative linear combinations of functions K_P. By Theorem 2 each function X that belongs to such a family is Ψ-positive. This answers Question 2 for a large class of families $\mathcal{F}$.

As an application we give positivity results and combinatorial interpretations for the coefficients in the expansion into (quasi)symmetric power sums for the following families of quasisymmetric functions:

- The chromatic quasisymmetric functions of J. Shareshian and M. Wachs, [SW16]. We prove a generalization of a recent result by B. Ellzey [Ell16], that applies to all directed graphs, and not only those with a symmetric chromatic quasisymmetric functions. Our result also extends to a q-generalization of so called k-balanced chromatic quasisymmetric functions that were introduced by B. Humpert in [Hum11].
- Unicellular and vertical-strip LLT polynomials, which are of special interest in [CM17] and in the study of diagonal harmonics. This generalizes an observation in [AP17, HW17] and answers an open problem in [AP17]. Furthermore, this result provides more supporting evidence regarding a related e-positivity conjecture.
- The multivariate Tutte polynomials introduced by R. Stanley [Sta98], and the more general B-polynomals on directed graphs due to J. Awan and O. Bernardi, [AB16].
- The quasisymmetric functions associated to matroids due to L. Billera, N. Jia and V. Reiner, [BJR09].

[2] To be precise, the elements of $\mathcal{L}^*_\alpha(P, w)$ lie in the Jordan–Hölder set of (P, w), that is, they are perhaps more accurately described as inverses of linear extensions of (P, w).

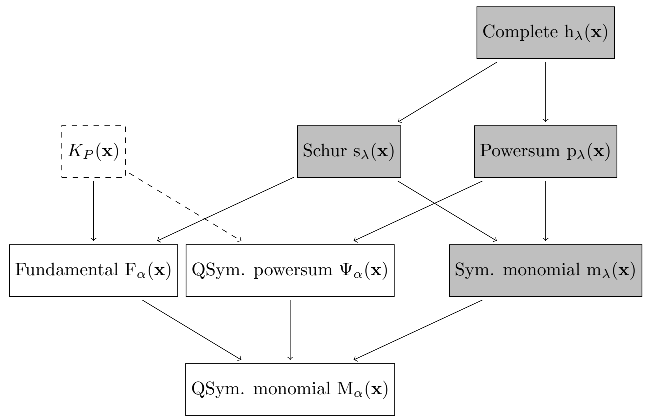

FIGURE 1. An overview of the families of functions we discuss. The shaded families are symmetric functions (and bases for the corresponding space). The remaining families are bases for the space of quasisymmetric functions, except for K_P which is too large to be a basis. The arrows represent the relation "expands positively in" (which of course is a transitive relation). The dashed line is the result in Theorem 2.

- Certain Eulerian quasisymmetric functions introduced by J. Shareshian and M. Wachs in [SW10].

Figure 1 gives an overview of some of the bases of symmetric and quasisymmetric functions that are mentioned in this paper.

1.1. **Outline.** In Section 2 we engage in the combinatorics of α-unimodal permutations and sets. We prove two new results which are instrumental in the proofs of our main theorems. In particular we define the set $\mathcal{L}^*_\alpha(P, w)$ attached to a labeled poset as the set of certain α-unimodal linear extensions of P. In Section 3 we give a short introduction to quasisymmetric functions and define quasisymmetric power sums. We proceed to prove Theorem 1 and conclude [Ath15, Prop. 3.2] as a corollary. In Section 4 we define reverse P-partitions and prove the first half of Theorem 2. Section 5 is dedicated to order-preserving surjections onto chains. It contains the definition of the set $\mathcal{O}^*_\alpha(P)$ and the proof of the second half of Theorem 2. In Section 6 we generalize Theorem 2 to include weighted posets or, equivalently, reverse P-partitions with forced equalities. This is perhaps the most technical section and it is not required to understand the rest of the paper. In Section 7 we use the developed tools to derive Ψ-expansions of some of the most commonly used bases of the space of symmetric functions (h_λ, p_λ and s_λ), including Roichman's formula [Roi97, Thm. 4]. Moreover we obtain the positivity results mentioned above. Finally, in Section 8 we mention several interesting direction that could be pursued in the future, as well as some ideas that, sadly, do not work.

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2. α-UNIMODAL COMBINATORICS

In this section we investigate α-unimodal permutations, sets and compositions. Our main objective is to prove two bijective results, namely Theorem 2.5 and Theorem 2.9, which we apply to the theory of quasisymmetric functions in the subsequent sections. However, we contend that α-unimodal combinatorics is an interesting topic in its own right.

The so called *α-unimodal* sets, where α is a composition, first appear in a recursive formula for Kazhdan–Lusztig characters of the Hecke algebra of type A_{n-1} due to Y. Roichman [Roi97, Thm. 4]. A bijective treatment of this formula was later given by A. Ram [Ram98]. The term α-unimodal was coined in [AR15]. We refer to [ER13, Thi01] for more results on unimodal permutations. More recently α-unimodal sets were used in the works of C. Athanasiadis [Ath15] and B. Ellzey [Ell16] to derive the power sum expansions of certain families of symmetric functions.

A word $\sigma_1 \cdots \sigma_n$ is *unimodal* if there exists an index $k \in [n]$ such that

$$\sigma_1 > \cdots > \sigma_k < \cdots < \sigma_n. \tag{3}$$

For instance, note that a permutation $\sigma \in \mathfrak{S}_n$ is unimodal if and only if the set $\{\sigma^{-1}(1), \ldots, \sigma^{-1}(i)\}$ is a subinterval of $[n]$ for all $i \in [n]$. We remark that this definition, which is borrowed from [AR15, Ath15, Ell16], is not standard in the study of unimodal sequences, where one would usually assume $\sigma_1 < \cdots < \sigma_k > \cdots > \sigma_n$ instead. However, (3) is more natural if one wants to use descents rather than ascents, which is common practice when working with tableaux and quasisymmetric functions. In Sections 7 and 8 we discuss unimodality of polynomials. In that case the standard definition is used, that is, a polynomial $a_0 + a_1 q + \cdots + a_d q^d$ is *unimodal* if there exists $k \in \{0, \ldots, d\}$ with $a_0 \leq \cdots \leq a_k \geq \cdots \geq a_d$.

Given a composition α of n with ℓ parts define

$$S_\alpha \coloneqq \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell-1}\}.$$

The correspondence $\alpha \mapsto S_\alpha$ is a well-known bijection between compositions of n and subsets of $[n-1]$. Sometimes it is more convenient to work with the set $S_\alpha^0 \coloneqq S_\alpha \cup \{0\}$ instead. Let α, β be compositions of n. We say α *refines* β if $S_\beta \subseteq S_\alpha$. Refinement on compositions is denoted by $\alpha \leq \beta$. Equivalently, $\alpha \leq \beta$ if β can be obtained by adding contiguous parts of α. We also say β is *coarser* than α.

Define the *blocks* of α as the sets

$$B_1^\alpha \coloneqq \{1, 2, \ldots, \alpha_1\}, \quad B_2^\alpha \coloneqq \{\alpha_1 + 1, \ldots, \alpha_1 + \alpha_2\},$$

and so on, such that $|B_i^\alpha| = \alpha_i$ for all $i \in [\ell]$ and $\{B_1^\alpha, B_2^\alpha, \ldots, B_\ell^\alpha\}$ forms a set partition of $[n]$.

Example 2.1. Let $\alpha = 11213311$ and $\beta = 535$. Then $S_\alpha = \{1, 2, 4, 5, 8, 11, 12\}$, $S_\beta = \{5, 8\}$ and $\alpha \leq \beta$. The blocks of α are

$$\{1\}, \{2\}, \{3, 4\}, \{5\}, \{6, 7, 8\}, \{9, 10, 11\}, \{12\} \text{ and } \{13\}. \qquad \square$$

Let α be a composition of n. A permutation $\sigma \in \mathfrak{S}_n$ is *α-unimodal* if the word obtained by restricting σ to the block B_i^α is unimodal for all $i \in [\ell]$. That is, for all $i \in [\ell]$ if $B_i^\alpha = [a, b]$ then there exists $k \in [a, b]$ such that $\sigma_a > \cdots > \sigma_k < \cdots \sigma_b$.

The *descent set* of a permutation $\sigma \in \mathfrak{S}_n$ is defined as

$$\mathrm{DES}(\sigma) \coloneqq \{i \in [n-1] : \sigma(i) > \sigma(i+1)\},$$

Furthermore, set $\mathrm{des}(\sigma) \coloneqq |\mathrm{DES}(\sigma)|$.

A set $S \subseteq [n-1]$ is *α-unimodal* if it is the descent set of an α-unimodal permutation. Equivalently, $S \subseteq [n-1]$ is α-unimodal if for all $i \in [\ell]$ the intersection of S with $B_i^\alpha \setminus S_\alpha$ is a prefix of the latter.

There is yet another equivalent description of α-unimodal sets. Define the binary sequence $a_1, \dots, a_{n-1}$ by letting $a_i = 1$ if $i \in S$, and $a_i = 0$ otherwise. Similarly define $b_1, \dots, b_{n-1}$ by $b_i = 1$ if $i \in S_\alpha$, and $b_i = 0$ otherwise. Then S is α-unimodal if and only if the two-line arrangement

$$\begin{matrix} a_1 & a_2 & \cdots & a_{n-1} \\ b_1 & b_2 & \cdots & b_{n-1} \end{matrix} \quad \text{avoids the pattern} \quad \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix},$$

that is,

$$\begin{matrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{matrix} \neq \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$$

for all $i \in [n-2]$.

Let U_α denote the set of α-unimodal subsets of $[n-1]$.

Example 2.2. Let $\alpha = 33$. Then $B_1^\alpha \setminus S_\alpha = \{1,2\}$ and $B_2^\alpha \setminus S_\alpha = \{4,5,6\}$. The α-unimodal subsets of $[5]$ are the following sets:

$\emptyset$	1	3	4	12	13	14	34	45
123	124	134	145	345	1234	1245	1345	12345

In short, if 2 appears, then 1 must also appear and, similarly, if 5 appears then 4 must also be there. Equivalently, if the two-line arrangement

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 1 & 0 & 0 \end{matrix}$$

avoids the forbidden pattern, then $a_2 = 1$ implies $a_1 = 1$, and $a_5 = 1$ implies $a_4 = 1$.

□

We start out by collecting some basic combinatorial facts on α-unimodality. All of them are straightforward to prove and should be viewed as a warm up to get acquainted with α-unimodal sets.

Proposition 2.3. *Let $n \in \mathbb{N}$ and α, β be compositions of n with ℓ parts.*

(i) A subset $S \subseteq [n-1]$ is α-unimodal if and only if for all $k \in S \setminus S_\alpha$ we have $k - 1 \in S \cup S_\alpha^0$.
(ii) If $\alpha \leq \beta$, then $U_\beta \subseteq U_\alpha$.
(iii) If $\alpha \leq \beta$ then S_β is α-unimodal.
(iv) In particular, S_α is α-unimodal.
(v) The number of α-unimodal subsets of $[n-1]$ is given by

$$|U_\alpha| = 2^{\ell-1} \cdot \alpha_1 \cdot \alpha_2 \cdots \alpha_\ell. \tag{4}$$

(vi) The set U_α is closed under unions and intersections and therefore forms a sublattice of the Boolean lattice.

(vii) The lattice U_α is the direct product of chains. Its Möbius function is given by

$$\mu(S) = \begin{cases} (-1)^{|S|} & \text{if } S \subseteq S_\alpha \cup \{k+1 : k \in S_\alpha^0\}, \\ 0 & \text{otherwise.} \end{cases}$$

(viii) Let $V_\alpha \coloneqq \{\gamma \vDash n : S_\alpha \in U_\gamma\}$. Then

$$|V_\alpha| = 2^{n-1} \cdot \left(\frac{3}{4}\right)^m , \tag{5}$$

where m denotes the number of indices $i \in [\ell - 1]$ such that $\alpha_i > 1$.

(ix) The set V_α defined in (viii) *is an order ideal and thus a meet-semilattice in the lattice of compositions ordered by refinement. However, V_α is generally not a sublattice.*

(x) The q, t-generating function of α-unimodal sets is given by

$$F(q,t,z) \coloneqq \sum_{n \geq 1} \sum_{\alpha \vDash n} \sum_{S \in U_\alpha} q^{|S|} t^{\ell(\alpha)} z^n = \frac{tz}{1 - (1+q)(1+t)z + qz^2} .$$

Proof. Claims (i), (ii), (iii), (iv) and (vi) follow directly from the definition of α-unimodal sets. Claim (ix) is an immediate consequence of Claim (ii).

Given $S \subseteq [n-1]$ let $\varphi(S) \coloneqq (S \cap S_\alpha, r_1, \ldots, r_\ell)$, where $r_i \coloneqq |S \cap (B_i^\alpha \setminus S_\alpha)|$ for all $i \in [\ell]$. It is not difficult to see that this defines a bijection

$$\varphi : U_\alpha \to \{T \subseteq S_\alpha\} \times \prod_{i=1}^{\ell} \{0, \ldots, \alpha_i - 1\} ,$$

which yields (v). In fact, the map φ is an isomorphism of partially ordered sets, where by definition $(T, r_1, \ldots, r_\ell) \leq (T', r'_1, \ldots, r'_\ell)$ if and only if $T \subseteq T'$ and $r_i \leq r'_i$ for all $i \in [\ell]$. Thus Claim (vii) follows from standard techniques for computing the Möbius function of finite posets, see [Sta11, Sec. 3.8].

The remaining two claims are best understood using the definition of α-unimodality via two-line arrangements of zeroes and ones. To see Claim (viii) let $a_1, \ldots, a_{n-1}$ be a fixed binary sequence encoding the set S_α. We are looking to determine the number of compatible sequences $b_1, \ldots, b_{n-1}$. Each part of α except the last part corresponds to a pattern $a_i a_{i+1} = 01$ if it is greater than 1. There are therefore three choices for $b_i b_{i+1}$, namely $01, 10$ or 11. In total this contributes the factor of 3^m in (5). All other entries of b can be chosen arbitrarily, contributing a factor of 2^{n-2m-1}.

In order to see Claim (x) note that a pair of compatible sequences $a, b \in \{0,1\}^n$ is either empty, or it can be obtained from a pair of shorter compatible sequences $a', b' \in \{0,1\}^{n-1}$ by appending one of the four patterns

$$\begin{matrix} 0 \\ 0 \end{matrix} \quad \begin{matrix} 1 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 1 \end{matrix} \quad \text{or} \quad \begin{matrix} 1 \\ 1 \end{matrix} .$$

Sequences a, b that are obtained in this way and do contain the forbidden pattern are of the form

$$\begin{matrix} a'' & 0 & 1 \\ b'' & 0 & 0 \end{matrix}$$

where $a'', b'' \in \{0,1\}^{n-2}$ avoid the forbidden pattern. Thus we conclude the recursion

$$F(q,t,z) = zt + (1+q+t+qt)zF(q,t,z) - qz^2F(q,t,z)$$

and Claim (x) follows. ■

The relation defined on the subsets of $[n-1]$ by letting $S \prec S_\alpha$ if and only if S is α-unimodal is neither symmetric, antisymmetric nor transitive. It follows from Proposition 2.3 (x) that the total number $f(n)$ of such relations satisfies the recursion $f(n) = 4f(n-1) - f(n-2)$ with $f(0) = 0$ and $f(1) = 1$, and is therefore equal to the sequence A001353 in [Slo16]. This relates α-unimodal sets to, for example, spanning trees in a $2 \times n$ grid.

It is also easy to obtain q-analogues of Proposition 2.3 (v) and (viii) — this is left as an exercise.

In order to state the first main result of this section we need one more definition, which is due to C. Ballantine et al. and appears in the study of quasisymmetric analogues of the power sum symmetric functions [BDH+17].

Let $\alpha \leq \beta$ be compositions of n. Given a permutation $\sigma \in \mathfrak{S}_n$ and $i \in [\ell(\alpha)]$ define the subword $\sigma^{(i)} := \sigma_a \cdots \sigma_b$ of σ where $B_i^\alpha = [a,b]$. A permutation $\sigma \in \mathfrak{S}_n$ is called *consistent with* $\alpha \leq \beta$ if the following two conditions are satisfied:

(i) For each $i \in [\ell(\alpha)]$ the maximum of $\sigma^{(i)}$ is in last position.
(ii) For each $k \in [\ell(\beta)]$ the subwords $\sigma^{(i)}, \dots, \sigma^{(j)}$ are sorted increasingly with respect to their maximal elements, where $i, j \in [\ell(\alpha)]$ are determined by

$$\bigcup_{r=i}^{j} B_r^\alpha = B_k^\beta .$$

Let $\textsc{Cons}(\alpha,\beta)$ denote the set of permutations $\sigma \in \mathfrak{S}_n$ that are consistent with $\alpha \leq \beta$.

Example 2.4. Let $\sigma = 438756219$, $\alpha = 12123$ and $\beta = 315$. We shall see that $\sigma \in \textsc{Cons}(\alpha,\beta)$. We separate β-blocks by vertical lines and put α-blocks into parentheses:

$$(4)(38)|(7)|(56)(219)$$

In each α-block the maximum is in last position. Moreover the maxima are increasing within each β-block:

$$4 < 8 \mid 7 \mid 6 < 9$$

Thus $\sigma \in \textsc{Cons}(\alpha,\beta)$. On the other hand

$$(4)(38)|(7)|(65)(219) \notin \textsc{Cons}(\alpha,\beta)$$

because the maximum of $\sigma^{(4)} = 65$ is not in last position. Similarly

$$(4)(38)|(7)|(59)(216) \notin \textsc{Cons}(\alpha,\beta)$$

because the maxima in the third β-block are not in increasing order. □

The definitions of both α-unimodal permutations and consistent permutations are somewhat out of the blue at first glance, however, these two concepts interplay in an interesting fashion. The following theorem will allow us to expand Gessel's fundamental basis into quasisymmetric power sums in Section 3.

