

ON TRIANGULAR PAPERFOLDING PATTERNS

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ABSTRACT. We introduce patterns on a triangular grid generated by paperfolding operations. We show that in case these patterns are defined using a periodic sequence of foldings, they can also be generated using substitution rules and compute eigenvalues and eigenvectors of corresponding matrices. We also prove that densities of all basic triangles are equal in these patterns.

1. INTRODUCTION

A classical paperfolding sequence can be constructed by taking a strip of paper of length 2^n and folding it n times at the center. If we then unfold the resulted strip of length 1 back into the strip of length 2^n , then we can read sequence of valleys and peaks. The one sided paperfolding sequence is defined as taking the limit as n goes to infinity of the pattern read from left (see [13, A014577]). The two-sided paperfolding sequence is defined as the same pattern read from the center of the strip again as n goes to infinity (see [3, Sect. 4.5]).

The properties of the paperfolding sequences have been studied by many authors [1, 2, 6, 8, 12]. In particular, the paper [8] by Dekking, Mendès-France, and van der Poorten exhibits a set of substitution rules to generate the paperfolding sequence.

The substitution sequence was generalized to higher-dimensional case by Ben-Abraham, Quandt and Shapiraa [4]. The paper [11] by Gähler and Nilsson provides substitution rules for the corresponding paperfolding structures and studies various properties of the corresponding patterns.

In this paper we study a family of patterns that can be generated using a paperfolding approach. The patterns are formed by peaks and valleys on a triangular grid and obtained by repeated folding of a regular triangle along its midsegments. On each stage the foldings are done through upper or lower halfspace of the ambient 3-dimensional space, but the choice of upper or lower halfspace is independent for different stages in general.

The main results of the paper are the following. In theorem 4.2 we prove that if we always perform the foldings through upper halfspace, then the resulted pattern can be generated using substitution rules. In theorem 5.1 we generalize this result for the case of periodic choices between upper and lower halfspaces. In theorem 7.4 we obtain all eigenvalues and eigenvectors for corresponding substitution matrices. Finally, in theorem 6.5 we prove that for any choice of halfspaces (not necessarily periodic) densities of all types of unit triangles constituting the corresponding pattern are equal.

The paper is organized as follows. Section 2 contains basic definitions of triangular folding patterns and related triangular tilings.

Section 3 contains preliminary lemmas that describe the structure of these patterns.

In Section 4 we prove that the triangular folding pattern defined by all-up sequence of elementary foldings can be defined using a substitution rule. Section 5 is devoted to the proof of existence of a substitution rule for any periodic sequence of elementary foldings.

In Section 6 we prove that for any sequence of elementary foldings (periodic or not) the densities of different types of triangles present in the corresponding triangular tiling exist and equal. The types of triangles here are defined using Definition 2.6.

In Section 7 we obtain the eigenvalues and the eigenvectors of the substitution matrix for a periodic sequence of elementary foldings.

2. BASIC NOTIONS

Definition 2.1. We will work only with regular triangles of a fixed triangular lattice (or grid) \mathcal{L} obtained from a grid of horizontal lines and its rotations by $\pi/3$ and $2\pi/3$ (see figure 1). We assume that the smallest triangle in this grid has side 1.

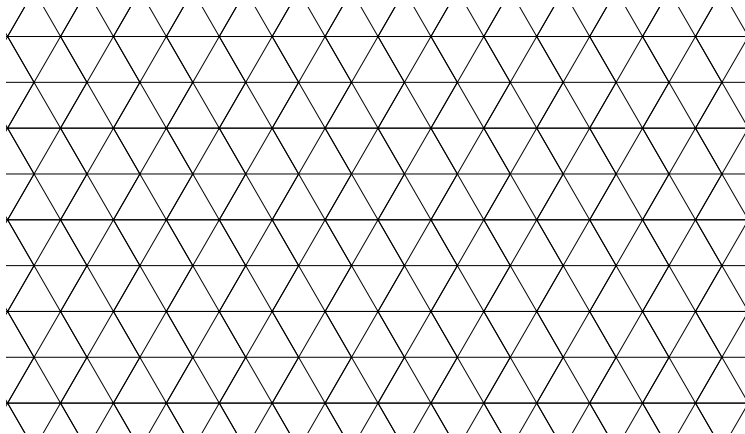


FIGURE 1. Triangular grid \mathcal{L} .

A regular triangle with integer side $n \geq 1$ of the grid \mathcal{L} is called *positive* if its third vertex is higher than its horizontal side and it is called *negative* otherwise.



FIGURE 2. A positive triangle with side 2 and a negative triangle with side 3.

Definition 2.2. By an *elementary triangular folding* we mean a folding of a regular triangle, positive or negative, with an even side $2a$ into a regular triangle, negative or positive respectively, with side a (covered four times) by folding it in the midsegments (see figure 3).

In general the separate foldings in each midsegment are independent, so each can be done through the upper or through the lower halfspace of the ambient three-dimensional space. In general we will study only the elementary foldings with all foldings in each midsegment done through the same halfspace. If all three foldings are done through the upper halfspace, then we call the elementary folding a *folding up*. Alternatively we call the elementary folding a *folding down* if all three foldings are done through the lower halfspace.

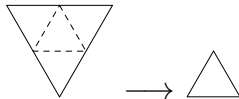


FIGURE 3. Elementary triangular folding.

If we unfold the four-layered triangle with side a back into triangle with side $2a$, then the midsegments will form peaks if the elementary folding was a folding down, or valleys if the elementary folding was a folding up. In the former case we will color the midsegments with blue, and in the latter case with red.

Suppose $\mathcal{F} = \{a_1, \dots, a_k\}$ is a sequence of elementary foldings, so each a_i encodes whether the corresponding folding should be done through upper or lower halfspace forming valley or peaks respectively.

We construct a *folding pattern* corresponding to \mathcal{F} in the following way. We take a regular triangle T (made from paper) with side 2^k which is positive if k is even and negative if k is odd. Then we fold T according to the elementary foldings in \mathcal{F} performing a_k first, then a_{k-1} , and so on, finishing with a_1 . In the end we get a positive triangle with side 1 consisting of 2^{2k} layers (of paper).

If we unfold the triangle back to its initial size, then every unit segment of the grid \mathcal{L} in T will become a valley or a peak. Note, that we unfold a triangle with side 2^n ($n < k$) that has a folding pattern of peaks and valleys, then the resulting triangle with side 2^{n+1} will have the same pattern inside its central part, but the patterns in the “side” parts will be different because each valley will become a peak after a single unfolding, and each peak will become a valley. We will describe this dependence in more details later.

Definition 2.3. The *folding pattern* $P_{\mathcal{F}}$ corresponding to \mathcal{F} is the coloring of all peaks in the resulted unfolded triangle T with blue, and all the valleys with red. An example of the folding pattern corresponding to $\mathcal{F} = \{a_1, \dots, a_6\}$ where each a_i is done through upper halfspace, or $\mathcal{F} = \{+, +, +, +, +, +\}$ for short, is shown on Figure 4.

