NONASSOCIATIVITY MEASUREMENTS OF SOME BINARY OPERATIONS

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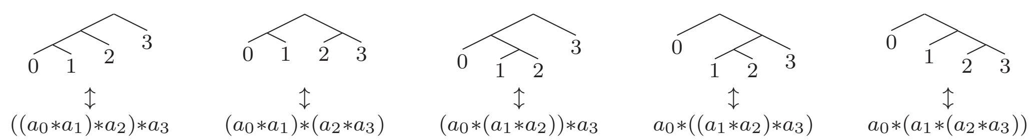
Abstract. We investigate certain nonassociative binary operations that satisfy a four-parameter generalization of the associative law. From this we obtain variations of the ubiquitous Catalan numbers and connections to many interesting combinatorial objects such as binary trees, plane trees, lattice paths, and permutations.

1. Introduction

Binary operations are widely used in mathematics and other fields. Some operations are associative, including addition, multiplication, union, intersection, and function composition. Others are not, such as subtraction, division, exponentiation, vector cross product, and Lie algebra multiplication. We consider a natural yet overlooked question: to what degree is a given operation nonassociative?

We use $*$ to denote a binary operation on a set A and a_i, for $i \in \mathbb{N}$, to denote an A-valued indeterminate. Let $\mathcal{P}_{*,n}$ be the set of all parenthesizations of the otherwise ambiguous expression $a_0 * \cdots * a_n$. The set $\mathcal{P}_{*,n}$ is in bijection the set of (full) binary trees with $n+1$ leaves, denoted by $\mathcal{T}_n$. We illustrate $\mathcal{P}_{*,3} \leftrightarrow \mathcal{T}_3$ below.



One reason $\mathcal{P}_{*,n}$ and $\mathcal{T}_n$ must be in bijection is that they are each enumerated by the *Catalan number* $C_n := \frac{1}{n+1}\binom{2n}{n}$ which does not depend on $*$.

In this paper we introduce and investigate two nonassociativity measurements. As nonassociativity is an inequivalence of parenthesizations, we broaden our investigation to include other Catalan objects . We say trees $t, t' \in \mathcal{T}_n$ are *$(*, n)$-equivalent*, written $t \sim_* t'$, if the corresponding parenthesizations are equal as functions from A^{n+1} to A. This is an equivalence relation on a set of Catalan objects, and for brevity we say its equivalence classes are *$(*, n)$-classes*. We define $C_{*,n}$ to be the number of $(*, n)$-classes, and we immediately observe $1 \le C_{*,n} \le C_n$. We now have an alternate definition of associativity which agrees with the traditional meaning: $*$ is associative if $C_{*,n} = 1$ for all $n \in \mathbb{N}$. Thus $C_{*,n}$ measures the failure of $*$ to be associative. We say $*$ is *totally nonassociative* if our measure for nonassociativity attains its theoretical upper bound, $C_{*,n} = C_n$.

Alternatively, one may quantify nonassociativity by computing the cardinality of the largest $(*, n)$-class for each n. We write $\widetilde{C}_{*,n}$ for that cardinality. As with our other measure of nonassociativity, we have $1 \le \widetilde{C}_{*,n} \le C_n$. One may see that $\widetilde{C}_{*,n} = 1$ if and only if $*$ is totally nonassociative and $\widetilde{C}_{*,n} = C_n$ if and only if $*$ is associative. Moreover, $2 \le C_{*,n} + \widetilde{C}_{*,n} \le C_n + 1$.

Lord [10] introduced a measurement of nonassociativity, called the *depth of nonassociativity*, which is given by $\inf\{n+1 : C_{*,n} < C_n\} = \inf\{n+1 : \widetilde{C}_{*,n} > 1\}$. Each of the measurements we propose refine depth of nonassociativity.

Key words and phrases. Binary operation, binary tree, Catalan number, nonassociativity.

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In previous work [6], we investigated the nonassociativity of a 1-parameter family of binary operations which generalize addition and subtraction. Here, we further generalize to a 4-parameter family (depending on d, e, k, ℓ) that gives a richer class of examples of operations that are neither associative nor totally nonassociative. We are mainly interested in the sequences of numbers $C_{*,n}$ and $\widetilde{C}*, n$ for $*$ in the 4-parameter family. Our prototypical example is the binary operation $*$ on $\mathbb{C}[x,y]/I$ given by

$$f * g := xf + yg, \quad \forall f, g \in \mathbb{C}[x,y]/I \tag{1}$$

where $I = (x^{d+k} - x^d, y^{e+\ell} - y^e)$ is an ideal of the polynomial ring $\mathbb{C}[x,y]$. Though we are presently interested in the ideal given above, one may more generally study binary operations defined by (1) with I being any ideal. A parenthesization corresponding to $t \in \mathcal{T}_n$ has the form

$$x^{\delta_0(t)} y^{\rho_0(t)} f_0 + \cdots + x^{\delta_n(t)} y^{\rho_n(t)} f_n. \tag{2}$$

Here we list the leaves of t as $0, 1, \ldots, n$ according to *preorder* and define the *left depth $\delta_i(t)$* (resp., *right depth $\rho_i(t)$*) of i to be the number of left (resp., right) steps along the unique path from the root of t down to i. The map sending each $t \in \mathcal{T}_n$ to its *left depth $\delta(t)$* $:= (\delta_0(t), \ldots, \delta_n(t))$ is one-to-one [6, §2.1]. Symmetrically, the map sending each $t \in \mathcal{T}_n$ to its *right depth $\rho(t)$* $:= (\rho_0(t), \ldots, \rho_n(t))$ is also one-to-one.

To characterize $(*, n)$-equivalence for $*$ defined by (1), we define some equivalence relations between two sequences $\mathbf{b} = (b_0, \ldots, b_n)$ and $\mathbf{c} = (c_0, \ldots, c_n)$ of nonnegative integers:

- $\mathbf{b} \sim_k \mathbf{c}$ if $b_i \equiv c_i \pmod{k}$ for $i = 0, \ldots, n$,
- $\mathbf{b} \sim^d \mathbf{c}$ if $\min\{b_i, c_i\} < d$ implies $b_i = c_i$ for $i = 0, \ldots, n$, and
- $\mathbf{b} \sim_k^d \mathbf{c}$ if $\mathbf{b} \sim_k \mathbf{c}$ and $\mathbf{b} \sim^d \mathbf{c}$.

If $*$ is defined by (1) then comparing expressions for $t, t' \in \mathcal{T}_n$ of the form (2) implies

$$t \sim_* t' \quad \text{if and only if} \quad \delta(t) \sim_k^d \delta(t') \quad \text{and} \quad \rho(t) \sim_\ell^e \rho(t'). \tag{3}$$

More generally, when every equivalent pair of binary trees $t \sim_* t'$ satisfies both $\delta(t) \sim_k^d \delta(t')$ and $\rho(t) \sim_\ell^e \rho(t')$, we say $*$ is *(k,ℓ)-associative at depth (d,e)*. Note that $(1,1)$-associativity at depth $(1,1)$ is the usual associativity. We write $C_{k,\ell,n}^{d,e} := C_{*,n}$ and $\widetilde{C}_{k,\ell,n}^{d,e} := \widetilde{C}_{*,n}$ for any binary operation $*$ satisfying (3).

We observe that $\mathbf{b} \sim_k^d \mathbf{c}$ implies $\mathbf{b} \sim_{k'}^{d'} \mathbf{c}$ if $d \le d'$ and $k \mid k'$. Thus if $d \le d'$, $e \le e'$, $k \mid k'$, and $\ell \mid \ell'$, then $C_{k,\ell,n}^{d,e} \le C_{k',\ell',n}^{d',e'}$ and $\widetilde{C}_{k,\ell,n}^{d,e} \ge \widetilde{C}_{k',\ell',n}^{d',e'}$, and (d,e)-associativity at depth (k,ℓ) implies (d',e')-associativity at depth (k',ℓ').

Also, the $(*, n)$-equivalence classes that determine $C_{k,\ell,n}^{d,e}$ and $\widetilde{C}_{k,\ell,n}^{d,e}$ are the same as the classes that determine $C_{\ell,k,n}^{e,d}$ and $\widetilde{C}_{\ell,k,n}^{e,d}$, but with each binary tree in each class reflected about a vertical line. Thus for $d, e, k, \ell \ge 1$ we have $C_{k,\ell,n}^{d,e} = C_{\ell,k,n}^{e,d}$ and $\widetilde{C}_{k,\ell,n}^{d,e} = \widetilde{C}_{\ell,k,n}^{e,d}$.

We now describe the relationship between $\sim_k^d$ in (3) and associativity. First, note the relation $\sim_1^d$ coincides with $\sim^d$ as all integers are congruent modulo 1. Next, observe, since $\delta_i(t) = 0 \Leftrightarrow i = n$ and $\rho_i(t) = 0 \Leftrightarrow i = 0$ for all $t \in \mathcal{T}_n$, we see $\sim_k^1$ coincides with $\sim_k$ on left and right depths of binary trees in $\mathcal{T}_n$. In earlier work [6], we determined $C_{k,n} := C_{k,1,n}^{1,1}$ using plane trees, Dyck paths, and Lagrange inversion. We call $C_{k,n}$ a *(k-)modular Catalan number* as for any binary operation $*$ satisfying (3) with $d = e = \ell = 1$, the $(*, n)$-relation is the same as the congruence relation modulo k on left depths of binary trees in $\mathcal{T}_n$. We also determined $\widetilde{C}_{k,n} := \widetilde{C}_{k,1,n}^{1,1}$ and enumerated $(*, n)$-classes with this largest size. By our earlier result [6, Proposition 2.11], the "if" part of (3) with $d = e = \ell = 1$ is equivalent to *k-associativity*, given by the rule $(a_0 * \cdots * a_k) * a_{k+1} = a_0 * (a_1 * \cdots * a_{k+1})$, where the $*$'s in parentheses are evaluated from left to right. This gives a one-parameter generalization of the usual associativity (i.e., 1-associativity).