Theorem 2.5. *Let $n \in \mathbb{N}$ and β, γ be compositions of n. Then*

$$\sum_{\alpha} |\textsc{Cons}(\alpha,\beta)|\,(-1)^{|S_\gamma \setminus S_\alpha|} = \begin{cases} n! & \text{if } \beta \leq \gamma, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where the sum ranges over all compositions α of n such that $\alpha \leq \beta$ and S_γ is α-unimodal.

Proof. Let $R(\beta,\gamma)$ denote the set of all compositions α of n such that $\alpha \leq \beta$ and S_γ is α-unimodal.

Case 1. First assume $\beta \leq \gamma$. Then S_γ is α-unimodal for all $\alpha \leq \beta$ by Proposition 2.3 (iii). Moreover $S_\gamma \setminus S_\alpha = \emptyset$ for all $\alpha \in R(\beta,\gamma)$. Thus it suffices to give a bijection

$$\varphi : \bigsqcup_{\alpha \leq \beta} \textsc{Cons}(\alpha,\beta) \to \mathfrak{S}_n\,.$$

This is accomplished simply by reversing each subword $\sigma^{(i)}$. The same idea appears in the well-known method for switching between cycle notation and one line notation by forgetting the parentheses, see [Sta11, Sec. 1.3]. Note that φ^{-1} depends on β.

For example, consider $\beta = 3$ when α varies over all compositions of 3.

$$\begin{array}{ccc@{\qquad\qquad}ccc} (123) & \to & 321 & (1)(23) & \to & 132 \\ (213) & \to & 312 & (2)(13) & \to & 231 \\ (12)(3) & \to & 213 & (1)(2)(3) & \to & 123 \end{array}$$

Case 2. On the other hand, if $\beta \nleq \gamma$ then let i be the minimal element of $S_\gamma \setminus S_\beta$.

If $i-1 \in S^0_\beta$ then $i-1 \in S^0_\alpha$ for all $\alpha \in R(\beta,\gamma)$. Moreover the set $R(\beta,\gamma)$ is partitioned into pairs $\{\alpha,\alpha'\}$ such that $S_{\alpha'} = S_\alpha \cup \{i\}$ and $S_\alpha = S_{\alpha'} \setminus \{i\}$. We claim that

$$\textsc{Cons}(\alpha,\beta) = \textsc{Cons}(\alpha',\beta).$$

This becomes clear via the correspondence

$$\cdots)|(\sigma_i\sigma_{i+1}\cdots\sigma_j)\cdots \qquad \longleftrightarrow \qquad \cdots)|(\sigma_i)(\sigma_{i+1}\cdots\sigma_j)\cdots$$

where we use the notation of Example 2.4. Moreover,

$$(-1)^{|S_\gamma \setminus S_\alpha|} = -(-1)^{|S_\gamma \setminus S_{\alpha'}|}\,.$$

Thus the left hand side of (6) vanishes.

If $i-1 \notin S^0_\beta$ then $i-1 \notin S_\gamma$ by minimality of i. Thus for S_γ to be α-unimodal we must have $i-1 \in S_\alpha$ or $i \in S_\alpha$ by Proposition 2.3 (i). In fact, the set $R(\beta,\gamma)$ is partitioned into sets of three, $\{\alpha,\alpha',\alpha''\}$, such that

$$S_\alpha = T \cup \{i-1\}\,, \qquad S_{\alpha'} = T \cup \{i\}\,, \qquad S_{\alpha''} = T \cup \{i-1,i\}$$

for some $T \subseteq [n-1] \setminus \{i-1,i\}$. We claim that there exists a bijection

$$\varphi : \textsc{Cons}(\alpha,\beta) \to \textsc{Cons}(\alpha',\beta) \sqcup \textsc{Cons}(\alpha'',\beta)\,.$$

Define the map φ as follows. For $\sigma \in \textsc{Cons}(\alpha,\beta)$ set

$$\varphi(\sigma) := \begin{cases} \sigma \circ (i-1,i) \in \textsc{Cons}(\alpha',\beta) & \text{if } \sigma_{i-1} > \sigma_i, \\ \sigma \in \textsc{Cons}(\alpha'',\beta) & \text{if } \sigma_{i-1} < \sigma_i. \end{cases}$$

With the notation from Example 2.4 this amounts to the following:

$$\cdots(\sigma_h\cdots\sigma_{i-1})(\sigma_i\sigma_{i+1}\cdots\sigma_j)\cdots \quad\longleftrightarrow\quad \begin{cases}\cdots(\sigma_h\cdots\sigma_i\sigma_{i-1})(\sigma_{i+1}\cdots\sigma_j)\cdots\\ \cdots(\sigma_h\cdots\sigma_{i-1})(\sigma_i)(\sigma_{i+1}\cdots\sigma_j)\cdots\end{cases}$$

The claim follows from

$$(-1)^{|S_\gamma\setminus S_\alpha|} = -(-1)^{|S_\gamma\setminus S_{\alpha'}|} = -(-1)^{|S_\gamma\setminus S_{\alpha''}|}\,. \qquad \blacksquare$$

We now turn to labeled posets, for which we adopt the same conventions as in [Sta11]. A *labeled poset* (P,w) is a finite poset P equipped with a bijection $w : P \to [n]$. We call (P,w) a *naturally labeled poset* if w is order-preserving, that is, $w(x) < w(y)$ for all $x, y \in P$ with $x <_P y$.

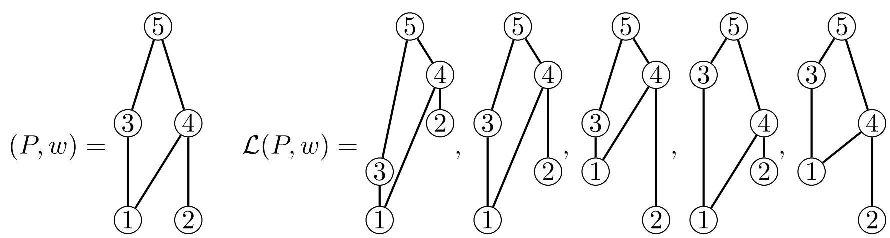
The *Jordan–Hölder set* of a labeled poset (P,w) with n elements is defined as

$$\mathcal{L}(P,w) := \big\{\sigma\in\mathfrak{S}_n : \sigma^{-1}\circ w(x) < \sigma^{-1}\circ w(y) \text{ for all } x,y\in P \text{ with } x <_P y\big\}.$$

That is, $\sigma\in\mathcal{L}(P,w)$ if and only if $\sigma^{-1}\circ w$ is a linear extension of P.

To avoid ambiguity we refer to the values $w(x)$, where $x\in P$, as *labels*. Other functions $f : P\to\mathbb{N}$ (such as $\sigma^{-1}\circ w$, where $\sigma\in\mathcal{L}(P,w)$) we sometimes call *colorings* and their values $f(x)$ *colors*. With this convention the elements $\sigma\in\mathcal{L}(P,w)$ map colors to labels.

Example 2.6. For example, let (P,w) be the labeled poset below:



13245 12345 21345 12435 21435

Thus $\mathcal{L}(P,w) = \{13245, 12345, 21345, 12435, 21435\}$. $\square$

Given a labeled poset (P,w) with n elements and a composition α of n with ℓ parts, let

$$\mathcal{L}_\alpha(P,w) := \{\sigma\in\mathcal{L}(P,w) : \sigma \text{ is } \alpha\text{-unimodal}\}.$$

Furthermore, given $\sigma\in\mathfrak{S}_n$, define the subposets

$$P_i^\alpha(\sigma) := \{w^{-1}(\sigma_j) : j\in B_i^\alpha\}\subseteq P\,. \tag{7}$$

Finally denote by $\mathcal{L}^*_\alpha(P,w)$ the subset of $\mathcal{L}_\alpha(P,w)$ that consists of the elements $\sigma\in\mathcal{L}_\alpha(P,w)$ such that $P_i^\alpha(\sigma)$ contains a unique minimal element for all $i\in[\ell]$.

Example 2.7. Let (P,w) be as in Example 2.6. If $\alpha = (2,3)$, then

$$\mathcal{L}_{23}(P,w) = \{\hat{1}3|\hat{2}45, \hat{1}\hat{2}|\hat{3}\hat{4}5, \hat{2}\hat{1}|\hat{3}\hat{4}5, \hat{1}\hat{2}|\hat{4}\hat{3}5, \hat{2}\hat{1}|\hat{4}\hat{3}5\} \quad\text{and}\quad \mathcal{L}^*_{23}(P,w) = \{13245\},$$

where we have marked the labels of the minimal elements in each subposet. Similarly, if $\alpha = (4,1)$ then

$$\mathcal{L}_{41}(P,w) = \{\hat{1}\hat{2}34|\hat{5}, \hat{2}\hat{1}34|\hat{5}\} \quad\text{and}\quad \mathcal{L}^*_{41}(P,w) = \emptyset. \qquad \square$$

The set $\mathcal{L}^*_\alpha(P,w)$ associated to a naturally labeled poset, turns out to be a highly useful concept in the study of power sum symmetric functions. The following lemma gives a necessary condition for a permutation σ to lie in $\mathcal{L}^*_\alpha(P,w)$.

Lemma 2.8. *Let (P,w) be a naturally labeled poset with n elements, let α be a composition of n, and let $\sigma \in \mathcal{L}^*_\alpha(P,w)$. Then $\mathrm{DES}(\sigma) \subseteq S_\alpha$.*

Proof. Let $i \in [\ell(\alpha)]$ and $B^\alpha_i = [a,b]$. Suppose $x, y \in P^\alpha_i(\sigma)$ with $x <_P y$. Then $w(x) < w(y)$ since w is natural, and $\sigma^{-1} \circ w(x) < \sigma^{-1} \circ w(y)$ by definition of $\mathcal{L}(P,w)$. Consequently the pair $(\sigma^{-1} \circ w(x), \sigma^{-1} \circ w(y))$ is a noninversion of σ. Since $P^\alpha_i(\sigma)$ has a unique minimal element we obtain $\sigma_a < \sigma_j$ for all $j \in [a+1,b]$. Thus by α-unimodality $\sigma_a < \cdots < \sigma_b$. ■

Example 2.7 shows that $\mathrm{DES}(\sigma) \subseteq S_\alpha$ is not a sufficient condition for $\sigma \in \mathcal{L}^*_\alpha(P,w)$.

We now come to the second main result of this section. The following theorem will be used to prove that several families of quasisymmetric or symmetric functions expand positively into power sums.

Theorem 2.9. *Let (P,w) be a naturally labeled poset with n elements, and let α be a composition of n. Then*

$$\sum_{\sigma \in \mathcal{L}_\alpha(P,w)} (-1)^{|\,\mathrm{DES}(\sigma) \setminus S_\alpha|} = |\mathcal{L}^*_\alpha(P,w)|\,. \tag{8}$$

In particular, the left-hand side of (8) is nonnegative.

Proof. We prove the theorem by the use of a sign-reversing involution

$$\varphi : \mathcal{L}_\alpha(P,w) \setminus \mathcal{L}^*_\alpha(P,w) \to \mathcal{L}_\alpha(P,w) \setminus \mathcal{L}^*_\alpha(P,w)$$

similar to an involution due to B. Ellzey [Ell16, Thm. 4.1].

Let ℓ be the number of parts of α, and let $\sigma \in \mathcal{L}_\alpha(P,w) \setminus \mathcal{L}^*_\alpha(P,w)$. Then there exists a minimal index $i \in [\ell]$ such that $P^\alpha_i(\sigma)$ has at least two minimal elements. Suppose $B^\alpha_i = [a,b]$. Since σ is α-unimodal there exists $k \in B^\alpha_i$ with

$$\sigma_a > \cdots > \sigma_k < \cdots < \sigma_b.$$

Define $M \in [n]$ as

$$M \coloneqq \max\left\{w(x) : x \in P^\alpha_i(\sigma) \text{ and } x \leq_P y \text{ for all } y \in P^\alpha_i(\sigma)\right\}.$$

That is, M is the maximal label of a minimal element of $P^\alpha_i(\sigma)$. Since σ is α-unimodal, there exist indices $j, m \in B^\alpha_i$ with $j < m$ and $j \leq k \leq m$ such that

$$[j,m] = \{r \in B^\alpha_i : \sigma_r \leq M\}.$$

In particular, $M = \sigma_j$ or $M = \sigma_m$. We distinguish between these two cases.

Case 1. If $M = \sigma_j$ then set $\varphi(\sigma) \coloneqq \sigma \circ (j, \ldots, m)$.

Case 2. If $M = \sigma_m$ then set $\varphi(\sigma) \coloneqq \sigma \circ (j, \ldots, m)^{-1}$.

It is straightforward to verify that $\varphi(\sigma)$ is α-unimodal in both cases. Moreover, $P^\alpha_r(\sigma) = P^\alpha_r(\varphi(\sigma))$ for all $r \in [\ell]$, so by the definition of $\mathcal{L}^*_\alpha(P,w)$ we have $\varphi(\sigma) \notin \mathcal{L}^*_\alpha(P,w)$. Another consequence is that $\varphi(\sigma)^{-1} \circ w(x) < \varphi(\sigma)^{-1} \circ w(y)$ for all

$x \in P_r^\alpha(\sigma)$ and $y \in P_s^\alpha(\sigma)$ with $r < s$. To show that φ is well-defined, it therefore suffices to verify that the restriction of $\varphi(\sigma)^{-1} \circ w$ to $P_i^\alpha(\sigma)$ is a linear extension.

Suppose we are in Case 1 and set $x = w^{-1}(M)$. In order to prove $\varphi(\sigma) \in \mathcal{L}(P, w)$ we need to verify
$$\varphi(\sigma)^{-1} \circ w(x) < \varphi(\sigma)^{-1} \circ w(y)$$
for all $y \in P_i^\alpha(\sigma)$ with $x <_P y$. Thus assume that $y \in P_i^\alpha(\sigma)$ satisfies $x <_P y$. Then $M = w(x) < w(y)$ implies $\sigma^{-1} \circ w(y) > j$. By the defining property of j and m we also have $\sigma^{-1} \circ w(y) > m$, and therefore
$$\varphi(\sigma)^{-1} \circ w(x) = (j, \dots, m)^{-1} \circ \sigma^{-1}(M) = m < \sigma^{-1} \circ w(y) = \varphi(\sigma)^{-1} \circ w(y),$$
as claimed.

Next assume we are in Case 2 and set $y = w^{-1}(M)$. To show that $\varphi(\sigma) \in \mathcal{L}(P, w)$ we need to verify
$$\varphi(\sigma)^{-1} \circ w(x) < \varphi(\sigma)^{-1} \circ w(y)$$
for all $x \in P_i^\alpha(\sigma)$ with $x <_P y$. But this is trivially true because y is a minimal element of $P_i^\alpha(\sigma)$.

To see that φ is an involution note that $M > \sigma_k$ since $P_i^\alpha(\sigma)$ has at least two minimal elements by assumption. Thus σ belongs to Case 1 if and only if $\varphi(\sigma)$ belongs to Case 2.

It is also clear that φ is sign-reversing. Indeed
$$|\operatorname{DES}(\varphi(\sigma)) \setminus S_\alpha| = |\operatorname{DES}(\sigma) \setminus S_\alpha| - 1$$
whenever σ belongs to Case 1, and equivalently,
$$|\operatorname{DES}(\varphi(\sigma)) \setminus S_\alpha| = |\operatorname{DES}(\sigma) \setminus S_\alpha| + 1$$
if σ belongs to Case 2. Hence
$$\sum_{\sigma \in \mathcal{L}_\alpha(P,w)} (-1)^{|\operatorname{DES}(\sigma) \setminus S_\alpha|} = \sum_{\sigma \in \mathcal{L}^*_\alpha(P,w)} (-1)^{|\operatorname{DES}(\sigma) \setminus S_\alpha|}.$$
The claim in (17) now follows from Lemma 2.8, which guarantees that $\operatorname{DES}(\sigma) \subseteq S_\alpha$ for all $\sigma \in \mathcal{L}^*_\alpha(P, w)$. ∎

3. Quasisymmetric functions

The main result of this section, Theorem 3.1, is the expansion of Gessel's fundamental basis into quasisymmetric power sums. As an immediate consequence we obtain a new proof of a recent result of C. Athanasiadis [Ath15, Prop. 3.2] (Corollary 3.4 below). We start out with a brief introduction to quasisymmetric functions. For more background the reader is referred to [Sta01, LMvW13].

A *quasisymmetric function* f is a formal power series $f \in \mathbb{Q}[[x_1, x_2, \dots]]$ such that the degree of f is finite, and for every composition $(\alpha_1, \dots, \alpha_\ell)$ the coefficient of $\mathbf{x}_{i_2}^{\alpha_2} \cdots \mathbf{x}_{i_\ell}^{\alpha_\ell}$ in f is the same for all integer sequences $1 \leq i_1 < i_2 < \cdots < i_\ell$.

Given a composition α with ℓ parts, the *monomial quasisymmetric function* M_α is defined as
$$\mathrm{M}_\alpha(\mathbf{x}) \coloneqq \sum_{i_1 < i_2 < \cdots < i_\ell} \mathbf{x}_{i_1}^{\alpha_1} \mathbf{x}_{i_2}^{\alpha_2} \cdots \mathbf{x}_{i_\ell}^{\alpha_\ell}.$$

The functions M_α, where α ranges over all compositions of n, constitute a basis for the space of homogeneous quasisymmetric functions of degree n.

