On Certain Reciprocal sums

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Abstract

In this note we associate a sequence of non-negative integers to any convergent series of positive real numbers and study this sequence for the series $\sum_{n\geq 1} n^{-k}$ where k is an integer ≥ 2.

1 Introduction

Let $(x_i)_{i=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{i=1}^{\infty} x_i$ converges. Given such a sequence one can associate a sequence of non-negative integers $(a_n)_{n=1}^{\infty}$ by defining

$$a_n = \left[\frac{1}{\sum_{i=n+1}^{\infty} x_i}\right]$$

where $[x]$ = the largest integer $\leq x$, for a real number x.

This problem has been studied for some special class of sequences. For example, Ohtsuka and Nakamura [1] derived a formula for $x_i = \frac{1}{F_i}$, where F_i denotes the i^{th} Fibonacci number. Since then several results have been discovered about the case in which x_i s are reciprocals of a sequence given by linear recurrence relations (for example see [2]).

In this note we consider the case $x_i = i^{-k}$ where k is a positive integer ≥ 2.

The first theorem that we prove is following:

Theorem 1.1 : Let k be an integer ≥ 2. Then there is a polynomial $f(X) \in \mathbb{Q}[X]$ of degree $(k-1)$ (unique upto the constant term) and an integer N_0 (depending on f) such that

$$f(n) < \frac{1}{\sum_{i=n+1}^{\infty} i^{-k}} < f(n) + 1$$

holds for all $n \geq N_0$.

In section-2 we prove a lemma which is central to our treatment. In section-3 we prove theorem 1.1 and indicate how to compute a closed form formula for a_n and compute it for $k = 2, 3, 4, 5$. In section-4 we prove a generalization which is as follows :

Let $P(X)$ be a polynomial over $\mathbb{R}$ of degree ≥ 2 such that the leading coefficient is positive. Let $i_0 \in \mathbb{R}$ be large enough so that $P(x) > 0$ for all $x > i_0$. Put $x_i = \frac{1}{P(i+i_0)}$ for all $i \geq 1$. Clearly $\sum_{i\geq 0} x_i < \infty$. Then we have an analogus result :

Theorem 1.2 : There is a polynomial $f(X) \in \mathbb{R}[X]$ depending on P (unique upto constant term), an integer N_0 depending on f and i_0 so that degree of f is $(k-1)$ and

$$f(n + i_0) < \frac{1}{\sum_{i=n+1}^{\infty} x_i} < f(n + i_0) + 1$$

holds for all $n \geq N_0$.

2 An Important Lemma

We begin by proving a useful lemma.

Lemma 2.1: Fix an integer k, $k \geq 2$. Let $x_0, x_1, \cdots, x_{k-1}$ be k unknowns. Consider $F(X, x_0, \cdots, x_{k-1}) = x_0 X^{k-1} + x_1 X^{k-2} + \cdots + x_{k-2} X + x_{k-1} \in \mathbb{R}[X, x_0, \cdots, x_{k-1}]$. Let $g(X) \in \mathbb{R}[X]$ be a polynomial of degree k. Assume that

$$g(X + 1) = a_0 X^k + \cdots + a_k.$$

Put

$$F((X+1), x_0, \cdots, x_{k-1}) = x_0 X^{k-1} + (x_1 + y_1) X^{k-2} + \cdots + (x_{k-2} + y_{k-2}) X + (x_{k-1} + y_{k-1}),$$

$$H(X) = F((X+1), x_0, \cdots, x_{k-1})F(X, x_0, \cdots, x_{k-1}) = p_0 X^{2k-2} + \cdots + p_{2k-3}X + p_{2k-2},$$

$$G(X) = g(X+1)(F((X+1), x_0, \cdots, x_{k-1}) - F(X, x_0, \cdots, x_{k-1})) = q_0 X^{2k-2} + \cdots + q_{2k-3}X + q_{2k-2}$$

where $y_i, p_j, q_l \in \mathbb{R}[x_0, \cdots, x_{k-1}]$ for all $1 \leq i \leq k-1$, $0 \leq j \leq 2k-2$ and $0 \leq l \leq 2k-2$.

Consider the system of k equations

$$p_i = q_i, \ \forall\ 0 \leq i \leq k-1$$

in k unknowns $x_0, \cdots, x_{k-1}$.