On the other hand, we will see in Section 2 that, for $e = k = \ell = 1$, the "if" part of the $(*, n)$-relation (3) can be viewed as *associativity at left depth d*, that is, $t \sim_* t'$ if t can be obtained from t' by a finite sequence of moves, each of which replaces the maximal subtree rooted at a node of left depth at least $d-1$ by another binary tree with the same number of leaves.

The two 1-parameter generalizations of associativity given above justify the terminology "(k,ℓ)-associativity at depth (d,e)" for the "if" part of (3). In this paper we focus on two special cases, $k = \ell = 1$ and $e = \ell = 1$, each giving a two-parameter generalization of the usual associativity with connections to many interesting integer sequences and combinatorial objects.

In Section 2 we study the case $k = \ell = 1$. In this case the "if" part of (3) with $k = \ell = 1$ can be viewed as *associativity at left depth d and right depth e*, which recovers the associativity at left depth d when $e = 1$. Define $C_n^{d,e} := C_{1,1,n}^{d,e}$ and $\widetilde{C}_n^{d,e} := \widetilde{C}_{1,1,n}^{d,e}$. We determine $\widetilde{C}_n^{d,e}$ and enumerate $(*, n)$-classes with this size for arbitrary binary operations $*$ satisfying (3) with $k = \ell = 1$. We show that the cardinality of each $(*, n)$-class is a product of Catalan numbers in Corollary 2.3. We also provide a recursive formula for the generating function $C^{d,e}(x)$ of $C_n^{d,e}$. Then we give closed formulas for $C^{d,e}(x)$ and $C_n^{d,e}$ when $e = 1, 2$. It turns out that $C^{d,1}(x)$ is a well-known continued fraction, and $\{C_n^{d,1} : d \geq 1, n \geq 0\}$ coincides with an array in OEIS [11, A080934], which enumerates various families of objects, including

- binary trees with $n+1$ leaves of left depth at most d (by Proposition 3.9),
- plane trees with $n+1$ nodes of depth at most d (de Bruijn, Knuth, and Rice [3]),
- Dyck paths of length $2n$ with height at most d (Flajolet [4] and Kreweras [7, page 38]),
- permutations in $\mathfrak{S}_n$ avoiding 132 and $123\cdots(d+1)$ (Kitaev, Remmel, and Tiefenbruck [9]),
- ad-nilpotent ideals of the Borel subalgebra of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of order at most $d-1$ (Andrews, Krattenthaler, Orsina, and Papi [1]).

There are previously known closed formulas for $C_n^{d,1}$ [1, 3], but our formula (Prop. 2.7) is apparently different. The number $C_n^{d,2}$ occurs in OEIS only for $d \leq 3$ [11, A045623,A142586]; we find no result on $C^{d,2}$ for $d \geq 4$ or $C_n^{d,3}$ for $d \geq 3$ in OEIS [11].

In Section 3 we study the case $e = \ell = 1$. In this case the "if" part of (3) can be viewed as *k-associativity of left depth d*, which recovers the k-associativity when $d = 1$ and recovers the associativity of left depth d when $k = 1$. We give a few families of combinatorial objects enumerated by $C_{k,n}^d := C_{k,1,n}^{d,1}$, including binary trees, plane trees, and Dyck paths with certain constraints, and establish a recursive formula for the generating function $C_k^d(x)$ of $C_{k,n}^d$. Then we study $C_{k,n}^d$ when $d = 1, 2, 3$ or $k = 1, 2$. We have $C_1^d(x) = C^{d,1}(x)$ and $C_2^d(x) = C^{d+1,1}(x)$ for $d, n \geq 0$. The sequence $\{C_{3,n}^d\}$ has been studied by Barcucci, Del Lungo, Pergola, and Pinzani [2] in terms of pattern avoidance in permutations; see also [11, A005773, A054391–A054394] for $d = 1, \ldots, 5$. There is a closed formula for $C_3^d(x)$ but no closed formula for $C_{3,n}^d$ given in [2]. We provide a different formula for $C_3^d(x)$ and derive a closed formula for $C_{3,n}^d$ from it. We also give closed formulas for the number $C_{k,n}^d$ when $d = 2$, using Lagrange inversion and our ealier work on $C_{k,n}^1$ [6].

Another two-parameter specialization of (3) can be obtained by taking $d = e = 1$. Let $C_{k,\ell,n} := C_{k,\ell,n}^{1,1}$. Computations suggest a conjecture: $C_{k,\ell,n} = C_{k+\ell-1,n}$ for all $k, \ell \geq 1$ and $n \geq 0$.

We will also explore the case $d, e, k, \ell > 1$ in the future.

Contents

2. Associativity at depth (d, e): the case $k = \ell = 1$

In this section we assume $*$ is a binary operation satisfying (3) with $k = \ell = 1$ and $d, e \geq 1$. This means, for $t, t' \in \mathcal{T}_n$, $t \sim_* t$ if and only if both $\delta(t) \sim^d \delta(t')$ and $\rho(t) \sim^e \rho(t')$. We study $(*, n)$-classes and the nonassociativity measurements $C_n^{d,e} := C_{*,n}$ and $\widetilde{C}_n^{d,e} := \widetilde{C}_{*,n}$ of $*$ arising from these classes.

2.1. **Equivalence classes.** We first introduce some notation. If a node in a binary tree has left depth $\delta \geq d-1$ and right depth $\rho \geq e-1$ then we say this node is *(d, e)-contractible*, or simply *contractible* if d and e are clear from the context. We call a contractible node *maximal* if its parent is not contractible. One sees that a node with left depth δ and right depth ρ is a maximal contractible node if and only if $\delta = d-1$ and $\rho \geq e-1$ when v is the left child of its parent, or $\delta \geq d-1$ and $\rho = e-1$ when v is the right child of its parent.

Let $t \in \mathcal{T}_n$ and assign each leaf weight one. For each maximal contractible node v, we contract its subtree to a single node and assign v a weight equal to the number of leaves in this subtree. Denote by $\phi(t)$ the resulting weighted binary tree. This gives a map $\phi : \mathcal{T}_n \to \mathcal{T}_n^{d,e}$ by $t \mapsto \phi(t)$, where $\mathcal{T}_n^{d,e}$ is the set of all leaf-weighted binary trees such that

- every contractible leaf is maximal and has a positive integer weight,
- every non-contractible leaf has weight one, and
- the sum of leaf weights is $n + 1$.

Conversely, let $\bar{t} \in \mathcal{T}_n^{d,e}$ have leaves $v_0, \ldots, v_r$ with weights $m_0, \ldots, m_r$ respectively. We replace v_i by an arbitrary binary tree t_i with m_i leaves for $i = 0, \ldots, r$. Write $\phi^{-1}(\bar{t}; t_0, \ldots, t_r)$ for the resulting tree.

Lemma 2.1. *(i) We have a surjection $\phi : \mathcal{T}_n \twoheadrightarrow \mathcal{T}_n^{d,e}$.*
(ii) For each $\bar{t} \in \mathcal{T}_n^{d,e}$ whose leaves $v_0, \ldots, v_r$ are weighted $m_0, \ldots, m_r$, its fiber is

$$\phi^{-1}(\bar{t}) = \{\phi^{-1}(\bar{t}; t_0, \ldots, t_r) : t_i \in \mathcal{T}_{m_i - 1}\}.$$

(iii) We have $t \sim_* t'$ whenever $\phi(t) = \phi(t')$.

Proof. We first prove the third statements, and the others quickly follow. Let $\bar{t} \in \mathcal{T}_n^{d,e}$ with leaves $v_0, \ldots, v_r$ having weight $m_0, \ldots, m_r$. As $\sim_*$ is an equivalence relation, so, to prove (iii) it suffices to show $t = \phi^{-1}(\bar{t}; t_0, \ldots, t_r) \sim_* t' = \phi^{-1}(\bar{t}; t'_0, \ldots, t'_r)$, where t_j are t'_j are distinct binary trees with m_j leaves for some $j \in \{0, \ldots, r\}$ and t_i and t'_i are the same binary tree with m_i leaves for all $i \neq j$. Suppose the jth leaf v_j has left depth δ and right depth ρ. Since v_j is contractible by the definition of $\mathcal{T}_n^{d,e}$, we have $\delta \geq d-1$ and $\rho \geq e-1$. We also know $m_j \geq 3$ as $t_j \neq t'_j$, so in t we have:

- the first leaf of t_j has left depth at least $\delta + 1 \geq d$ and right depth equal to ρ,
- the last leaf of t_j has left depth equal to δ and right depth at least $\rho + 1 \geq e$, and
- all other leaves of t_j have left depths at least $\delta + 1 \geq d$ and right depths at least $\rho + 1 \geq e$.

Similarly, the leaves of t'_j satisfy the same properties. Thus $\delta(t) \sim^d \delta(t')$ and $\rho(t) \sim^e \rho(t')$. This proves (iii).

The argument above implies that each descendant of a contractible node is contractible, so (ii) also holds. Furthermore, each fiber $\phi^{-1}(\bar{t})$ is nonempty as one can choose trees t_i with m_i leaves for each i. This implies the surjectivity of ϕ. $\square$

Theorem 2.2. *Let $*$ be a binary operation satisfying* (3) *with $k = \ell = 1$. Then the fibers of ϕ are precisely the $(*, n)$-classes.*

Proof. Let $\bar{s} \in \mathcal{T}_n^{d,e}$ whose leaves $u_0, \ldots, u_a$ are weighted $m_0, \ldots, m_a$. Let $\bar{t} \in \mathcal{T}_n^{d,e}$ whose leaves $v_0, \ldots, v_b$ are weighted $n_0, \ldots, n_b$. Assume $\bar{s}$ and $\bar{t}$ are distinct. Let $s = \phi^{-1}(\bar{s}; s_1, \ldots, s_a) \in \phi^{-1}(\bar{s})$ and $t = \phi^{-1}(\bar{t}; t_1, \ldots, t_b) \in \phi^{-1}(\bar{t})$. Assume for a contradiction that $s \sim_* t$.

Let j be the smallest integer such that $m_j \neq n_j$, say $m_j < n_j$. Then $m_i = n_i$ for all $i < j$. Moreover, $n_j > 1$ implies that v_j is a maximal contractible node, i.e., $\delta(v_j) \geq d - 1$, $\rho(v_j) \geq e - 1$, and equality holds in at least one of these two inequalities.