Another basis for the space of quasisymmetric functions are the fundamental quasisymmetric functions. The *fundamental quasisymmetric functions* of degree n are indexed by subsets $S \subseteq [n-1]$ and defined as

$$\mathrm{F}_{n,S}(\mathbf{x}) \coloneqq \sum_{\substack{j_1 \le j_2 \le \cdots \le j_n \\ i \in S \Rightarrow j_i < j_{i+1}}} \mathbf{x}_{j_1} \cdots \mathbf{x}_{j_n} \, .$$

Alternatively, given a composition α of n, we sometimes write $\mathrm{F}_\alpha \coloneqq \mathrm{F}_{n,S_\alpha}$. The expansion of fundamental quasisymmetric functions into monomial quasisymmetric functions is given by

$$\mathrm{F}_\alpha(\mathbf{x}) = \sum_{\beta \le \alpha} \mathrm{M}_\beta(\mathbf{x}) \, . \tag{9}$$

Given a composition α set $z_\alpha \coloneqq \prod_{i \ge 1} i^{m_i} m_i!$, where m_i denotes the number of parts of α that are equal to i. The following definitions appear in [BDH+17]. Given compositions $\alpha \le \beta$, let

$$\pi(\alpha) \coloneqq \prod_{i=1}^{\ell(\alpha)} (\alpha_1 + \alpha_2 + \cdots + \alpha_i) \qquad \text{and} \qquad \pi(\alpha, \beta) \coloneqq \prod_{i=1}^{\ell(\beta)} \pi\left(\alpha^{(i)}\right) , \tag{10}$$

where $\alpha^{(i)}$ is the composition of β_i that consists of the parts α_j with $B_j^\alpha \subseteq B_i^\beta$. The *quasisymmetric power sum* Ψ_α is defined as

$$\Psi_\alpha(\mathbf{x}) \coloneqq z_\alpha \sum_{\beta \ge \alpha} \frac{1}{\pi(\alpha, \beta)} \mathrm{M}_\beta(\mathbf{x}). \tag{11}$$

For example,

$$\Psi_{231} = \frac{1}{10}\mathrm{M}_6 + \frac{1}{4}\mathrm{M}_{24} + \frac{3}{5}\mathrm{M}_{51} + \mathrm{M}_{231}.$$

The quasisymmetric power sums refine the power sum symmetric functions as

$$\mathrm{p}_\lambda(\mathbf{x}) = \sum_{\alpha \sim \lambda} \Psi_\alpha(\mathbf{x}) \, , \tag{12}$$

where the sum ranges over all compositions α whose parts rearrange to λ. This is shown in [BDH+17, Thm. 3.11] and we also give an alternative proof in Section 7.2.

Let ω be the automorphism on quasisymmetric functions defined by $\omega(\mathrm{F}_S) = \mathrm{F}_{[n-1]\setminus(n-S)}$. In particular, on the classical symmetric functions ω acts as $\omega(\mathrm{e}_\lambda) = \mathrm{h}_\lambda$, $\omega(\mathrm{s}_\lambda) = \mathrm{s}_{\lambda'}$, and $\omega(\mathrm{p}_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} \mathrm{p}_\lambda$. We remark that

$$\omega\left(\Psi_\alpha\right) = (-1)^{|\alpha| - \ell(\alpha)} \Psi_{\alpha^r} \, , \tag{13}$$

where α^r denotes the reverse of α, see [BDH+17, Sec. 4].

The following expansion of the fundamental basis into quasisymmetric power sums is the main result of this section.

Theorem 3.1. *Let $n \in \mathbb{N}$ and $S \subseteq [n-1]$. Then*

$$\mathrm{F}_{n,S}(\mathbf{x}) = \sum_{\alpha} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} (-1)^{|S \setminus S_\alpha|} \, , \tag{14}$$

where the sum ranges over all compositions α of n such that S is α-unimodal.

Proof. Expanding both sides of (14) in the monomial basis according to (9) and (11) we obtain

$$\sum_{\beta \leq \gamma} \mathrm{M}_\beta(\mathbf{x}) = \sum_{\substack{\alpha \vDash n \\ S_\gamma \in U_\alpha}} \sum_{\beta \geq \alpha} \frac{1}{\pi(\alpha,\beta)} \mathrm{M}_\beta(\mathbf{x}) \, (-1)^{|S_\gamma \setminus S_\alpha|}\,.$$

Comparing coefficients of M_β, it suffices to prove that for all compositions β and γ of n, we have

$$\sum_{\alpha \in R(\beta,\gamma)} \frac{1}{\pi(\alpha,\beta)} (-1)^{|S_\gamma \setminus S_\alpha|} = \begin{cases} 1 & \text{if } \beta \leq \gamma, \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

where $R(\beta,\gamma)$ denotes the set of all compositions $\alpha \leq \beta$ such that S_γ is α-unimodal.

It is shown in [BDH+17, Lemma 3.7] that $n! = |\textsc{Cons}(\alpha,\beta)| \cdot \pi(\alpha,\beta)$. We give a short alternative proof of this identity in Proposition 3.2 below using the hook-length formula for forests. After multiplying both sides of (15) by $n!$ the claim follows from Theorem 2.5. ■

Proposition 3.2. *Let $n \in \mathbb{N}$ and $\alpha \leq \beta$ be compositions of n. Then*

$$|\textsc{Cons}(\alpha,\beta)| \cdot \pi(\alpha,\beta) = n!\,.$$

Proof. To the pair (α,β) we associate a (labeled) rooted forest on the vertices $[n]$. For $i \in [\ell(\alpha)]$ set $s_i \coloneqq \alpha_1 + \cdots + \alpha_i$. For all $i \in [\ell(\alpha)]$ and all $j \in B_i^\alpha \setminus \{s_i\}$ add an edge from s_i to j. Moreover if $s_i, s_{i+1} \in B_k^\beta$ for some $k \in [\ell(\beta)]$ then add an edge from s_{i+1} to s_i. For example, if $\alpha = 2312$ and $\beta = 62$ then the forest is shown in Figure 2.

The *hook length* of a vertex in the forest is defined as the size of the subtree rooted at that vertex, see Figure 2. From the definitions in (10) it is straightforward to see that the product of all hook lengths in the forest is $\pi(\alpha,\beta)$. Furthermore it is immediate from the definition that the linear extensions of the forest can be identified with $\textsc{Cons}(\alpha,\beta)$. The hook-length formula for counting linear extensions of forests by D. Knuth [Knu98, Chap. 5.1.4, Ex. 20] then implies that $|\textsc{Cons}(\alpha,\beta)| \cdot \pi(\alpha,\beta) = n!$ as claimed. ■

Theorem 3.1 immediately implies the following result.

Corollary 3.3. *Let X be a quasisymmetric function with*

$$X(\mathbf{x}) = \sum_{S \subseteq [n-1]} c_S \mathrm{F}_{n,S}(\mathbf{x}).$$

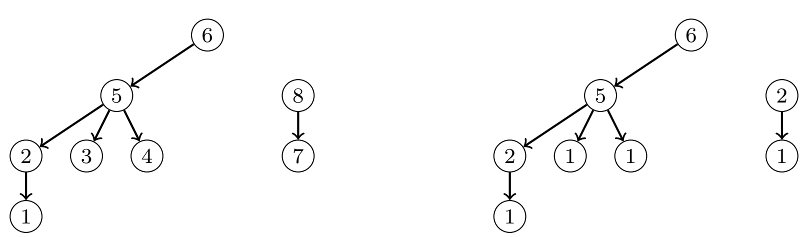


Figure 2. Left: The forest associated with the compositions $\alpha = 2312$ and $\beta = 62$. Right: The hook lengths of the vertices.

Then

$$X(\mathbf{x}) = \sum_{\alpha \vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} \sum_{S \in U_\alpha} (-1)^{|S \setminus S_\alpha|} c_S \,.$$

From Corollary 3.3 and (12) we obtain a new proof of the following result due to C. Athanasiadis for symmetric functions, which has served as inspiration for Theorem 3.1.

Corollary 3.4. ([Ath15, Prop. 3.2], see also [AR15, Prop. 9.3]) *Let X be a symmetric function with*

$$X(\mathbf{x}) = \sum_{S \subseteq [n-1]} c_S \mathrm{F}_{n,S}(\mathbf{x}).$$

Then

$$X(\mathbf{x}) = \sum_{\lambda \vdash n} \frac{\mathrm{p}_\lambda(\mathbf{x})}{z_\lambda} \sum_{S \in U_\lambda} (-1)^{|S \setminus S_\lambda|} c_S.$$

4. Reverse P-partitions

In this section it is shown that the generating function of reverse P-partitions expands positively into quasisymmetric power sums for all posets P.

The theory of P-partitions was developed by R. Stanley [Sta72], and has numerous applications in the world of quasisymmetric functions. Let (P, w) be a labeled poset. A function $f : P \to \mathbb{N}^+$ is called *reverse (P, w)-partition* if it satisfies the following two properties for all $x, y \in P$:

(i) If $x \leq_P y$ then $f(x) \leq f(y)$.
(ii) If $x <_P y$ and $w(x) > w(y)$ then $f(x) < f(y)$.

Let $\mathcal{A}^r(P, w)$ be the set of reverse (P, w)-partitions, and denote the generating function of reverse (P, w)-partitions by

$$K_{P,w}(\mathbf{x}) \coloneqq \sum_{f \in \mathcal{A}^r(P,w)} \prod_{x \in P} \mathbf{x}_{f(x)}. \tag{16}$$

If (P, w) is naturally labeled, then reverse (P, w)-partitions are just order-preserving maps $f : P \to \mathbb{N}^+$, which are also called *reverse P-partitions*. Denote the set of reverse P-partitions by $\mathcal{A}^r(P)$ and the corresponding generating function by

$$K_P(\mathbf{x}) = \sum_{\substack{f : P \to \mathbb{N}^+ \\ x <_P y \Rightarrow f(x) \leq f(y)}} \prod_{x \in P} \mathbf{x}_{f(x)}.$$

If instead w is order-reversing then reverse (P, w)-partitions are *strict reverse P-partitions*.

The expansion of $K_{P,w}$ into fundamental quasisymmetric functions is a well-known result.

Lemma 4.1 ([Sta01, Cor. 7.19.5])**.** *Let (P, w) be a labeled poset with n elements. Then the series $K_{P,w}(\mathbf{x})$ is quasisymmetric and its expansion into fundamental* *quasisymmetric functions is given by*

$$K_{P,w}(\mathbf{x}) = \sum_{\sigma \in \mathcal{L}(P,w)} \mathrm{F}_{n,\mathrm{DES}(\sigma)}(\mathbf{x})\,.$$

The following theorem is the main result of this section.

Theorem 4.2. *Let (P, w) be a naturally labeled poset with n elements. Then*

$$K_P(\mathbf{x}) = \sum_{\alpha \vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} |\mathcal{L}^*_\alpha(P, w)|\,. \tag{17}$$

In particular, the quasisymmetric function K_P is Ψ-positive.

Proof. By Lemma 4.1

$$K_P(\mathbf{x}) = K_{P,w}(\mathbf{x}) = \sum_{\sigma \in \mathcal{L}(P,w)} \mathrm{F}_{n,\mathrm{DES}(\sigma)}(\mathbf{x})\,.$$

Expanding the right hand side into quasisymmetric power sums according to Theorem 3.1, we obtain

$$\begin{aligned} K_P(\mathbf{x}) &= \sum_{\sigma \in \mathcal{L}(P,w)} \sum_{\substack{\alpha \vDash n \\ \mathrm{DES}(\sigma) \in U_\alpha}} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} (-1)^{|\,\mathrm{DES}(\sigma) \setminus S_\alpha|} \\ &= \sum_{\alpha \vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} \sum_{\sigma \in \mathcal{L}_\alpha(P,w)} (-1)^{|\,\mathrm{DES}(\sigma) \setminus S_\alpha|}\,. \end{aligned}$$

The claim follows directly from Theorem 2.9. ∎

The following is an immediate consequence of Theorem 4.2 and (12).

Corollary 4.3. *Let $\mathscr{P}$ be a finite set of posets. If the sum*

$$\sum_{P \in \mathscr{P}} K_P(\mathbf{x})$$

is symmetric, then it is p-positive.

We obtain analogous results for the generating functions of strict reverse P-partitions.

Lemma 4.4 (See also [LMvW13, Chap. 3.3.2])**.** *Let (P, w) be a labeled poset with n elements. Then the automorphism ω acts in an order-reversing manner as*

$$\omega K_{P,w}(\mathbf{x}) = K_{P,n+1-w}(\mathbf{x})$$

where $(n + 1 - w)(x) \coloneqq n + 1 - w(x)$.

Proof. This follows from the fact that

$$\sigma^{-1} \in \mathcal{L}(P, w) \iff \tau^{-1} \in \mathcal{L}(P, n + 1 - w)\,,$$

where $\tau_i = n + 1 - \sigma_i$, and that

$$S = \mathrm{DES}(\sigma^{-1}) \iff \mathrm{DES}(\tau^{-1}) = [n - 1] \setminus (n - S)\,.$$ ∎

In particular, the generating function of *strict* reverse P-partitions is given by $\omega K_P(\mathbf{x})$.

Corollary 4.5. *Let (P, w) be a labeled poset such that w is order-reversing. Then $\omega K_{P,w}$ is Ψ-positive.*

Corollary 4.6. *Let $\mathscr{P}$ be a finite family of labeled posets (P, w), all of which are equipped with an order-reversing map w. If*

$$X(\mathbf{x}) = \sum_{(P,w)\in\mathscr{P}} K_{P,w}(\mathbf{x})$$

is symmetric, then ωX is p-positive.

Note that $K_{P,w}$ is not in general Ψ-positive if w is not a natural labeling. Moreover positive linear combinations of quasisymmetric functions $K_{P,w}$ are not in general p-positive whenever they are symmetric.

For example, Corollary 4.6 shows that $K_{P,w}$ is not in general Ψ-positive if w is order-reversing, because the automorphism ω can introduce signs.

Another example is provided by Schur functions s_λ, which are shown to be special cases of the functions $K_{P,w}$ in [Sta01, Sec. 7.19]. The expansion of Schur functions into power sum symmetric functions is given by the celebrated Murnaghan–Nakayama rule, which is not positive, see [Sta01, Cor. 7.17.5].

It is conjectured [Sta72, p. 81] that $K_{P,w}$ is symmetric if and only if it is a skew Schur function. Among these, w is order-preserving only if $K_{P,w}$ is equal to a complete homogeneous symmetric function h_λ. It is an interesting question whether the expansion of the quasisymmetric functions $K_{P,w}$ into quasisymmetric power sums can be used to obtain new insights in this regard.

5. Order-preserving surjections

The purpose of this section is to present a different characterisation of the set $\mathcal{L}^*_\alpha(P, w)$ associated to a naturally labeled poset (P, w). In the process we eschew α-unimodal linear extensions in exchange for order-preserving surjections. This new point of view is then used to compute three examples that are related to matroid quasisymmetric functions, chromatic quasisymmetric functions and Eulerian quasisymmetric functions. Moreover in Theorem 5.4 below we formulate an equivalent version of Theorem 4.2 using order-preserving surjections.

Let P be a poset with n elements. Denote by $\mathcal{O}(P)$ the set of order-preserving surjections $f : P \to [k]$ for some $k \in \mathbb{N}$. Define the *type* of a surjection $f : P \to [k]$ as

$$\alpha(f) := (\left|f^{-1}(1)\right|, \dots, \left|f^{-1}(k)\right|).$$

Thus $\alpha(f)$ is a composition of $n = |P|$ with k parts. Denote by $\mathcal{O}_\alpha(P)$ the set of order-preserving surjections of P with type α.

Furthermore, let $\mathcal{O}^*(P)$ denote the set of order-preserving surjections $f \in \mathcal{O}(P)$ such that $f^{-1}(i)$ contains a unique minimal element for all $i \in [\ell(\alpha(f))]$. That is, for all $y, z \in P$ with $f(y) = f(z)$ there exists $x \in P$ with $x \leq_P y$ and $x \leq_P z$ and $f(x) = f(y)$. Set $\mathcal{O}^*_\alpha(P) := \mathcal{O}_\alpha(P) \cap \mathcal{O}^*(P)$.

Note that $\mathcal{O}_{(1^n)}(P) = \mathcal{O}^*_{(1^n)}(P)$ is just the set of linear extensions of P. Compared to linear extensions, order-preserving surjections onto chains have not received much explicit attention. Nevertheless some combinatorial objects can be regarded as order-preserving surjections in disguise. For example, $\mathcal{O}^*_{(2^m)}(\lambda)$, where λ denotes the Young diagram of a partition of $n = 2m$, is just the set of domino tableaux of shape λ, see [vL00]. The set $\mathcal{O}^*_{(2^m)}(P)$ can be taken as the definition of P-domino tableaux [Sta05, Sec. 4]. Similarly, ribbon tableaux which appear in the study of Schur functions can be seen as certain order-preserving surjections on Young diagrams. We do not pursue this direction any further in this paper.

Let (P, w) be a naturally labeled poset with n elements, and let α be a composition of n with ℓ parts. Given a permutation $\sigma \in \mathcal{L}^*_\alpha(P, w)$ define the map $f_\sigma : P \to [\ell]$ by $f_\sigma(x) := i$ for all $x \in P^\alpha_i(\sigma)$ and all $i \in [\ell]$. Here we use the notation from Equation (7).

The following proposition is the key result of this section and relates the set $\mathcal{L}^*_\alpha(P, w)$ to order-preserving surjections onto chains.

Proposition 5.1. *Let (P, w) be a naturally labeled poset with n elements, and let α be a composition of n. Then the correspondence $\sigma \mapsto f_\sigma$ defines a bijection $\varphi : \mathcal{L}^*_\alpha(P, w) \to \mathcal{O}^*_\alpha(P)$.*

Proof. Let ℓ denote the number of parts of α. It is not difficult to see that the map $\varphi : \mathcal{L}^*_\alpha(P, w) \to \mathcal{O}^*_\alpha(P)$ is well-defined.

We next show that φ is injective. Suppose $\sigma, \tau \in \mathcal{L}^*_\alpha(P, w)$ with $f_\sigma = f_\tau$. Then $P^\alpha_i(\sigma) = P^\alpha_i(\tau)$ for all $i \in [\ell]$. Consequently $\sigma^{-1} \circ w(P^\alpha_i(\sigma)) = B^\alpha_i = \tau^{-1} \circ w(P^\alpha_i(\sigma))$, and therefore $\sigma(B^\alpha_i) = \tau(B^\alpha_i)$ for all $i \in [\ell]$. Lemma 2.8 implies $\sigma = \tau$.

To see that φ is surjective let $f \in \mathcal{O}^*_\alpha(P)$. Define $\sigma \in \mathfrak{S}_n$ as the unique permutation such that $\sigma(B^\alpha_i) = w(f^{-1}(i))$ for all $i \in [\ell]$ and $\mathrm{DES}(\sigma) \subseteq S_\alpha$. That is, the word $\sigma_1 \cdots \sigma_n$ is obtained by first listing the numbers in $w(f^{-1}(1))$ in increasing order, then the numbers in $w(f^{-1}(2))$ in increasing order, and so on.