If a sequence $\mathcal{F}' = \{a_i\}_{i=1}^m$ is a subsequence of $\mathcal{F} = \{a_i\}_{i=1}^k$, ($m < k$), then the pattern $P_{\mathcal{F}'}$ is a subpattern of $P_{\mathcal{F}}$. Namely, the central triangle with side 2^m of $P_{\mathcal{F}}$ will be colored according to the pattern $P_{\mathcal{F}'}$. This allows to introduce us a coloring of the grid \mathcal{L} as the pattern corresponding to an infinite sequence of elementary foldings.

Definition 2.4. Let $\mathcal{A} = \{a_i\}_{i=1}^{\infty}$ be a sequence of elementary foldings. We fix a positive unit triangle T_0 of \mathcal{L} and define $P_{\mathcal{A}}$ as the limit of patterns $P_{\mathcal{A}_n}$ where $\mathcal{A}_n = \{a_i\}_{i=1}^n$ and each $P_{\mathcal{A}_n}$ has T_0 as its central triangle.

The limit is correctly defined because as n increases the patterns $P_{\mathcal{A}_n}$ cover larger and larger neighborhoods of T_0 , and any two such patterns agree on the common neighborhood.

Definition 2.5. Using the pattern $P_{\mathcal{A}}$ we can define the hull $H_{\mathcal{A}}$ corresponding to an infinite sequence \mathcal{A} as a topological space of all patterns on \mathcal{L} that possess same local patterns as $P_{\mathcal{A}}$ and its translations. It is clear that the resulted hull does not depend on the choice of T_0 . We refer to [3, Sect. 5.4] for more detailed definition of geometric hull associated with a given pattern.

The pattern $P_{\mathcal{A}}$ is given as a coloring of the grid \mathcal{L} , however we can transform it in the tiling of the plane with decorated regular triangles of eight types defined by one of four colors and positivity/negativity. Namely, we have four colors for positive triangles based on

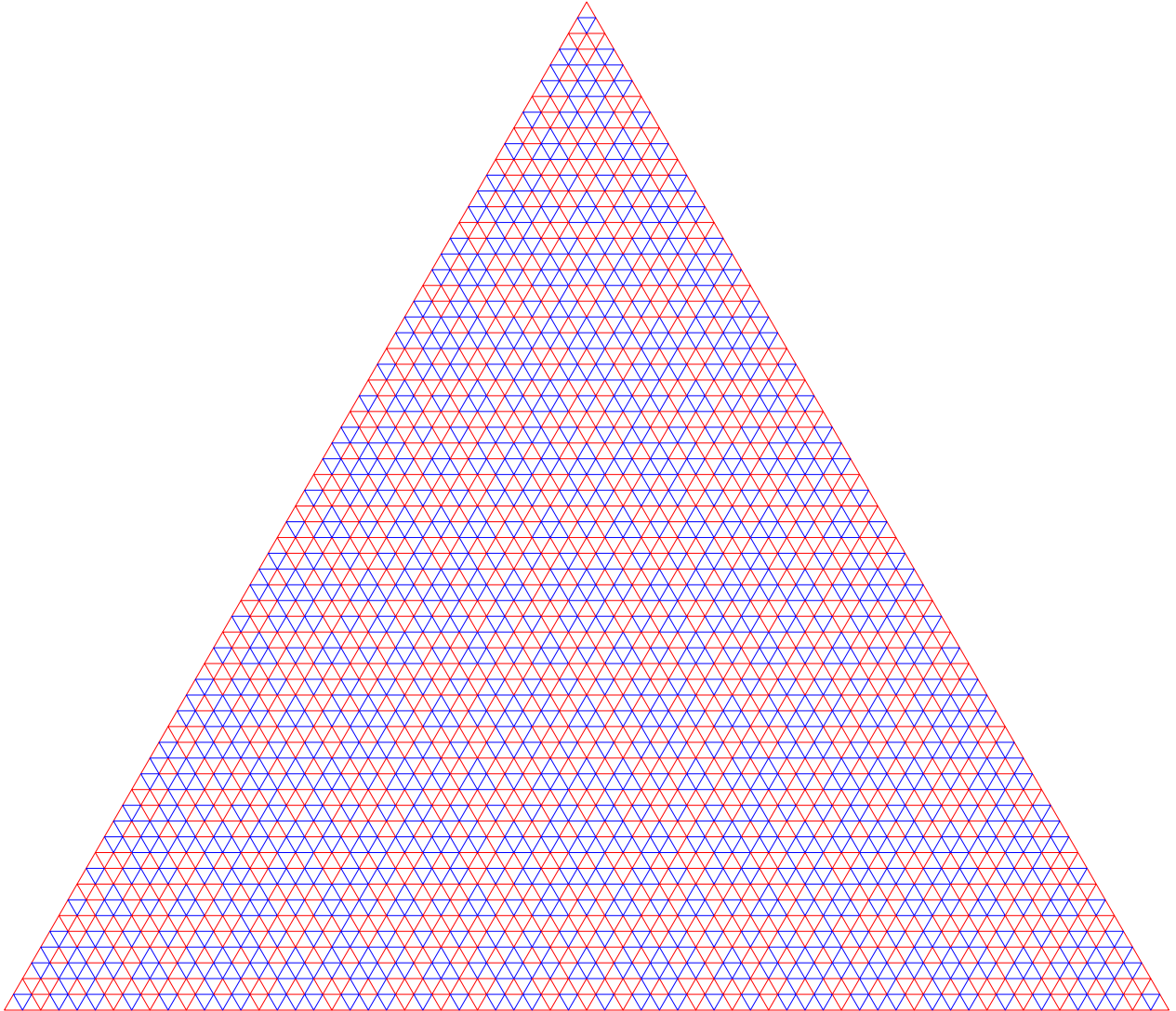


FIGURE 4. A folding pattern for sequence of six elementary foldings up (for certain choice of boundary).

the number of red sides from 0 to 3 with decoration for the triangles with 1 or 2 red sides opposite to the only red or blue side of the triangle, see Figure 5. Similarly we have four colors for negative triangles, see Figure 5 as well.

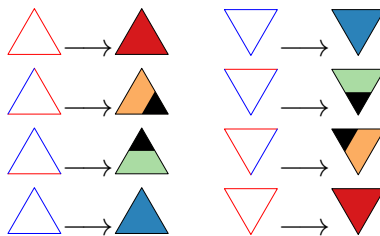


FIGURE 5. The rules for constructing the tiling $T_{\mathcal{A}}$ from the pattern $P_{\mathcal{A}}$.

Definition 2.6. We will call the resulted tiling $T_{\mathcal{A}}$ the *folding tiling* associated with \mathcal{A} .