The last leaf of s_j has left depth equal to $\delta(u_j)$ in s and the m_jth leaf of t_j has left depth at least $\delta(v_j) + 1 \geq d$ in t. Then $s \sim_* t$ implies $\delta(u_j) \geq d$. Since the parent of u_j is not contractible, we have $\rho(u_j) < e - 1$ if u_j is a left child or $\rho(u_j) < e$ if u_j is a right child.

One also sees that the first leaf of s_j has right depth equal to $\rho(u_j)$ in s and the first leaf of t_j has right depth equal to $\rho(v_j)$ in t. Combining this with $\rho(u_j) < e$ and $\rho(v_j) \geq e - 1$ we have $\rho(u_j) = \rho(v_j) = e - 1$. Thus u_j must be a right child. This implies that u_{j+1} has right depth at most $\rho(u_j) = e - 1$ in $\bar{s}$. Then the first leaf of s_{j+1} has right depth at most $e - 1$ in s. On the other hand, the $(m_j + 1)$th leaf of t_j has right depth at least $\rho(v_j) + 1 = e$ in t. This gives a contradiction. $\square$

Corollary 2.3. *The cardinality of each $(*, n)$-class is a product of Catalan numbers $C_{m_0-1} \cdots C_{m_r-1}$ with $m_0 + \cdots + m_r = n + 1$.*

Proof. By Theorem 2.2, every $(*, n)$-class can be written as $\phi^{-1}(\bar{t})$ for some $\bar{t} \in \mathcal{T}_n^{d,e}$. If the leaves of $\bar{t}$ have weights $m_0, \ldots, m_r$ then $|\phi^{-1}(\bar{t})| = C_{m_0-1} \cdots C_{m_r-1}$ by Lemma 2.1. We have $m_0 + \cdots + m_r = n + 1$ by the definition of $\mathcal{T}_n^{d,e}$. $\square$

2.2. **Nonassociativity measurements.** We first determine the largest size $\widetilde{C}_n^{d,e}$ of a $(*, n)$-class.

Theorem 2.4. *Let $d, e \geq 1$. If $0 \leq n < d + e$ then $\widetilde{C}_n^{d,e} = 1$. If $n \geq d + e$ then $\widetilde{C}_n^{d,e} = C_{n+2-d-e}$ and the number of $(*, n)$-classes with this size is $\binom{d+e-2}{d-1}$.*

Proof. It is well known that the Catalan sequence $\{C_n\}$ is log-convex, i.e., $C_m C_n \leq C_{m+1} C_{n-1}$ for $m \geq n \geq 1$. Hence for each $r \geq 0$ the largest result from products of the form $C_{m_0-1} \cdots C_{m_r-1}$ with $m_0 + \ldots + m_r = n + 1$ is C_{n-r}, which is attained when all but one of $m_0, \ldots, m_r$ equal one.

Now let $\bar{t} \in \mathcal{T}_n^{d,e}$ with leaves $v_0, \ldots, v_r$ weighted $m_0, \ldots, m_r$. If $m_0 = \cdots = m_r = 1$ then $C_{m_0-1} \cdots C_{m_r-1} = 1$. Assume $m_i > 1$ for some i. Then v_i is contractible, i.e. $\delta(v_i) \geq d - 1$ and $\rho(v_i) \geq e - 1$. Hence the unique path from the root of $\bar{t}$ to v_i has length at least $d + e - 2$. This implies that $\bar{t}$ has at least $d + e - 2$ internal nodes and thus $r \geq d + e - 2$. It follows that $|\phi^{-1}(\bar{t})| \leq C_{n+2-d-e}$, in which equality holds only if $r = d + e - 2$ and all but one of $m_0, \ldots, m_r$ equal one. We have $n + 1 = m_0 + \cdots + m_r \geq r + 2 \geq d + e$, i.e., $n \geq d + e - 1$. If $n = d + e - 1$ then $C_{n+2-d-e} = C_1 = 1$. Thus we assume $n \geq d + e$ below.

It remains to show that there are precisely $\binom{d+e-2}{d-1}$ many trees in $\mathcal{T}_n^{d,e}$ with $d + e - 1$ leaves, one having weight larger than one and all others having weight one. Suppose that $\bar{t}$ is such a tree and let v be its unique leaf with weight larger than one. Then the unique path from the root of $\bar{t}$ down to v has length $d + e - 2$. This path has $d - 1$ left steps and $e - 1$ right steps. Thus there are precisely $\binom{d+e-2}{d-1}$ many choices for this path. The entire tree $\bar{t}$ is determined by this path as all nodes not on this path must be leaves. $\square$

Now we study the other nonassociativity measurement $C_n^{d,e}$ and the generating function

$$C^{d,e}(x) := \sum_{n\geq 0} C_n^{d,e} x^{n+1}.$$

By symmetry we have $C^{d,e}(x) = C^{e,d}(x)$. If d or e is zero then we treat it as one. Theorem 2.2 implies a recurrence relation for $C^{d,e}(x)$.

Proposition 2.5. *For $d, e \geq 1$ we have*

$$C^{d,e}(x) = x + C^{d-1,e}(x)C^{d,e-1}(x)$$

Proof. By Theorem 2.2, $C_n^{d,e} = |\mathcal{T}_n^{d,e}|$. For any $\bar{t} \in \mathcal{T}_n^{d,e}$, let $\bar{t}_L$ and $\bar{t}_R$ denote the (maximal) weighted subtrees rooted at the left and right children of the root of $\bar{t}$. Then $\bar{t}_L \in \mathcal{T}_m^{d-1,e}$ and $\bar{t}_R \in \mathcal{T}_{n-m-1}^{d,e-1}$ for some m. Two trees $\bar{s}$ and $\bar{t}$ in $\mathcal{T}_n^{d,e}$ are equal if and only if $\bar{s}_L = \bar{t}_L$ and $\bar{s}_R = \bar{t}_R$. The result follows. $\square$

2.3. **Associativity of left depth d: The case $e = k = \ell = 1$.** We apply our previous results to the case $e = k = \ell = 1$. In this case the $(*, n)$-relation (3) can be regarded as *associativity at left depth d*, since Theorem 2.2 implies that in this case two trees $t, t' \in \mathcal{T}_n$ satisfy $t \sim_* t'$ if and only if t can be obtained from t' by a finite sequence of moves, each of which replaces the maximal subtree rooted at a node of left depth at least $d-1$ by another binary tree containing the same number of leaves.

Using Proposition 2.5 we can determine the number $C_n^d := C_n^{d,1}$ of $(*, n)$-classes and the generating function $C^d(x) := C^{d,1}(x)$. To state a formula for $C^d(x)$, we need the *Fibonacci polynomials* defined by $F_n(x) := F_{n-1}(x) - xF_{n-2}(x)$ for $n \geq 2$ with $F_0(x) := 0$ and $F_1(x) := 1$. For $n \geq 1$ we have [3, (8), (9), (10)]

$$\begin{aligned} F_n(x) &= \frac{1}{\sqrt{1-4x}}\left(\left(\frac{1+\sqrt{1-4x}}{2}\right)^n - \left(\frac{1-\sqrt{1-4x}}{2}\right)^n\right) \\ &= \sum_{0\leq i\leq (n-1)/2} \binom{n-1-i}{i}(-x)^i = \prod_{1\leq j\leq (n-1)/2} (1-4x\cos^2(j\pi/n)). \end{aligned}$$

For example, we have $F_2(x) = 1$, $F_3(x) = 1-x$, $F_4(x) = 1-2x$, $F_5(x) = 1-3x+x^2$, $F_6(x) = 1-4x+3x^2$, $F_7(x) = 1-5x+6x^2-x^3$, and so on.

Corollary 2.6 (Kreweras [7])**.** *For $d \geq 1$ we have (with $C^0(x) := x$)*

$$C^d(x) = \frac{x}{1-C^{d-1}(x)} = \frac{xF_{d+1}(x)}{F_{d+2}(x)}.$$

Proof. Observe that $C^1(x) = x/(1-x)$. By Proposition 2.5, we have $C^d(x) = x + C^{d-1}(x)C^d(x)$ and thus $C^d(x) = x/(1-C^{d-1}(x))$. By induction on d we have $C^d(x) = xF_{d+1}(x)/F_{d+2}(x)$. $\square$

Corollary 2.6 implies that $C^d(x)$ is a well-known continued fraction [3, 4]. For example, we have

$$C^1(x) = \frac{x}{1-x}, \quad C^2(x) = \frac{x}{1-\frac{x}{1-x}} = \frac{x(1-x)}{1-2x}, \quad C^3(x) = \frac{x}{1-\frac{x}{1-\frac{x}{1-x}}} = \frac{x(1-2x)}{1-3x+x^2}, \quad \ldots.$$

Hence $(C_n^d)_{d\geq 1, n\geq 0}$ coincides with an array in OEIS [11, A080934] and enumerates

- plane trees with $n+1$ nodes of depth at most d (de Bruijn, Knuth, and Rice [3]),
- Dyck paths of length $2n$ with height at most d (Flajolet [4] and Kreweras [7, page 38]),
- permutations in $\mathfrak{S}_n$ avoiding 132 and $123\cdots(d+1)$ (Kitaev, Remmel, and Tiefenbruck [9]),
- ad-nilpotent ideals of the Borel subalgebra of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of order at most $d-1$ (Andrews, Krattenthaler, Orsina, and Papi [1]).

Note that plane trees with $n+1$ nodes of depth at most d correspond to binary trees with $n+1$ leaves of left depth at most d. The latter is a family of objects more relevant to our current work and can also be obtained from our later result Proposition 3.9 by setting $k=1$.