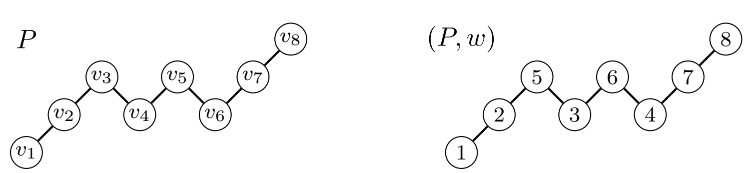
Clearly σ is α-unimodal, and each subposet $P^\alpha_i(\sigma) = f^{-1}(i)$ contains a unique minimal element. It remains to show that $\sigma^{-1} \circ w : P \to [n]$ is a linear extension.

Let $x, y \in P$ with $x <_P y$. If there exists $i \in [\ell]$ with $x, y \in f^{-1}(i)$ then $w(x), w(y) \in \sigma(B^\alpha_i)$ and $\sigma^{-1} \circ w(x) < \sigma^{-1} \circ w(y)$ since $w(x) < w(y)$ and σ restricted to the set B^α_i has no descents.

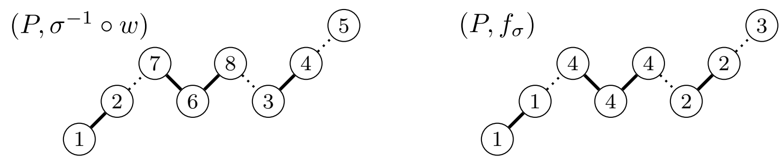
Otherwise $x \in f^{-1}(i)$ and $y \in f^{-1}(j)$ for some $i, j \in [\ell]$ with $i < j$ because f is order-preserving. We conclude $\sigma^{-1} \circ w(x) \in B^\alpha_i$ and $\sigma^{-1} \circ w(y) \in B^\alpha_j$ where $i < j$, and therefore $\sigma^{-1} \circ w(x) < \sigma^{-1} \circ w(y)$.

Hence $\sigma \in \mathcal{L}^*_\alpha(P, w)$ and $f_\sigma = f$. ■

Example 5.2. Let $\alpha = 2213$ and consider the following labeled poset.



Let $\sigma = 12|47|8|356 \in \mathcal{L}^*_\alpha(P,w)$. Then $P^\alpha_1(\sigma) = \{v_1, v_2\}$, $P^\alpha_4(\sigma) = \{v_3, v_4, v_5\}$, $P^\alpha_2(\sigma) = \{v_6, v_7\}$ and $P^\alpha_3(\sigma) = \{v_8\}$. Hence $f_\sigma : P \to \{1,2,3,4\}$ is as shown below.



$\square$

An immediate consequence of Proposition 5.1 is the following.

Corollary 5.3. *Let (P,w) be a naturally labeled poset with n elements, and let α be a composition of n. Then the cardinality $|\mathcal{L}^*_\alpha(P,w)|$ is independent of w. That is, if $w' : P \to [n]$ is another natural labeling, then $|\mathcal{L}^*_\alpha(P,w)| = |\mathcal{L}^*_\alpha(P,w')|$.*

Note that Corollary 5.3 also follows from Theorem 4.2. Proposition 5.1 even provides a bijection between $\mathcal{L}^*_\alpha(P,w)$ and $\mathcal{L}^*_\alpha(P,w')$.

Another advantage of order-preserving surjections over unimodal linear extensions is that the sets $\mathcal{O}_\alpha(P,w)$ for $\alpha \vDash n$ are disjoint whereas the sets $\mathcal{L}^*_\alpha(P,w)$ intersect. It is therefore much more convenient to work, say, with the set $\mathcal{O}^*(P)$ than with its pendant in the world of α-unimodal linear extensions.

The main result of the previous section, Theorem 4.2, has an equivalent formulation in terms of order-preserving surjections.

Theorem 5.4. *Let P be poset with n elements. Then*

$$K_P(\mathbf{x}) = \sum_{f \in \mathcal{O}^*(P)} \frac{\Psi_{\alpha(f)}(\mathbf{x})}{z_{\alpha(f)}} = \sum_{\alpha \vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} \, |\mathcal{O}^*_\alpha(P)| \, .$$

Theorem 5.4 yields the intriguing identity

$$K_P(\mathbf{x}) = \sum_{f \in \mathcal{O}(P)} \mathrm{M}_{\alpha(f)} = \sum_{f \in \mathcal{O}^*(P)} \frac{\Psi_{\alpha(f)}(\mathbf{x})}{z_{\alpha(f)}}. \tag{18}$$

Multiplying (18) by $n!$ and taking coefficients of M_β using (11) and Proposition 3.2, leads to the following open problem.

Problem 5.5. Let P be a poset with n elements and β a composition of n. Find a bijection

$$\varphi : \mathfrak{S}_n \times \mathcal{O}_\beta(P) \to \bigcup_{\alpha \leq \beta} \textsc{Cons}(\alpha, \beta) \times \mathcal{O}^*_\alpha(P)\,.$$

We are currently unable to provide such bijections, although we suspect it might not be hopelessly difficult to find them. Note that the first part of the proof of Theorem 2.5 solves Problem 5.5 in the case where P is a chain. A full solution to Problem 5.5 should give an independent proof of Theorems 4.2 and 5.4.[3] We also suspect that guessing (18) without proving Theorem 4.2 first would have been very difficult.

[3] It woud be particularly appealing to the combinatorialist if such a proof were cancellation free and avoided sign-reversing involutions.

We now compute the numbers $|\mathcal{O}^*_\alpha(P)|$ for three examples. In later sections we use these examples and apply Theorem 5.4 to three families of quasisymmetric functions. The first example is related to the quasisymmetric functions of uniform matroids, see Section 7.8.

Example 5.6 (The complete bipartite graph)**.** Let P be the poset with ground set $\{x_1,\dots,x_r,y_1,\dots,y_m\}$ and cover relations $x_i < y_j$ for all $i\in[r]$ and $j\in[m]$. Thus the Hasse diagram of P is the complete bipartite graph $K_{r,m}$.

Suppose $f\in\mathcal{O}^*(P)$. Then f restricts to a bijection $f:\{x_i : i\in[r]\}\to[r]$. Moreover, there exists a subset $S\subseteq[m]$ such that $f(y_j)=r$ for all $j\in S$ and f restricts to a bijection

$$f:\{y_j : j\in[m]\setminus S\}\to\{r+s : s\in[m-k]\},$$

where $k\coloneqq|S|$. We conclude that

$$|\mathcal{O}^*_\alpha(P)| = \begin{cases} r!m!/k! & \text{if } \alpha=(1^{r-1},k+1,1^{m-k}),\\ 0 & \text{otherwise.}\end{cases} \qquad \square$$

The next example is connected to Eulerian quasisymmetric functions and chromatic quasisymmetric functions of paths. This is explained in Section 7.9.

Example 5.7 (The path)**.** Let α be a composition of n with ℓ parts. We want to compute

$$\sum_{S\subseteq[n-1]} q^{|S|}\,|\mathcal{O}^*_\alpha(P_S)|\ ,$$

where P_S denotes the poset on $\{x_1,\dots,x_n\}$ with cover relations $x_i<x_{i+1}$ for $i\in[n-1]\setminus S$ and $x_i>x_{i+1}$ for $i\in S$. Thus the Hasse diagram of P_S viewed as an (undirected) graph is a path. Arranging the vertices $x_1,\dots,x_n$ from left to right, we can think of the Hasse diagram of P_S as a word $W_S\in\{\mathtt{u},\mathtt{d}\}^{n-1}$ with *up-* and *down-steps*, where the down-steps correspond to elements of S.

In this way the poset P in Example 5.2 corresponds to the set $S=\{3,5\}$ and the word $\mathtt{uududuu}$.

Let $S\subseteq[n-1]$ and fix some $f\in\mathcal{O}^*_\alpha(P_S)$.

For each $i\in[\ell]$ the subgraph of the Hasse diagram of P_S induced by $f^{-1}(i)$ can be identified with a word $W(i)$ that consists of r_i down-steps followed by α_i-r_i-1 up-steps. This follows from the fact that $f^{-1}(i)$ has a unique minimal element. Moreover, reading the values of f from left to right without repetitions we obtain a permutation π.

For instance, the order-preserving surjection f_σ from Example 5.2 yields $r_1=0$, $r_2=0$, $r_3=0$, $r_4=1$ and $\pi=1423$.

We claim that the map ψ defined by $f\mapsto(\pi,r_1,\dots,r_\ell)$ is a bijection

$$\psi:\bigsqcup_{S\subseteq[n-1]}\mathcal{O}^*_\alpha(P_S)\to\mathfrak{S}_\ell\times\prod_{i=1}^{\ell}\{0,\dots,\alpha_i-1\}\,.$$

To see this we construct the inverse of ψ. Given $(\pi,r_1,\dots,r_\ell)$ first form words $W(i)$ consisting of r_i down-steps followed by α_i-r_i-1 up-steps. We recover S from the identity

$$W_S = W(\pi_1)a_1W(\pi_2)\cdots W(\pi_{\ell-1})a_{\ell-1}W(\pi_\ell)\,,$$

where $a_i = \mathtt{d}$ if $i \in \mathrm{DES}(\pi)$ and $a_i = \mathtt{u}$ otherwise. Once α, S and π are known, it is easy to recover f.

From $|S| = \mathrm{des}(\pi) + \sum_{i=1}^{\ell} r_i$ it follows that

$$\sum_{S\subseteq[n-1]} q^{|S|}\,|\mathcal{O}^*_\alpha(P_S)| = A_\ell(q)\prod_{i=1}^{\ell}[\alpha_i]_q\,,$$

where $A_k(q) \coloneqq \sum_{\sigma\in\mathfrak{S}_k} q^{\mathrm{des}(\sigma)}$ is the *Eulerian polynomial* and $[a]_q \coloneqq \frac{1-q^a}{1-q}$ is the commonly used *q-integer.* □

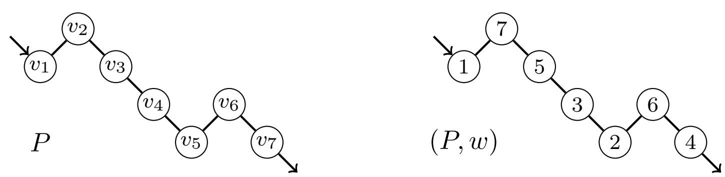
Our final example is related to chromatic quasisymmetric functions of cycles and cycle Eulerian quasisymmetric functions, see Section 7.9. The same computation was previously done by B. Ellzey with different notation.

Example 5.8 (The cycle, [Ell16, Thm. 4.4])**.** Let α be a composition of n with ℓ parts. We want to compute

$$\sum_{\substack{S\subseteq[n]\\0<|S|<n}} q^{|S|}\,|\mathcal{O}^*_\alpha(P_S)|\,,$$

where P_S denotes the poset on $\{x_1,\dots,x_n\}$ with cover relations $x_i < x_{i+1}$ for $i \in [n]\setminus S$ and $x_i > x_{i+1}$ for $i \in S$. Here indices are to be understood modulo n, thus the Hasse diagram of P_S viewed as an (undirected) graph is a cycle. Arranging the vertices $x_1,\dots,x_n$ from left to right, we can think of the Hasse diagram of P_S as a word $W_S \in \{\mathtt{u},\mathtt{d}\}^n$, where the last letter determines the relation between x_1 and x_n. Note that we have to exclude the cases $S = \emptyset$ and $S = [n]$ since they do not give rise to partial orders.

For example, the following poset corresponds to the set $S = \{2,3,4,6,7\}$ and the word `udddudd`.

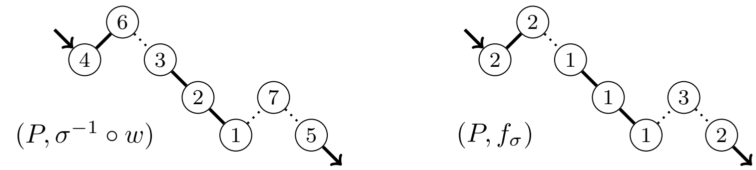


Let $S \subseteq [n]$ with $0 < |S| < n$. We have to distinguish two cases.

Case 1. First assume that $\ell \geq 2$.

Fix some $f \in \mathcal{O}^*_\alpha(P_S)$. For each $i \in [\ell]$ the subgraph of the Hasse diagram of P_S induced by $f^{-1}(i)$ is a path and can be identified with a word $W(i)$ that consists of r_i down-steps followed by $\alpha_i - r_i - 1$ up-steps. Again this follows from the fact that $f^{-1}(i)$ has a unique minimal element. Reading the values of f from left to right without repetitions we obtain a long cycle $(\pi_1,\dots,\pi_{\ell-1},\ell) \in \mathfrak{S}_\ell$ which we identify with the permutation $\pi_1\cdots\pi_{\ell-1} \in \mathfrak{S}_{\ell-1}$. Moreover, let k be the position of the minimal element of $f^{-1}(1)$.

For instance, consider the composition $\alpha = 331$ and the labeled poset (P, w) above, and let $\sigma = 235|147|6 \in \mathcal{L}^*_\alpha(P, w)$. Then $P^\alpha_2(\sigma) = \{v_1, v_2, v_7\}$, $P^\alpha_1(\sigma) = \{v_3, v_4, v_5\}$ and $P^\alpha_3(\sigma) = \{v_6\}$. Hence f_σ is as shown below.



This yields $r_1 = 2$, $r_2 = 1$, $r_3 = 0$, $\pi = 21$, and $k = 5$.

We claim that the map ψ defined by $f \mapsto (k, \pi, r_1, \dots, r_\ell)$ is a bijection

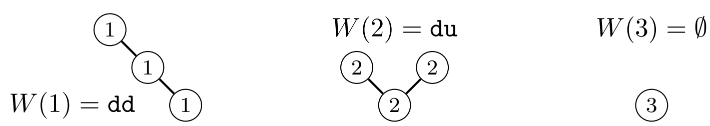
$$\psi : \bigsqcup_{\substack{S \subseteq [n] \\ 0 < |S| < n}} \mathcal{O}^*_\alpha(P_S) \to [n] \times \mathfrak{S}_{\ell-1} \times \prod_{i=1}^{\ell} \{0, \dots, \alpha_i - 1\}\,.$$

To see this we construct the inverse of ψ in a similar fashion as in Example 5.7. Given $(k, \pi, r_1, \dots, r_\ell)$ first form words $W(i)$ consisting of r_i down-steps followed by $\alpha_i - r_i - 1$ up-steps. Then W_S is a cyclic shift of the word

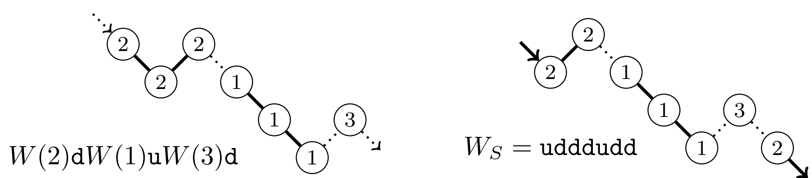
$$W(\pi_1) a_1 W(\pi_2) \cdots a_{\ell-2} W(\pi_{\ell-1}) \mathtt{u} W(\ell) \mathtt{d}\,,$$

where $a_i = \mathtt{d}$ if $i \in \mathrm{DES}(\pi)$ and $a_i = \mathtt{u}$ otherwise. The number k contains precisely the information needed to recover S. As before, f can be obtained once α, S and π are known.

For example, suppose we are given the data $(k, \pi, r_1, r_2, r_3) = (5, 21, 2, 1, 0)$. We first form the posets corresponding to the words $W(i)$.



Next join the words $W(i)$ according to π (with $W(3)$ in last position).



Since the minimal element of $f^{-1}(1)$ is currently in sixth position, we have to cyclically shift until the position is $k = 5$ to obtain W_S.

From $|S| = 1 + \mathrm{des}(\pi) + \sum_{i=1}^{\ell} r_i$ it follows that

$$\sum_{S \subseteq [n-1]} q^{|S|} \, |\mathcal{O}^*_\alpha(P_S)| = nqA_{\ell-1}(q) \prod_{i=1}^{\ell} [\alpha_i]_q\,,$$

where $A_k(q)$ and $[a]_q$ are defined as in Example 5.7.

Case 2. Now suppose $\alpha = (n)$.

Note that $\mathcal{O}^*_{(n)}(P_S)$ contains a unique element if P_S has a unique minimal element, and $\mathcal{O}^*_{(n)}(P_S)$ is empty otherwise. It is not difficult to see that

$$\sum_{\substack{S\subseteq[n]\\0<|S|<n}} q^{|S|}|\mathcal{O}^*_{(n)}(P_S)| = nq[n-1]_q\,. \qquad \square$$

6. Reverse P-partitions with forced equalities

In this section we consider the generating function of reverse P-partitions that assign equal colors to certain given elements. In Theorem 6.2 below we show that these functions are Ψ-positive, thus extending Theorem 4.2. We hope this will enable (or simplify) future applications of our results to families of quasisymmetric functions that are not covered by Section 4. A simple example of such a family are the power sum symmetric functions p_λ, which we discuss in Section 7.2. Reverse P-partitions with forced equalities have not received much attention to date. The results of this section might be an indication that they are worth investigating further.

A *partitioned poset* (P, w, E) is a naturally labeled poset (P, w) endowed with an equivalence relation E on P. Given a partitioned poset (P, w, E) with n elements, let

$$\mathcal{A}^r(P,E) := \left\{f \in \mathcal{A}^r(P) : f(x) = f(y) \text{ for all } x, y \in P \text{ with } x \overset{E}{\sim} y\right\}$$

and define the generating function

$$K_{P,E}(\mathbf{x}) := \sum_{\substack{f\in\mathcal{A}^r(P)\\x\overset{E}{\sim}y\Rightarrow f(x)=f(y)}} \prod_{x\in P} \mathbf{x}_{f(x)}\,.$$

This is a homogeneous quasisymmetric function of degree n. For example, if all equivalence classes of E are singletons then $K_{P,E} = K_P$. On the other hand if $x \overset{E}{\sim} y$ for all $x, y \in P$, then $K_{P,E}$ equals the power sum symmetric function p_n.

