A one-variable bracket polynomial for some Turk's head knots

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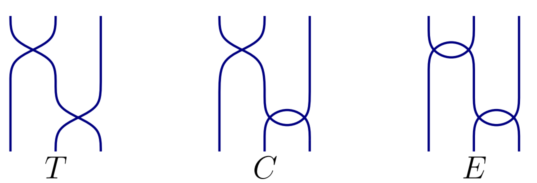
Abstract

We compute the Kauffman bracket polynomial of *the three-lead Turk's head*, *the chain sinnet* and *the figure-eight chain* shadow diagrams. Each of these knots can in fact be constructed by repeatedly concatenating the same 3-tangle, respectively, then taking the closure. The bracket is then evaluated by expressing the state diagrams of the concerned 3-tangle by means of the Kauffman monoid diagram's elements.

Keywords: bracket polynomial, tangle shadow, Kauffman state, flat sinnet.

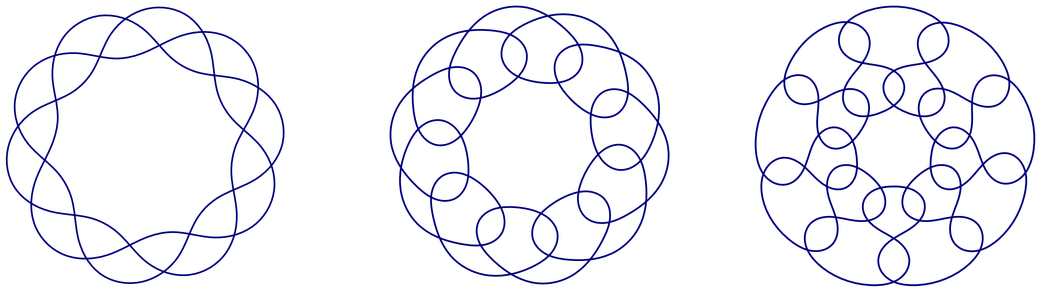
1 Introduction

The present paper is a follow-up on our previous work which aims at collecting statistics on knot shadows [5]. We would like to establish the bracket polynomial for knot diagram generated by the 3-tangle shadows below:



(1)

The knot diagrams under consideration are those obtained by repeatedly multiplying (or concatenating) the same 3-tangle, then taking the closure of the resulting 3-tangle (i.e., connecting the endpoints in a standard way, without introducing further crossings between the strands). Knots obtained from the 3-tangles pictured in (1) belong to the Ashley's *Turk's head* family [1, p. 226, Chap. 17]: the *three-lead Turk's head* [1, #1305], the *chain sinnet* [1, #1374] and the *figure-eight chain* [1, #1376], respectively (e.g. see Figure 1).



Three-lead Turk's head　　Chain sinnet　　Figure-eight chain

Figure 1: Some flat Turk's-head knot diagrams.

The remainder of this paper is arranged as follows. In section 2, we establish the expression of the bracket polynomial for any 3-tangle shadow diagram. Then in section 3, we apply those results to the flat sinnet Turk's heads mentioned earlier.

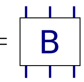
2 The Kauffman bracket of a 3-tangle shadow

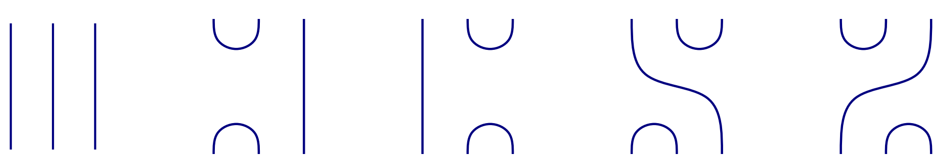
In this paper, the Kauffman bracket maps a shadow diagram D to $\langle D \rangle \in \mathbb{Z}[x]$ and is constructed from the following rules:

(**K1**): $\langle \bigcirc \rangle = x$;

(**K2**): $\langle \bigcirc \sqcup D \rangle = x \, \langle D \rangle$;

(**K3**): $\langle \times \rangle = \langle)(\rangle + \langle \asymp \rangle$.