This system of equations has a unique solution $(c_0, \cdots, c_{k-1}) \in \mathbb{R}^k$ with $c_0 \neq 0$.

Further, if g is defined over $\mathbb{Q}$ then $(c_0, \cdots, c_{k-1}) \in \mathbb{Q}^k$.

Proof: First notice that the coefficient of X^k in $F((X+1), x_0, \cdots, x_{k-1})$ is indeed x_0 and degrees of $G(X), H(X)$ in X are indeed at most $(2k-2)$. Hence the hypothesis of the lemma is justified.

Now an application of binomial theorem gives

$$y_i = \binom{k-i}{1} x_{i-1} + \binom{k-i+1}{2} x_{i-2} + \cdots + \binom{k-1}{i} x_0 \tag{2.1}$$

for each $1 \leq i \leq k-1$.

A direct calculation gives

$$p_j = \sum_{r=0}^{j} x_r (x_{j-r} + y_{j-r}) \tag{2.2}$$

for all $0 \leq j \leq k-1$ where $y_0 = 0$.

Similarly,

$$q_l = \sum_{r=0}^{l} a_r y_{l-r+1} \tag{2.3}$$

for all $0 \leq l \leq k-1$ where $y_k = 0$.

Thus $p_0 = x_0^2$ and $q_0 = y_1 = a_o \binom{k-1}{1} x_0 = a_0(k-1)x_0$.

Hence $x_0 = a_0(k-1)$ is a solution to $p_0 = q_0$. Clearly $a_0(k-1) \neq 0$.

Now notice that p_i depends only on $\{x_0, \cdots, x_i, y_0, \cdots, y_i\}$.

By (2.1), this observation implies p_i depends only on $\{x_0, \cdots, x_i\}$. This holds for all $0 \leq i \leq k-1$. Similarly q_i depends only on $\{y_1, \cdots, y_{i+1}\}$ i.e. only on $\{x_0, \cdots, x_i\}$. This is true for all $0 \leq i \leq k-1$.

So we can take an inductive approach to solve the system of equations.

We have already found a c_0 (namely $a_0(k-1)$) such that $x_0 = c_0$ solves $p_0 = q_0$ and $c_0 \neq 0$. Note that this is the only non-zero solution to $p_0 = q_0$ and if $a_0 \in \mathbb{Q}$ then $c_0 \in \mathbb{Q}$.

Assume that we have found $(c_0, \cdots, c_i) \in \mathbb{R}^{i+1}$ such that $c_0 \neq 0$ and this is the unique tuple solving the system of equations

$$p_0 = q_0, \cdots, p_i = q_i$$

for some i in the range $0 \leq i \leq k-2$. Further if g is defined over $\mathbb{Q}$ then $\mathbb{Q}^{i+1}$.

Now the goal is to find a $c_{i+1} \in \mathbb{R}$ such that $(c_0, \cdots, c_{i+1})$ solves $p_{i+1} = q_{i+1}$.

We consider two cases:

Case I: $i < k-2$

Here $i+1 \leq k-2$.

Coefficient of x_{i+1} in p_{i+1} is $= 2x_0$ (one x_0 arises from term $x_0(x_{i+1} + y_{i+1})$ and other x_0 arises from the term $x_{i+1}(x_0 + y_0)$).(Follows from (2.1) and (2.2))

Coefficient of x_{i+1} in q_{i+1} is $= a_0$(coefficient of x_{i+1} in y_{i+2}) $= a_0\binom{k-(i+2)}{1} = a_0(k-(i+2))$ (follows from (2.1) and (2.2)).

Hence $p_{i+1} = q_{i+1}$ can be rewritten as

$$\{2x_0 - a_0(k-i-2)\}x_{i+1} = \text{some polynomial in } x_0, \cdots, x_i \tag{2.4}$$

Note that $\{2c_0 - a_0(k-i-2)\} = a_0(k+i) \neq 0$. Thus we can put $x_0 = c_0, \cdots, x_i = c_i$ in (2.4) and solve for x_{i+1} to get a tuple $(c_0, \cdots, c_{i+1}) \in \mathbb{R}^{i+2}$ which is a solution for the system of equations

$$p_0 = q_0, \cdots, p_{i+1} = q_{i+1}.$$

Now if $(d_0, \cdots, d_{i+1})$ is another solution with $d_0 \neq 0$ then by induction hypothesis $(c_0, \cdots, c_i) = (d_0, \cdots, d_i)$. From (2.4) it follows that $d_{i+1} = c_{i+1}$. Hence the uniqueness.