In the above definition the equivalence relation E is completely arbitrary. We now define a very special kind of equivalence relation.

Let P be a poset. An equivalence relation E on P is a *chain congruence* on P if the following two conditions are satisfied:

(i) The equivalence class $[x]_E$ is a chain in P for all $x \in P$.
(ii) For all $x, y \in P$ with $x <_P y$ and $x \overset{E}{\nsim} y$ we have $\max\,[x]_E <_P \min\,[y]_E$.

In order to prove Ψ-positivity of the functions $K_{P,E}$ it suffices to consider chain congruences. This is the content of the next lemma.

Lemma 6.1. *Let (P, w, E) be a partitioned poset. Then there exists a partitioned poset (P', w', E') such that*

$$K_{P,E}(\mathbf{x}) = K_{P',E'}(\mathbf{x})\,,$$

the poset P' is obtained from P by adding order relations, E' is obtained from E by joining equivalence classes, and E' is a chain congruence on P'.

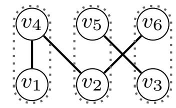


FIGURE 3. A poset P with an equivalence relation E given by $v_1 \overset{E}{\sim} v_4$ and $v_2 \overset{E}{\sim} v_5$ and $v_3 \overset{E}{\sim} v_6$. The equivalence relation E' has two classes $\{v_1, v_4\}$ and $\{v_2, v_3, v_5, v_6\}$. Suppose $w : P \to [6]$ sends v_i to i. Then $v_2 \leq_{P'} v_3 \leq_{P'} v_5 \leq_{P'} v_6 \leq_{P'} v_1 \leq_{P'} v_4$.

Proof. Define a relation E' on P as follows. For $x, y \in P$ let $x \overset{E'}{\sim} y$ if and only if the following two (symmetric) conditions are satisfied:

(i) There exist $k \in \mathbb{N}$ and $x_1, y_1, \dots, x_k, y_k \in P$ such that $x = x_1$, $y = y_k$, $x_i \leq_P y_i$ for all $i \in [k]$, and $y_i \overset{E}{\sim} x_{i+1}$ for all $i \in [k-1]$.
(ii) There exist $k \in \mathbb{N}$ and $x_1, y_1, \dots, x_k, y_k \in P$ such that $y = x_1$, $x = y_k$, $x_i \leq_P y_i$ for all $i \in [k]$, and $y_i \overset{E}{\sim} x_{i+1}$ for all $i \in [k-1]$.

See Figure 3 for an example.

Clearly E' is an equivalence relation and $x \overset{E}{\sim} y$ implies $x \overset{E'}{\sim} y$. We claim that $K_{P,E} = K_{P,E'}$. To see this note that $\mathcal{A}^r(P, E') \subseteq \mathcal{A}^r(P, E)$. Thus let $f \in \mathcal{A}^r(P, E)$ and $x, y \in P$ with $x \overset{E'}{\sim} y$. The definition of E' yields $f(x) \leq f(y) \leq f(x)$ and therefore $f \in \mathcal{A}^r(P, E')$.

Next define a relation $\leq_{P'}$ on P as follows. For all $x, y \in P$ let $x \leq_{P'} y$ if and only if one of the following two mutually exclusive conditions is satisfied:

(i) We have $x \overset{E'}{\sim} y$ and $w(x) \leq w(y)$.
(ii) We have $x \overset{E'}{\nsim} y$ and there exist $k \in \mathbb{N}$ and $x_1, y_1, \dots, x_k, y_k \in P$ such that $x = x_1$, $y = y_k$, $x_i \leq_P y_i$ for all $i \in [k]$, and $y_i \overset{E'}{\sim} x_{i+1}$ for all $i \in [k-1]$.

Then P' is a poset with the same ground set as P, and $x \leq_P y$ implies $x \leq_{P'} y$ for all $x, y \in P$. Let w' be an arbitrary natural labeling of P'.

We claim that $K_{P',E'} = K_{P,E'}$. To see this note that $\mathcal{A}^r(P', E') \subseteq \mathcal{A}^r(P, E')$. Thus let $f \in \mathcal{A}^r(P, E')$ and $x, y \in P$ with $x \leq_{P'} y$. The definition of P' implies $f(x) \leq f(y)$ and therefore $f \in \mathcal{A}^r(P', E')$.

It remains to show that E' is a chain congruence on P'. Clearly $[x]_{E'}$ is a chain by definition of P' for all $x \in P'$. Secondly suppose $x <_P y$ and $x \overset{E'}{\nsim} y$ for some $x, y \in P'$. Then

$$\max{[x]_{E'}} \leq_P \max{[x]_{E'}} \overset{E'}{\sim} x \leq_P y \overset{E'}{\sim} \min{[y]_{E'}} \leq_P \min{[y]_{E'}}.$$

Hence $\max{[x]_{E'}} \leq_{P'} \min{[y]_{E'}}$and the proof is complete. ∎

Let (P, w, E) be a partitioned poset. If we are only interested in $K_{P,E}$, then by Lemma 6.1 we may assume that E is a chain congruence on P. If this is the case we can form the *quotient poset* P/E, that is, the partial order on the set of equivalence classes $\{[x]_E : x \in P\}$ defined by $[x]_E \le [y]_E$ if and only if $x \le_P y$. Then

$$K_{P,E}(\mathbf{x}) = \sum_{f \in \mathcal{A}^r(P/E)} \prod_{x \in P} \mathbf{x}_{f([x]_E)} = \sum_{f \in \mathcal{A}^r(P/E)} \prod_{C \in P/E} \mathbf{x}_{f(C)}^{|C|}.$$

This leads to a second, equivalent, way of thinking about partitioned posets.

A *weighted poset* (P, w, d) consists of a naturally labeled poset (P, w) and a vector $d = (d_x)_{x \in P}$ of positive integers — a weight on P. Define the weighted generating function

$$K_P^d(\mathbf{x}) \coloneqq \sum_{f \in \mathcal{A}^r(P)} \prod_{x \in P} \mathbf{x}_{f(x)}^{d_x}.$$

This is a homogeneous quasisymmetric function of degree $|d|$. For example, if P is a poset with one element then $K_P^d(\mathbf{x})$ is equal to the power sum symmetric function $\mathrm{p}_{|d|}(\mathbf{x})$.

The following theorem is the main result of this section and generalizes Theorem 4.2.

Theorem 6.2. *Let (P, w) be a naturally labeled poset with n elements, and let E be a chain congruence on P. Then*

$$K_{P,E}(\mathbf{x}) = \sum_{\alpha \vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} \sum_{\sigma \in \mathcal{L}_\alpha^*(P,w,E)} \prod_{i=1}^{\ell(\alpha)} \left|[\min P_i^\alpha(\sigma)]_E\right|, \tag{19}$$

where $\mathcal{L}_\alpha^*(P, w, E)$ denotes the set of permutations $\sigma \in \mathcal{L}_\alpha^*(P, w)$ such that for each $x \in P$ there exists $i \in [\ell(\alpha)]$ with $[x]_E \subseteq P_i^\alpha(\sigma)$. Note that $\min P_i^\alpha(\sigma)$ *is well-defined for all $\sigma \in \mathcal{L}_\alpha^*(P, w)$.*

Equivalently,

$$K_{P,E}(\mathbf{x}) = \sum_{\alpha \vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} \sum_{f \in \mathcal{O}_\alpha^*(P,E)} \prod_{i=1}^{\ell(\alpha)} \left|[\min f^{-1}(i)]_E\right|, \tag{20}$$

where $\mathcal{O}_\alpha^*(P, E)$ denotes the set of order-preserving surjections $f \in \mathcal{O}_\alpha^*(P)$ that satisfy $f(x) = f(y)$ for all $x, y \in P$ with $x \overset{E}{\sim} y$. Note that $\min f^{-1}(i)$ *is well-defined for all $f \in \mathcal{O}_\alpha^*(P)$.*

In particular, $K_{P,E}$ is Ψ-positive.

Theorem 6.2 has an equivalent formulation in terms of weighted posets.

Theorem 6.3. *Let (P, w, d) be a weighted poset with n elements. Then*

$$K_P^d(\mathbf{x}) = \sum_{\alpha \vDash n} \sum_{\sigma \in \mathcal{L}_\alpha^*(P,w)} \frac{\Psi_\beta(\mathbf{x})}{z_\beta} \prod_{i=1}^{\ell(\alpha)} d_{\min P_i^\alpha(\sigma)}, \tag{21}$$

where $\beta = \beta(d, \alpha, \sigma)$ is the composition of $|d|$ defined by

$$\beta_i = \sum_{x \in P_i^\alpha(\sigma)} d_x$$

for all $i \in [\ell(\alpha)]$.

Equivalently,

$$K_P^d(\mathbf{x}) = \sum_{\alpha \vDash n} \sum_{f \in \mathcal{O}_\alpha^*(P)} \frac{\Psi_\beta(\mathbf{x})}{z_\beta} \prod_{j=1}^{\ell(\alpha)} d_{\min f^{-1}(j)} \,, \tag{22}$$

where $\beta = \beta(d, \alpha, f)$ *is the composition of* $|d|$ *defined by*

$$\beta_i = \sum_{x \in f^{-1}(i)} d_x \tag{23}$$

for all $i \in [\ell(\alpha)]$.

In particular, K_P^d *is* Ψ*-positive.*

Proof of equivalence of the statements in Theorems 6.2 and 6.3. To see that (19) and (20) are equivalent, note that the bijection $\sigma \mapsto f_\sigma$ from Proposition 5.1 restricts to a bijection from $\mathcal{L}_\alpha^*(P, w, E)$ to $\mathcal{O}_\alpha^*(P, E)$. The claimed equivalence follows from $P_i^\alpha(\sigma) = f_\sigma^{-1}(i)$.

The equivalence of (21) and (22) follows from Proposition 5.1 as above.

To see that Theorem 6.3 implies Theorem 6.2 let E be a chain congruence on a finite poset P, and let P/E be the associated quotient poset. Every order-preserving surjection $f \in \mathcal{O}_\beta^*(P, E)$ yields an order-preserving surjection $\bar{f} \in \mathcal{O}_\alpha^*(P/E)$ where the type β of f is related to the type α of $\bar{f}$ by (23). Indeed, this correspondence defines a bijection from $\mathcal{O}_\beta^*(P, E)$ to $\mathcal{O}_\alpha^*(P/E)$. Define a weight d on P/E by assigning to each equivalence class $[x]_E$ its cardinality. Then the formula (22) for $K_{P/E}^d$ implies (20).

Conversely, let (P, w, d) be a weighted poset. Define a poset Q by replacing each element $x \in P$ by a chain C_x with d_x elements. That is, $Q = \bigsqcup_{x \in P} C_x$ and for $y_1 \in C_{x_1}$ and $y_2 \in C_{x_2}$, where $x_1 \neq x_2$, we have $y_1 \leq_Q y_2$ if and only if $x_1 \leq_P x_2$. The poset Q comes equipped with a chain congruence E, namely $y_1 \overset{E}{\sim} y_2$ if and only if $y_1, y_2 \in C_x$ for some element $x \in P$. Given an order-preserving surjection $f \in \mathcal{O}_\alpha^*(P)$ define an order-preserving surjection $\hat{f} \in \mathcal{O}_\beta^*(Q, E)$ by $\hat{f}(y) = f(x)$ if $y \in C_x$. This defines a bijection from $\mathcal{O}_\alpha^*(P)$ to $\mathcal{O}_\beta^*(Q, E)$ where α and β are related by (23). Hence the formula (20) for $K_{Q,E}$ implies (22), and Theorem 6.3 follows from Theorem 6.2. ∎

Proof of Theorem 6.2 (20). Given an equivalence relation R let $\mathscr{C}(R)$ denote the set of its equivalence classes, and let $c(R) = |\mathscr{C}(R)|$ denote the number of equivalence classes.

We prove the claim by induction on $n - c(E)$.

Clearly the case $c(E) = n$ reduces to Theorem 4.2, so we may assume that there exists $C \in \mathscr{C}(E)$ with $|C| \geq 2$, and that the claim holds for all chain congruences on P with more equivalence classes. Let $C = \{x_0, \ldots, x_k\}$ for some $k \in [n-1]$, so that $x_0 = \min C$ and $x_k = \max C$.

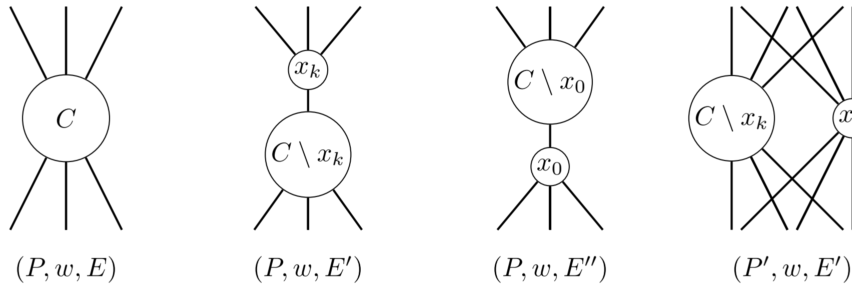


FIGURE 4. The four partitioned posets from the proof of Theorem 6.2.

Define two new equivalence relations on P by

$$\mathscr{C}(E') := \mathscr{C}(E) \setminus \{C\} \cup \{\{x_0\}, C \setminus \{x_0\}\},$$
$$\mathscr{C}(E'') := \mathscr{C}(E) \setminus \{C\} \cup \{\{x_k\}, C \setminus \{x_k\}\}.$$

Moreover define a new naturally labeled poset (P', w) by removing all order relations $x_i < x_k$ for $i \in \{0, \dots, k-1\}$, see Figure 4.

Note that E' and E'' are chain congruences on P, and that E' is a chain congruence on P'. Thus $K_{P,E'}$, $K_{P,E''}$ and $K_{P',E'}$ satisfy (20) by the induction hypothesis.

The function $K_{P,E}$ satisfies the recursion

$$K_{P,E}(\mathbf{x}) = K_{P,E'}(\mathbf{x}) + K_{P,E''}(\mathbf{x}) - K_{P',E'}(\mathbf{x}). \tag{24}$$

This identity follows directly from the definition of $K_{P,E}$ and is analogous to the fact that

$$\left|\{(i,j) \in [n]^2 : i = j\}\right| = \left|\{(i,j) \in [n]^2 : i \leq j\}\right| + \left|\{(i,j) \in [n]^2 : i \geq j\}\right| - \left|\{(i,j) \in [n]^2\}\right| .$$

Now fix a composition α of n with ℓ parts and a surjective map $f : P \to [\ell]$ of type α. We need to show that the contribution of f predicted by (20) is the same on both sides of (24).

First assume that $f \in \mathcal{O}^*_\alpha(P, E)$. Then there exists $i \in [\ell]$ with $f(x) = i$ for all $x \in C$. It follows that

$$f \in \mathcal{O}^*_\alpha(P, E') \cap \mathcal{O}^*_\alpha(P, E'')$$

and that $f \in \mathcal{O}_\alpha(P')$ is an order-preserving surjection of type α.

If $x_0 = \min f^{-1}(i)$ then $f \notin \mathcal{O}^*_\alpha(P', E')$ because $f^{-1}(i)$ has two minimal elements in P'. The contribution of f on both sides of (24) is

$$|C| \cdot \prod_{j \neq i} \left|[\min f^{-1}(j)]_E\right| = ((|C| - 1) + 1 - 0) \cdot \prod_{j \neq i} \left|[\min f^{-1}(j)]_E\right| .$$

On the other hand, if $\min f^{-1}(i) <_P x_0$ then $f \in \mathcal{O}^*_\alpha(P', E')$. The contribution of f on both sides of (24) is

$$\prod_{j=1}^{\ell} \left|[\min f^{-1}(j)]_E\right| = (1 + 1 - 1) \cdot \prod_{j=1}^{\ell} \left|[\min f^{-1}(j)]_E\right| .$$

To finish the proof we need to show that the contributions of surjective maps $f : P \to [\ell]$ of type α that do not lie in $\mathcal{O}^*_\alpha(P, E)$ on the right hand side of (24) cancel.

Let
$$\mathcal{O}^+_\alpha(P', E') \coloneqq \{f \in \mathcal{O}^*_\alpha(P', E') : f(x_0) < f(x_k)\},$$
and similarly
$$\mathcal{O}^-_\alpha(P', E') \coloneqq \{f \in \mathcal{O}^*_\alpha(P', E') : f(x_0) > f(x_k)\}.$$

If $f \in \mathcal{O}^*_\alpha(P, E')$ then
$$f \notin \mathcal{O}^*_\alpha(P, E) \quad \Longleftrightarrow \quad f(x_{k-1}) < f(x_k).$$
In this case $f \in \mathcal{O}^+_\alpha(P', E')$, and the terms corresponding to f cancel.

If $f \in \mathcal{O}^*_\alpha(P, E'')$ then
$$f \notin \mathcal{O}^*_\alpha(P, E) \quad \Longleftrightarrow \quad f(x_0) < f(x_1).$$
If this is the case define $f' : P \to [\ell]$ by $f'(x_0) = f(x_k)$, $f'(x_k) = f(x_0)$ and $f'(x) = f(x)$ for all $x \in P \setminus \{x_0, x_k\}$. It is not difficult to see that $f' \in \mathcal{O}^-_\alpha(P', E')$, and that the terms coming from f and f' cancel.

Finally, if $f \in \mathcal{O}^*_\alpha(P', E')$ then
$$f \notin \mathcal{O}_\alpha(P, E) \quad \Longleftrightarrow \quad f(x_0) \neq f(x_k).$$
Hence we are in one of the cases above, and the proof is complete. ■

An immediate consequence of Lemma 6.1 and Theorem 6.2 is the following result.

Corollary 6.4. *Let (P, w, E) be a partitioned poset. Then $K_{P,E}$ is Ψ-positive.*

Note that Theorem 6.2 does not apply unless E is a chain congruence. Furthermore, Theorem 6.3 requires that the weight d is indeed a vector of *positive* integers. For example, K^d_P given by the poset $x < y < z$ with weights $d_x = 1$, $d_y = 0$ and $d_z = 2$ is not Ψ-positive.