The diagram $\bigcirc$ in (**K1**) represents that of a single loop, and the symbol $\sqcup$ in (**K2**) denotes the disjoint union operation. Formula in (**K3**) expresses the splitting of a crossing. Recall that the choice of such splittings for any single crossing is referred to as the so-called *Kauffman state*. Rules (**K1**), (**K2**) and (**K3**) can be summarized by the summation which is taken over all the states for D, namely $\langle D \rangle = \sum_S x^{|S|}$, where $|S|$ gives the number of loops in the state S. Kauffman shows that the states elements of a 3-tangle diagram $B :=$  are generated by the product of a loop and the following 5 elements of the 3-*strand diagram monoid* $\mathcal{D}_3$ [2, 8]:



1_3　　U_1　　U_2　　r　　s .

In other words, given a state S, there exist a nonnegative integer k and an element U in $\mathcal{D}_3$ such that one writes $S = \bigcirc^k \sqcup U$, where $\bigcirc^k = \bigcirc \sqcup \bigcirc \sqcup \cdots \sqcup \bigcirc$ denotes the disjoint union of k loops [3, p. 100]. The bracket of the 3-tangle B becomes $\langle B\rangle = \sum_S \langle S\rangle$, where $\langle S\rangle = x^{|S|}\langle U\rangle$ for certain $U \in \mathcal{D}_3$.

Therefore $\langle B\rangle$ is a linear combination of the brackets $\langle 1_3\rangle$, $\langle U_1\rangle$, $\langle U_2\rangle$,$\langle r\rangle$ and $\langle s\rangle$, i.e., there exist five polynomials a, b, c, d, e in $\mathbb{Z}[x]$ such that

$$\langle B\rangle = a\langle 1_3\rangle + b\langle U_1\rangle + c\langle U_2\rangle + d\langle r\rangle + e\langle s\rangle\,. \tag{2}$$

Lemma 1. *Given two* 3*-tangles B and D, we have*

$$\begin{aligned}\langle BD\rangle = a_B a_D\langle 1_3\rangle &+ (b_B a_D + (a_B + b_B x + d_B)\, b_D + (d_B x + b_B)\, e_D)\langle U_1\rangle \\ &+ (c_B a_D + (a_B + c_B x + e_B)\, c_D + (c_B + e_B x)\, d_D)\langle U_2\rangle \\ &+ (d_B a_D + (d_B x + b_B)\, c_D + (a_B + b_B x + d_B)\, d_D)\langle r\rangle \\ &+ (e_B a_D + (c_B + e_B x)\, b_D + (a_B + c_B x + e_B)\, e_D)\langle s\rangle\,.\end{aligned}$$

.

Proof. We first establish the states of B leaving D intact, and then in D:

$$\begin{aligned}\langle BD\rangle = a_B a_D\left\langle 1_3^2\right\rangle &+ a_B b_D\langle 1_3 U_1\rangle + a_B c_D\langle 1_3 U_2\rangle + a_B D\langle 1_3 r\rangle + a_B e_D\langle 1_3 s\rangle \\ &+ b_B a_D\langle U_1 1_3\rangle + b_B b_D\left\langle U_1^2\right\rangle + b_B c_D\langle U_1 U_2\rangle + b_B d_D\langle U_1 r\rangle + b_B e_D\langle U_1 s\rangle \\ &+ c_B a_D\langle U_2 1_3\rangle + c_B b_D\langle U_2 U_1\rangle + c_B c_D\left\langle U_2^2\right\rangle + c_B d_D\langle U_2 r\rangle + c_B e_D\langle U_2 s\rangle \\ &+ d_B a_D\langle r 1_3\rangle + d_B b_D\langle r U_1\rangle + d_B c_D\langle r U_2\rangle + d_B d_D\left\langle r^2\right\rangle + d_B e_D\langle rs\rangle \\ &+ e_B a_D\langle s 1_3\rangle + e_B b_D\langle s U_1\rangle + e_B c_D\langle s U_2\rangle + e_B d_D\langle sr\rangle + e_B e_D\left\langle s^2\right\rangle\,.\end{aligned}$$

The brackets for the pairs in the right-hand side can be evaluated by applying the following multiplication table.

.	1_3	U_1	U_2	r	s
1_3	1_3	U_1	U_2	r	s
U_1	U_1	$\bigcirc \sqcup U_1$	s	U_1	$\bigcirc \sqcup s$
U_2	U_2	r	$\bigcirc \sqcup U_2$	$\bigcirc \sqcup r$	U_2
r	r	$\bigcirc \sqcup r$	U_2	r	$\bigcirc \sqcup U_2$
s	s	U_1	$\bigcirc \sqcup s$	$\bigcirc \sqcup U_1$	s

Table 1: Multiplication of elements in $\mathcal{D}_3$.

