arXiv:1807.03005v1 [math.CO] 9 Jul 2018

# Action Graphs, Rooted Planar Forests, and Self-Convolutions of the Catalan Numbers

Julia E. Bergner, Cedric Harper, Ryan Keller, and Mathilde Rosi-Marshall
Department of Mathematics
University of Virginia
Charlottesville, VA 22904
jeb2md@virginia.edu
cbh2ta@virginia.edu
rjk7bb@virginia.edu
mgr3km@virginia.edu

**Abstract**

We show that families of action graphs, with initial graphs which are linear of varying length, give rise to self-convolutions of the Catalan sequence. We prove this result via a comparison with planar rooted forests with a fixed number of trees.

## 1 Introduction

In a recent paper [1], the first-named author, Alvarez, and Lopez showed that a certain family of directed graphs, called *action graphs*, give a new way to produce the Catalan numbers. These graphs arose in work of the first-named author and Hackney, with the goal of understanding the structure of a rooted category action on another category [2]. However, in that work we considered a much larger collection of such graphs than those considered in the paper with Alvarez and Lopez. In this paper, we take up the question of what sequences arise from the other families of action graphs, and we prove that they produce self-convolutions of the Catalan sequence.

The Catalan sequences arises in many contexts in combinatorics A000108 [4]; a long list of ways to obtain the Catalan numbers is found in Stanley's books [6], [7] and online addendum [5]; see also Koshy's book [3]. Recall that the 0th Catalan number is $C_0 = 1$, and, for any $n \geq 1$, the $(n+1)$st Catalan number $C_{k+1}$ is given by the formula

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

Here, we are interested in convolutions of this sequence with itself. To this end, let us recall the definition of the convolution of sequences.

**Definition 1.1.** Given two sequences $A$ and $B$, their *convolution* is given by

$$(A * B)_n = \sum_{m=0}^{n} A_m B_{n-m}$$

The *self-convolution* of a sequence $A$ is the convolution $A * A$, which we denote by $A^1$. More generally, we let

$$A^k = \underbrace{A * \cdots * A}_{k+1},$$

and in particular $A^0$ is simply the original sequence $A$.

**Remark 1.2.** The notation we have chosen is arguably confusing, as one might prefer to have $A^1$ denote the sequence itself and $A^2$ the convolution $A * A$. However, we want to think of the index $k$ as telling us how many convolutions we have done, rather than how many copies of the sequence have been convolved. Furthermore, our indexing is convenient for the comparison we undertake here.

Observe that $C^1 = C * C$ is just a reindexing of the Catalan numbers (omitting the zero term and shifting the other terms down an index). Further convolutions are more interesting, and in particular $C^2$ and $C^3$ have appeared in other contexts A001003 and A002057 [4].

In Section 2, we define the general families of action graphs that we want to consider. Then in Section 3 we introduce planar rooted forests and prove a connection with self-convolutions of Catalan numbers. Finally, in Section 4 we give a bijection between certain sets of planar rooted forests and our general action graphs.

## 2 $k$-Extended action graphs

In this section, we recall the definition of action graph as it appeared in previous work [1] and generalize to the setting we wish to consider. To begin, let us recall the definition of a directed graph that we use here.

**Definition 2.1.** A *directed graph* is a pair $G = (V, E)$ where $V$ is a set whose elements are called *vertices* and $E$ is a set of ordered pairs of vertices, called *edges*. Given an edge $e = (v, w)$, we call $v$ the *source* of $e$, denoted by $v = s(e)$, and call $w$ the *target* of $e$, denoted by $w = t(e)$.

For the directed graphs that we consider here, we assume that, for every $v \in V$, we have $(v, v) \in E$. While we could think of these edges as loops at each vertex, we prefer to regard them as *trivial edges* given by the vertices. Otherwise, we assume there are no loops or multiple edges, so there is no ambiguity in the definition as we have given it.