7. Applications

In this section we apply Theorems 5.4 and 6.3 to derive the expansions of various families of quasisymmetric functions into quasisymmetric power sums.

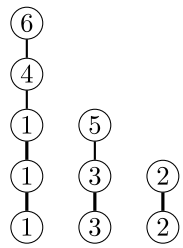
7.1. **Complete homogeneous symmetric functions.** As a warm up we now derive the expansion of the complete homogeneous symmetric function h_λ into power sum symmetric functions.

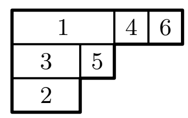
Let P_λ be the poset consisting of ℓ disjoint chains of lengths λ_i. Then $\mathrm{h}_\lambda = K_{P_\lambda}$ and
$$\mathrm{h}_\lambda(\mathbf{x}) = \sum_{\mu \vdash n} \frac{\mathrm{p}_\mu(\mathbf{x})}{z_\mu} |\mathcal{O}^*_\mu(P_\lambda)|.$$
by Theorem 5.4.

Another interpretation of these coefficients is given by so called ordered μ-bricks, see [ER91]. An *ordered μ-brick tabloid* of shape λ is a Young diagram of shape λ

filled with labeled *bricks* of sizes given by μ. The bricks are placed in the diagram such that the bricks in each row are sorted with increasing label. Let $OB_{\mu\lambda}$ be the set of such ordered μ-brick tabloids of shape λ.

It is easy to see that $\mathcal{O}^*_\mu(P_\lambda)$ is in bijection with $OB_{\mu\lambda}$. For instance, let $\lambda = 532$ and $\mu = 322111$. Then $\mathcal{O}^*_\mu(P_\lambda)$ contains the following order-preserving surjection, which is in natural correspondence with the shown ordered brick tabloid.





7.2. **Power sum symmetric functions.** We have already stated in (12) the expansion of power sum symmetric functions p_λ into quasisymmetric power sums Ψ_α. We now give an independent proof of this result using the tools we have developed.

Let λ be a partition of n with ℓ parts, and let A denote the antichain with elements $1, \ldots, \ell$. Then λ defines a weight on A. By Theorem 6.3 we have

$$\mathrm{p}_\lambda(\mathbf{x}) = K^\lambda_A(\mathbf{x}) = \sum_{\sigma \in \mathfrak{S}_\ell} \frac{\Psi_{\sigma(\lambda)}(\mathbf{x})}{z_{\sigma(\lambda)}} \prod_{i=1}^{\ell} \lambda_i = z_\lambda \sum_{\alpha \sim \lambda} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} = \sum_{\alpha \sim \lambda} \Psi_\alpha(\mathbf{x})\,.$$

Here $\sigma(\lambda)$ denotes the composition $(\lambda_{\sigma^{-1}(1)}, \ldots, \lambda_{\sigma^{-1}(\ell)})$, and the last two sums are taken over all compositions α whose parts rearrange to λ.

7.3. **Schur functions.** While Schur functions are not p-positive, as is evident from the famous Murnaghan–Nakayama rule, we still obtain an expansion of Schur functions into quasisymmetric power sums from the results in Section 3. The well-known formula below expresses the Schur functions in the fundamental quasisymmetric functions (see for example [Sta01, Thm. 7.19.7]):

$$\mathrm{s}_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SYT}(\lambda)} \mathrm{F}_{n,\mathrm{DES}(T)}(\mathbf{x}). \tag{25}$$

The set $\mathrm{SYT}(\lambda)$ is the set of standard Young tableaux of shape $\lambda \vdash n$, and the descent set of such a standard Young tableau is the set of entries i such that $i+1$ appears in a row with a higher index. The following result due to Y. Roichman is an immediate consequence of (25) and Corollary 3.4.

Theorem 7.1 ([Roi97, Thm. 4])**.** *Let λ be a partition of n. Then the expansion of the Schur function s_λ into power sum symmetric functions is given by*

$$\mathrm{s}_\lambda(\mathbf{x}) = \sum_{\mu \vdash n} \frac{\mathrm{p}_\mu(\mathbf{x})}{z_\mu} \sum_{\substack{T \in \mathrm{SYT}(\lambda) \\ \mathrm{DES}(T) \in U_\mu}} (-1)^{\mathrm{DES}(T) \setminus S_\mu}\,. \tag{26}$$

Example 7.2. Let $\lambda = (3,3)$. Then $\mathrm{SYT}(\lambda)$ is the following set of standard Young tableaux, where the descents have been marked bold.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & \mathbf{3} \\ \hline 4 & 5 & 6 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & \mathbf{2} & \mathbf{4} \\ \hline 3 & 5 & 6 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline \mathbf{1} & 3 & \mathbf{4} \\ \hline 2 & 5 & 6 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & \mathbf{2} & \mathbf{5} \\ \hline 3 & 4 & 6 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline \mathbf{1} & \mathbf{3} & \mathbf{5} \\ \hline 2 & 4 & 6 \\ \hline \end{array}$$

If $\mu = (2,2,2)$ then $S_\mu = \{2,4\}$. We can check that the descent sets of all five tableaux are μ-unimodal, and contribute the signs -1, $+1$, -1, -1 and -1 respectively. Furthermore, U_μ is the set of all subsets of $[5]$. The formula in (26) then gives

$$\mathrm{s}_{33}(\mathbf{x}) = \cdots + (-3)\frac{\mathrm{p}_{222}(\mathbf{x})}{z_{222}} + \cdots .$$

In contrast, the Murnaghan–Nakayama rule — given as a sum over so called rim-hook tableaux — is cancellation-free for this choice of λ and μ, and is given as a sum over exactly three rim-hook tableaux.

□

7.4. **Chromatic quasisymmetric functions.** In 1995, R. Stanley introduced a symmetric function generalization of the chromatic polynomal, [Sta95]. This definition was later refined by J. Shareshian and M. Wachs in [SW16], where a q-parameter was introduced.

Definition 7.3. Let G be a directed graph (no loops, but multiple edges are allowed) on the vertex set $[n]$. A *coloring* of G is an assignment of colors in $\mathbb{N}^+$ to the vertices. A coloring is *proper* if vertices connected by an edge are assigned different colors. An *ascent*[4] of a coloring κ is a directed edge (i,j) of G such that $\kappa(i) < \kappa(j)$. The number of ascents of a coloring is denoted $\mathrm{asc}(\kappa)$.

The *chromatic quasisymmetric function* of G is defined in [SW16][5] as

$$\mathrm{X}_G(\mathbf{x};q) = \sum_{\substack{\kappa:G\to\mathbb{N}^+ \\ \kappa \text{ proper}}} \mathbf{x}_{\kappa(1)} \cdots \mathbf{x}_{\kappa(n)} q^{\mathrm{asc}(\kappa)}. \qquad \square$$

When $q = 1$ we obtain the *chromatic symmetric function* $\mathrm{X}_G(\mathbf{x})$ in [Sta95]. The function $\mathrm{X}_G(\mathbf{x})$ is a symmetric function, and does not depend on the orientation of G. R. Stanley proves that $\omega\mathrm{X}_G(\mathbf{x})$ is p-positive for any (undirected) graph G.

For some choices of directed graphs G, $\mathrm{X}_G(\mathbf{x};q)$ is a symmetric function. A class of such graphs is characterized by B. Ellzey in [Ell16, Thm. 5.5]. In particular, if G is the incomparability graph of a $3+1$ and $2+2$-avoiding poset (together with a certain associated orientation), then the function $\mathrm{X}_G(\mathbf{x};q)$ is symmetric — these graphs are referred to as *unit-interval graphs.*

B. Ellzey also proves that $\mathrm{X}_G(\mathbf{x};q)$ has the property that $\omega\mathrm{X}_G(\mathbf{x};q)$ expands positively in the power-sum symmetric functions whenever it is symmetric. The only part of her proof that requires the restriction to symmetric functions is the application of Corollary 3.4. We can now prove the following generalization of the main result in [Ell16, Thm. 4.1]:

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[4]In the case of multiple edges (i,j), the contribution to asc is the multiplicity.

[5] The definition in [SW16] is slightly less general, and uses the acyclic orientation of the edges determined by the labeling of the graph. The more general definition was introduced in [Ell16].

Theorem 7.4. *Let G be a directed graph and consider the expansion into quasisymmetric power sums*

$$\omega \mathrm{X}_G(\mathbf{x}; q) = \sum_{\alpha} c_\alpha^G(q) \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha}. \tag{27}$$

Then $c_\alpha^G(q) \in \mathbb{N}[q]$ for all compositions α.

Proof. Let $AO(G)$ denote the set of acyclic orientations of the graph G viewed as an undirected graph. For an orientation θ, we let $\operatorname{asc}(\theta)$ be the number of edges oriented in the same direction as in G. It is straightforward to prove (see [AP17, Ell16]) that

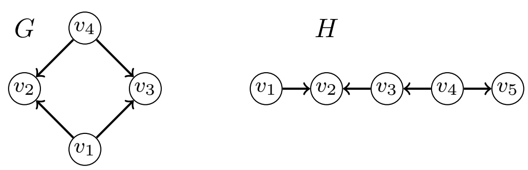
$$\mathrm{X}_G(\mathbf{x}; q) = \sum_{\theta \in AO(G)} q^{\operatorname{asc}(\theta)} K_{P(\theta), w}(\mathbf{x})$$

where $P(\theta)$ is the poset obtained from θ by taking the transitive closure of the directed edges, and $w = w(\theta)$ is order-reversing. The statement now follows from Corollary 4.5. ∎

Note that different types of combinatorial interpretations for the coefficients $c_\alpha^G(q)$ in (7.4) are known in certain special cases, see [Ath15, Ell16]. Our approach yields a combinatorial interpretation in the more general setting, which is similar to [Ell16].

It was conjectured in [SW16, Conj. 7.6] that the coefficients $c_\alpha^G(q)$ are unimodal for unit-interval graphs. This conjecture is still open. However, this conjecture does not extend to the general quasisymmetric setting.

Example 7.5. Consider the following directed graphs G and H.



Then

$$\begin{aligned}
\omega \mathrm{X}_G(\mathbf{x}; q) = {} & (4q + 4q^2 + 4q^3) \frac{\Psi_4}{z_4} + (2 + 4q + 4q^3 + 2q^4) \frac{\Psi_{13}}{z_{13}} \\
& + (4q + 8q^2 + 4q^3) \frac{\Psi_{22}}{z_{22}} + (4q + 4q^2 + 4q^3) \frac{\Psi_{31}}{z_{31}} \\
& + (4q + 8q^2 + 4q^3) \frac{\Psi_{112}}{z_{112}} + (4 + 4q + 4q^3 + 4q^4) \frac{\Psi_{121}}{z_{121}} \\
& + (4q + 8q^2 + 4q^3) \frac{\Psi_{211}}{z_{211}} + (4 + 4q + 8q^2 + 4q^3 + 4q^4) \frac{\Psi_{1111}}{z_{1111}}.
\end{aligned}$$

Note that the coefficient of Ψ_{121} is not unimodal.

For the graph H we have

$$\omega \mathrm{X}_H(\mathbf{x}; q) = \cdots + (2 + 5q + 4q^2 + 5q^3 + 2q^4) \frac{\Psi_{131}}{z_{131}} + \cdots,$$

where the coefficient of Ψ_{131} is not unimodal. □

7.5. **k-balanced chromatic quasisymmetric functions.** B. Humpert introduces another quasisymmetric generalization of chromatic symmetric functions in [Hum11].

Definition 7.6. Let G be an oriented graph (no loops or multiple edges) on the vertex set $[n]$ and let $k \in \mathbb{N}^+$. An orientation θ of G is said to be *k-balanced* if for every undirected cycle in G, walking along the cycle one traverses at least k edges forward, and at least k edges backwards. Thus an orientation is acyclic if and only if it is 1-balanced.

A proper coloring κ of G induces an acyclic orientation $\theta(\kappa)$ of G by orienting edges towards the vertex with larger color.

The *k-balanced chromatic quasisymmetric function* ([Hum11] defined this only for $q = 1$) is defined as

$$\mathrm{X}^k_G(\mathbf{x}; q) = \sum_{\substack{\kappa: G \to \mathbb{N}^+ \\ \kappa \text{ proper} \\ \theta(\kappa) \text{ is } k\text{-balanced}}} \mathbf{x}_{\kappa(1)} \cdots \mathbf{x}_{\kappa(n)} q^{\operatorname{asc}(\kappa)}. \qquad \square$$

Note that for $k = 1$ we recover the quasisymmetric function in Definition 7.3 as $\mathrm{X}^1_G(\mathbf{x}; q) = \mathrm{X}_G(\mathbf{x}; q)$.

Proposition 7.7 ([Hum11, Thm. 3.4])**.** *The k-balanced chromatic quasisymmetric function of an oriented graph G has the expansion*

$$\mathrm{X}^k_G(\mathbf{x}; q) = \sum_{\substack{\theta \in O(G) \\ \theta \text{ is } k\text{-balanced}}} q^{\operatorname{asc}(\theta)} K_{P(\theta),w}(\mathbf{x})$$

where the sum taken is over all k-balanced orientations of G, $P(\theta)$ is the transitive closure of the directed edges and $w = w(\theta)$ is order-reversing.

From Proposition 7.7 and Corollary 4.5 we obtain the following consequence.

Corollary 7.8. *Let G be any oriented graph and consider the expansion into quasisymmetric power sums*

$$\omega \mathrm{X}^k_G(\mathbf{x}; q) = \sum_{\alpha} c^G_\alpha(q) \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha}.$$

Then $c^G_\alpha(q) \in \mathbb{N}[q]$ for all compositions α.

7.6. **LLT polynomials.** The LLT polynomials were introduced by A. Lascoux, B. Leclerc and J.-Y. Thibon in [LLT97] using *ribbon tableaux*. The LLT polynomials can be seen as q-deformations of products of Schur functions and there are several open problems regarding LLT polynomials. A different combinatorial model for the LLT polynomials was considered in [HHL05], where each k-tuple of skew shapes index an LLT polynomial. When each such skew shape is a skew Young diagram with a single box, we say that the LLT polynomial is *unicellular*. The unicellular LLT polynomials have a central role in the work of E. Carlsson and A. Mellit [CM17], in which they introduced a combinatorial model for the unicellular LLT polynomials using Dyck paths. In [AP17] this Dyck path model was extended to certain directed graphs.

By modifying the definition of the chromatic symmetric functions slightly, we recover the unicellular LLT polynomials considered in [AP17]:

Definition 7.9. Let G be a directed graph (no loops, but multiple edges are allowed) on the vertex set $[n]$. The *unicellular graph LLT polynomial* is defined as

$$\mathrm{G}_G(\mathbf{x};q) = \sum_{\kappa:G\to\mathbb{N}^+} \mathbf{x}_{\kappa(1)}\cdots\mathbf{x}_{\kappa(n)} q^{\mathrm{asc}(\kappa)}.$$

Note that we now sum over all colorings. □

The $\mathrm{G}_G(\mathbf{x};q)$ are in general only quasisymmetric, but for certain choices of G (the same choices as for the chromatic quasisymmetric functions) they turn out to be symmetric and contain the family of unicellular LLT polynomials, see [AP17].

It was observed in [AP17, HW17] that $\omega\mathrm{G}_G(\mathbf{x};q+1)$ is p-positive whenever $\mathrm{G}_G(\mathbf{x};q)$ is a unicellular LLT polynomial. We can now give a proof of the following much stronger statement.

Theorem 7.10. *Let G be a directed graph and consider the expansion into quasisymmetric power sums*

$$\omega\mathrm{G}_G(\mathbf{x};q+1) = \sum_{\alpha} c^G_\alpha(q)\frac{\Psi_\alpha(\mathbf{x})}{z_\alpha}.$$

Then $c^G_\alpha(q) \in \mathbb{N}[q]$ for all compositions α.

Proof. Let $O(G)$ denote the set of orientations of the graph G viewed as an undirected graph. For $\theta \in O(G)$ we let $\mathrm{asc}(\theta)$ be the number of edges oriented in the same direction as in G. Similar as in the proof of Theorem 7.4, we have

$$\mathrm{G}_G(\mathbf{x};q+1) = \sum_{\theta\in O(G)} q^{\mathrm{asc}(\theta)} K_{P(\theta),w}(\mathbf{x})$$

where $P(\theta)$ is the transitive closure of *only* the edges of θ oriented in the same manner as in G and $w = w(\theta)$ is order-reversing. Note that we let $K_{P(\theta),w}(\mathbf{x}) := 0$ when $P(\theta)$ has a cycle — this can only happen if G has a directed cycle.

Again the result follows from Corollary 4.5. ■

We can enlarge the family of unicellular graph LLT polynomials.

Definition 7.11. Let G be a directed graph on the vertex set $[n]$ and let S be a subset of the edges of G. The *vertical strip graph LLT polynomial* is defined as

$$\mathrm{G}_{G,S}(\mathbf{x};q) = \sum_{\substack{\kappa:G\to\mathbb{N}^+\\ (i,j)\in S\Rightarrow\kappa(i)<\kappa(j)}} \mathbf{x}_{\kappa(1)}\cdots\mathbf{x}_{\kappa(n)} q^{\mathrm{asc}(\kappa)-|S|},$$

where we sum over all colorings such that $\kappa(i) < \kappa(j)$ whenever (i,j) is a (directed) edge in S. □

The name "vertical strip graph LLT polynomials" is motivated as follows. For some choices of G and S we recover the family of LLT polynomials that are, in the model introduced in [HHL05], indexed by k-tuples of vertical strips. Vertical strip LLT polynomials occur naturally in the study of the delta operator and diagonal harmonics. The family of vertical strip LLT polynomials contains (a version of) modified Hall–Littlewood polynomials. See [AP17] for an explicit construction of the correspondence between the above model and the model in [HHL05].

Theorem 7.12. *Let G be a directed graph, S a subset of the edges of G, and let*

$$\omega \mathrm{G}_{G,S}(\mathbf{x}; q+1) = \sum_{\alpha} c^{G,S}_{\alpha}(q) \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} \tag{28}$$

be the expansion into quasisymmetric power sums. Then $c^G_\alpha(q) \in \mathbb{N}[q]$ for all compositions α.