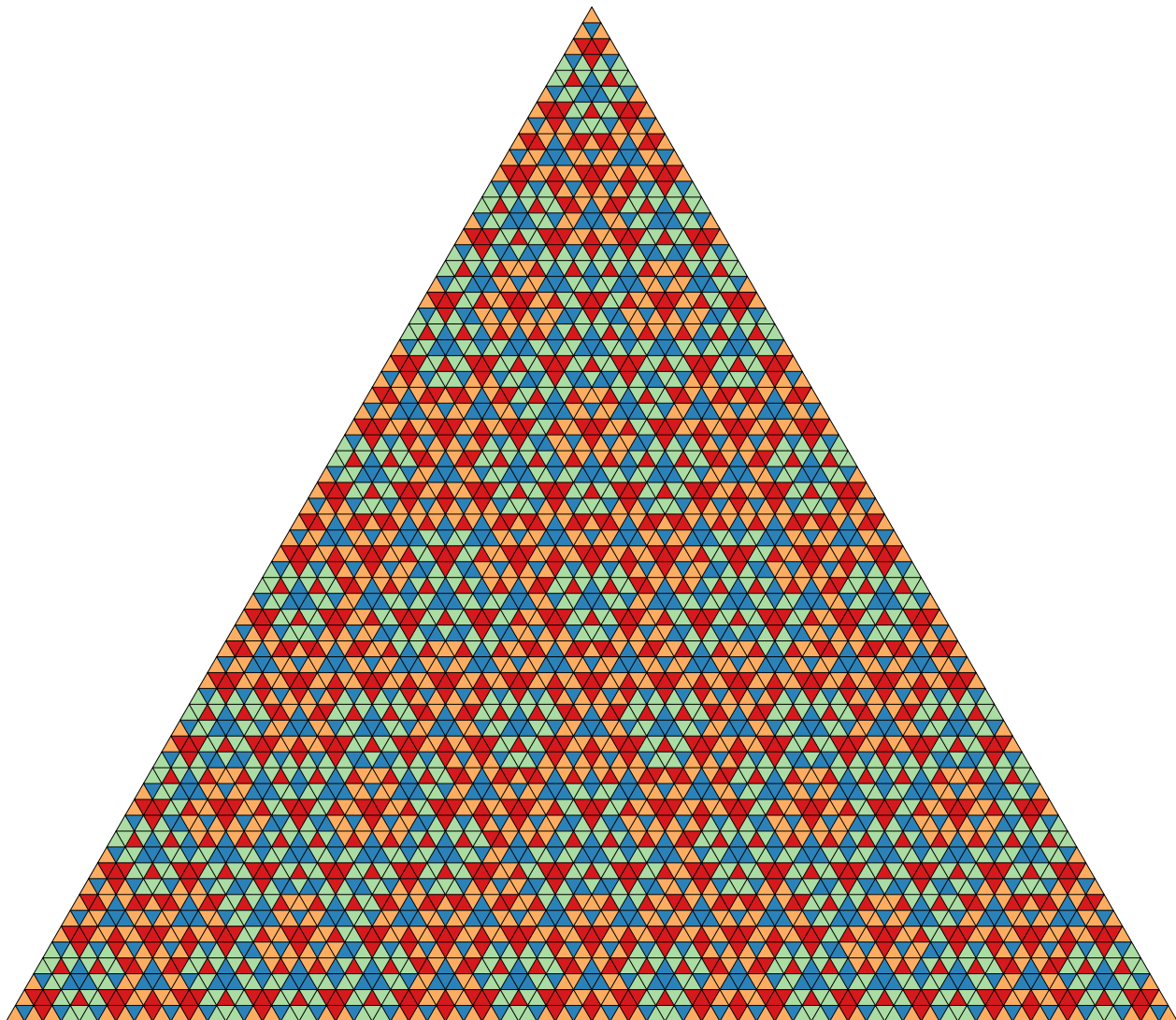


FIGURE 6. Tiling for sequence of six elementary foldings up without decoration (for certain choice of boundary).

It is clear that the pattern $P_{\mathcal{A}}$ and tiling $T_{\mathcal{A}}$ belong to the same MLD class (mutually locally derivable, see [3, Sect. 5.2] for definition and details). So as the corresponding hulls are MLD as well. Later we will show that removal of the decoration on triangles preserves the MLD class.

3. PRELIMINARY LEMMAS

Recall that T_0 is the positive unit triangle we use as our “reference point”, that is if we construct a folding pattern for a finite sequence, then T_0 is exactly the triangle we get after performing all elementary foldings. Let O denote the center of T_0 . Let ξ_1 be the vertical vector of length $2\sqrt{3}$ pointing downwards and let ξ_2 and ξ_3 be counterclockwise rotations of

ξ_1 by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ respectively. Then the grid \mathcal{L} can be written as the following disjoint union of lines

$$\mathcal{L} = \bigsqcup_{i=1}^3 \bigsqcup_{n \in \mathbb{Z}} \ell_{i,n}$$

where $\ell_{i,n} = \{\mathbf{x} \in \mathbb{R}^2 \mid (\mathbf{x}, \xi_i) = 3n + 1\}$.

Definition 3.1. For $k \in \mathbb{N}$ by k th layer \mathcal{L}_k of \mathcal{L} we denote the union

$$\mathcal{L}_k = \bigsqcup_{i=1}^3 \bigsqcup_{n \in \Lambda_k} \ell_{i,n}$$

where $\Lambda_k = 2^k \mathbb{Z} + \frac{1}{3}((-2)^{k-1} - 1)$.

The layer \mathcal{L}_1 is shown on the figure 7.

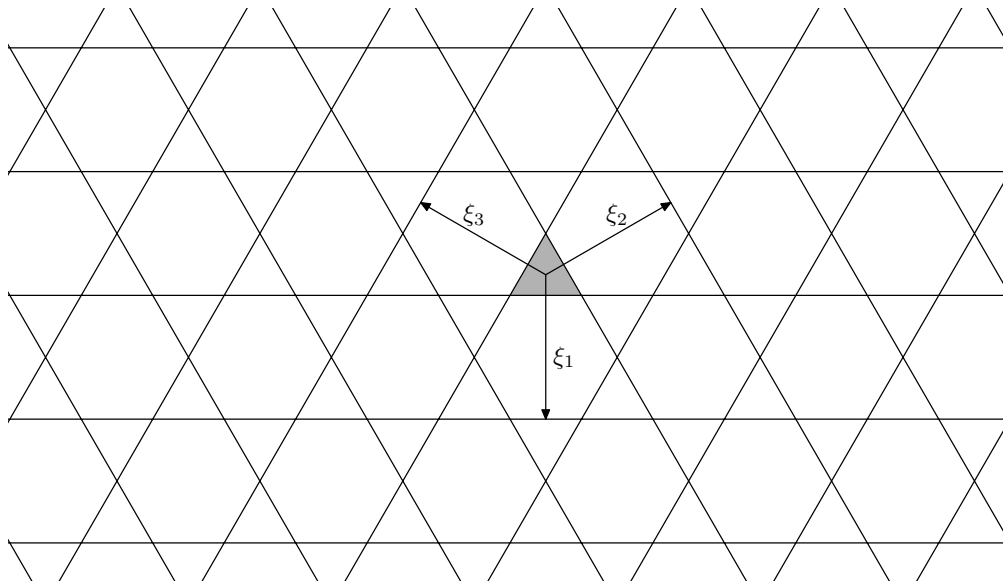


FIGURE 7. Layer \mathcal{L}_1 with shaded triangle T_0 and the vectors ξ_i .