There are many known closed formulas for the number C_n^d, such as

$$\begin{aligned} C_n^d &= \sum_{i\in\mathbb{Z}} \frac{2i(d+2)+1}{2n+1}\binom{2n+1}{n-i(d+2)} && \text{[1, Thm. 4.5]} \\ &= \det\left[\binom{i-\max\{-1,j-d\}}{j-i+1}\right]_{i,j=1}^{n-1} && \text{[1, Thm. 4.5]} \\ &= \sum_{0=i_0\le i_1\le\cdots\le i_{d-1}\le i_d=n}\ \prod_{0\le j\le d-2}\binom{i_{j+2}-i_j-1}{i_{j+1}-i_j} && \text{[1, Cor. 4.3]} \\ &= \frac{2^{2n+1}}{d+2}\sum_{1\le j\le d+1}\sin^2(j\pi/(d+2))\cos^{2n}(j\pi/(d+2)). && \text{[3, (14)]} \end{aligned}$$

When d is small the number C_n^d satisfies simpler formulas: $C_n^2=2^{n-1}$, $C_n^3=F_{2n-1}$, and $C_n^4=\frac{1}{2}(1+3^{n-1})$ for $n\ge 1$ [1, 9].

Now we derive a closed formula for C_n^d from the generating function $C^d(x)$, which seems different from other known formulas for this number. We write $\alpha\models n$ if α is a *composition* of n, i.e., if $\alpha=(\alpha_1,\dots,\alpha_\ell)$ is a sequence of positive integer such that $\alpha_1+\cdots+\alpha_\ell=n$. We also define $\ell(\alpha):=\ell$ and $\max(\alpha):=\max\{\alpha_1,\dots,\alpha_\ell\}$.

Proposition 2.7. *For $n,d\ge 1$ we have*

$$C_n^d = \sum_{\substack{\alpha\models n\\ \max(\alpha)\le(d+1)/2}} (-1)^{n-\ell(\alpha)}\binom{d-\alpha_1}{\alpha_1-1}\prod_{2\le r\le\ell(\alpha)}\binom{d+1-\alpha_i}{\alpha_i}$$

Proof. For $d\ge 1$ we have

$$\begin{aligned} C^d(x) &= x+\frac{x^2F_d(x)}{F_{d+2}(x)} = x+\frac{\sum_{0\le i\le(d-1)/2}\binom{d-1-i}{i}(-x)^{i+2}}{1+\sum_{1\le i\le(d+1)/2}\binom{d+1-i}{i}(-x)^i} \\ &= x+\sum_{0\le i\le(d-1)/2}\binom{d-1-i}{i}(-x)^{i+2}\sum_{j\ge0}\left(\sum_{1\le i\le(d+1)/2}\binom{d+1-i}{i}(-1)^{i-1}x^i\right)^j \end{aligned}$$

Extracting the coefficient of x^{n+1} from $C^d(x)$ gives

$$\begin{aligned} C_n^d &= \sum_{0\le i\le(d-1)/2}(-1)^i\binom{d-1-i}{i}\sum_{j\ge0}\sum_{\substack{1\le i_1,\dots,i_j\le(d+1)/2\\ i_1+\cdots+i_j=n-1-i}}\prod_{1\le r\le j}(-1)^{i_r-1}\binom{d+1-i_r}{i_r} \\ &= \sum_{j\ge0}(-1)^{n-1-j}\sum_{\substack{1\le i_0,i_1,\dots,i_j\le(d+1)/2\\ i_0+i_1+\cdots+i_j=n}}\binom{d-i_0}{i_0-1}\prod_{1\le r\le j}\binom{d+1-i_r}{i_r}. \end{aligned}$$

Viewing $(i_0,\dots,i_j)$ as a composition α of n gives the desired formula. □

Next, we give a new interpretation of the number C_n^d, which is very similar to the one obtained by Andrews, Krattenthaler, Orsina, and Papi [1]. We do not know any quick way to convert from one to the other.

Let $\mathcal{U}_n$ be the algebra of n-by-n upper triangular matrices over a field $\mathbb{F}$, with the usual matrix product. Using column operations one can write a (two-sided) ideal I of $\mathcal{U}_n$ as an upper triangular matrix $[a_{ij}]_{1\le i,j\le n}$ with $a_{ij}\in\{0,*\}$ such that $a_{ij}=*$ implies $a_{ij'}=*$ for all $j'\ge j$ and $a_{i'j}=*$ for all $i'\le i$. The elements of I are all matrices in $\mathcal{U}_n$ whose (i,j)-entry is arbitrary if $a_{ij}=*$ or zero if $a_{ij}=0$. The ideal $I=[a_{ij}]_{1\le i,j\le n}$ is nilpotent if and only if $a_{ii}=0$ for all $i\in[n]$. Thus the stars in the matrix form of a nilpotent ideal I give a partition inside the staircase $(n-1,\dots,1,0)$. It follows that the number of nilpotent ideals of $\mathcal{U}_n$ is the Catalan number C_n.

The *order* of a nilpotent ideal I is $\inf\{d : I^d=0\}$. Observe that an ideal I of $\mathcal{U}_n$ is commutative if and only if $I^2=0$. Shapiro [12] showed that the number commutative ideals of $\mathcal{U}_n$ is 2^{n-1}. We generalize this result below, using the number C_n^d.

Proposition 2.8. *For $d\ge 1$, nilpotent ideals of order at most d in $\mathcal{U}_n$ are enumerated by C_n^d.*

Proof. [1] A nilpotent ideal I of $\mathcal{U}_n$ is determined by the lower boundary D of the stars in its matrix form, which can be identified as a Dyck path of length $2n$. We say a sequence $(i_1,i_2,\dots,i_d)$ of d integers between 1 and n is *D-admissible* if the matrix entry (i_j,i_{j+1}) lies to the northeast of D for all $j=1,2,\dots,d-1$. For any positive integer m, we have $I^m\ne 0$ if and only if there exists a D-admissible sequence $(i_1,i_2,\dots,i_{m+1})$. Thus the order of I is the largest integer d such that there exists a D-admissible sequence $(i_1,\dots,i_d)$.

Next, we construct the *bounce path* of D by starting from the northwest corner, going east until hitting a south step of D, then turning south and bouncing off the main diagonal to proceed east, and repeating this process all the way to the southeast corner. The bounce path must be of the form $E^{b_1}S^{b_1}E^{b_2}S^{b_2}\cdots E^{b_k}S^{b_k}$, where E is an east step, S is a south step, and $(b_1,b_2,\dots,b_k)$ is a composition of n. We say this bounce path has k *parts*.

For $j=1,\dots,k-1$, the matrix entry (a_j,a_{j+1}), defined by $a_1=1$ and $a_{j+1}=a_j+b_j$, is immediately to the right of the first south step of the segment S^{b_j}, which also lies to the northeast of D. Thus $(a_1,\dots,a_k)$ is D-admissible. This shows that $d\ge k$.

On the other hand, the longest D-admissible sequence $(i_1,\dots,i_d)$ must satisfy $i_1=1=a_1$; otherwise $(1,i_1,\dots,i_d)$ would be an even longer D-admissible sequence. Let m be the largest integer such that $a_j=i_j$ for $j=1,2,\dots,m$. If $m<k$ then $a_{m+1}<i_{m+1}$ and $(a_1,\dots,a_m,a_{m+1},i_{m+2},\dots,i_d)$ is also D-admissible. Repeating this process gives a D-admissible sequence $(a_1,a_2,\dots,a_k,i_{k+1},\dots,i_d)$. But there is no entry on the a_k-th row to the northeast of D. This implies $d=k$.

There exists a bijection between Dyck paths of length $2n$ with exactly h parts in their bounce paths and Dyck paths of length $2n$ with height h; see, e.g., Haglund [5, Theorem 3.15 and Remark 3.16]. Thus nilpotent ideals of order at most d in $\mathcal{U}_n$ are in bijection with Dyck paths of length $2n$ with height at most d; the latter family is known to be enumerated by C_n^d [7, page 38]. This establishes the result. □

2.4. **Associativity at depth $(d,2)$.** Now we give closed formulas for $C^{d,2}(x)$ and $C_n^{d,2}$.

Proposition 2.9. *For $d\ge 2$ we have*

$$C^{d,2}(x)=C^d(x)+\frac{x^{d+2}}{(1-2x)F_{d+2}(x)}.$$

Consequently, for $n,d\ge 2$ we have

$$C_n^{d,2}=C_n^d+\sum_{1\le i\le n-d}2^{i-1}\sum_{\substack{\alpha\models n-d-i\\ \max(\alpha)\le(d+1)/2}}(-1)^{n-d-i-\ell(\alpha)}\prod_{1\le j\le\ell(\alpha)}\binom{d+1-\alpha_j}{\alpha_j}.$$

Proof. By Proposition 2.5 and Corollary 2.6, we can verify the formula for $d=2$,

$$C^{2,2}(x)=x+\frac{x^2(1-x)^2}{(1-2x)^2}=\frac{x(1-2x)(1-x)+x^4}{(1-2x)^2}.$$

[1] We are grateful to Brendon Rhoades for giving this proof and allowing us to include it here.

For $d \geq 3$ we obtain the desired formula for $C^{d,2}(x)$ by induction on d, Proposition 2.5, and Corollary 2.6. It follows that

$$C^{d,2}(x) = C^d(x) + \frac{\sum_{i\geq 0} 2^i x^{d+i+2}}{1+\sum_{1\leq i\leq (d+1)/2}\binom{d+1-i}{i}(-x)^i}$$
$$= C^d(x) + \sum_{i\geq 0} 2^i x^{d+i+2} \sum_{j\geq 0}\left(\sum_{1\leq i\leq (d+1)/2}\binom{d+1-i}{i}(-1)^{i-1}x^i\right)^j.$$

Hence

$$C_n^{d,2} = C_n^d + \sum_{i\geq 0} 2^i \sum_{j\geq 0}(-1)^{n-1-d-i-j} \sum_{\substack{1\leq i_1,\ldots,i_j\leq (d+1)/2\\ i_1+\cdots+i_j=n-1-d-i}} \prod_{1\leq r\leq j}\binom{d+1-i_r}{i_r}.$$

This implies the desired formula for $C_n^{d,2}$. □

Using Proposition 2.9 we find $C_n^{2,2}$ and $C_n^{3,2}$ in OEIS. The sequence $\{C_n^{2,2} : n \geq 0\}$ is the binomial transformation of $1,1,2,2,3,3,\ldots$, has a simple formula $C_n^{2,2} = (n+2)2^{n-3}$ for $n \geq 2$, and enumerates a few families of objects, such as copies of r in all compositions of $n+r$ for any positive integer r, weak compositions of $n-1$ with exactly one zero, triangulations of a regular $(n+3)$-gon in which each triangle contains at least one side of the polygon, and so on [11, A045623]. The sequence $(C_n^{3,2})_{n\geq 0}$ is the binomial transformation of $(\lfloor(\frac{1+\sqrt{5}}{2})^n\rfloor)_{n\geq 0}$ and satisfies the formula $C_n^{3,2} = \left(\frac{1+\sqrt{5}}{2}\right)^{2n-2} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n-2} - 2^{n-2}$ for $n \geq 2$ [11, A142586]. We do not see $C_n^{4,2}$ in OEIS, but it also satisfies a simple formula.