Proof. The same technique as above (also in [AP17]) shows that

$$\mathrm{G}_{G,S}(\mathbf{x}; q+1) = \sum_{\theta \in O_S(G)} q^{\mathrm{asc}(\theta) - |S|} K_{P(\theta),w}(\mathbf{x})$$

where $O_S(G)$ is now the subset of orientations of G such that edges in S are oriented as in G. ■

A special case of Theorem 7.12 was proved in [AP17].

It is conjectured by P. Alexandersson and G. Panova in [AP17] that the coefficients $c^G_\alpha(q)$ in Theorem 7.10 are unimodal whenever G is a unit interval graph. Computer experiments suggests that this conjecture extends to the more general setting in Theorem 7.10.

Conjecture 7.13. *Let G be an oriented graph (no loops or multiple edges). Then the coefficients $c^G_\alpha(q) \in \mathbb{N}[q]$ in the expansion*

$$\omega \mathrm{G}_{G}(\mathbf{x}; q+1) = \sum_{\alpha} c^{G}_{\alpha}(q) \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha}$$

are unimodal for all compositions α.

Conjecture 7.13 has been verified for all oriented graphs with six or fewer vertices.[6] In contrast, we note that the coefficients $c^{G,S}_\alpha(q)$ in (28) are *not* unimodal in general. For example,

$$G = \{1 \to 2, 1 \to 3, 2 \to 4, 2 \to 5\}, \qquad S = \{1 \to 2, 1 \to 3\}$$

gives

$$\mathrm{G}_{G,S}(\mathbf{x}; q+1) = \cdots + (1+q^2)\frac{\Psi_{12}}{z_{12}} + \cdots .$$

It is possible to refine Theorems 7.4, 7.10 and 7.12 by assigning a different q-weight to each edge of G, so that for a coloring κ we let

$$\mathbf{q}^{\mathrm{asc}(\kappa)} := \prod_{\substack{(i,j) \in E(G) \\ \kappa(i) < \kappa(j)}} q_{i,j} .$$

The resulting functions are again quasisymmetric and the analogues of the above theorems can be proved in the same manner. We leave out the details.

[6] See [Slo16, A001174] for the number of such graphs.

7.7. **Tutte quasisymmetric functions.** In [Sta98, Definition 3.1] R. Stanley defines the *multivariate Tutte polynomial* of a graph G with vertex set $[n]$ as

$$\mathrm{Tutte}_G(\mathbf{x}; q) \coloneqq \sum_{\kappa: G \to \mathbb{N}^+} \mathbf{x}_{\kappa(1)} \cdots \mathbf{x}_{\kappa(n)} (1+q)^{m(\kappa)},$$

where the sum ranges over all vertex colorings of G and $m(\kappa)$ denotes the number of monochromatic edges — edges $\{i, j\}$ such that $\kappa(i) = \kappa(j)$. It is evident that this is a symmetric function and it is straightforward to prove (see [Sta98]) that

$$\mathrm{Tutte}_G(\mathbf{x}; q) = \sum_{S \subseteq E(G)} q^{|S|} \mathrm{p}_{\lambda(S)}(\mathbf{x}) \tag{29}$$

where the sum ranges over all subsets of the edges of G, and $\lambda(S)$ is the partition whose parts are the sizes of the connected components of the subgraph of G spanned by the edges in S.

J. Awan and O. Bernardi [AB16] define a quasisymmetric generalization of the Tutte polynomial, which they call the B-polynomial.

Definition 7.14. Let G be a directed graph on the vertices $[n]$. Let

$$B_G(\mathbf{x}; y, z) \coloneqq \sum_{\kappa: G \to \mathbb{N}^+} \mathbf{x}_{\kappa(1)} \cdots \mathbf{x}_{\kappa(n)} y^{\mathrm{asc}(\kappa)} z^{\mathrm{inv}(\kappa)} \tag{30}$$

where $\mathrm{asc}(\kappa)$ and $\mathrm{inv}(\kappa)$ are defined as

$$\mathrm{asc}(\kappa) = |\{(i,j) : \kappa(i) < \kappa(j)\}| \qquad \text{and} \qquad \mathrm{inv}(\kappa) = |\{(i,j) : \kappa(i) > \kappa(j)\}|. \quad \square$$

Notice that the chromatic quasisymmetric function can be obtained as $\mathrm{X}_G(\mathbf{x}; q) = [z^n] B_G(\mathbf{x}; qz, z)$, and that the unicellular graph LLT polynomials can be obtained as $\mathrm{G}_G(\mathbf{x}; q) = B_G(\mathbf{x}; q, 0)$. Furthermore, for any directed graph G, the Tutte polynomials satisfies the relationship

$$y^{|E(G)|} \mathrm{Tutte}_{\underline{G}}(\mathbf{x}; \tfrac{1}{y} - 1) = B_G(\mathbf{x}; y, y).$$

where $\underline{G}$ denotes the undirected version of G.

Theorem 7.15. *Let G be a directed graph and consider the expansion*

$$\omega B_G(\mathbf{x}; y+1, z+1) = \sum_{\alpha} c^G_\alpha(y, z) \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha}.$$

Then $c^G_\alpha(y, z) \in \mathbb{N}[y, z]$ for all compositions α.

Proof. Let $E(G)$ be the set of directed edges of G. Then we have that

$$B_G(\mathbf{x}; y+1, z+1) = \sum_{\substack{A, I \subseteq E(G) \\ A \cap I = \emptyset}} y^{|A|} z^{|I|} K_{P(A,I),w}(\mathbf{x})$$

where $P(A, I)$ is the transitive closure of the directed edges

$$A \cup \{(j, i) : (i, j) \in I\} \tag{31}$$

and w is an order-reversing labeling of $P(A, I)$. Here we let $K_{P(A,I),w} \coloneqq 0$ if some edges in (31) form a cycle. By Corollary 4.5, the statement follows. ■

7.8. **Matroid quasisymmetric functions.** In 2009 L. Billera, N. Jia and V. Reiner introduced a quasisymmetric function associated to matroids as a new matroid invariant, see [BJR09]. The definition is as follows:

Definition 7.16. Let M be a matroid with ground set E and bases $\mathcal{B}(M)$. A map $f : E \to \mathbb{N}^+$ is said to be M-*generic* if the sum $f(B) \coloneqq \sum_{e\in B} f(e)$ is minimized by a unique $B \in \mathcal{B}(M)$. An M-generic function f must also be injective.

The *matroid quasisymmetric function* is then defined as

$$F(M,\mathbf{x}) \coloneqq \sum_{f\ M\text{-generic}} \prod_{e\in E} \mathbf{x}_{f(e)}. \qquad \square$$

In [BJR09] it is proved that $F(M,\mathbf{x})$ is indeed a quasisymmetric function.

Let $M = (E,\mathcal{B}(M))$ be a matroid. Given a basis $B \in \mathcal{B}(M)$ let $B^* \coloneqq E \setminus B$ and define the poset P_B on the vertex set $E = B \sqcup B^*$ such that

$$e \prec e' \text{ if and only if } e \in B \text{ and } (B \setminus \{e\}) \cup \{e'\} \text{ is in } \mathcal{B}(M).$$

Theorem 7.17 ([BJR09, Thm. 5.2])**.** *Let $M = (E,\mathcal{B}(M))$ be a matroid. Then*

$$F(M,\mathbf{x}) = \sum_{B\in\mathcal{B}(M)} K_{P_B,w}(\mathbf{x})$$

where w is any order-reversing labeling of P_B.

Using Corollary 4.5 we get the following corollary.

Corollary 7.18. *Let M be a matroid. Then $\omega F(M,\mathbf{x})$ is Ψ-positive.*

We compute the Ψ-expansion of the matroid quasisymmetric function $F(M,\mathbf{x})$ explicitly in the case where M is the uniform matroid.

Example 7.19 (Uniform matroid)**.** The uniform matroid $U = U_n^r$ has ground set $E = [n]$ and every r-element subset of E constitutes a basis. That is, $\mathcal{B}(U) = \binom{[n]}{r}$. In this case the poset P_B is given by $e \prec e'$ for all $e \in B$ and $e' \in B^*$. In particular all posets P_B for $B \in \mathcal{B}(U)$ are isomorphic. The Hasse diagram of P_B is the complete bipartite graph $K_{r,m}$, where $m = n - r$.

Fix a basis $B \in \mathcal{B}(U)$ and a natural labeling w of P_B. It follows from Theorem 7.17, Theorem 5.4 and Example 5.6 that

$$\begin{aligned}
\omega F(U,\mathbf{x}) &= |\mathcal{B}(U)| \sum_{\alpha\vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} |\mathcal{O}^*_\alpha(P_B)| \\
&= \binom{n}{r} \sum_{k=0}^{m} \frac{\Psi_{(1^{r-1},k+1,1^{m-k})}(\mathbf{x})}{(k+1)(n-k-1)!} \cdot \frac{r!m!}{k!} \\
&= \sum_{k=0}^{m} \binom{n}{k+1} \Psi_{(1^{r-1},k+1,1^{m-k})}(\mathbf{x}) . \qquad \square
\end{aligned}$$

7.9. **Eulerian quasisymmetric functions.** The aim of this section is to explain how the tools developed in this paper can be used to prove known p-expansions of the Eulerian quasisymmetric functions and the cycle Eulerian quasisymmetric functions of J. Shareshian and M. Wachs.

The *Eulerian polynomials* are defined as the descent generating functions of the symmetric group

$$A_n(q) \coloneqq \sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{des}(\sigma)}\,,$$

where $\operatorname{des}(\sigma) \coloneqq |\operatorname{DES}(\sigma)|$. By convention $A_0(q) \coloneqq 1$.

In [SW10] J. Shareshian and M. Wachs generalize the classical identity for the exponential generating function of Eulerian polynomials

$$\sum_{n \geq 0} A_n(q) \frac{z^n}{n!} = \frac{1-q}{\exp(z(1-q)) - q}\,, \tag{32}$$

and introduced Eulerian quasisymmetric functions.

Let $\sigma \in \mathfrak{S}_n$ be a permutation. Define the set of *exceedences* of σ as

$$\operatorname{EXC}(\sigma) \coloneqq \{i \in [n-1] : \sigma_i > i\}\,,$$

and set $\operatorname{exc}(\sigma) \coloneqq |\operatorname{EXC}(\sigma)|$. For $n \in \mathbb{N}$ let $[\bar{n}] \coloneqq \{\bar{1}, \dots, \bar{n}\}$ be a disjoint copy of the set $[n]$. Define a total order on the alphabet $[\bar{n}] \cup [n]$ by

$$\bar{1} < \cdots < \bar{n} < 1 < \cdots < n. \tag{33}$$

Given a permutation $\sigma = \sigma_1 \cdots \sigma_n$, define the word $\bar{\sigma}$ in the alphabet $[\bar{n}] \cup [n]$ by replacing σ_i with $\bar{\sigma_i}$ whenever $i \in \operatorname{EXC}(\sigma)$ is an exceedence. Let

$$\operatorname{DEX}(\sigma) \coloneqq \operatorname{DES}(\bar{\sigma}),$$

where descents of $\bar{\sigma}$ are computed with respect to the order in (33). For example, let $\sigma = 613542$. Then $\operatorname{EXC}(\sigma) = \{1, 4\}$, $\bar{\sigma} = \bar{6}13\bar{5}42$ and $\operatorname{DEX}(\sigma) = \{3, 5\}$.

The *Eulerian quasisymmetric functions* are defined as

$$Q_{n,j}(\mathbf{x}) \coloneqq \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{exc}(\sigma) = j}} \mathrm{F}_{n,\operatorname{DEX}(\sigma)}(\mathbf{x})\,.$$

By definition $Q_{n,j}$ is quasisymmetric. It turns out that $Q_{n,j}$ is in fact symmetric.

In [SW10, Thm. 1.2] Eulerian quasisymmetric functions are shown to have the following generating function that specializes to (32).

$$\sum_{n,j \geq 1} Q_{n,j}(\mathbf{x}) q^j z^n = \frac{(1-q)H(\mathbf{x}; z)}{H(\mathbf{x}; qz) - qH(\mathbf{x}; z)} \tag{34}$$

Here $H(\mathbf{x}; z) \coloneqq \sum_{n \geq 0} \mathrm{h}_n(\mathbf{x}) z^n$ denotes the generating function of complete homogeneous symmetric functions.

From (34) J. Shareshian and M. Wachs deduce the following expansion of Eulerian quasisymmetric functions into power sum symmetric functions.

Proposition 7.20 ([SW10, Prop. 6.6])**.** *Let $n \in \mathbb{N}$. Then*

$$\sum_{j=0}^{n-1} q^j Q_{n,j}(\mathbf{x}) = \sum_{\lambda \vdash n} \frac{\mathrm{p}_\lambda(\mathbf{x})}{z_\lambda} A_{\ell(\lambda)}(q) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_q \,,$$

where $[a]_q \coloneqq \frac{1-q^a}{1-q}$ denotes the usual q-integer.

We present an alternative proof of Proposition 7.20 using the theory of order-preserving surjections and an interpretation of Eulerian quasisymmetric functions as generating functions of banners, which was obtained in [SW10, Sec. 3.2]. Note that this also offers a different route to proving that $Q_{n,j}$ is symmetric and satisfies (34).

Let X and $\bar{X}$ denote disjoint copies of the positive integers, that is,

$$X \coloneqq \{1, 2, 3, \dots\} \qquad \text{and} \qquad \bar{X} \coloneqq \{\bar{1}, \bar{2}, \bar{3}, \dots\}\,. \tag{35}$$

Moreover define $|\cdot| : X \cup \bar{X} \to \mathbb{N}$ by $|a| = |\bar{a}| = a$. A *banner* of length n is a word $b = b_1 \cdots b_n$ in the alphabet $X \cup \bar{X}$ such that the following three conditions are satisfied:

(i) If $b_i \in \bar{X}$ then $|b_i| \geq |b_{i+1}|$ for all $i \in [n-1]$.
(ii) If $b_i \in X$ then $|b_i| \leq |b_{i+1}|$ for all $i \in [n-1]$.
(iii) We have $b_n \in X$.

Let $\mathfrak{B}_{n,j}$ denote the set of banners of length n that contain exactly j barred letters. Given a banner $b \in \mathfrak{B}_{n,j}$ define its *weight* as $\mathbf{x}^b \coloneqq \mathbf{x}_{|b_1|} \cdots \mathbf{x}_{|b_n|}$.

It can be shown [SW10, Thm. 3.6] that

$$Q_{n,j}(\mathbf{x}) = \sum_{b \in \mathfrak{B}_{n,j}} \mathbf{x}^b\,. \tag{36}$$

As was observed by R. Stanley (see [SW10, Thm. 7.2] and the remarks thereafter), it is an immediate consequence of (36) that Eulerian quasisymmetric functions are related to reverse P-partitions and chromatic symmetric functions.

For $S \subseteq [n-1]$ let $\mathfrak{B}_{n,S}$ denote the set of banners b such that $b_i \in \bar{X}$ if and only if $i \in S$. Then

$$\sum_{b \in \mathfrak{B}_{n,S}} \mathbf{x}^b = K_{P_S}(\mathbf{x})\,,$$

where P_S is the poset defined in Example 5.7. Consequently

$$\sum_{j=0}^{n-1} q^j Q_{n,j}(\mathbf{x}) = \sum_{S \subseteq [n-1]} q^{|S|} K_{P_S}(\mathbf{x}) = \omega X_G(\mathbf{x}; q)\,, \tag{37}$$

where G denotes the directed path of length $n-1$.

Proof of Proposition 7.20. Let $S \subseteq [n-1]$. By Theorem 5.4

$$K_{P_S}(\mathbf{x}) = \sum_{\alpha \vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} \left|\mathcal{O}^*_\alpha(P_S)\right|\,.$$

In combination with (37) and Example 5.7 we obtain

$$\begin{aligned}\sum_{j=0}^{n-1} q^j Q_{n,j}(\mathbf{x}) &= \sum_{\alpha \vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} \sum_{S \subseteq [n-1]} q^{|S|} \, |\mathcal{O}^*_\alpha(P_S)| \\ &= \sum_{\alpha \vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} A_{\ell(\alpha)}(q) \prod_{i=1}^{\ell(\alpha)} [\alpha_i]_q \,.\end{aligned}$$

The claim follows from (12). ■

Using a few tricks one can also deal with the more challenging *cycle Eulerian quasisymmetric functions* $Q_{(n),j}$, which are defined as

$$Q_{(n),j}(\mathbf{x}) \coloneqq \sum_{\substack{\sigma \in \mathfrak{S}_n \text{ is a long cycle} \\ \mathrm{exc}(\sigma)=j}} \mathrm{F}_{n,\mathrm{DEX}(\sigma)}(\mathbf{x}) \,.$$

The Eulerian quasisymmetric functions $Q_{n,j}$ can be expressed in terms of cycle Eulerian quasisymmetric functions $Q_{(n),j}$ (and vice versa) via plethysm. J. Shareshian and M. Wachs conjectured and later proved together with B. Sagan the following expansion into power sum symmetric functions.

Theorem 7.21 ([SW10, Conj. 6.5], [SSW11, Thm. 4.1]). *Let $n \in \mathbb{N}$. Then*

$$\sum_{j=0}^{n-1} q^j Q_{(n),j}(\mathbf{x}) = \sum_{\lambda \vdash n} \frac{\mathrm{p}_\lambda(\mathbf{x})}{z_\lambda} \sum_{d \mid \gcd(\lambda)} \mu(d) \, d^{\ell(\lambda)-1} \, q^d \, A_{\ell(\lambda)-1}(q^d) \prod_{i=1}^{\ell(\lambda)} \left[\frac{\lambda_i}{d} \right]_{q^d} , \quad (38)$$

where μ denotes the number theoretic Möbius function, and $\gcd(\alpha) \coloneqq \gcd(\alpha_1, \ldots, \alpha_\ell)$ *for all compositions α with ℓ parts.*

Note that our statement of Theorem 7.21 differs slightly from the given references. See [SSW11, Lem. 4.2] for a proof that the statements are equivalent.