Another equivalent description of layers is the following. We put the line $\ell_{i,n}$ in the layer \mathcal{L}_k if and only if the largest power of 2 that divides $3n + 1$ is 2^{k-1} . It is easy to see from that description that

$$\mathcal{L} = \bigsqcup_{k \in \mathbb{N}} \mathcal{L}_k.$$

Moreover, \mathcal{L}_{k+1} is the (-2) -dilation of \mathcal{L}_k with respect to O . Indeed, the index n for the line $\ell_{i,n}$ corresponds to the line with equation $(\mathbf{x}, \xi_i) = 3n + 1$. After (-2) -dilation we get the line $(\mathbf{x}, \xi_i) = -2(3n + 1) = -6n - 2 = 3(-2n - 1) + 1$ or the line $\ell_{i,-2n-1}$. If $\ell_{i,n} \in \mathcal{L}_k$, then the largest power of 2 that divides $3n + 1$ is 2^{k-1} and therefore the largest of power of 2 that divides $3(-2n - 1) + 1 = -2(3n + 1)$ is 2^k .

From the figure 7 we can see that \mathcal{L}_1 can be viewed as a collection of triangles with side 1. Similarly, since the layer \mathcal{L}_2 is a (-2) -dilation of \mathcal{L}_1 , it can be viewed as a collection of triangles with side 2. In general, the layer \mathcal{L}_k is a collection of triangles with side 2^{k-1} .

Definition 3.2. We will call these triangles with side 2^{k-1} *positive or negative triangles of the layer \mathcal{L}_k* .

The remaining regular hexagons with side 2^{k-1} are called *hexagons of the layer \mathcal{L}_k* .

We note that six segments at a vertex of \mathcal{L} are organized as shown in figure 8 (rotations are allowed). In order to see that, we claim that a vertex v of \mathcal{L}_k is a vertex of some layer \mathcal{L}_k . Indeed, it is true if a vertex belongs to \mathcal{L}_0 , and if it doesn't, then it is in the union $\bigsqcup_{k=2}^{\infty} \mathcal{L}_k = (-2)\mathcal{L}$, and we can repeat these arguments. Then four of six segments at v are segments of \mathcal{L}_k and they form two pairs of opposite segments such that two neighbor segments are colored with red, and two segments opposite to these red segments are colored with blue. The remaining two segments are opposite and belong to a side of triangle of some layer \mathcal{L}_n for $n > k$, therefore they are colored both with red or both with blue. Both these cases are exactly the cases from figure 8.



FIGURE 8. Two possible arrangements at a vertex of \mathcal{L} .

Lemma 3.3. Let $\mathcal{A} = \{a_i\}_{i=1}^{\infty}$ be an infinite sequence of elementary foldings, and let $P_{\mathcal{A}}$ be the corresponding folding pattern. Then for every $k \in \mathbb{N}$ the coloring of \mathcal{L}_k in $P_{\mathcal{A}}$ is defined by a_k only.

Namely, if k is odd and a_k is a folding up or if k is even and a_k is a folding down, then positive triangles of \mathcal{L}_k are colored with red and negative triangles if \mathcal{L}_l are colored with blue. Alternatively, if k is odd and a_k is a folding down or if k is even and a_k is a folding up, then positive triangles of \mathcal{L}_k are colored with blue and negative triangles if \mathcal{L}_l are colored with red.

Proof. We will prove by induction that for any $n \geq k$ the intersection of \mathcal{L}_k with the triangle with side 2^n centered at O (positive if n is even, negative if n is odd) is colored according to the description in the statement.

The basis of induction for $n = k$ is obvious because in this case the pattern inside the triangle with side 2^k is exactly the pattern $P_{\mathcal{A}_k}$ defined by the sequence $\mathcal{A}_k = \{a_i\}_{i=1}^k$, and the elementary folding a_k defines only the coloring of the midsegments of this triangle, red if a_k is a folding up, and blue if a_k is a folding down. These midsegments form a positive triangle if k is odd and a negative triangle if k is even.

The induction step $n \rightarrow n + 1$ can be shown by analyzing an additional unfolding of the triangle with sides 2^n into a triangle with sides 2^{n+1} . The pattern inside the central triangle stays the same and it is also unfolded in each of three “side” triangles, see figure 9 for an illustration with $k = 1$ and $n = 3$.

In order to construct the pattern in the three “side” triangles with side 2^n we need to perform reflections in midsegments of the triangle with side 2^{n+1} and change the colors of the new segments because in each of the unfolded “side” peaks become valleys and vice versa. Also each positive triangle of \mathcal{L}_k unfolds into a negative triangle of \mathcal{L}_k and each negative triangle of \mathcal{L}_k unfolds into a positive triangle of \mathcal{L}_k . Together these two observations (change of color and change of positiveness of triangle) complete the induction step. \square

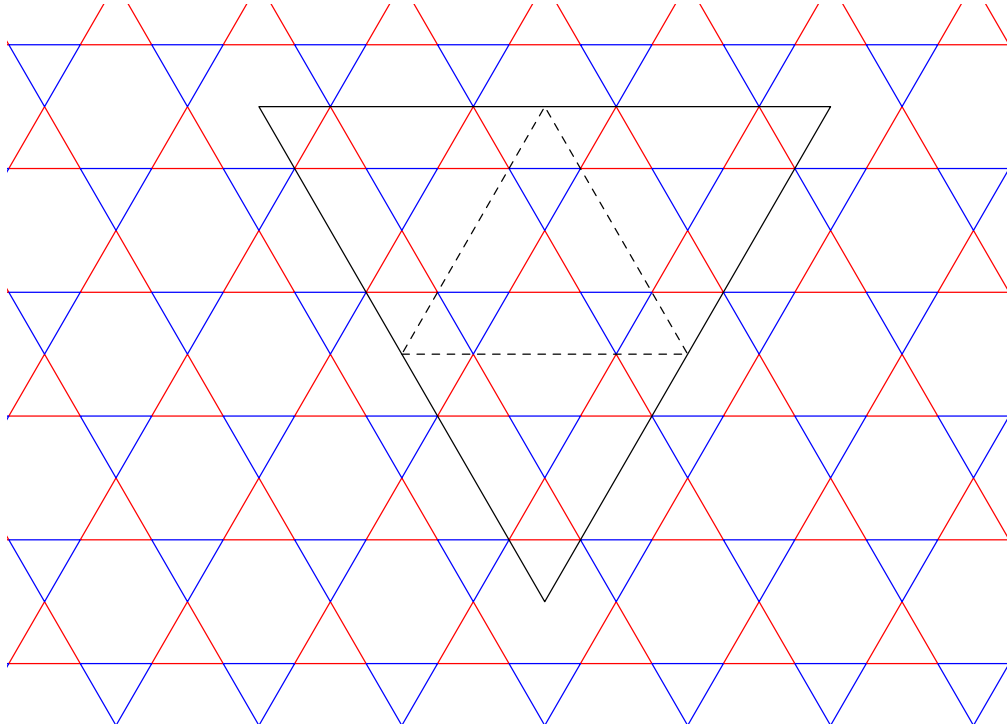


FIGURE 9. A coloring of the layer \mathcal{L}_1 if a_1 is a folding up. The triangles with side 8 (solid lines) and 4 (dashed lines) are shown to illustrate the induction step from lemma 3.3.

Definition 3.4. Let P be a pattern in \mathbb{R}^2 . We say that P is *non-periodic* if the equality $P + \mathbf{t} = P$ for a vector $\mathbf{t} \in \mathbb{R}^2$ implies $\mathbf{t} = 0$.