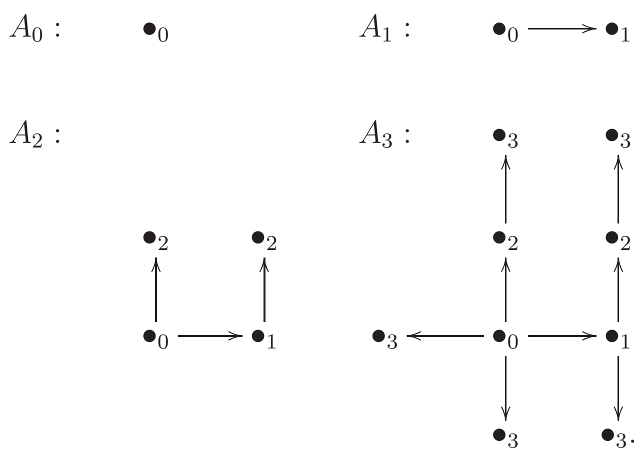
**Definition 2.2.** A *(directed) path* in a directed graph is a sequence of edges $e_1, \ldots, e_n$ such that for each $1 \leq i < k$, $t(e_i) = s(e_{i+1})$. For paths consisting of more than one edge, we require all these edges to be nontrivial. We call $s(e_1)$ the *initial vertex* of the path and $t(e_k)$ the *terminal vertex* of the path.

We also equip directed graphs with a labeling by a set $S$, i.e., a specified function $V \to S$. Here, $S$ is either the set of natural numbers $\mathbb{N}$, which we assume to include 0, or $\mathbb{N} \cup \{-k, -k+1, \ldots, -1\}$ for some $k \geq 1$.

We begin with the definition of action graph from [1], using some additional terminology to distinguish it from later examples.

**Definition 2.3.** For each natural number $n$, the (*0-extended*) *action graph* $A_n^0$ is the directed graph labeled by $\mathbb{N}$ which is defined inductively as follows. The action graph $A_0^0$ is defined to be the graph with one vertex labeled by 0 and no nontrivial edges. Inductively, given $A_n^0$, define $A_{n+1}^0$ by freely adjoining new edges by the following rule. For any vertex $v$ of $A_n^0$ labeled by $n$, consider all paths in $A_n^0$ with terminal vertex $v$. For each such path, adjoin a new edge whose source is the initial vertex of that path, and whose target is a new vertex which is labeled by $n+1$.

Thus, the first few 0-extended action graphs can be depicted as follows:




The first main result of [1] is the following theorem.

**Theorem 2.4.** *[1, 2.4] The number of new vertices (and edges) adjoined to $A_n^0$ to obtain $A_{n+1}^0$ is given by the $(n+1)$st Catalan number $C_{n+1}$.*

In this paper, we want to prove an analogous result for more general families of action graphs. Above, we defined a sequence $A^0$ of 0-extended action graphs; we now want to give more general starting graphs which, using the same inductive procedure, produce sequences $A^k$ for any $k \geq 1$.

**Definition 2.5.** For any $k \geq 1$, let $A_0^k$ be the directed graph labeled by $\mathbb{N} \cup \{-k, \dots, -1\}$ given by

$$\bullet_{-k} \longrightarrow \bullet_{-k+1} \longrightarrow \cdots \longrightarrow \bullet_{-1} \longrightarrow \bullet_0.$$

Then for any $n \geq 1$ the directed graph $A_n^k$ is defined using the same inductive rule as before. We call such an action graph $k$-*extended*.

The first few 1-extended action graphs can be depicted as follows:

$$A_0^1: \quad \bullet_{-1} \longrightarrow \bullet_0 \qquad\qquad A_1^1: \quad \begin{tikzcd} \bullet_1 & \bullet_1 \\ \bullet_{-1} \arrow[u] \arrow[r] & \bullet_0 \arrow[u] \end{tikzcd}$$

$$A_2^1: \quad \begin{tikzcd} & \bullet_2 & \bullet_2 \\ \bullet_2 & \bullet_{-1} \arrow[l] \arrow[u] \arrow[r] \arrow[d] & \bullet_0 \arrow[u] \arrow[d] \\ & \bullet_2 & \bullet_2. \end{tikzcd}$$

Observe that the graphs themselves look like the 0-extended action graphs, but the index of each graph is shifted by one and likewise there is a shift in the labeling.