If g is defined over $\mathbb{Q}$ then by induction hypothesis $(c_0, \cdots, c_i) \in \mathbb{Q}^{i+1}$. Since p_{i+1} and q_{i+1} are defined over $\mathbb{Q}$ using (2.4) we conclude that $c_{i+1} \in \mathbb{Q}$.

So for this case we are done.

Case II: $i = k-2$

This is essentially similar to previous case. Only difference is coefficient of x_{k-1} in q_{k-1} is 0.

So $p_{k-1} = q_{k-1}$ can be rewritten as $2x_0x_{k-1} =$ some polynomial in $x_0, \cdots, x_{k-2}$ and from here the arguments of previous case goes through since $c_0 \neq 0$.

Thus inductively we can find $(c_0, \cdots, c_{k-1}) \in \mathbb{R}^k$ such that this tuple is the unique solution to the system of equations under consideration with $c_0 \neq 0$. Further if g is defined over $\mathbb{Q}$ then $(c_0, \cdots, c_{k-1}) \in \mathbb{Q}^k$.

This completes the proof of lemma. □

3 Proof of theorem 1.2

At first we prove theorem 1.2 and deduce theorem 1.1 as a corollary.

We shall use lemma 2.1 with $g(X) = P(X)$. Say, leading co-efficient of P is $a_0 > 0$.

Let k be an integer ≥ 2.

Let $(c_0, \cdots, c_{k-1}) \in \mathbb{R}^k$ be the tuple as in lemma 2.1.

We continue to use notations from lemma 2.1.

Put $f(X) = F(X, c_0, \cdots, c_{k-2}, c) = c_0X^{k-1} + \cdots + c_{k-2}X + c$ where c is any real number satisfying $c < c_{k-1} < c + \frac{k}{k-1}$.

Now

$$\frac{1}{f(X)} - \frac{1}{f(X+1)} - \frac{1}{g(X+1)} = \frac{g(X+1)(f(X+1) - f(X)) - f(X)f(X+1)}{f(X)f(X+1)g(X+1)}$$

Consider the expression on the numerator.

By choice of $(c_0, .., c_{k-2})$ the coefficients of $X^{2k-2}, \cdots, X^k$ vanishes (i.e. $p_i = q_i$ holds for all $0 \leq i \leq k-2$).

Coefficient of X^{k-1} is $(q_{k-1}(c_0, \cdots, c_{k-2}, c) - p_{k-1}(c_0, \cdots, c))$.

From proof of lemma 2.1 we have q_{k-1} does not depends on x_{k-1}.

Hence $q_{k-1}(c_0, .., c_{k-2}, c) = q_{k-1}(c_0, \cdots, c_{k-2}, c_{k-1}) = p_{k-1}(c_0, \cdots, c_{k-2}, c_{k-1})$. (By choice of the tuple$(c_0, \cdots, c_{k-1})$)

Again from the proof of lemma 2.1 the coefficient of x_{k-1} in p_{k-1} is $2x_0$. Thus

$q_{k-1}(c_0, \cdots, c_{k-2}, c) - p_{k-1}(c_0, \cdots, c)$

$= p_{k-1}(c_0, \cdots, c_{k-2}, c_{k-1}) - p_{k-1}(c_0, \cdots, c_{k-2}, c)$

$= 2c_0(c_{k-1} - c)$

Hence the coefficient of the leading term of the polynomial in the numerator is $2c_0(c_{k-1} - c)$. But $c_0 = a_0(k-1) > 0$. So $2c_0(c_{k-1} - c) > 0$.
The coefficient of the leading term of the polynomial in the denominator is $c_0^2 > 0$.
Hence the is a large enough natural number N_1 such that for all $i \geq N_1$

$$\frac{1}{f(i+i_0)} - \frac{1}{f(i+i_0+1)} - \frac{1}{g(i+i_0+1)} > 0$$

i.e.