Any vector \mathbf{t} that satisfies $P + \mathbf{t} = P$ is called a *period* of P .

Theorem 3.5. For any sequence $\mathcal{A} = \{a_i\}_{i=1}^{\infty}$ of elementary foldings, the pattern $P_{\mathcal{A}}$ is non-periodic.

Proof. Suppose \mathbf{t} is a period of $P_{\mathcal{A}}$. We note that each layer \mathcal{L}_k is a collection of lines and the maximum non-extendable segment of one color (red or blue) in \mathcal{L}_k has length exactly 2^{k-1} . Thus $\mathcal{L}_k + \mathbf{t} = \mathcal{L}_k$ because no line from the translation $\mathcal{L}_k + \mathbf{t}$ can belong to the layer other than \mathcal{L}_k . Thus \mathbf{t} is a period for every layer \mathcal{L}_k .

It is easy to see that any non-zero period of \mathcal{L}_k has length at least 2^k , thus if $\mathbf{t} \neq 0$, then $|\mathbf{t}| \geq 2^k$ for every k which is impossible. \square

Remark. The pattern $P_{\mathcal{A}}$ is limit periodic because it can be represented as disjoint union of periodic patterns, layers \mathcal{L}_k with commensurate periods. We refer to [3] (section 4.5 in particular) for more examples of tilings and patterns that are limit periodic and properties such tilings. We also refer to [10] for another example of limit periodic pattern that shares many properties with the patterns under study.

Corollary 3.6. The tiling $T_{\mathcal{A}}$ as well as the hull $H_{\mathcal{A}}$ are aperiodic.

Proof. Recall, that a hull is aperiodic if it doesn't contain a periodic pattern/tiling (see [3, Def. 5.12], for example). If P is a pattern from $H_{\mathcal{A}}$, then we can use the same arguments to find a contradiction as in theorem 3.5 as the arguments use only local properties of $P_{\mathcal{A}}$.

For the tiling $T_{\mathcal{A}}$ we use that it is MLD equivalent to $P_{\mathcal{A}}$ (see the proof in theorem 3.8 below), and therefore is aperiodic. \square

Now we will show that the decoration of the tiling $T_{\mathcal{A}}$ can be reconstructed from undecorated tiling.

Definition 3.7. Let $T'_{\mathcal{A}}$ denote the *undecorated folding tiling* corresponding to the infinite sequence \mathcal{A} of elementary foldings. The tiling $T'_{\mathcal{A}}$ is obtained from $T_{\mathcal{A}}$ by removing the decoration of triangles with both red and blue sides.

It is worth noting that the tiling $T_{\mathcal{A}}$ has 16 translation types of different triangles, and the tiling $T'_{\mathcal{A}}$ has only 8 different translation types.

Theorem 3.8. *The tilings $T_{\mathcal{A}}$ and $T'_{\mathcal{A}}$ can be locally reconstructed from each other, that is they belong to the same MLD equivalence class.*

Proof. The tiling $T'_{\mathcal{A}}$ can be reconstructed from $T_{\mathcal{A}}$ by removing decoration, and this is clearly a local operation.

In order to reconstruct $T_{\mathcal{A}}$ from $T'_{\mathcal{A}}$ we suppose that we are given a tiling T' that is constructed as $T'_{\mathcal{A}}$ for some (unknown) sequence \mathcal{A} of elementary foldings, and our goal is to construct the corresponding unique tiling pattern $P_{\mathcal{A}}$ using local patches of T' . After that the tiling $T_{\mathcal{A}}$ can be constructed from the pattern $P_{\mathcal{A}}$ using the definition of a folding tiling.

All triangles of the layer \mathcal{L}_1 have sides of one color, so the coloring of their sides can be reconstructed from the tiling T' . Also the layer \mathcal{L}_1 can be identified as well. Indeed, if there is a segment of length 3 colored in alternating pattern red-blue-red or blue-red-blue, then this segment belongs to \mathcal{L}_1 because all other layers must contain segments of length at least 2 of single color. Note, that since the length of the segment we use is bounded, this process is local. Thus, the only part left is to reconstruct the coloring inside the hexagons of \mathcal{L}_1 .

Let H be such a hexagon, then its sides are colored alternatively with red and blue and we can reconstruct their colors. For the six triangles inside H the tiling T' gives us information about number of red and blue sides for each, therefore, if we will identify the color of at least one segment inside H , then all six will be recovered.

Now we refer to figure 8. We can see that from six segments at the center of H , there are three (actually four) consecutive segments of one color. Since the sides of H have alternating colors, there will be a triangle inside H with all sides of the same color, and those sides can be recognized locally from the tiling T' inside H (provided we already identified the layer \mathcal{L}_0).

Thus we described a local algorithm of finding $P_{\mathcal{A}}$ from T' . \square

Theorem 3.9. *If two sequences \mathcal{A} and \mathcal{B} of elementary foldings differ only in finitely many terms, then the corresponding patterns $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ belong to the same MLD equivalence class.*

Proof. Suppose that $\mathcal{A} = \{a_i\}$ and $\mathcal{B} = \{b_i\}$ and $a_i = b_i$ for $i > n$ for some natural n . We reconstruct the color of a fixed segment X in $P_{\mathcal{B}}$ using its $2^n + 1$ (open) neighborhood in $P_{\mathcal{A}}$.

This neighborhood contains the extension of T by 2^n in both directions. If the longest extension colored with single color has length 2^k with $k < n$, then T belongs to \mathcal{L}_{k+1} . In that case if $a_{k+1} = b_{k+1}$, then we keep the color of T , because the layer \mathcal{L}_{k+1} is colored in the same way in $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ according to lemma 3.3. If $a_{k+1} \neq b_{k+1}$, then we change the color of T by the same reason.

In case the longest extension of T of a single color has length 2^n or greater, then we keep the color of T as in that case T belongs to \mathcal{L}_{k+1} with $k + 1 > n$ and $a_{k+1} = b_{k+1}$. \square

4. SEQUENCE OF ALL FOLDINGS UP

In this section we study the tiling $T_{\mathcal{A}}$ and the pattern $P_{\mathcal{A}}$ for the sequence $\mathcal{A} = \{a_i\}_{i=1}^{\infty}$ where each a_i is a folding up. Namely we show that the tiling $T_{\mathcal{A}}$ and the pattern $P_{\mathcal{A}}$ are substitution tiling and pattern respectively. And then we use the Perron-Frobenius theorem to find densities of all types of tiles in $T_{\mathcal{A}}$.

We refer to [3, Ch. 6] and [9] for the exact definition(s) of the substitution tiling/pattern, and to [3, Sect 2.4] for more details on Perron-Frobenius theory.

Definition 4.1. We define the “pattern” substitution rule F_P and the “tiling” substitution rule F_T using figure 10.