The proof is then completed by factoring with respect to the resulting brackets, eventually simplified according to (**K2**). ☐

Notation 2. Let $B_n := BB\cdots B$ denote the 3-tangle obtained by multiplying the 3-tangle B n times, with $B_0 := 1_3$. For convenience, we shall identify the bracket formal expression in (2) by the 5-tuple $[a, b, c, d, e]^T$. Similarly, assume that $\langle B_n\rangle$ is identified by $[a_n, b_n, c_n, d_n, e_n]^T$.

Lemma 3. *The bracket* 5*-tuple for* B_n *is given by*

$$\begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ b & a+bx+d & 0 & 0 & dx+b \\ c & 0 & a+cx+e & c+ex & 0 \\ d & 0 & dx+b & a+bx+d & 0 \\ e & c+ex & 0 & 0 & a+cx+e \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3}$$

Proof. We write $B_{n+1} = BB_n$, then from Lemma 1 we have

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \\ e_{n+1} \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ b & a+bx+d & 0 & 0 & dx+b \\ c & 0 & a+cx+e & c+ex & 0 \\ d & 0 & dx+b & a+bx+d & 0 \\ e & c+ex & 0 & 0 & a+cx+e \end{bmatrix} \begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \end{bmatrix}. \tag{4}$$

We conclude by unfolding the recurrence and taking into consideration the initial condition $[a_0, b_0, c_0, d_0, e_0]^T = [1, 0, 0, 0, 0]^T$. □

We let M_B denote the 5×5 matrix in (3), and we will later refer to it as the *states matrix* for the 3-tangle B. Using the standard method for computing (3) we obtain the characteristic polynomial for M_B

$$\chi(M_B, \lambda) = -(\lambda - a)\left(\lambda - \frac{1}{2}(p-q)\right)^2 \left(\lambda - \frac{1}{2}(p+q)\right)^2,$$

then

$$a_n = a^n, \tag{5}$$

$$b_n = \frac{-1}{2q(x^2-1)}\left(2a^n qx + \left(\frac{p-q}{2}\right)^n \left((b-c)x^2 + (-d-e-q)x - 2b\right) + \left(\frac{p+q}{2}\right)^n \left((-b+c)x^2 + (d+e-q)x + 2b\right)\right), \tag{6}$$

$$c_n = \frac{-1}{2q(x^2-1)}\left(2a^n qx + \left(\frac{p+q}{2}\right)^n \left((b-c)x^2 + (d+e-q)x + 2c\right) + \left(\frac{p-q}{2}\right)^n \left((-b+c)x^2 + (-d-e-q)x - 2c\right)\right), \tag{7}$$

$$d_n = \frac{1}{2q(x^2-1)}\left(2a^n q + \left(\frac{p-q}{2}\right)^n \left(-2dx^2 + (-b-c)x + d - e - q\right) + \left(\frac{p+q}{2}\right)^n \left(2dx^2 + (b+c)x - d + e - q\right)\right), \tag{8}$$

$$e_n = \frac{1}{2q(x^2-1)}\left(2a^n q + \left(\frac{p-q}{2}\right)^n \left(-2ex^2 + (-b-c)x - d + e - q\right) + \left(\frac{p+q}{2}\right)^n \left(2ex^2 + (b+c)x + d - e - q\right)\right), \tag{9}$$

where

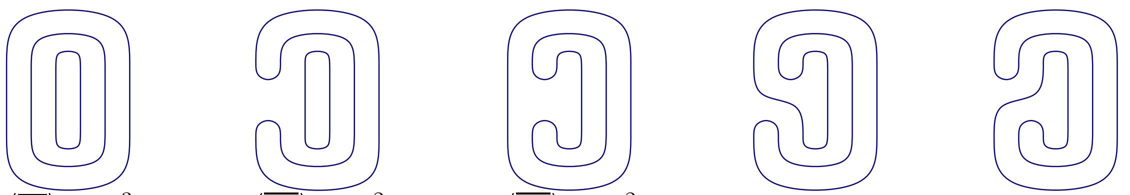
$$p := (b+c)x + 2a + d + e, \tag{10}$$
$$q := \sqrt{\left(b^2 - 2bc + c^2 + 4de\right)x^2 + (2bd + 2cd + 2be + 2ce)\,x + 4bc + d^2 - 2de + e^2}. \tag{11}$$

Now let $\overline{B_n}$ denote the tangle closure of B_n. In order to evaluate $\left\langle \overline{B_n} \right\rangle$ from formula (3) we need to apply the closure to the elements of $\mathcal{D}_3$.