In [SSW11] Theorem 7.21 is derived from Proposition 7.20 by the use of plethystic calculus and the manipulation of formal power series. We present a proof based on the theory of order-preserving surjections, Möbius inversion and an interpretation of the cycle Eulerian quasisymmetric functions as the generating functions of primitive necklaces due to J. Shareshian and M. Wachs. The application of Möbius inversion to problems related to primitive necklaces is very classical and dates back (at least) to [MR83]. In this way the Möbius function and the power q^d on the right hand side of (38) appear naturally.

Let $X, \bar{X}$ and $|\cdot|$ be as in (35). A *bicolored necklace* of length n is a *circular word* $o_1 \cdots o_n$ in the alphabet $X \cup \bar{X}$ that satisfies the three conditions below. Circular means that we do not distinguish between $o_1 \cdots o_n$ and $o_2 \cdots o_n o_1$.

(i) If $o_i \in X$ then $|o_i| \leq |o_{i+1}|$ for all $i \in [n]$, where indices are viewed modulo n.
(ii) If $o_i \in \bar{X}$ then $|o_i| \geq |o_{i+1}|$ for all $i \in [n]$, where indices are viewed modulo n.
(iii) If $n = 1$ then $o_1 \in X$.

A bicolored necklace o is *primitive* if the words $o_k \cdots o_{k+n-1}$ are distinct for all $k \in [n]$, where indices are once more viewed modulo n. Let $\mathfrak{N}_{(n),j}$ denote the set of primitive bicolored necklaces of length n that contain exactly j barred letters. For example, the primitive bicolored necklaces in $\mathfrak{N}_{(3),1}$ with letters $1, 2, \bar{1}, \bar{2}$ are

$$11\bar{1}, \quad 22\bar{2}, \quad 12\bar{2}, \quad 11\bar{2}.$$

Given a bicolored necklace $o_1 \cdots o_n$, define its *weight* as $\mathbf{x}^o \coloneqq \mathbf{x}_{|o_1|} \cdots \mathbf{x}_{|o_n|}$. In [SW10, Sec. 3.1] J. Shareshian and M. Wachs show that cycle Eulerian quasisymmetric functions are the generating functions of primitive bicolored necklaces.

$$Q_{(n),j}(\mathbf{x}) = \sum_{o \in \mathfrak{N}_{(n),j}} \mathbf{x}^o$$

The quasisymmetric functions $Q_{(n),j}$ are not positive linear combinations of reverse P-partition enumerators K_P for some simple set of posets P. Instead, let $\mathfrak{B}'_{n,j}$ be the set of words $b_1 \cdots b_n$ in the alphabet $X \cup \bar{X}$ that satisfy the following three properties:

(i) If $b_i \in X$ then $|b_i| \leq |b_{i+1}|$ for all $i \in [n]$, where indices are viewed modulo n.
(ii) If $b_i \in \bar{X}$ then $|b_i| \geq |b_{i+1}|$ for all $i \in [n]$, where indices are viewed modulo n.
(iii) The word b contains exactly j barred letters.

Denote the generating function of $\mathfrak{B}'_{n,j}$ by

$$F_{n,j}(\mathbf{x}) \coloneqq \sum_{b \in \mathfrak{B}'_{n,j}} \mathbf{x}^b \,.$$

Note that every word $b_1 \cdots b_n$ can be written uniquely as $(a_1 \cdots a_d)^{n/d}$ where d divides n and $a_1 \cdots a_d$ is a primitive word of length d. Moreover each primitive circular word $o_1 \cdots o_d$ gives rise to d distinct primitive words. Therefore the quasisymmetric functions $F_{n,j}$ are related to the cycle Eulerian quasisymmetric functions by

$$\sum_{j=0}^{n-1} q^j F_{n,j}(\mathbf{x}) = \sum_{d|n} \sum_{j=0}^{d-1} d q^{jn/d} Q_{(d),j}(\mathbf{x}^{n/d}) \,, \tag{39}$$

where $\mathbf{x}^k$ denotes the variables $\mathbf{x}_1^k, \mathbf{x}_2^k, \ldots$.

Contrary to the cycle Eulerian quasisymmetric functions $Q_{(n),j}$, the generating functions $F_{n,j}$ are immediately related to reverse P-partitions and chromatic quasisymmetric functions of cycles. More precisely,

$$\sum_{j=0}^{n-1} q^j F_{n,j}(\mathbf{x}) = \mathrm{p}_n(\mathbf{x}) + \sum_{\substack{S \subseteq [n] \\ 0<|S|<n}} q^{|S|} K_{P_S}(\mathbf{x}) = \mathrm{p}_n(\mathbf{x}) + \omega X_G(\mathbf{x}; q) \,, \tag{40}$$

where P_S is defined as in Example 5.8, and G denotes the directed cycle of length n.

Proposition 7.22 ([Ell16, Thm. 4.4])**.** *Let $n \in \mathbb{N}$. Then*

$$\sum_{j=0}^{n-1} q^j F_{n,j}(\mathbf{x}) = \mathrm{p}_n(\mathbf{x})[n]_q + \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \geq 2}} \frac{\mathrm{p}_\lambda(\mathbf{x})}{z_\lambda} \, nq \, A_{\ell(\lambda)-1}(q) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_q \,.$$

Proof. By (40) and Theorem 5.4 we obtain

$$\sum_{j=0}^{n-1} q^j F_{n,j}(\mathbf{x}) = \mathrm{p}_n(\mathbf{x}) + \sum_{\alpha \vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} \sum_{\substack{S \subseteq [n] \\ 0<|S|<n}} q^{|S|} \left|\mathcal{O}^*_\alpha(P_S)\right| . \tag{41}$$

By Example 5.8 the right hand side of (41) is equal to

$$\mathrm{p}_n(\mathbf{x})(1+q[n-1]_q) + \sum_{\substack{\alpha \vDash n \\ \ell(\alpha)\geq 2}} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha}\, nq\, A_{\ell(\alpha)-1}(q) \prod_{i=1}^{\ell(\alpha)} [\alpha_i]_q .$$

The claim follows from (12). ∎

Note that in particular Proposition 7.22 implies that $F_{n,j}(\mathbf{x})$ is symmetric.

Proof of Theorem 7.21. We can generalize (39) as follows: For all $k \in [n]$ with $k \mid n$ we have

$$\sum_{j=0}^{k-1} q^{jn/k} F_{k,j}(\mathbf{x}^{n/k}) = \sum_{d|k} \sum_{j=0}^{d-1} d q^{jn/d} Q_{(d),j}(\mathbf{x}^{n/d}) .$$

The Möbius inversion formula for the interval $[1, n]$ in the divisor lattice yields

$$\sum_{j=0}^{n-1} q^j Q_{(n),j}(\mathbf{x}) = \frac{1}{n} \sum_{d|n} \mu(d) \sum_{j=0}^{n/d-1} q^{dj} F_{n/d,j}(\mathbf{x}^d) . \tag{42}$$

Given a partition λ of length ℓ, let $d\lambda \coloneqq (d\lambda_1, \dots, d\lambda_\ell)$. By Proposition 7.22 the right hand side of (42) is equal to

$$\sum_{d|n} \frac{\mu(d)}{d} \left(\frac{\mathrm{p}_n(\mathbf{x})}{z_{(n/d)}} [n/d]_{q^d} + \sum_{\substack{\lambda \vdash n/d \\ \ell(\lambda)\geq 2}} \frac{\mathrm{p}_{d\lambda}(\mathbf{x})}{z_\lambda}\, q^d\, A_{\ell(\lambda)-1}(q^d) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_{q^d} \right)$$

$$= \sum_{\lambda \vdash n} \frac{\mathrm{p}_\lambda(\mathbf{x})}{z_\lambda} \sum_{d|\gcd(\lambda)} \mu(d)\, d^{\ell(\lambda)-1}\, B_{\ell(\lambda)-1}(q^d) \prod_{i=1}^{\ell(\lambda)} \left[\frac{\lambda_i}{d}\right]_{q^d} ,$$

where $B_k(q) \coloneqq qA_k(q)$ if $k > 0$, and $B_0(q) \coloneqq 1$. The claim follows from the amusing identity

$$\sum_{d|n} \mu(d)[n/d]_{q^d} = \sum_{d|n} \mu(d) q^d [n/d]_{q^d} .$$ ∎

8. Further directions

8.1. **Symmetry.** There is an almost trivial necessary and sufficient condition for when a linear combination of quasisymmetric power sums is a symmetric function that follows from (12).

Proposition 8.1. *Let X be a quasisymmetric function with*

$$X(\mathbf{x}) = \sum_\alpha c_\alpha \Psi_\alpha(\mathbf{x}).$$

Then X is symmetric if and only if $c_\alpha = c_\beta$ for all compositions α and β such that β can be obtained by permuting the parts of α.

Proposition 8.1 offers a method for proving (or disproving) that a given quasisymmetric function is symmetric. As mentioned at the end of Section 4, it would be particularly interesting if this idea was applicable to the generating functions $K_{P,w}$.

Furthermore by Proposition 8.1 any symmetric function for which the expansion into quasisymmetric power sums is known, immediately gives rise to a set of symmetries on its coefficients c_α. These symmetries might not at all be obvious from a purely combinatorial point of view. For example, consider the expansion of Schur functions into quasisymmetric power sums given in Section 7.3. It follows from the symmetry of Schur functions that the sum

$$\sum_{\substack{T \in \mathrm{SYT}(\lambda) \\ \mathrm{DES}(T) \in U_\alpha}} (-1)^{\mathrm{DES}(T) \setminus S_\alpha}$$

is invariant under the permutation of the parts of α. The authors are unaware of a proof of this fact that does not appeal to the theory of symmetric functions. We expect that many potentially interesting combinatorial problems can be obtained in similar fashion.

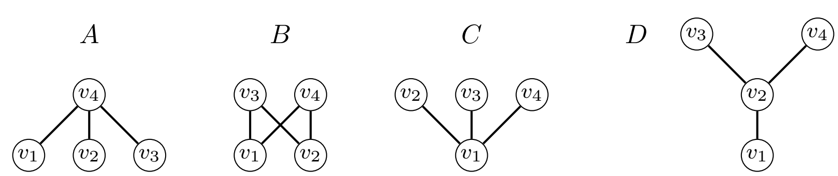
Lastly, if a combinatorial statistic c on compositions appears to satisfy the symmetry properties of Proposition 8.1, then it might be worth a try to investigate the quasisymmetric function

$$X_n(\mathbf{x}) \coloneqq \sum_{\alpha \vDash n} c(\alpha) \frac{\Psi_\alpha}{z_\alpha}.$$

Proving that X_n is symmetric, for example by deriving its expansion into the fundamental or monomial bases, will also prove the symmetry of c.

8.2. **Schur-positivity and h-positivity.** There are open problems regarding a combinatorial proof of the Schur-positivity of LLT polynomials [HHL05], as well as proving e-positivity of chromatic symmetric functions [SS93].

Since both families of polynomials are related to the $K_P(\mathbf{x})$, it is natural to ask if every symmetric positive linear combination of such functions is h-positive (or weaker, Schur-positive). This is not the case. A computer search gave us the following counterexamples:



For the above posets

$$\begin{aligned} 2K_A(\mathbf{x}) + 3K_B(\mathbf{x}) + 2K_C(\mathbf{x}) &= 7\mathrm{s}_4 + 7\mathrm{s}_{31} + \mathrm{s}_{22} + 2\mathrm{s}_{211} \\ &= 2h_4 + 4h_{31} - h_{22} + 2h_{1111} \end{aligned}$$

is Schur-positive but not h-positive, and

$$K_A(\mathbf{x}) + 3K_B(\mathbf{x}) + K_C(\mathbf{x}) + 3K_D(\mathbf{x}) = 8\mathrm{s}_4 + 5\mathrm{s}_{31} - \mathrm{s}_{22} + \mathrm{s}_{211}$$

is not even Schur-positive.

8.3. **Murnaghan–Nakayama-type formula.** There are several quasisymmetric analogues of Schur functions. Here is an incomplete list of functions that can be viewed as such:

- The *fundamental quasisymmetric functions*.
- The *quasisymmetric Schur functions*, $\mathcal{QS}_\alpha(\mathbf{x})$, introduced by J. Haglund, K. Luoto, S. Mason and S. van Willgienburg in [HLMvW11].
- The *row-strict Schur functions*, $\mathcal{RS}_\alpha(\mathbf{x})$ by S. Mason and J. Remmel which are related to the quasisymmetric Schur functions as $\mathcal{RS}_\alpha(\mathbf{x}) = \omega(\mathcal{QS}_\alpha(\mathbf{x}))$.
- The *Young quasisymmetric Schur functions*, which are also closely related to the two above quasisymmetric variants of Schur functions, see [LMvW13].
- The *dual immaculate quasisymmetric functions*, which expands positively into the Young quasisymmetric Schur functions, see [AHM18].
- The *extended Schur functions*, by S. Assaf and D. Searles, see [AS17]. This family includes the usual Schur functions.

All of the above families expand positively into the fundamental quasisymmetric functions, and are bases for the space of quasisymmetric functions. Using Theorem 3.1 one can give analogues of Roichman's formula, see Theorem 7.1, for these bases.

The classical Murnaghan–Nakayama rule [Mur37, Nak40] states that

$$\mathrm{p}_r(\mathbf{x})\,\mathrm{s}_\lambda(\mathbf{x}) = \sum_{\mu} (-1)^{\mathrm{ht}(\mu/\lambda)}\,\mathrm{s}_\mu(\mathbf{x})\,,$$

where the sum is over all μ such that μ/λ is a ribbon of size r. A natural future direction is then to seek quasisymmetric refinements or analogues of the Murnaghan–Nakayama rule. Let $\{X_\alpha\}$ be any of the above families of quaissymmetric functions indexed by partitions. The problem is then to find a rule that gives the coefficients $\chi^\gamma_{\alpha\beta}$ in the expansion

$$\Psi_\alpha(\mathbf{x})\,X_\beta(\mathbf{x}) = \sum_{\gamma} \chi^\gamma_{\alpha\beta}\,X_\gamma(\mathbf{x})\,.$$

Possible research in this direction is also discussed in [BDH$^+$17, Sec. 7.1].

8.4. **Poset invariants.** Whenever a class of combinatorial objects has a nontrivial isomorphism problem (such as posets, graphs or knots) it immediately becomes an interesting task to find invariants that might be used to distinguish such objects.

In [MW14] P. McNamara and R. Ward ask for a necessary and sufficient condition that two labeled posets (P, w) and (Q, w') have the same P-partition generating function, that is, $K_{P,w} = K_{Q,w'}$.

For naturally labeled posets Theorem 5.4 yields that $K_P = K_Q$ if and only if $|\mathcal{O}^*_\alpha(P)| = |\mathcal{O}^*_\alpha(Q)|$ for all compositions α. That is, the numbers of certain order-preserving surjections onto chains agree. Note that this includes (in the case of naturally labeled posets) the observation [MW14, Prop. 3.2] that $K_P = K_Q$ implies $|\mathcal{L}(P,w)| = |\mathcal{L}(Q,w')|$. The judgment whether our answer is more useful than the trivial answer, "$K_P = K_Q$ if and only if the multisets of descent sets $\{\mathrm{DES}(\sigma) : \sigma \in \mathcal{L}(P,w)\}$ and $\{\mathrm{DES}(\sigma) : \sigma \in \mathcal{L}(Q,w')\}$ agree", is left to the reader.

P. McNamara and R. Ward pose several other problems in this direction, many of which where recently solved for naturally labeled posets by R. Liu and M. Weselcouch [LW18]. It could be worth investigating whether the Ψ-expansion of K_P has applications in this regard.

Another open question that was first raised by R. Stanley in [Sta95, p. 170] is whether the chromatic symmetric function $\mathrm{X}_G(\mathbf{x})$ defined in Section 7.4 distinguish trees. This was investigated, for instance, by J. Martin, M. Morin and J. Wagner in [MMW08].

Let T be a rooted tree. We interpret T as a poset by declaring the edges to be cover relations, and the root to be the unique minimal element. It was recently shown by T. Hasebe and S. Tsujie that the generating function of reverse P-partitions distinguishes rooted trees [HT17]. The following is a straightforward consequence of this result.

Proposition 8.2. *The chromatic quasisymmetric function $\mathrm{X}_G(\mathbf{x};q)$ distinguishes rooted trees with all edges directed away from the root.*

Proof. Let T be a rooted tree on n vertices. It follows from the proof of Theorem 7.4 that the coefficient of q^{n-1} in $\mathrm{X}_T(\mathbf{x};q)$ is just $K_{T,w}$, where T is viewed as a poset as above and w is an order-reversing labeling. Thus the claim is a consequence of [HT17, Thm. 1.3]. ■

It is an open problem whether $\mathrm{X}_G(\mathbf{x};q)$ distinguishes all oriented trees.

Let P and Q be (disjoint) posets. Let $P + Q$ denote the *direct sum* of P and Q, that is, the partial order on $P \sqcup Q$ defined by $x < y$ if $x <_P y$ or $x <_Q y$. Let $P \oplus Q$ denote the *ordinal sum* of P and Q, that is, the partial order on $P \sqcup Q$ defined by $x < y$ if (i) $x <_P y$, or (ii) $x <_Q y$, or (iii) $x \in P$ and $y \in Q$. Clearly all rooted trees can be obtained by successively taking direct sums and adding a minimal element, that is, forming the poset $\{\hat{0}\} \oplus P$. A poset is called *series parallel* if it can be built from sigletons using only the two operations direct and ordinal sum.

A very natural question raised by T. Hasebe and S. Tsujie is whether K_P distinguishes series parallel posets. The ideas in this paper might offer a new angle to attack this problem, since it is not too difficult to compute the expansions of K_{P+Q} respectively $K_{P\oplus Q}$ into quasisymmetric power sums recursively using Theorem 5.4.

Another possible question is whether other constructions on posets have a simple interpretation in the basis of quasisymmetric power sums.

8.5. **Type B_n analogues.** There is a type B analogue of Theorem 7.1 given in [AAER17]. This setting uses Schur functions, power sum symmetric functions and fundamental quasisymmetric functions in two sets of variables $\mathbf{x}$ and $\mathbf{y}$. It is natural to ask if this result extends to an analogue of Theorem 3.1 that uses some kind of quasisymmetric power sums in two sets of variables.

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