Proposition 2.10. *For $n \geq 3$ we have $C_n^{4,2} = 1 + 5\cdot 3^{n-3} - 2^{n-3}$.*

Proof. By Proposition 2.9 we have

$$C^{4,2}(x) = x + \frac{x^2(1-2x)}{1-4x+3x^2} + \frac{x^6}{(1-2x)(1-4x+3x^2)}$$
$$= x + \left(\frac{1}{1-x}+\frac{1}{1-3x}\right)\frac{x^2}{2} + \left(\frac{1}{1-x}-\frac{8}{1-2x}+\frac{9}{1-3x}\right)\frac{x^6}{2}.$$

This implies the desired formula for $C_n^{4,2}$. □

We find no result on $C_n^{d,3}$ for $d \geq 3$ in OEIS.

3. k-associativity of left depth d: the case $e = \ell = 1$

In this section we assume $*$ is a binary operation satisfying (3) with $e = \ell = 1$, i.e., $t \sim_* t'$ if and only if $\delta(t) \sim_k^d \delta(t')$ for all $t, t' \in \mathcal{T}_n$. For $d, k \geq 1$, we study $C_{k,n}^d := C_{*,n}$ and $\widetilde{C}_{k,n}^d := \widetilde{C}_{*,n}$.

3.1. **Equivalence classes.** We first study $(*, n)$-classes using rotations on binary trees. Given a node v in a binary tree t, a *subtree rooted at v* is a subtree of t whose root is v, and the *maximal subtree rooted at v* is the subtree consisting of all descendants of v, including v itself. Given two binary trees s and t, we say *t contains s at left depth d* if t contains s as a subtree rooted at a node of left depth d and *t avoids s at left depth d* otherwise.

Given binary trees s and t, write $s \wedge t$ for the binary tree whose root has left and right maximal subtrees s and t, respectively. There is a natural bijection between the set of parenthesizations of $x_0 * \cdots * x_n$ and the set $\mathcal{T}_n$ of binary trees with n internal nodes (i.e., with $n+1$ leaves) by replacing each x_i by a leaf labeled i and replacing each $*$ by $\wedge$.

Let t be a binary tree and v a node. Suppose the maximal subtree of t rooted at v can be written as $t_0 \wedge \cdots \wedge t_{k+1}$, where $t_0, \ldots, t_{k+1}$ are binary trees and the operations $\wedge$ are performed left-to-right. Replacing this subtree with $t_0 \wedge (t_1 \wedge \cdots \wedge t_{k+1})$ in t gives another binary tree t'. We call the operation $t \mapsto t'$ a *right k-rotation at v*, and call the inverse operation $t' \mapsto t$ a *left k-rotation at v*.

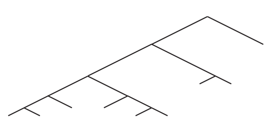
Lemma 3.1. *A left or right k-rotation at a node v in a binary tree t produces another binary tree t' satisfying $t \sim_k^d t'$ if and only if the left depth of v in t is at least $d-1$.*

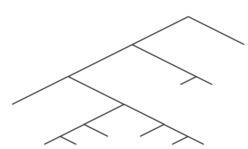
Proof. A right k-rotation $t \mapsto t'$ at v replaces the maximal subtree $t_0 \wedge \cdots \wedge t_{k+1}$ of t rooted at v with $t_0 \wedge (t_1 \wedge \cdots \wedge t_{k+1})$. This corresponds to a change in left depth by subtracting k from the left depth of each leaf of t_1 and leaving the left depths of all other leaves of t invariant. If $\delta(v)$ is the left depth of v in t, then the rightmost leaf of t_1 has left depth $\delta(v)+1+k$ in t, which is the smallest among all leaves of t, and has left depth $\delta(v)+1$ in t'. Hence $t \sim_k^d t'$ if and only if $\delta(v) \geq d-1$. For a left k-rotation the proof is similar. $\square$

If $s \in \mathcal{T}_n$ can be obtained from $t \in \mathcal{T}_n$ by finitely many left k-rotations at nodes of left depths at least $d-1$ then we say $s \leq_k^d t$. We call this partial order on $\mathcal{T}_n$ the *$\binom{d}{k}$-order*, which includes the k-associative order as a special case ($d=1$). A binary tree minimal or maximal under the $\binom{d}{k}$-order is called *$\binom{d}{k}$-minimal* or *$\binom{d}{k}$-maximal*. The following result is straightforward.

Proposition 3.2. *Let t be a binary tree. For $d, k \geq 1$ we have*
(i) t is $\binom{d}{k}$-minimal if and only if it avoids comb_k^1 *at any left depth at least $d-1$, and*
(ii) t is $\binom{d}{k}$-maximal if and only if it avoids comb_{k+1} *at any left depth at least $d-1$.*

Let $t \in \mathcal{T}_n$. The *left border* of t is the set of all nodes with right depth zero in t. Let $r_1, r_2, \ldots, r_h$ be the nodes on the left border of t so that r_1 is the root and r_{i+1} is the left child of r_i for all $i = 1, \ldots, h-1$. For $2 \leq i \leq h$ we define a tree $t^{+i} \in \mathcal{T}_{n+1}$ by first cutting t at r_i to get the maximal subtree s of t rooted at r_i and another subtree s' of t with leftmost leaf r_i, and then inserting the tree $u \in \mathcal{T}_1$, identifying the root of u with the leaf r_i in s' and the right leaf of u with the root r_i of s. The tree t^{+1} is obtained analogously by attaching t to the right leaf of the tree $u \in \mathcal{T}_1$. By construction, $\delta(t^{+i}) = (i, \delta_0(t), \ldots, \delta_n(t))$ for all $i = 1, 2, \ldots, h$. For example, the left picture below is a tree $t \in \mathcal{T}_8$ with $\delta(t) = (5,4,3,3,3,2,2,1,0)$ and the right picture below is the tree $t^{+3} \in \mathcal{T}_9$ with $\delta(t^{+3}) = (3,5,4,3,3,3,2,2,1,0)$.





Conversely, write t_- for the binary tree in $\mathcal{T}_{n-1}$ obtained from $t \in \mathcal{T}_n$ by contracting the leftmost leaf, its sibling, and their parent to a single node. One sees that $\delta(t^-) = (\delta_1, \ldots, \delta_n)$. Thus $s \sim_k^d t$ implies $s^- \sim_k^d t^-$. Furthermore, if $\delta_0(t) = i$ then $t_-^{+i} := (t_-)^{+i} = t$.

For each $k \geq 1$ we define $\mathrm{comb}_k := t_0 \wedge \cdots \wedge t_k$ and $\mathrm{comb}_k^1 := t_0 \wedge \mathrm{comb}_k$ where $t_0 = \cdots = t_k \in \mathcal{T}_0$.

Proposition 3.3. *Assume $*$ satisfies* (3) *with $e = \ell = 1$. Then each $(*, n)$-class has a unique $\binom{d}{k}$-minimal element.*

Proof. Since $\mathcal{T}_n$ is a finite set, each $(*, n)$-class must contain a $\binom{d}{k}$-minimal element. We prove by induction on n that each $(*, n)$-class has only one $\binom{d}{k}$-minimal element. Let s and t are distinct binary trees in $\mathcal{T}_n$ such that $s \sim_k^d t$. We need to show that either s or t is not $\binom{d}{k}$-minimal, i.e., contains comb_k^1 at left depth at least $d-1$.

First assume $s_- \neq t_-$. Since $s \sim_k^d t$ implies $s_- \sim_k^d t_-$, it follows from the induction hypothesis that, either s_- or t_-, say the former, contains comb_k^1 at left depth at least $d-1$. Hence s also contains comb_k^1 at left depth at least $d-1$.

Next assume $s_- = t_-$. This together with $s \neq t$ implies that $i = \delta_0(s)$ and $j = \delta_0(t)$ are distinct. Assume $i < j$, without loss of generality. Then $s = s_-^{+i} = t_-^{+i}$ contains comb_{j-i}^1 at left depth $i-1$. Since $s \sim_k^d t = t_-^{+j}$, we have $i \geq d$ and $j = i + km$ for some $m > 0$. Thus s contains comb_k^1 at left depth $i - 1 \geq d - 1$. □

Connected components of the Hasse diagram of the $\binom{d}{k}$-order on $\mathcal{T}_n$ are called *$\binom{d}{k}$-components*.

Theorem 3.4. *Let $d, k \geq 1$ and $n \geq 0$. Assume $*$ is a binary operation satisfying* (3) *with $e = \ell = 1$. Then the $(*, n)$-classes are precisely the $\binom{d}{k}$-components of $\mathcal{T}_n$.*

Proof. By Lemma 3.1, each $\binom{d}{k}$-component of $\mathcal{T}_n$ is contained in some $\binom{d}{k}$-class. We have the quality holds by Proposition 3.3. □

A subpath L' of a lattice path L is *at height* h if the initial point of L' has height h. We say L *avoids* L' *at height* h is L contains no subpath L' at height h.