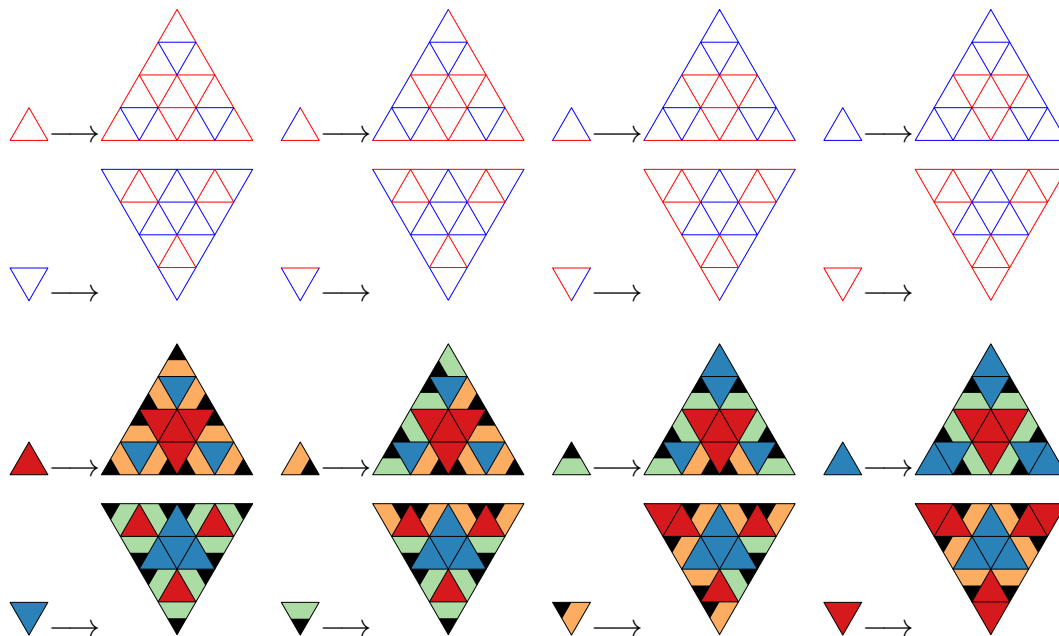


FIGURE 10. Substitution rules F_P and F_T .

Note, that these rules are “derivable” from each other in a sense, that triangles of side 4 of F_T can be reconstructed from triangles of side 4 of F_P using the same approach we use in the proof of theorem 3.8. We can also use triangles of side 4 of F_T to reconstruct all segments in triangles of sides 4 of F_P except 6 segments incident to the vertices, but these segments can be reconstructed because each side of each triangle of side 4 in F_P has one color.

Theorem 4.2.

- The substitution rule F_P with a (legal) seed \triangle centered at the origin generates the pattern $P_{\mathcal{A}}$ for the sequence \mathcal{A} of all elementary foldings up.
- The substitution rule F_T with a (legal) seed \blacktriangle centered at the origin generates the (decorated) tiling $T_{\mathcal{A}}$ for the sequence \mathcal{A} of all elementary foldings up.

Proof. We prove the statement for the pattern substitution rule F_P . Then the statement for the tiling substitution rule will follow immediately from the remark before the current theorem.

The rule F_P preserves the coloring on the external sides of a triangle, therefore after applying the rule F_P to the positive triangle of side 1 with red sides centered at O k times

we get a pattern inside a positive triangle T_k of side 2^{2k} centered at O with red sides. We will prove by induction that the pattern inside T_k obtained via F_P coincides with the pattern P_A inside the triangle of the same size centered at O .

The basis of induction is evident. For the step $k \rightarrow k+1$ of induction we notice, that the 4-dilation of T_k with respect to O gives the coloring of the subdivision of T_{k+1} into triangles of side 4 that are constructed from unit triangles of T_k using the rule F_P .

The coloring of these triangles with side 4 coincides with the coloring of corresponding segments in P_A because 4-dilation a positive (negative) triangle of \mathcal{L}_i with respect to O gives a positive (negative) triangle of \mathcal{L}_{i+2} and they are colored identically due to lemma 3.3.

The segments inside each triangle of side 4 are formed by three triangles of the layer \mathcal{L}_1 and one triangle of the layer \mathcal{L}_2 , see figure 11. The coloring of these triangles are determined by the coloring of the internal segments from the rule F_P and this coloring coincides with the coloring of the positive/negative triangles in the corresponding layers from lemma 3.3.

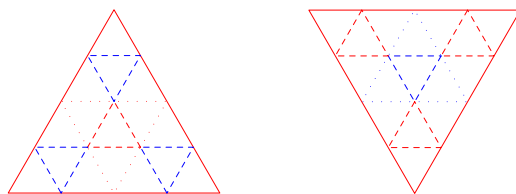


FIGURE 11. The triangles from layer \mathcal{L}_1 (dashed) and from layer \mathcal{L}_2 (dotted).

□

As a consequence we can use Perron-Frobenius theory for substitution tilings to compute densities of all triangles in the tilings T_A and P_A .

Example 4.3. The rule F_T uses 8 different types of triangles, so the corresponding substitution matrix M_{++} is an 8×8 integer matrix. Namely, if we use the same order of triangles as in F_T (see figure 10), then

$$M_{++} = \begin{pmatrix} 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 \\ 9 & 5 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 & 3 & 3 & 0 \\ 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 9 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 6 & 6 \\ 3 & 3 & 3 & 3 & 0 & 0 & 1 & 3 \end{pmatrix}$$

where the first 4 columns correspond to positive triangles, and the last 4 to negative. Since

$$M_{++}^2 = \begin{pmatrix} 28 & 28 & 28 & 28 & 36 & 36 & 36 & 36 \\ 54 & 42 & 31 & 21 & 27 & 27 & 27 & 27 \\ 36 & 44 & 50 & 54 & 18 & 18 & 18 & 18 \\ 18 & 22 & 27 & 33 & 39 & 39 & 39 & 39 \\ 36 & 36 & 36 & 36 & 28 & 28 & 28 & 28 \\ 27 & 27 & 27 & 27 & 54 & 42 & 31 & 21 \\ 18 & 18 & 18 & 18 & 36 & 44 & 50 & 54 \\ 39 & 39 & 39 & 39 & 18 & 22 & 27 & 33 \end{pmatrix}$$

is a positive matrix we can see that the substitution is primitive. The Perron-Frobenius eigenvalue of M_{++} is $\lambda_{PF} = 16$ and all others are $4, 4, 4, 1, 1, 0, 0$.

The left and right λ_{PF} -eigenvectors of M_{++} are $(1, 1, 1, 1, 1, 1, 1)$ and $(1, 1, 1, 1, 1, 1, 1)^t$. The left PF -eigenvector of M_P shows that areas of all prototiles (eight triangles) can be chosen to be equal which we already know because they are equal regular triangles of different colors. The right PF -eigenvector shows that densities of all types of triangles are equal (to $\frac{1}{8}$) in the tiling $T_{\mathcal{A}}$ in case we say that two triangles are of the same type if they are translations of each other or rotations by a multiple of $\frac{2\pi}{3}$.

Since the tiling $T_{\mathcal{A}}$ has a three-fold rotational symmetry with respect to origin, then the densities of translationally different tiles in $T_{\mathcal{A}}$ can be found. Namely, all four types of triangles without decoration have density $\frac{1}{8}$, and all twelve types of triangles with decoration have density $\frac{1}{24}$.

5. PERIODIC SEQUENCES OF ELEMENTARY FOLDINGS

The main goal of this section is to prove that if a sequence \mathcal{A} of elementary foldings is periodic, then the corresponding pattern $P_{\mathcal{A}}$ and tiling $T_{\mathcal{A}}$ can be generated via substitution rules.