Lemma 4. *The expression of the bracket polynomial for the closure $\overline{B_n}$ is given by*

$$\left\langle \overline{B_n} \right\rangle = x^3 a_n + x^2\left(b_n + c_n\right) + x\left(d_n + e_n\right). \tag{12}$$

The splitting at each crossing do not conflict with the closing process, hence the only point remaining concerns the evaluation of the brackets to the closure of the elements of $\mathcal{D}_3$, namely



$\left\langle \overline{1_3} \right\rangle = x^3$ $\quad\left\langle \overline{U_1} \right\rangle = x^2$ $\quad\left\langle \overline{U_2} \right\rangle = x^2$ $\quad\left\langle \overline{r} \right\rangle = x$ $\quad\left\langle \overline{s} \right\rangle = x$.

Next, combining (3), (5)–(9) and (12), we obtain a better expression of the bracket:

Lemma 5. *The bracket polynomial for the knot $\overline{B_n}$ is given by*

$$\left\langle \overline{B_n} \right\rangle = x a^n \left(x^2 - 2\right) + x\left(\left(\frac{p-q}{2}\right)^n + \left(\frac{p+q}{2}\right)^n\right), \tag{13}$$

where p and q are expressions defined in (10) and (11).

Finally, we let $\overline{B}(x;y) := \sum_{n\geq 0} \left\langle \overline{B_n} \right\rangle y^n$ denote the generating function of $\left(\left\langle \overline{B_n} \right\rangle\right)_n$. By (13) we deduce

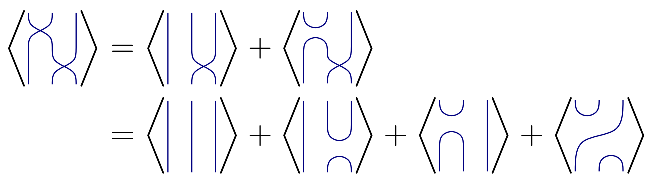
$$\overline{B}(x;y) = \frac{\left((b+c)x + 2a + d + e\right)y - 2}{\left(\left(de - bc\right)x^2 + \left(-ac - ab\right)x + \left(-d - a\right)e - ad + bc - a^2\right)y^2 + \left((b+c)x + 2a + d + e\right)y - 1} + \frac{x\left(x^2 - 2\right)}{1 - ay}.$$

3 Application

Throughout this section, let us refer to the 3-tangles in (1) as *generators*. Recall that in the expression $\left\langle \overline{B_n} \right\rangle = \sum_{k>0} s_B(n,k)x^k$ we have $b_{n,k} = \#\{S \mid S \textit{ is a state of } B_n \textit{ and } |S| = k\}$, with $B \in \{T, C, E\}$. For each flat sinnet Turk's head below, we will give the corresponding distribution $\left(s_B(n,k)\right)_{n,k}$ for small values of n and k.

1. **Three-lead Turk's head**. Let $\sum_{k\geq 0} s_T(n,k)x^k := \langle \overline{T_n} \rangle$.

   - Bracket for the generator T:



$$\langle T \rangle = \langle 1_3 \rangle + \langle U_1 \rangle + \langle U_2 \rangle + \langle s \rangle \, .$$

   - States matrix:

$$M_T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & x+1 & 0 & 0 & 1 \\ 1 & 0 & x+2 & x+1 & 0 \\ 0 & 0 & 1 & x+1 & 0 \\ 1 & x+1 & 0 & 0 & x+2 \end{bmatrix}.$$

   - Bracket for T_n:

$$\langle \overline{T_n} \rangle = x\left(x^2-2\right) + x\left(\left(\frac{2x+3-\sqrt{4x+5}}{2}\right)^n + \left(\frac{2x+3+\sqrt{4x+5}}{2}\right)^n\right).$$