Proposition 3.5. *For $k, d \geq 1$ and $n \geq 0$, the number $C_{k,n}^d$ enumerates*

(1) *binary trees with n internal nodes avoiding comb_k^1 at any left depth at least $d-1$,*
(2) *plane trees with multi-degree $(d_0, \ldots, d_n)$ satisfying $d_0 + \cdots + d_{i-1} - i \geq d \Rightarrow d_i < k$, $\forall i \in [n]$,*
(3) *Dyck paths of length $2n$ avoiding DU^k at height at least d,*

Proof. By Proposition 3.3 and Theorem 3.4, $C_{k,n}^d$ enumerates the $\binom{d}{k}$-minimal elements in $\mathcal{T}_n$. Combining this with Proposition 3.2 establishes (1).

A binary tree t with $n+1$ leaves corresponds to a plane tree T with $n+1$ nodes by contracting northeast-southwest edges in t. By [6, Proposition 2.10], the relation between the left depth $\delta(t) = (\delta_0, \ldots, \delta_n)$ and the multi-degree $d(T) = (d_0, \ldots, d_n)$ is given by

$$\delta_i = d_0 + \cdots + d_i - i, \quad \forall i \in \{0, 1, \ldots, n\}.$$

A left k-rotation at a node v in t corresponds to an up k-slide at a node of degree $d_i \geq k$ in T for some $i \in [n]$. One can check that the left depth of v equals $d_0 + \cdots + d_{i-1} - i$. Thus t is $\binom{d}{k}$-minimal if and only if $d_0 + \cdots + d_{i-1} - i \geq d \Rightarrow d_i < k$ for all $i \in [n]$. This implies (2).

Next, a plane tree with multi-degree $(d_0, \ldots, d_n)$ corresponds to a Dyck path $U^{d_0} D U^{d_1} \cdots D U^{d_n}$. The height of the initial point of the ith down-step in this Dyck path is $d_0 + \cdots + d_{i-1} - i + 1$. Thus (3) follows from (2). □

Remark 3.6. For any fixed n and k, the limit of $C_{k,n}^d$ as $d \to \infty$ is the Catalan number C_n since the constraints in Proposition 3.5 are redundant if d is large enough.

For $k \geq 1$ we define $M_{k-1,n}^d$ to be the number of binary trees in $\mathcal{T}_n$ avoiding comb_k at any left depth at least $d-1$. By Proposition 3.2, $M_{k,n}^d$ counts $\binom{d}{k}$-maximal elements in $\mathcal{T}_n$. The number $M_{k,n} := M_{k,n}^1$ is called a *generalized Motzkin number* in our earlier work [6] and also studied by Takács [14].

3.2. **Generating functions.** For $d, k \geq 1$ we define

$$C_k^d(x) := \sum_{n \geq 0} C_{k,n}^d x^{n+1} \quad \text{and} \quad M_{k-1}^d(x) := \sum_{n \geq 0} M_{k-1,n}^d x^{n+1}.$$

We also set $C_k^0(x) := M_{k-1}^1(x)$. To study these generating functions we need the following Lagrange inversion formula.

Theorem 3.7 (Stanley [13, Theorem 5.4.2])**.** *Suppose that $A(x)$ and $B(x)$ are formal power series such that $A(0) = B(0) = 0$ and $A(B(x)) = x$. Let $n, \ell \in \mathbb{Z}$. Then*

$$n[x^n]B(x)^\ell = \ell[x^{n-\ell}](x/A(x))^n.$$

Also recall the following binomial expansion for $m \geq 0$:

$$(1-x)^{-m} = \sum_{i\geq 0} \binom{m+i-1}{i} x^i.$$

Proposition 3.8. *For $m, n, d \geq 0$ and $k \geq 1$ we have*

$$C_k^{d+1}(x) = x \Big/ \Big(1 - C_k^d(x)\Big) \quad \textit{and}$$

$$[x^{n+m}]C_k^{d+1}(x)^m = \sum_{0\leq i\leq n} \binom{m+i-1}{i} [x^n]C_k^d(x)^i.$$

Proof. For $d = 0$ the first equation follows from [6, (6)]. Assume $d \geq 1$. Let $t \in \mathcal{T}_n$ with $n \geq 1$. Denote by t_L and t_R the maximal subtrees rooted at the left and right children of the root of t. By Proposition 3.2, t is $\binom{d+1}{k}$-minimal if and only if t_L is $\binom{d}{k}$-minimal and t_R is $\binom{d+1}{k}$-minimal. Combining this with Proposition 3.5 we have

$$C_k^{d+1}(x) = x + C_k^d(x)C_k^{d+1}(x)$$

This implies the first equation. Applying the binomial expansion gives the second equation. □

Proposition 3.9. *For $n, d \geq 0$ and $k \geq 1$ we have $M_{k-1}^{d+1}(x) = C_k^d(x)$ and $M_{k-1,n}^{d+1} = C_{k,n}^d$.*

Proof. [2] The result holds for $d = 0$ by definition. Similarly to the proof of Proposition 3.8, we have

$$M_k^{d+1}(x) = x + M_k^d(x)M_k^{d+1}(x), \quad \forall\, d, k \geq 0.$$

Hence $M_k^{d+1}(x) = x/(1 - M_k^d(x))$. The result then follows from induction on d. □

Corollary 3.10. *For $d, k \geq 1$ and $n \geq 0$ we have $M_{k-1,n}^d \leq C_{k,n}^d \leq M_{k,n}^d$.*

Proof. The first inequality follows from Proposition 3.2 and the second from Proposition 3.3. □

Combining Proposition 3.9 and Corollary 3.10 we have the following diagram for $d, k \geq 1$.

$$\begin{array}{ccccccccccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & & & \\
 & & \| & & & & \| & & & & \| & & & & & & \\
\cdots & \leq & M_{k-1,n}^d & \leq & C_{k,n}^d & \leq & M_{k,n}^d & \leq & C_{k+1,n}^d & \leq & M_{k+1,n}^d & \leq & C_{k+2,n}^d & \leq & \cdots & & \\
 & & & & \| & & & & \| & & & & \| & & & & \\
 & & \cdots & \leq & M_{k-1,n}^{d+1} & \leq & C_{k,n}^{d+1} & \leq & M_{k,n}^{d+1} & \leq & C_{k+1,n}^{d+1} & \leq & M_{k+1,n}^{d+1} & \leq & C_{k+2,n}^{d+1} & \leq & \cdots \\
 & & & & & & \| & & & & \| & & & & \| & & \\
 & & & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

Proposition 3.11. *For $d \geq 1$ and $n \geq 0$ we have $M_1^d(x) = C_1^d(x)$ and $M_{1,n}^d = C_{1,n}^d$.*

Proof. We have $M_1^1(x) = x/(1-x) = C_1^1(x)$. The result then follows from induction on d, using the recurrence relations in Proposition 3.8 and the proof of Proposition 3.9. □

[2] It would be interesting to have a bijective proof for this result.

3.3. **The case $k=3$ and $e=\ell=1$.** We already studied $C^d_{1,n}=C^d_n=C^{d,1}_n$ in Section 2.3. Now we determine $C^d_{2,n}$.

Proposition 3.12. *For $d,n\ge 0$ we have $C^d_2(x)=C^{d+1}_1(x)$ and $C^d_{2,n}=C^{d+1}_{1,n}$.*

Proof. By definition, $C^0_2(x)=x/(1-x)=C^1_1(x)$. By Proposition 3.8, we have the same recurrence relation $C^{d+1}_k(x)=x/(1-C^d_k(x))$ for all $k\ge 1$. The result follows from induction on d. ☐

Next, we study $C^d_{3,n}$. By our earlier work [6], $C^0_{3,n}$ is the Motzkin number [11, A001006], which has many closed formulas, and its generating function is

$$C^0_3(x)=\frac{1-x-\sqrt{1-2x-3x^2}}{2x}=\frac{x}{1-x-\frac{x^2}{1-x-\frac{x^2}{\cdots}}}.$$

As a warm-up, we derive some closed formulas for $C^0_{3,n}$, which are probably well known, from the generating function $C^0_3(x)$. We have

$$\sqrt{1-2x-3x^2}=\sqrt{1-3x}\cdot\sqrt{1+x}=\left(\sum_{i\ge 0}\binom{1/2}{i}(-3x)^i\right)\left(\sum_{j\ge 0}\binom{1/2}{j}x^j\right).$$

Thus for $n\ge 0$,

$$C^0_{3,n}=-\frac{1}{2}\sum_{0\le i\le n+2}(-3)^i\binom{1/2}{i}\binom{1/2}{n+2-i}.$$

The right hand side of this equation motivates the definition

$$C^0_{3,n}:=\begin{cases}1/2, & n=-1,\\ -1/2, & n=-2,\\ 0, & n\le -3,\end{cases}\tag{4}$$

On the other hand, we have

$$\sqrt{1-2x-3x^2}=\sum_{i\ge 0}\binom{1/2}{i}(-x)^i(2+3x)^i=\sum_{i\ge 0}\binom{1/2}{i}(-x)^i\sum_{0\le j\le i}\binom{i}{j}2^{i-j}3^jx^j.$$

Thus

$$C^0_{3,n}=-\frac{1}{2}\sum_{0\le j\le (n+2)/2}(-1)^{n+2-j}2^{n+2-2j}3^j\binom{1/2}{n+2-j}\binom{n+2-j}{j}.$$

Applying Proposition 3.8 to the generating function $C^0_3(x)$ gives

$$C^d_3(x)=\frac{x}{1-\frac{x}{1-\cdots\frac{x}{1-C^0_3(x)}}}\tag{5}$$

where the number of ones is d. Equation (5) is a special case of the generating function studied by Flajolet [4] for *labeled positive paths*. Such a path L starts at $(0,0)$ and stays weakly above the line $y=0$, with three kinds of steps $U=(1,1)$, $D=(1,-1)$, and $H=(1,0)$. Each step is labeled with some weight, and the *total weight* of L is the sum of all weights of the steps. The *height* of L is the largest y-coordinate of a point on L.

By Equation (5) or Remark 3.6, the number $C^d_{3,n}$ interpolates between the Motzkin number $C^0_{3,n}$ and the Catalan number $C_n=\lim_{d\to\infty}C^d_{3,n}$. For $d=1,\ldots,5$, the sequences $\{C^d_{3,n}\}$ are recorded in The OEIS [11, A005773, A054391–A054394]. For an arbitrary d, Barcucci, Del Lungo, Pergola, and Pinzani [2] studied $C^d_{3,n}$ in terms of permutations avoiding certain *barred patterns*.