Theorem 5.1. *If $\mathcal{A} = \{a_i\}_{i=1}^d$ is a periodic sequence of elementary foldings, then the pattern $P_{\mathcal{A}}$ and the tiling $T_{\mathcal{A}}$ can be generated using substitution rules.*

Proof. Let $2n$ be the smallest even period of \mathcal{A} , so that $a_i = a_{i+2n}$. Similarly to the proof of theorem 4.2, we prove the statement for the pattern first. The statement for tilings will follow immediately.

We note that 2^{2n} -dilation of \mathcal{L}_k with respect to O is the layer \mathcal{L}_{k+2n} , and that the dilation preserves the colors due to lemma 3.3. Therefore we define the substitution rule for 8 types of unit triangles with dilation factor 2^{2n} in the following way:

- (1) The sides of the dilated triangle are colored in the same way as the sides of the initial unit triangle.
- (2) The interior segments of a positive triangle are colored in the same way as the interior segments of the positive triangle with side 2^{2n} of $P_{\mathcal{A}}$ centered at O (see figure 12).
- (3) The interior segments of a negative triangle are colored in the same way as the interior segments of the negative triangle with side 2^{2n} of $P_{\mathcal{A}}$ that is a one “side part” of the negative triangle with side 2^{2n+1} centered at O (see figure 12).

The patterns in the “side parts” are obtained from the central part by reflection with respect to corresponding midsegments and flipping the colors. The pattern inside the central triangle with side 2^{2n} has 3-fold rotational symmetry, therefore all three “side parts” have the same pattern inside.

After that the proof follows the ideas of the proof of theorem 4.2. Here the induction statement is that the coloring of the pattern inside the triangle T_k with side 2^{2kn} centered at O is the same as the coloring of the segments inside the triangle of side 2^{2kn} obtained after using the substitution rule k times.

The basis of induction is trivial, and the step of induction $k \rightarrow k+1$ is similar to proof of theorem 4.2. Again, we notice that 2^{2n} dilation of T_k gives a subdivision of triangle T_{k+1} with triangles with side 2^{2n} . The sides of these triangles are colored according to the coloring in $P_{\mathcal{A}}$ because each layer \mathcal{L}_i is dilated into \mathcal{L}_{i+2n} and these layers are colored in same way due to lemma 3.3 and $2n$ -periodicity of \mathcal{A} .

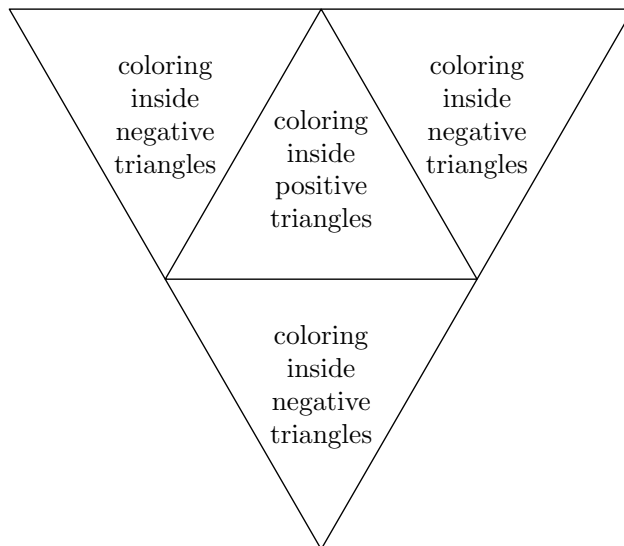


FIGURE 12. The triangle with side 2^{2n+1} divided into four triangles with interior colorings defining substitution rules for $2n$ -periodic sequence.

The segments inside each triangle of the subdivision of T_{k+1} are colored according to $P_{\mathcal{A}}$ because now these segments can be represented as union of several triangles from the layers $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{2n}$, and the substitution rule forces them to be colored accordingly to lemma 3.3. See figure 13 for an example with $n = 2$.

□

Below we compute the substitution matrices, their eigenvalues, and the densities of all triangles in $T_{\mathcal{A}}$ for all 2- and 4-periodic sequences \mathcal{A} of elementary foldings.

Example 5.2 (Sequences with period 2). There are four sequences with period 2, they can be encoded as $\mathcal{A}_{++}, \mathcal{A}_{+-}, \mathcal{A}_{-+}$ and \mathcal{A}_{--} where $+$ denotes an elementary folding up, and $-$ denotes an elementary folding down. Each of the sequences has all odd elementary foldings equal to the first symbol in the subscript, and all even elementary foldings equal to the second symbol in the subscripts.

The sequence \mathcal{A}_{++} is just the sequence of all foldings up from section 4, and the sequence \mathcal{A}_{--} generates the same pattern after swapping the colors of peaks and valleys. Thus the substitution matrix of the corresponding tilings is

$$M_{++} = \begin{pmatrix} 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 \\ 9 & 5 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 & 3 & 3 & 0 \\ 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 9 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 6 & 6 \\ 3 & 3 & 3 & 3 & 0 & 0 & 1 & 3 \end{pmatrix},$$

its eigenvalues are $16, 4, 4, 4, 1, 1, 0$, and the densities of all types of triangles in the tilings $T'_{\mathcal{A}_{++}}$ and $T'_{\mathcal{A}_{--}}$ are equal.

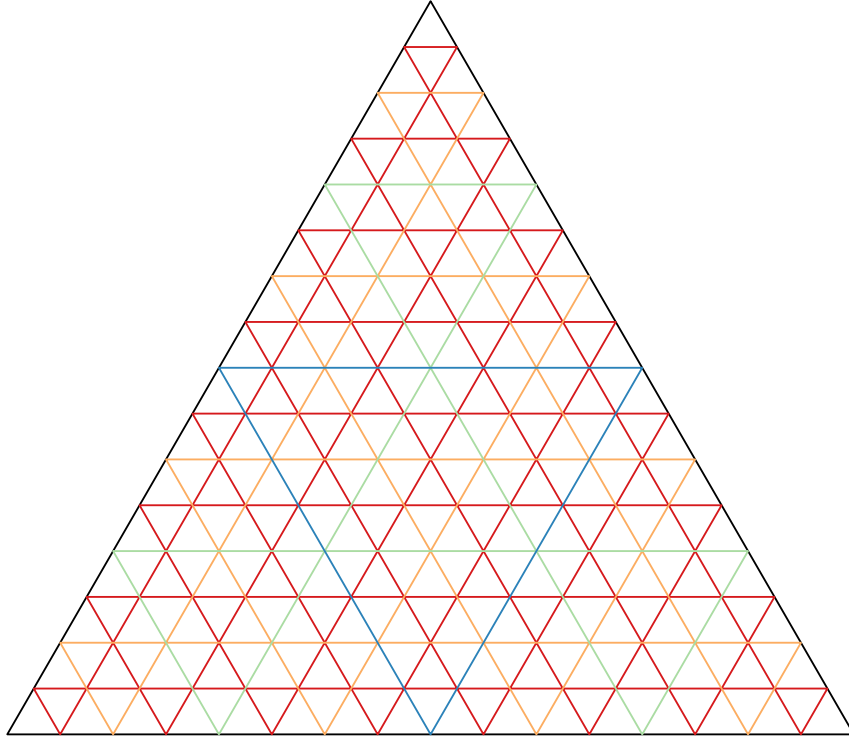


FIGURE 13. Triangles from layers \mathcal{L}_1 (red), \mathcal{L}_2 (orange), \mathcal{L}_3 (green) and \mathcal{L}_4 (blue) inside a triangle with side 16.