$$\frac{1}{f(i+i_0)} - \frac{1}{f(i+i_0+1)} > \frac{1}{g(i+i_0+1)}$$

From here telescoping we get

$$\frac{1}{f(n+i_0)} > \sum_{i=n+1}^{\infty} \frac{1}{g(i+i_0)} \tag{3.1}$$

for all $n \geq N_1 + i_0$. Now

$$\frac{1}{f(X)+1} - \frac{1}{f(X+1)+1} - \frac{1}{g(X+1)}$$
$$= \frac{g(X+1)(f(X+1) - f(X)) - (f(X)+1)(f(X+1)+1)}{(f(X)+1)(f(X+1)+1)g(X+1)}$$

Note that $(f(X)+1)(f(X+1)+1) - f(X)f(X+1)$ has degree equal to $(k-1)$. Hence co-efficient of X^{k-1} in $(f(X)+1)(f(X+1)+1)$ is $p_{k-1}(c_0, \cdots c_{k-2}, c) + 2a_0$. Similar calculation suggests that the coefficient of the leading term in the numerator is $2c_0(c_{k-1} - c - 1) - 2a_0 = 2c_0(c_{k-1} - c - \frac{k}{k-1}) < 0$.
But the coefficient of the leading term in the denominator is $c_0^2 > 0$.
Thus there is a large enough integer N_2 such that for all $i \geq N_2$

$$\frac{1}{f(i+i_0)+1} - \frac{1}{f(i+i_0+1)+1} < \frac{1}{g(i+i_0+1)}$$

Again by telescoping

$$\frac{1}{f(n)+1} < \sum_{i=n+1}^{\infty} \frac{1}{g(i+i_0)} \tag{3.2}$$

for all $n \geq N_2 + i_0$.

Put $N_0 = \max\{N_1, N_2\} + i_0$.
Then

$$\frac{1}{f(n)+1} < \sum_{i=n+1}^{\infty} \frac{1}{g(i)} < \frac{1}{f(n)} \tag{3.3}$$

for all $n \geq N_0$.

From (3.3) the theorem 1.2 follows. □

Remark 3.1: i) The proof of lemma 2.1 and proof of theorem 1.1 gives an algorithm to compute the polynomial $f(X)$ mentioned in the statement of the theorem. Note that this polynomial is not unique. We can choose infinitely many distinct values for the constant term. The integer N_0 depends on the choice of the polynomial f.
ii) It is natural to ask whether we can take $c = c_{k-1}$.
Put

$$f_1(X) = c_0 X^{k-1} + c_1 X^{k-2} + \cdots + c_{k-2} X + c_{k-1}.$$

We consider three possible cases:

Case I: $p_i(c_0, \cdots, c_{k-1}) = q_i(c_0, \cdots, c_{k-1})$ for all $0 \leq i \leq 2k-2$.
Then

$$\frac{1}{g(X+1)} = \frac{1}{f_1(X)} - \frac{1}{f_1(X+1)}.$$

Then by telescoping $\sum_{i=n+1}^{\infty} \frac{1}{g(i+i_0)} = \frac{1}{f_1(n+i_0)}$ holds for all positive integer n.
Since $c_{k-1} < \frac{k}{k-1} + c_{k-1}$ arguments as in proof of the theorem implies $\sum_{i=n+1}^{\infty} \frac{1}{g(n+i_0)} > \frac{1}{f_1(n+i_0)+1}$ for large enough n.
Thus

$$f_1(n+i_0) \leq \Big(\sum_{i \geq n+1} \frac{1}{g(i+i_0)} \Big)^{-1} < f_1(n+i_0) + 1$$

holds for large enough n.

Now say, case I does not hold. Then there is a i with $0 \le i \le 2k-2$ such that $p_i \ne q_i$ at the point $(c_0, \cdots, c_{k-1})$.
Let i_0 be the minimum of such i s. Note that here we must have $i_0 \ge k$. Now

Case II: $q_{i_0}(c_0, \cdots, c_{k-1}) > p_{i_0}(c_0, \cdots, c_{k-1})$
Then $\sum_{i=n+1}^{\infty} \frac{1}{g(i+i_0)} < \frac{1}{f_1(n+i_0)}$ for large enough n by arguments similar to the proof of the theorem.
Since $c_{k-1} < c_{k-1} + \frac{k}{k-1}$, so we have $\sum_{i=n+1}^{\infty} \frac{1}{g(n+i_0)} > \frac{1}{f_1(n+i_0)+1}$ for large enough n.
Thus $f_1(X)$ satisfies the required property of f in the theorem.