We get more a more interesting family of 2-extended action graphs:

$$A_0^2: \quad \bullet_{-2} \longrightarrow \bullet_{-1} \longrightarrow \bullet_0$$

$$A_1^2: \quad \begin{tikzcd} \bullet_1 & \bullet_1 & \bullet_1 \\ \bullet_{-2} \arrow[u] \arrow[r] & \bullet_{-1} \arrow[u] \arrow[r] & \bullet_0 \arrow[u] \end{tikzcd}$$

$$A_2^2: \quad \begin{tikzcd} & \bullet_2 & \bullet_2 & \bullet_2 & \\ & \bullet_1 \arrow[u] & \bullet_1 \arrow[u] & \bullet_1 \arrow[u] & \\ \bullet_2 & \bullet_{-2} \arrow[l] \arrow[u] \arrow[r] \arrow[d] \arrow[dl] & \bullet_{-1} \arrow[u] \arrow[r] \arrow[d] \arrow[dr] & \bullet_0 \arrow[u] \arrow[r] & \bullet_2 \\ \bullet_2 & \bullet_2 & \bullet_2 & \bullet_2 & . \end{tikzcd}$$

Our goal in this paper is to prove the analogue of Theorem 2.4 for $k$-extended action graphs. Specifically, we would like to identify the sequences given by the number of new vertices adjoined to form each new graph in the sequences $A^k$.

# 3 Planar rooted forests

In the earlier paper [1], the authors give a direct proof of Theorem 2.4, but they also show that the new vertices in each $A^0_n$ are in bijection with planar rooted trees with $n$ edges. The latter set is one of the standard ways to obtain the $n$-th Catalan number $C_n$.

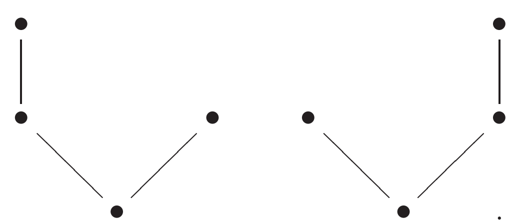
As we would like an analogue for $k$-extended action graphs, we generalize planar rooted trees to planar rooted forests consisting of $k+1$-trees. Indeed, in this secction we show that the family of such forests gives the sequence $C^k$. Let us begin with definitions.

**Definition 3.1.** A *tree* is a connected graph with no loops or multiple edges. A *rooted tree* is a tree with a specified vertex called the *root*.

**Definition 3.2.** A *leaf* of a rooted tree is a vertex of valence 1 which is not the root. In the special case of a tree consisting of a single vertex with no edges, we consider this vertex to be both the root and a leaf.

**Remark 3.3.** Applying this terminology to action graphs, note that they can be regarded as (directed) trees, and that the vertices of $A^k_{n+1}$ which are not in $A^k_n$ are precisely the leaves of $A^k_{n+1}$.

Here we consider *planar* rooted trees. In particular, if the bottom vertex is the root, we regard the following two trees as different:

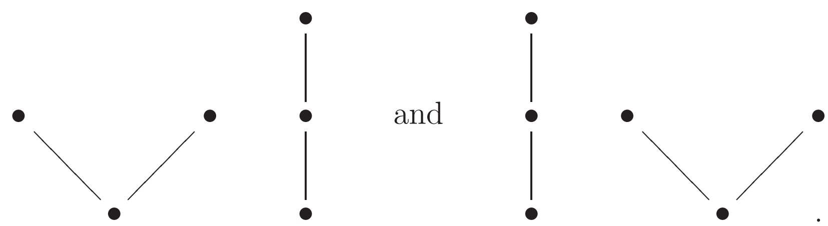


**Theorem 3.4.** *[6, §1.5] The set of all planar rooted trees with $n$ edges contains $C_n$ elements.*

Here, we want to consider the set of forests of rooted planar trees with $k$ trees containing $n$ total edges. We seek to establish that the set contains $C^k_n$ elements.

**Definition 3.5.** A *planar rooted forest* is a set of planar rooted trees. We assume that the trees in a planar rooted forest are given a specified ordering.