Proposition 3.13. *For $d,n\ge 0$ the number $C^d_{3,n}$ enumerates the following families of objects.*

- *Labeled positive paths with total weight n and no H-step strictly below $y = d$, where each U-step or D-step weakly below $y = d$ has a weight $1/2$ and each other step has a weight 1.*
- *Permutations of $1, 2, \ldots, n$ avoiding 321 and $(d+3)\bar{1}(d+4)23\cdots(d+2)$ (barred pattern).*

Proof. A specialization of work of Flajolet [4, Thm. 1] gives the first family of objects enumerated by $C_{3,n}^d$. The second one follows from Barcucci, Del Lungo, Pergola, and Pinzani [2, (12)]. □

Barcucci, Del Lungo, Pergola, and Pinzani [2, p. 47] provided a closed formula for $C_3^d(x)$ but no formula for $C_{3,n}^d$. We will provide a different closed formula for $C_3^d(x)$ and derive a closed formula for $C_{3,n}^d$ from that.

Theorem 3.14. *For $d \geq 0$ we have*

$$C_3^d(x) = \frac{2xF_{d+1}(x)F_{d+2}(x) - x^d - x^{d+1} + x^d\sqrt{1-2x-3x^2}}{2(F_{d+2}(x)^2 - x^d - x^{d+1})}.$$

Proof. We induct on d. The result is trivial if $d = 0$. For $d \geq 1$, it follows from Proposition 3.8 and the induction hypothesis that

$$\begin{aligned}
C_3^d(x) &= \frac{x}{1 - C_3^{d-1}(x)} \\
&= \frac{2x(F_{d+1}(x)^2 - x^{d-1} - x^d)}{2F_{d+1}(x)^2 - 2x^{d-1} - 2x^d - 2xF_d(x)F_{d+1}(x) + x^{d-1} + x^d - x^{d-1}\sqrt{1-2x-3x^2}} \\
&= \frac{2x(F_{d+1}(x)^2 - x^{d-1} - x^d)}{2F_{d+1}(x)F_{d+2}(x) - x^{d-1} - x^d - x^{d-1}\sqrt{1-2x-3x^2}} \\
&= \frac{2x(F_{d+1}(x)^2 - x^{d-1} - x^d)(2F_{d+1}(x)F_{d+2}(x) - x^{d-1} - x^d + x^{d-1}\sqrt{1-2x-3x^2})}{(2F_{d+1}(x)F_{d+2}(x) - x^{d-1} - x^d)^2 - x^{2d-2}(1-2x-3x^2)} \\
&= \frac{(2F_{d+1}(x)^2 - 2x^{d-1} - 2x^d)(2xF_{d+1}(x)F_{d+2}(x) - x^d - x^{d+1} + x^d\sqrt{1-2x-3x^2})}{4F_{d+1}(x)^2F_{d+2}(x)^2 - 4(x^{d-1} + x^d)F_{d+1}(x)F_{d+2}(x) + 4x^{2d-1} + 4x^{2d}}.
\end{aligned}$$

This implies the desired expression of $C_3^d(x)$ since

$$\begin{aligned}
&(F_{d+1}(x)^2 - x^{d-1} - x^d)(F_{d+2}(x)^2 - x^d - x^{d+1}) \\
=&F_{d+1}(x)^2F_{d+2}(x)^2 - (x^{d-1} + x^d)(F_{d+2}(x)^2 + xF_{d+1}^2(x)) + x^{2d-1} + 2x^{2d} + x^{2d+1} \\
=&F_{d+1}(x)^2F_{d+2}(x)^2 - (x^{d-1} + x^d)F_{d+2}(x)F_{d+1}(x) + x^{2d-1} + x^{2d}.
\end{aligned}$$

Here the last step follows from

$$F_{d+2}(x)^2 + xF_{d+1}^2(x) = F_{d+2}(x)F_{d+1}(x) - xF_{d+2}(x)F_d(x) + xF_{d+1}^2(x) = F_{d+2}(x)F_{d+1}(x) + x^{d+1}$$

as one can show $F_{d+1}(x)^2 - F_d(x)F_{d+2}(x) = x^d$ by induction on d. □

Some examples are given below.

$$C_3^1(x) = \frac{x - 3x^2 + x\sqrt{1-2x-3x^2}}{2(1-3x)}$$

$$C_3^2(x) = \frac{2x - 7x^2 + 3x^3 + x^2\sqrt{1-2x-3x^2}}{2(1-4x+3x^2-x^3)}$$

$$C_3^3(x) = \frac{2x - 10^2 + 13x^3 - 5x^4 + x^3\sqrt{1-2x-3x^2}}{2(1-6x+11x^2-7x^3)}$$

Next, we derive a closed formula for $C_{3,n}^d$ from the expression of $C_3^d(x)$ given by Theorem 3.14.

Theorem 3.15. *For $d \geq 1$ and $n \geq 0$ we have*

$$C_{3,n}^{d} = \sum_{\substack{\alpha \models n+1 \\ h>1 \Rightarrow \alpha_h \leq d+1}} -\left(C_{3,\alpha_1-d-2}^{0} + \frac{\delta_{\alpha_1,d}}{2} + (-1)^{\alpha_1} \sum_{i+j=\alpha_1-1} \binom{d-i}{i}\binom{d+1-j}{j} \right)$$
$$\cdot \prod_{h\geq 2} \left(\left(\delta_{\alpha_h,d} + (-1)^{\alpha_h-1} \sum_{i+j=\alpha_h} \binom{d+1-i}{i}\binom{d+1-j}{j} \right) \right)$$

where $C_{3,m}^{0}$ is the Motzkin number when $m \geq 0$ or defined by Equation (4) when $m < 0$ and

$$\delta_{m,d} := \begin{cases} 1, & m \in \{d, d+1\}, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. We have

$$F_r(x)F_s(x) = \sum_{0\leq i\leq (r-1)/2} (-x)^i \binom{r-1-i}{i} \sum_{0\leq j\leq (s-1)/2} (-x)^j \binom{s-1-j}{j},$$
$$\sqrt{1-2x-3x^2} = \sqrt{1-3x}\cdot\sqrt{1+x} = \left(\sum_{i\geq 0}\binom{1/2}{i}(-3x)^i\right)\left(\sum_{j\geq 0}\binom{1/2}{j}x^j\right).$$

Hence

$$\begin{aligned}
&2xF_{d+1}(x)F_{d+2}(x) - x^d - x^{d+1} + x^d\sqrt{1-2x-3x^2} \\
&= \sum_{n\geq 1} x^n \left(-\delta_{n,d} + (-1)^{n-1}2 \sum_{i+j=n-1}\binom{d-i}{i}\binom{d+1-j}{j} + \sum_{i+j=n-d}(-3)^i\binom{1/2}{i}\binom{1/2}{j} \right),
\end{aligned}$$
$$F_{d+2}(x)^2 - x^d - x^{d+1} = 1 - \sum_{1\leq n\leq d+1} x^n \left(\delta_{n,d} + (-1)^{n-1}\sum_{i+j=n}\binom{d+1-i}{i}\binom{d+1-j}{j} \right).$$

Substituting these expressions in the formula for $C_3^d(x)$ given by Theorem 3.14 and extracting the coefficient of x^{n+1} we obtain the desired formula for $C_{3,n}^{d}$. □

We have not found the sequences $\{C_{k,n}^{d}\}$ for $k \geq 4$ and $d \geq 2$ in the literature.

3.4. The case $d = 2$ and $e = \ell = 1$.

As Proposition 3.8 gives a way to obtain the numbers $C_{k,n}^{d+1}$ from $C_k^d(x)^m$, we investigate the sequences $[x^{n+m}]C_k^d(x)^m$ for fixed d.

We begin with $d = 0$ and generalize the closed formulas [6, (9) and (11)] for $C_k^0(x) = M_{k-1}(x)$. To state our result, we review some notation below.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with m_i parts equal to i for $i = 0, 1, 2, \ldots$. Then

- $|\lambda| = n$ if and only if $m_1 + 2m_2 + \cdots + km_k = n$, and
- $\lambda \subseteq k^n$ if and only if $m_0 + \cdots + m_k = n$ and $m_{k+1} = m_{k+2} = \cdots = 0$.

The *monomial symmetric function* $m_\lambda(x_1, \ldots, x_n)$ is the sum of $x_1^{a_1}\cdots x_n^{a_n}$ for all rearrangement $(a_1, \ldots, a_n)$ of $(\lambda_1, \ldots, \lambda_n)$. Taking $x_1 = \cdots = x_n = 1$ in m_λ gives the multinomial coefficient

$$m_\lambda(1^n) = \binom{n}{m_0, m_1, m_2, \ldots}.$$

One sees that

$$\prod_{1\leq i\leq n}(1 + x_i + x_i^2 + \cdots + x_i^k)^n = \sum_{\lambda\subseteq k^n} m_\lambda(x_1, \ldots, x_n). \tag{6}$$

Proposition 3.16. *For $k,m\ge 1$ and $n\ge 0$, the number of plane forests with m components and $n+m$ total nodes, each of degree less than k, is*

$$\begin{aligned}[x^{n+m}]M_{k-1}(x)^m &= \frac{m}{n+m}\sum_{0\le j\le n/k}(-1)^j\binom{n+m}{j}\binom{2n+m-jk-1}{n+m-1}\\ &= \frac{m}{n+m}\sum_{\substack{\lambda\subseteq(k-1)^{n+m}\\ |\lambda|=n}} m_\lambda(1^{n+m}).\end{aligned}$$

Proof. The result follows from [6, Proposition 4.5 and (12)]. For completeness we include a direct proof here. Plane forests with m components and $n+m$ total nodes, each of degree less than k, are enumerated by $[x^{n+m}]M_{k-1}(x)^m$. We have $M_{k-1}(x)=x(1-M_{k-1}(x)^k)/(1-M_{k-1}(x))$ by [6, (5)]. Applying Lagrange inversion to $A(x)=x(1-x)/(1-x^k)$ and $B(x)=M_{k-1}(x)$ gives

$$[x^n](M_{k-1}(x))^m = \frac{m}{n}[x^{n-m}]\frac{(1-x^k)^n}{(1-x)^n} \tag{7}$$

$$= \frac{m}{n}[x^{n-m}]\left(1+x+x^2+\cdots+x^{k-1}\right)^n. \tag{8}$$

Applying binomial expansion to (7) and replacing n with $n+m$ gives the first formula. Applying (6) to (8) and replacing n with $n+m$ gives the second formula. □

Remark 3.17. For $k=1$ we have $M_{k-1}(x)=x$. Thus $\{[x^{n+m}]M_{k-1}(x)^m\}=\{1,0,0,\ldots\}$ for any $m\ge 0$. For $k=2$ we have $M_{k-1}(x)=x/(1-x)$. Thus $[x^{n+m}]M_{k-1}(x)^m=\binom{m+n-1}{n}$ for $m,n\ge 0$. For $k=3$ the sequences $\{[x^{n+m}](M_{k-1}(x)^m\}$ form the diagonals of the Motzkin triangle [11, A026300]; see also [11, A002026, A005322–A005325] for $m=2,\ldots,6$. We have not found any result in OEIS for $k\ge 4$ and $m\ge 2$.