Case III: $q_{i_0}(c_0, \cdots, c_{k-1}) < p_{i_0}(c_0, \cdots, c_{k-1})$
Again by similar arguments $\sum_{i=n+1}^{\infty} \frac{1}{g(i+i_0)} > \frac{1}{f_1(n+i_0)}$ for large enough n.
Now since $c_{k-1} - 1 < c_{k-1}$, so we have $\sum_{i=n+1}^{\infty} \frac{1}{g(i+i_0)} < \frac{1}{f_1(n+i_0)-1}$ for large enough n. So we can not take $c = c_{k-1}$ but can take $c = c_{k-1} - 1$.

Corollary 3.2 (Theorem 1.1): Put $P(X) = X^k$, $i_0 = 0$. Using lemma-2.1 we conclude that $(c_0, \cdots, c_{k-1}) \in \mathbb{Q}^k$. Clearly one can choose c to be rational. Hence theorem 1.1. Note that considerations in remark 3.1 hold accordingly.

4 Computation of a_n

First we make a small observation:

Remark 4.1 : Let f be any polynomial given by theorem 1.2. Let N_0 be the corresponding integer. Then from the inequality in theorem 1.2 it follows that for all $n \ge N_0$
i) a_n is either $[f(n)]$ or $[f(n)] + 1$.
ii) If $f(n)$ is an integer for some n then $a_n = f(n)$.
iii) Conclusion in i) and ii) continue to hold if we replace the '$<$' sign in the inequality at the left hand side in the statement of the theorem by '$\le$'.

For the rest of the section we shall assume that P is defined over $\mathbb{Q}$. Further, we shift the polynomial so that we can take $i_0 = 0$. The polynomial X^k satisfies these properties.

4.1 A general algorithm

Fix a polynomial P as above.
Calculate $(c_0, \cdots, c_{k-1})$.
Write $c_i = \frac{u_i}{v_i}$ where u_i, v_i are integers with $v_i > 0$ and $\gcd(u_i, v_i) = 1$ for all $0 \leq i \leq k-1$.
Put $V = \text{lcm}\,(v_0, \cdots, v_{k-2})$.
Now consider two cases:

Case I : v_{k-1} does not divide V.
Write $c_{k-1} = [c_{k-1}] + \frac{r_{k-1}}{v_{k-1}}$ where r_{k-1} is a positive integer. Since $\gcd(u_{k-1}, v_{k-1}) = 1$, one has $\gcd(r_{k-1}, v_{k-1}) = 1$. So $\frac{r_{k-1}}{v_{k-1}} \neq \frac{n}{V}$ for any $n \in \mathbb{Z}$.
Let $r \in \{0, \cdots, V-1\}$ be fixed.
Then there is a unique integer $n(r)$ such that $n(r) - \frac{r}{V} < c_{k-1} < n(r) + 1 - \frac{r}{V}$.
Put

$$h(X) = c_0 X^{k-1} + \cdots + c_{k-2} X = \frac{h_0(X)}{V}$$

where $h_0(X) \in \mathbb{Z}[X]$.
Let

$$f_r(X) = h(X) + n(r) - \frac{r}{V}.$$

Now there is an integer $N(r)$ such that

$$f_r(n) < \frac{1}{\sum_{i=n+1}^{\infty} \frac{1}{P(i)}} < f_r(n) + 1$$

for all $n \geq N(r)$. Choose such a $N(r)$.
We do this for each $r \in \{0, \cdots, V-1\}$.
Put $N = \max\{N(0), \cdots, N(V-1)\}$.
Let $n \geq N$ and r be such that $r \in \{0, \cdots, V-1\}$ and $h_0(n) \equiv r \pmod{V}$. Clearly such r exists and is unique.
Then using remark 4.1 (ii) we have $a_n = f_r(n)$.

Now note that $n_1 \equiv n_2 \bmod(V)$ implies $h_0(n_1) \equiv h_0(n_2) \bmod(V)$.
Thus in this case we have a closed form formula for a_n depending on equivalence class of n modulo V whenever $n \geq N$.