In particular, the ordering distinguishes between the forests



Now, we want to show that we can obtain self-convolutions of the Catalan sequence from such forests. To do so, we use the following formula for the $n$th term of the $k$th self-convolution of the Catalan number.

**Proposition 3.6.** *[8, §2] The $n$th term of the sequence $C^k$ is given by the following formula:*

$$C_n^k = \sum_{\substack{0 \leq i_1, i_2, i_3, \ldots, i_k, i_{k+1} \leq n \\ \sum_{j=1}^{k+1} i_j = n}} C_{i_1} C_{i_2} C_{i_3} \ldots C_{i_k} C_{i_{k+1}}.$$

Using this formula, we can verify that the sequence $C^1$ begins as

$$C_0^1 = 1, C_1^1 = 2, C_2^1 = 5, C_3^1 = 14, \ldots,$$

which is simply a shift of the Catalan sequence, whereas $C^2$ begins to look quite different:

$$C_0^2 = 1, C_1^2 = 3, C_2^2 = 9, \ldots.$$

We apply this formula to prove our first main result, which is the following.

**Theorem 3.7.** *The set of all planar rooted forests with $k+1$ trees and $n$ edges contains exactly $C_n^k$ elements.*

*Proof.* Consider a planar rooted forest consisting of $k+1$ trees and $n$ total edges. For any $1 \leq j \leq k+1$, let $i_j$ denote the number of edges in the $j$th tree of this forest. Observe that $0 \leq i_j \leq n$ and

$$\sum_{j=1}^{k+1} i_j = n.$$

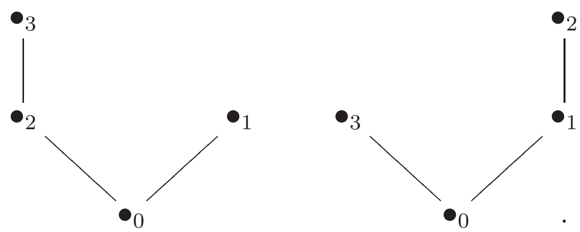
Given any $i_j$, we know from Theorem 2.4 that there are $C_{i_j}$ possible trees with $i_j$ edges. Thus, given any sequence $i_1, \ldots, i_{k+1}$, there are $C_{i_1} \cdots C_{i_{k+1}}$ forests with edges distributed among the trees via the sequence. Ranging over all such sequences, we obtain precisely the formula in Proposition 3.6. $\square$

# 4 Comparison of planar rooted forests with action graphs

In this last section, we establish a bijection between the number of new edges added to obtain $A_n^k$ from $A_{n-1}^k$ and the number of planar rooted forests with $k+1$ trees and $n$ edges. Combined with the theorem from the previous section, we thus show that there are $C_n^k$ vertices labeled by $n$ in the action graph $A_n^k$.

In the comparison between planar rooted trees and (0-extended) action graphs [1], the main idea is that the trees can be assembled in such a way as to obtain the corresponding action graph. We review the idea of this construction, so as to be able to generalize it.

We label the vertices of a planar rooted tree via the following rule. The root vertex is always given the label 0. Given a representative of the tree with the root at the bottom, label the vertices by successive natural numbers, moving upward from the root and from right to left. For example, we have the labelings

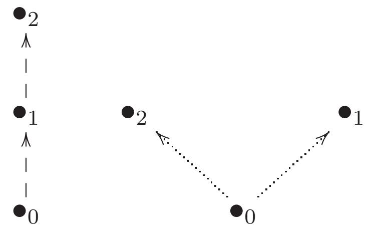



(Note that this convention differs slightly from the one used in previous work [1].) Because these labels are unique, they give a planar rooted tree a canonical directed structure, given by the usual ordering on the natural numbers.

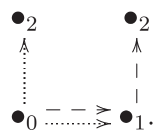
**Definition 4.1.** A *branch* of a rooted tree is a path from either the root or from a vertex of valence greater than 2 to a leaf. The *length* of the branch is the length of the path, namely the number of edges it contains.