We next generalize the closed formulas [6, (9) and (11)] for $C^1_{k,n}=C_{k,n}$.

Proposition 3.18. *For $k,m,n\ge 1$, the number of plane forests with m components and n non-root nodes, each of degree less than k, is*

$$\begin{aligned}[x^{n+m}]C_k(x)^m &= \frac{m}{n}\sum_{0\le j\le (n-1)/k}(-1)^j\binom{n}{j}\binom{2n+m-jk-1}{n+m}\\ &= \sum_{\lambda\subseteq(k-1)^n}\frac{n-|\lambda|}{n}\binom{m+n-|\lambda|-1}{n-|\lambda|}m_\lambda(1^n).\end{aligned}$$

Proof. Combining Proposition 3.8 with (7) we have

$$[x^{n+m}]C_k(x)^m = \sum_{1\le i\le n}\binom{m+i-1}{i}\frac{i}{n}[x^{n-i}]\frac{(1-x^k)^n}{(1-x)^n} \tag{9}$$

$$= \sum_{1\le i\le n}\frac{m}{n}\binom{m+i-1}{i-1}[x^{n-i}]\frac{(1-x^k)^n}{(1-x)^n}$$

$$= \frac{m}{n}[x^{n-1}]\frac{(1-x^k)^n}{(1-x)^{n+m+1}}. \tag{10}$$

Applying binomial expansion to (10) gives the first formula. Applying (6) to (9) gives the second formula. □

A *weak composition* of n into m parts is a sequence of m nonnegative integers whose sum is n. Our next result shows that the sequences $\{[x^{n+m}]C_k(x)^m\}_{n\ge 0}$ are related to weak compositions for $k=1,2$. The case $k=1$ is well known and the case $k=2$ has been studied by Janjić and

Petković [8] with a different approach from ours. For $k = 2$ and $1 \leq m \leq 10$ see also [11, A011782, A045623, A058396, A062109, A169792–A169797]. We have not found any result in the literature for $k \geq 3$ and $m \geq 2$ except the case $k = 3$ and $m = 2$ [11, A036908].

Corollary 3.19. *For $m, n \geq 0$, weak compositions of n into m parts are enumerated by*

$$[x^{n+m}]C_1(x)^m = \binom{m+n-1}{n} \quad \textit{and}$$

and weak compositions of n with $m-1$ zero parts are enumerated by

$$[x^{n+m}]C_2(x)^m = \sum_{0\leq i\leq m} \binom{m}{i}\binom{n-1}{n-i}2^{n-i} = \sum_{0\leq i\leq n} \binom{m+i-1}{i}\binom{n-1}{n-i}.$$

Proof. By Proposition 3.18, weak compositions of n into m parts are enumerated by

$$[x^{n+m}]C_1(x)^m = \binom{m+n-1}{n}$$

since they are in bijection with plane forests with m components and n non-root nodes, each of degree less than $k = 1$. In fact, such a forest is completely determined by the numbers of non-root nodes in its components. This implies the above bijection.

Similarly, weak compositions of n with $m-1$ zero parts are enumerated by $[x^{n+m}]C_2(x)^m$, since they are in bijection with plane forests with m components and n non-root nodes, each of degree less than $k = 2$. To see this bijection, let $v_1, \ldots, v_r$ be the children of the roots of such a forest. Since non-root nodes have degree at most one, the maximal subtree rooted at each v_i is a path consisting of a_i nodes, and this forest is determined by $a_1, \ldots, a_r$. We have $a_1 + \cdots + a_r = n$ and thus $(a_1, 0, a_2, 0, \ldots, a_{r-1}, 0, a_r)$ is a weak composition of n with $m-1$ zero parts.

Now using Proposition 3.8 and Proposition 3.12 we have

$$\begin{aligned}
[x^{n+m}]C_2(x)^m &= [x^n]\frac{(1-x)^m}{(1-2x)^m} \\
&= [x^n]\left(1+\frac{x}{1-2x}\right)^m \\
&= \sum_{0\leq i\leq m}\binom{m}{i}[x^{n-i}](1-2x)^{-i} \\
&= \sum_{0\leq i\leq m}\binom{m}{i}\binom{n-1}{n-i}2^{n-i}.
\end{aligned}$$

The second formula of $[x^{n+m}]C_2(x)^m$ follows from (9). □

Now we study the case $d = 2$.

Proposition 3.20. *For $m, n \geq 0$ and $k \geq 1$ we have*

$$\begin{aligned}
[x^{n+m}]C_k^2(x)^m &= \binom{m+n-1}{n} + \sum_{1\leq i\leq n-1}\binom{m+i-1}{i}\frac{i}{n-i}\sum_{0\leq j\leq \frac{n-i-1}{k}}(-1)^j\binom{n-i}{j}\binom{2n-i-jk-1}{n} \\
&= \binom{m+n-1}{n} + \sum_{1\leq i\leq n-1}\binom{m+i-1}{i}\sum_{\lambda\subseteq(k-1)^{n-i}}\frac{n-i-|\lambda|}{n-i}\binom{n-|\lambda|-1}{n-|\lambda|-i}m_\lambda(1^{n-i}).
\end{aligned}$$

In particular,

$$\begin{aligned} C^2_{k,n}(x) &= 1+\sum_{1\le i\le n-1}\frac{i}{n-i}\sum_{0\le j\le (n-i-1)/k}(-1)^j\binom{n-i}{j}\binom{2n-i-jk-1}{n} \\ &= 1+\sum_{1\le i\le n-1}\sum_{\lambda\subseteq(k-1)^{n-i}}\frac{n-i-|\lambda|}{n-i}\binom{n-|\lambda|-1}{n-|\lambda|-i}m_\lambda(1^{n-i}). \end{aligned}$$

Proof. Proposition 3.8 implies

$$\begin{aligned} [x^{n+m}]C^2_k(x)^m &= \sum_{0\le i\le n}\binom{m+i-1}{i}[x^n]C^1_k(x)^i \\ &= \binom{m+n-1}{n}+\sum_{1\le i\le n-1}\binom{m+i-1}{i}[x^n]C^1_k(x)^i. \end{aligned}$$

Substituting formulas from Proposition 3.18 establishes the result. □

We next study the special case when $d=k=2$.

Proposition 3.21. *Let $m,n\ge 0$. Then*

$$[x^{n+m}]C^2_2(x)^m=\sum_{0\le i\le n}\binom{m+i-1}{i}\sum_{0\le j\le n-i}\binom{i+j-1}{j}\binom{n-i-1}{n-i-j}.$$

In particular,

$$C^2_{2,n}=\sum_{0\le j\le n}\binom{n+j-1}{2j}=F_{2n-1}.$$

Proof. Let $m,n\ge 0$. Proposition 3.12 gives $C^2_1(x)=C^1_2(x)$ and thus Corollary 3.19 implies

$$[x^{n+m}]C^2_1(x)^m=\sum_{0\le i\le n}\binom{m+i-1}{i}\binom{n-1}{n-i}.$$

Combining this with Proposition 3.8 we have

$$[x^{n+m}]C^2_2(x)^m=\sum_{0\le i\le n}\binom{m+i-1}{i}\sum_{0\le j\le n-i}\binom{i+j-1}{j}\binom{n-i-1}{n-i-j}.$$

In particular, taking $m=1$ we have

$$\begin{aligned} C^2_{2,n} &= \sum_{0\le i\le n}\sum_{0\le j\le n-i}\binom{i+j-1}{j}\binom{n-i-1}{n-i-j} \\ &= \sum_{0\le j\le n}\sum_{0\le i\le n-j}\binom{i+j-1}{j}\binom{n-i-1}{n-i-j} \\ &= \sum_{0\le j\le n}\binom{n+j-1}{2j} \end{aligned}$$

where the last step follows from choosing $2j$ elements from $[n+j-1]$, assuming the $(j+1)$th chosen element is $i+j$ for some i. This sum is known to be the Fibonacci number F_{2n-1}, or one can use Proposition 3.12 and the discussion in Section 2.3 to conclude that $C^2_{2,n}=C^3_{1,n}=F_{2n-1}$. □

Remark 3.22. The limit of the sequence $\{[x^{n+m}]C_k^d(x)^m\}$ as $k \to \infty$ or $d \to \infty$ is $\{[x^{n+m}]C(x)^m\}$ where $C(x) := \sum_{n\geq 0} C_n x^{n+1}$. It is well known that $C(x) = x/(1-C(x))$. Thus for $m \geq 1$ and $n \geq 0$ applying Lagrange inversion to $A(x) := x(1-x)$ and $B(x) := C(x)$ gives

$$[x^{n+m}]C(x)^m = \frac{m}{n+m}[x^n](1-x)^{-(n+m)} = \frac{m}{n+m}\binom{2n+m-1}{n}.$$

Hence the sequences $\{[x^{n+m}]C(x)^m\}$ form the diagonals of Catalan's triangle [11, A009766].

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