   - Generating function:

$$\overline{T}(x;y) = \frac{x\left((-2x-3)\,y+2\right)}{\left(x^2+2x+1\right)y^2+(-2x-3)\,y+1} + \frac{x\left(x^2-2\right)}{1-y}.$$

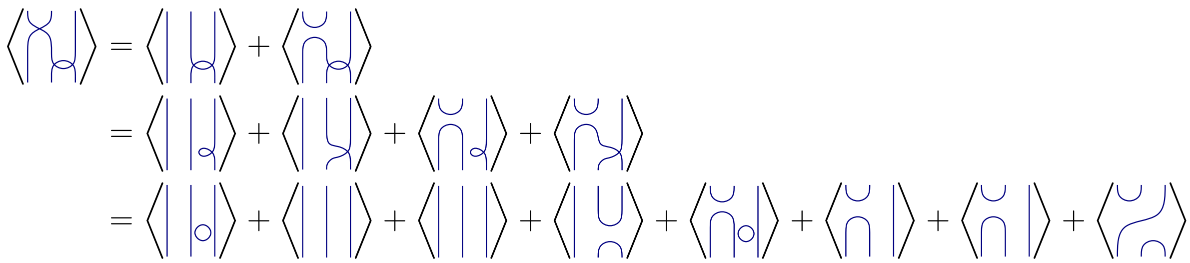
   - Distribution of $(s_T(n,k))_{n,k}$: [6, A316659]

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	1								
1	0	1	2	1								
2	0	5	8	3								
3	0	16	30	16	2							
4	0	45	104	81	24	2						
5	0	121	340	356	170	35	2					
6	0	320	1068	1411	932	315	48	2				
7	0	841	3262	5209	4396	2079	532	63	2			
8	0	2205	9760	18281	18784	11440	4144	840	80	2		
9	0	5776	28746	61786	74838	55809	26226	7602	1260	99	2	
10	0	15125	83620	202841	282980	249815	144488	54690	13080	1815	120	2

Table 2: Values of $s_T(n,k)$ for $0 \leq n \leq 10$ and $0 \leq k \leq 11$.

2. **Chain sinnet**. Let $\sum_{k\geq 0} s_C(n,k)x^k := \langle \overline{C_n} \rangle$.

- Bracket for the generator C:



$$\langle C \rangle = (x+2)\langle 1_3 \rangle + (x+2)\langle U_1 \rangle + \langle U_2 \rangle + \langle s \rangle \,.$$

- States matrix:

$$M_C = \begin{bmatrix} x+2 & 0 & 0 & 0 & 0 \\ x+2 & x^2+3x+2 & 0 & 0 & x+2 \\ 1 & 0 & 2x+3 & x+1 & 0 \\ 0 & 0 & x+2 & x^2+3x+2 & 0 \\ 1 & x+1 & 0 & 0 & 2x+3 \end{bmatrix}.$$

- Bracket for C_n:

$$\langle \overline{C_n} \rangle = x\left(x^2-2\right)(x+2)^n + x\left(\left(\frac{x^2+5x+5-\sqrt{x^4+2x^3+3x^2+10x+9}}{2}\right)^n + \left(\frac{x^2+5x+5+\sqrt{x^4+2x^3+3x^2+10x+9}}{2}\right)^n\right).$$

- Generating function

$$\overline{C}(x;y) = \frac{x\left(\left(-x^2-5x-5\right)y+2\right)}{\left(2x^3+8x^2+10x+4\right)y^2+\left(-x^2-5x-5\right)y+1} + \frac{x\left(x^2-2\right)}{1-(x+2)\,y}.$$

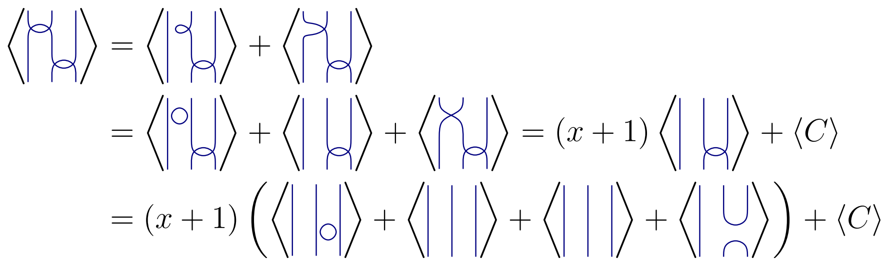
- Distribution of $(s_C(n,k))_{n,k}$:

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	1										
1	0	1	3	3	1									
2	0	9	22	21	10	2								
3	0	49	141	164	105	42	10	1						
4	0	225	796	1186	1008	569	232	67	12	1				
5	0	961	4115	7677	8400	6205	3393	1435	461	105	15	1		
6	0	3969	20106	45481	61630	57078	39298	21239	9198	3151	822	153	18	1

Table 3: Values of $s_C(n,k)$ for $0 \leq n \leq 6$ and $0 \leq k \leq 13$.