Case II: Case I does not hold.
Fix $r \in \{1, \cdots, V\}$.
If $\frac{r}{V} \neq 1 + [c_{k-1}] - c_{k-1}$, there is an unique integer $n(r)$ such that $n(r) - \frac{r}{V} < c_{k-1} < n(r) + 1 - \frac{r}{V}$.
Otherwise there is an integer n such that $n - \frac{r}{V} = c_{k-1}$.
Now we need to do further calculation and find out which case of remark 3.1 ii) holds. If case I or II holds then put $n(r) = n$. If case III holds put $n(r) = n - 1$.
Let

$$f_r(X) = h(X) + n(r) - \frac{r}{V}.$$

Using previous arguments and the discussion in remark 3.1 ii), there is an integer $N(r)$ such that

$$f_r(n) \leq \frac{1}{\sum_{i=n+1}^{\infty} \frac{1}{P(i)}} < f_r(n) + 1$$

for all $n \geq N(r)$.
Due to remark 3.1 iii) the arguments of case I goes through from here.

4.2 Explicit formulae

Consider the polynomial $P(X) = X^k$, $k \geq 2$.

For $k = 2$, $(c_0, c_1) = (1, \frac{1}{2})$.
Here $a_n = n$ for all $n \geq 1$ (ie one can take $N_0 = 1$).

For $k = 3$, $(c_0, c_1, c_2) = (2, 2, 1)$.
Here $a_n = 2n(n + 1)$ for all $n \geq 1$.

For $k = 4$, $(c_0, c_1, c_2, c_3) = (3, \frac{9}{2}, \frac{15}{4}, \frac{9}{8})$.

Here

$$a_n = \begin{cases} 3X^3 + \frac{9}{2}X^2 + \frac{15}{4}X + 1 \text{ if } n \equiv 0 (\bmod 4), \\ 3X^3 + \frac{9}{2}X^2 + \frac{15}{4}X + \frac{3}{4} \text{ if } n \equiv 1 (\bmod 4), \\ 3X^3 + \frac{9}{2}X^2 + \frac{15}{4}X + \frac{1}{2} \text{ if } n \equiv 2 (\bmod 4), \\ 3X^3 + \frac{9}{2}X^2 + \frac{15}{4}X + \frac{1}{4} \text{ if } n \equiv 3 (\bmod 4). \end{cases}$$

For $k = 5$, $(c_0, c_1, c_2, c_3, c_4) = (4, 8, \frac{28}{3}, \frac{16}{3}, -\frac{2}{9})$.
Here

$$a_n = \begin{cases} 4X^4 + 8X^3 + \frac{28}{3}X^2 + \frac{16}{3}X - 1 \text{ if } n \equiv 0 (\bmod 3), \\ 4X^4 + 8X^3 + \frac{28}{3}X^2 + \frac{16}{3}X - \frac{2}{3} \text{ if } n \equiv 1 (\bmod 3), \\ 4X^4 + 8X^3 + \frac{28}{3}X^2 + \frac{16}{3}X - 1 \text{ if } n \equiv 0 (\bmod 3). \end{cases}$$

Remark 3.2 : Results above answer two questions due to Kotesovec [3].

5 Concluding remarks

We end with some questions associated to the system of equations which come up in lemma 2.1 :

i) Consider the sequence of polynomials $\{P_k(X)\}_{k \geq 2}$ given by $P_k(X) = X^k$. With this sequence one can associate a sequence $(c_0, c_1, \cdots)$ where c_i is a function $\mathbb{N} - \{1, \cdots i+1\} \to \mathbb{R}$ such that $(c_0(k), \cdots, c_{k-1}(k))$ is the tuple associated to $P_k(X)$. From lemma 2.1 it follows that $c_i(k)$ must be a rational fuction of k. Computing first few elements of $(c_0(k), c_1(k), \cdots)$ one sees that it is actually a polynomial in k. This leads to the question if c_i is always a polynomial in k.

ii) One can consider a sequence qiven by $P_k(X) = X^k P_0(X)$ for some fixed polynomial $P_0(X)$ and ask similar question.

iii) Fix two polynomials $P(X), Q(X)$. Construct a sequence by $P_k(X) = P(X)Q(X)^k$. In this case c_i need not be a rational function but one may like to study the behaviour of the associated sequence of functions.

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