Now we can describe the comparison with action graphs. We begin with the action graph $A_{n-1}^0$ and use the planar rooted trees with $n$ edges to build $A_n^0$. Using the labeling described above, the set of planar rooted trees with $n$ edges can be partially ordered by the length of the unique path from the root to the vertex labeled by $n$. Beginning with the tree with longest such path length, namely, the tree with a branch of length $n$, for each directed tree, identify the edge from 0 to 1, and any branches that do not end in the vertex $n$, with vertices and edges already present in $A_{n-1}^0$. Adjoin new vertices and edges to the graph corresponding to the branch containing the vertex $n$. Using the partial ordering on the set of trees guarantees that, as we assemble the trees together, there is no ambiguity about placement.

This assembly can be depicted for the case when $n = 2$ as follows. The two planar rooted trees with two edges are given by



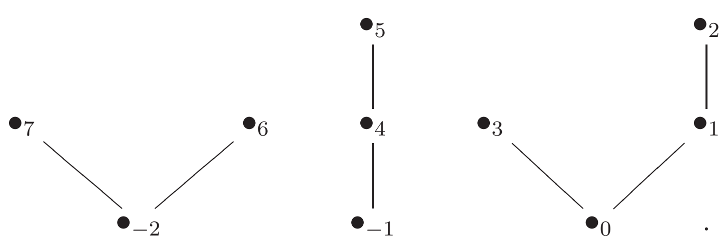
and can thus be assembled to form the action graph $A_2$, as given by



The precise statement of the comparison is as follows.

**Theorem 4.2.** *[1, 3.3] The function assigning any planar rooted tree with $n$ edges to the leaf that it contributes to the action graph $A_n^0$ defines a bijection.*

Here, we want to establish a bijection between $k$-extended action graphs and rooted planar forests with $k+1$ trees. For convenience for this comparison, in this section we assume that a planar rooted forest with $k+1$ trees has roots labeled, from left to right, by $-k, -k+1, \ldots, -1, 0$. Doing so faciliates the comparison with the vertices with the same labels in the $k$-extended action graphs. From there, we follow the same labeling convention as before, working from bottom to top and right to left. An example of such a labeling is the following:



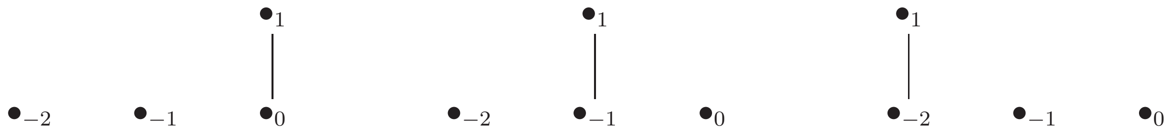
We claim that we can relate these forests to more general action graphs using the same kind of assembly process as before. To illustrate the process, let us look at the first few 2-extended action graphs. Our initial graph

$$\bullet_{-2} \longrightarrow \bullet_{-1} \longrightarrow \bullet_{0}$$

has one leaf, the vertex labeled by 0, which corresponds to the single planar rooted forest with 3 trees and no edges:

$$\bullet_{-2} \qquad \bullet_{-1} \qquad \bullet_{0}.$$

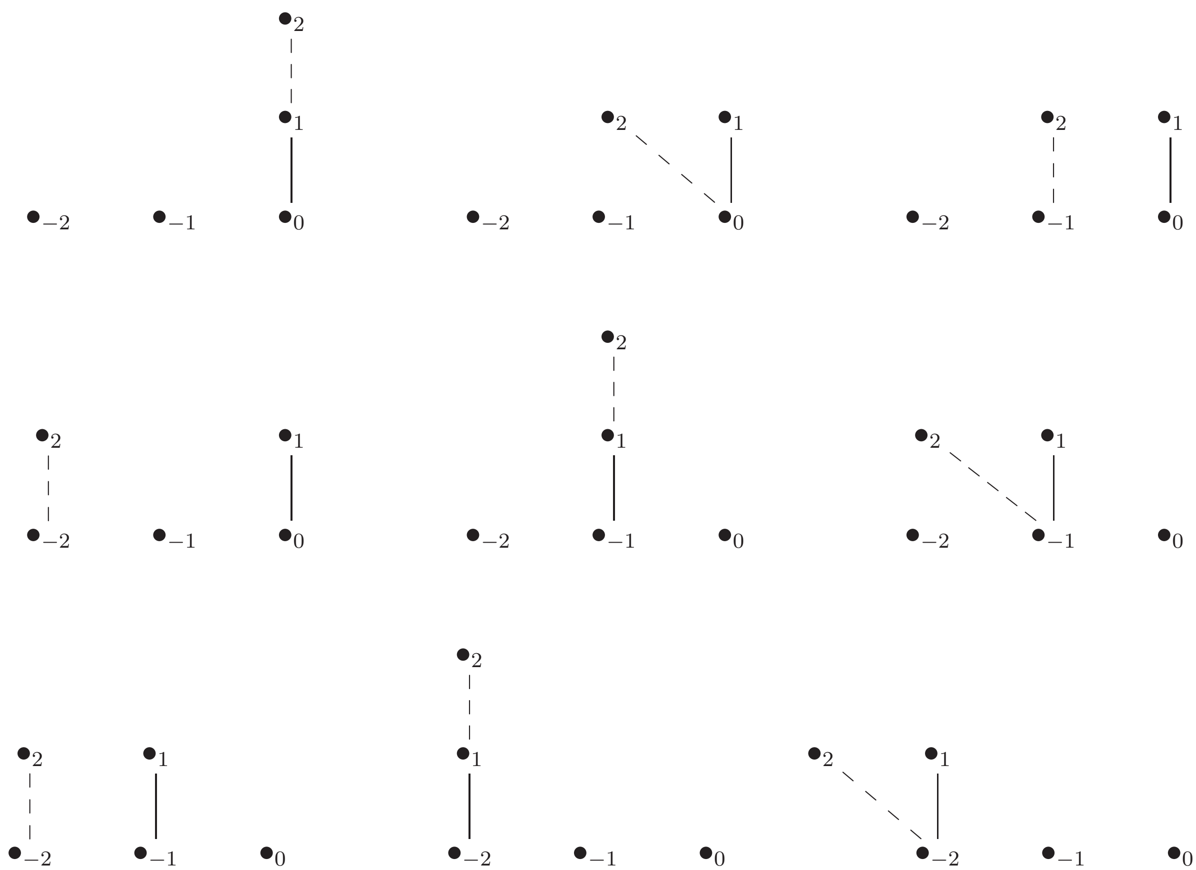
At the next stage, observe that there are three planar rooted forests with 3 trees and 1 edge:



which precisely assemble to give the leaves, labeled by 1, in the action graph $A_1^2$:

$$\begin{tikzcd}
\bullet_1 & \bullet_1 & \bullet_1 \\
\bullet_{-2} \arrow[u] \arrow[r] & \bullet_{-1} \arrow[u] \arrow[r] & \bullet_0 . \arrow[u]
\end{tikzcd}$$

We consider one more stage, at which things get more interesting. We have 9 planar rooted forests with 3 trees and 2 edges; the order in which we have listed them here suggests a canonical way for obtaining them from those with only one edge without repetition:



We invite the reader to check that these dotted edges assemble to the dotted edges in $A_2^2$:

$$
\begin{tikzcd}
 & \bullet_2 & \bullet_2 & \bullet_2 & \\
 & \bullet_1 \arrow[u, dashed] & \bullet_1 \arrow[u, dashed] & \bullet_1 \arrow[u, dashed] & \\
\bullet_2 & \bullet_{-2} \arrow[l, dashed] \arrow[u] \arrow[r] \arrow[d, dashed] \arrow[dl, dashed] & \bullet_{-1} \arrow[u] \arrow[r] \arrow[d, dashed] \arrow[dr, dashed] & \bullet_0 \arrow[u] \arrow[r, dashed] & \bullet_2 \\
\bullet_2 & \bullet_2 & \bullet_2 & \bullet_2 & .
\end{tikzcd}
$$

We now turn to the proof that this process always works.