3. **Figure-eight chain**. Let $\sum_{k\geq 0} s_E(n,k)x^k := \langle \overline{E_n} \rangle$.

- Bracket for the generator E:



$$\langle E \rangle = \left(x^2+4x+4\right)\langle 1_3 \rangle + (x+2)\langle U_1 \rangle + (x+2)\langle U_2 \rangle + \langle s \rangle .$$

- States matrix:

$$M_E = \begin{bmatrix} x^2+4x+4 & 0 & 0 & 0 & 0 \\ x+2 & 2x^2+6x+4 & 0 & 0 & x+2 \\ x+2 & 0 & 2x^2+6x+5 & 2x+2 & 0 \\ 0 & 0 & x+2 & 2x^2+6x+4 & 0 \\ 1 & 2x+2 & 0 & 0 & 2x^2+6x+5 \end{bmatrix}.$$

- Bracket for $\overline{E_n}$:

$$\langle \overline{E_n} \rangle = x(x^2-2)\left(x^2+4x+4\right)^n + x\left(\left(\frac{4x^2+12x+9-\sqrt{8x^2+24x+17}}{2}\right)^n + \left(\frac{4x^2+12x+9+\sqrt{8x^2+24x+17}}{2}\right)^n\right).$$

- Generating function

$$\overline{E}(x;y) = \frac{x\left(\left(-4x^2-12x-9\right)y+2\right)}{\left(4x^4+24x^3+52x^2+48x+16\right)y^2+\left(-4x^2-12x-9\right)y+1} + \frac{x\left(x^2-2\right)}{1-(x^2+4x+4)y}.$$

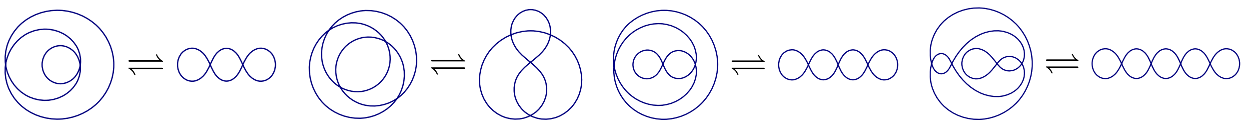
- Distribution of $(s_E(n,k))_{n,k}$:

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	1										
1	0	1	4	6	4	1								
2	0	17	56	80	64	30	8	1						
3	0	169	660	1120	1096	684	280	74	12	1				
4	0	1377	6640	14112	17504	14128	7808	3008	800	142	16	1		
5	0	10201	59660	156624	244280	252460	182544	94960	35904	9800	1880	242	20	1

Table 4: Values of $s_E(n,k)$ for $0 \leq n \leq 5$ and $0 \leq k \leq 13$.

Remark 6. Column 1 in Table 2 is sequence A004146 in the OEIS [6], the sequence of alternate Lucas numbers minus 2, which is the determinant of the Turk's Head Knots $THK(3, n)$ [4]. Column 2 is the x-coefficients of a generalized Jaco-Lucas polynomials for even indices [7] (see column 1 in triangle A122076) and is also a subsequence of a Fibonacci-Lucas convolution A099920 for odd indices. Column 1 in Table 3 is A060867 with a leading 0.

Rows 1 in Table 2, Table 3, Table 4 match the coefficients of the bracket for the 2-twist loop (see row 1 in A300184, A300192 and row 0 in A300454), the 3-twist loop and the 4-twist loop modulo planar isotopy and move on the 2-sphere [5], respectively (see Figure 2 (a), (b) and (d)). Row 2 in Table 2 gives those of the figure-eight knot (see Figure 2 (b) and row 1 in A300454).



(a) T_1 (b) T_2 (c) C_1 (d) E_1

Figure 2: Equivalent knot shadow diagrams.

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