**Theorem 4.3.** *There is an isomorphism between the set of leaves in $A_n^k$ and the set of rooted planar forests with $k+1$ trees and $n$ total edges.*

*Proof.* We assume $k$ is fixed and use induction on $n$. When $n=0$, we have the graph $A_0^k$, which is just

$$\bullet_{-k} \longrightarrow \bullet_{-k+1} \longrightarrow \cdots \longrightarrow \bullet_1 \longrightarrow \bullet_0.$$

This graph has one leaf, at the vertex labeled by 0. Correspondingly, there is exactly one planar rooted forest with $k+1$ trees and no edges, namely,

$$\bullet_{-k} \qquad \bullet_{-k+1} \qquad \cdots \qquad \bullet_1 \qquad \bullet_0.$$

Thus, we have the desired bijection when $n=0$.

Suppose the bijection holds for $n \geq 0$. Consider the set of planar rooted forests with $n+1$ edges and $k+1$ trees and the action graph $A_{n+1}^k$. Assume any such forest is given its canonical labeling, as described above. By the inductive hypothesis, the vertices of the forest labeled by $-k, \ldots, 0, 1, \ldots, n$ correspond to edges of the graph $A_n^k$. Consider the vertex labeled by $n+1$ and the unique branch from the appropriate root vertex to it. All but the last edge of this branch can be identified with a path in $A_n^k$; adjoin a new edge with final vertex labeled by $n+1$ to complete a path corresponding to the whole branch.

To complete the proof, then, it suffices to show that for $n \geq 0$ the number of planar rooted forests with $k+1$ trees and $n+1$ edges corresponds exactly to the number of directed paths in $A_n^k$ ending at vertices labeled by $n$. As illustrated by the examples above, we put the following ordering on the set of planar rooted forests with $n$ edges. When $n=0$, order the $k+1$ forests by the root at which the edge is placed, starting with the one labeled by 0 and moving left. Observe that these trees correspond exactly to directed paths to the vertex labeled by 0 in $A_0^k$.

For $n \geq 1$, start with the first forest in the inductively-defined ordering on planar rooted forests with $n$ edges. Obtain a forest with $n+1$ edges by adjoining a new edge to the vertex labeled by $n$. Moving top to bottom and left to right, obtain new forests by adjoining the extra edge to each vertex. Observe that these vertices are are in bijection with directed paths in $A_n^k$ ending at the appropriate vertex labeled by $n$. Repeat for the other planar rooted forests with $n$ edges, in order. Since this ordering was chosen to be compatible with the vertex labeling convention, we have assured that there is no repetition of graphs. Thus we see that planar rooted forests $k+1$ trees and $n+1$ edges are in bijection with paths in $A_n^k$ ending at a vertex labeled by $n$, as we wished to show. □

Combining Theorem 4.3 with Theorem 3.7, we obtain our desired result.

**Corollary 4.4.** *The number of vertices labeled by $n$ in the action graph $A_n^k$ is given by $C_n^k$.*

# 5 Acknowledgments

The first- and fourth-named authors were partially supported by NSF CAREER award DMS-1352298. The third-named author was partially supported by the USOAR program at the University of Virginia in 2017-18.

# References

[1] Gerardo Alvarez, Julia E. Bergner, and Ruben Lopez, Action graphs and Catalan numbers, *J. Integer Seq.* 18 (2015), Article 15.7.2.

[2] Julia E. Bergner and Philip Hackney, Reedy categories which encode the notion of category actions, *Fund. Math.*, **228** (2015), 193–222.

[3] Thomas Koshy, *Catalan Numbers with Applications*, Oxford University Press, 2009.

[4] N. J. A. Sloane, Online Encyclopedia of Integer Sequences, http://oeis.org/.

[5] Richard P. Stanley, Catalan Addendum, http://www-math.mit.edu/~rstan/ec/catadd.pdf.

[6] Richard P. Stanley, *Catalan Numbers*, *Cambridge University Press*, 2015.

[7] Richard P. Stanley, *Enumerative Combinatorics. Vol. 2.* Cambridge Studies in Advanced Mathematics, **62**. Cambridge University Press, 1999.

[8] Steven J. Tedford, Combinatorial interpretations of convolutions of the Catalan numbers, *Integers* 11 (2011), A3, 1–10.