The patterns generated by two remaining sequences \mathcal{A}_{+-} and \mathcal{A}_{-+} can be transformed into each other by swapping the colors as well, so we compute the data only for the sequence \mathcal{A}_{+-} . The substitution rules are sketched on the figure 14 and as we said in theorem 5.1 the colors of the sides are preserved and all positive/negative triangles share the same coloring inside so each rule on the figure 14 describes four actual rules depending on the coloring of the boundary.

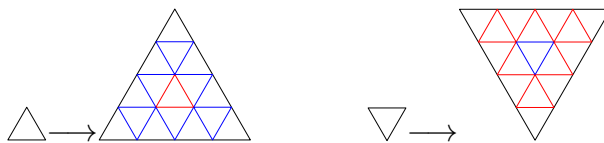


FIGURE 14. Substitution for $+-$ sequence.

The substitution matrix is given by

$$M_{+-} = \begin{pmatrix} 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 \\ 3 & 1 & 0 & 0 & 3 & 3 & 3 & 3 \\ 6 & 6 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 9 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 6 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 5 & 9 \end{pmatrix}.$$

The square of this matrix has only positive entries, so the substitution is primitive. Its eigenvalues are 16, 4, 4, 4, 1, 1, 0 again, and the right eigenvector corresponding to PF-eigenvalue 16 is $(1, 1, 1, 1, 1, 1, 1, 1)^T$ meaning that the densities of all triangles in $T_{A_{+-}}$ are equal.

Example 5.3 (Sequences with period 4). All four sequences with period 2 can be written as sequences with period four as well. The remaining 12 4-periodic sequences can be split in pairs of equivalent under swap of colors. These pairs (assuming similar notations) are

- \mathcal{A}_{++++} and \mathcal{A}_{----} ;
- \mathcal{A}_{++--} and \mathcal{A}_{--++} ;
- \mathcal{A}_{+-+-} and $\mathcal{A}_{-+ -+}$;
- \mathcal{A}_{+--+} and $\mathcal{A}_{-+ -+}$;
- \mathcal{A}_{+---} and \mathcal{A}_{-+++} ;
- \mathcal{A}_{+---} and \mathcal{A}_{-+++} .

For the sequence \mathcal{A}_{++++} the substitution rules are given on the figure 15. The substitution

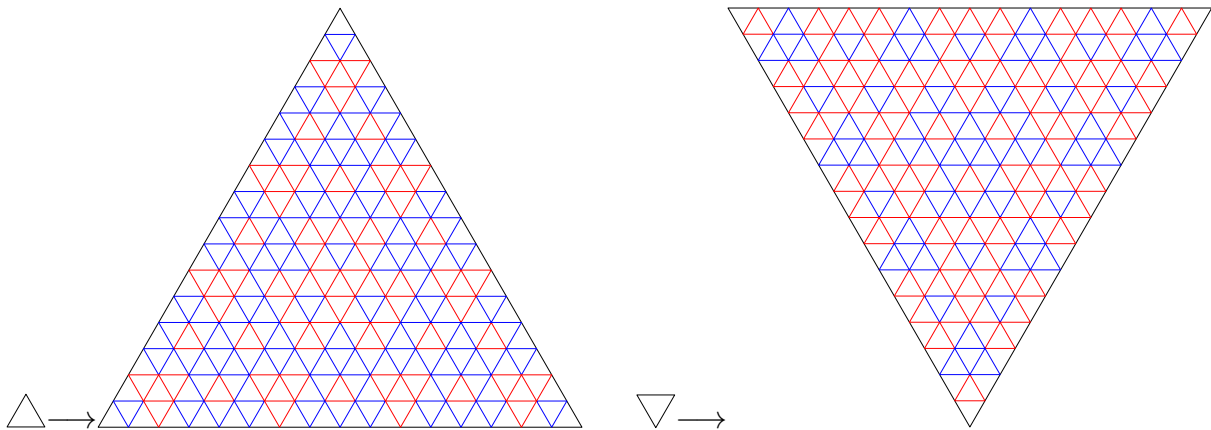


FIGURE 15. Substitution for +++- sequence.

matrix is

$$M_{++++} = \begin{pmatrix} 28 & 28 & 28 & 28 & 36 & 36 & 36 & 36 \\ 36 & 26 & 17 & 9 & 42 & 42 & 42 & 42 \\ 48 & 52 & 54 & 54 & 12 & 12 & 12 & 12 \\ 24 & 30 & 37 & 45 & 30 & 30 & 30 & 30 \\ 36 & 36 & 36 & 36 & 28 & 28 & 28 & 28 \\ 42 & 42 & 42 & 42 & 36 & 26 & 17 & 9 \\ 12 & 12 & 12 & 12 & 48 & 52 & 54 & 54 \\ 30 & 30 & 30 & 30 & 24 & 30 & 37 & 45 \end{pmatrix}.$$

For the sequence \mathcal{A}_{++++} the substitution rules are given on the figure 16. The substitution

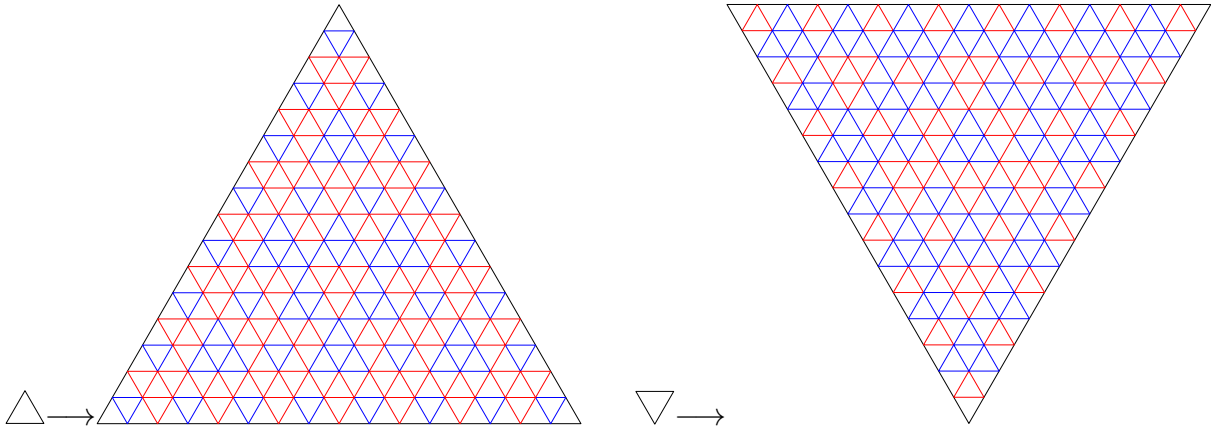


FIGURE 16. Substitution for ++-+ sequence.

matrix is

$$M_{+++-} = \begin{pmatrix} 28 & 28 & 28 & 28 & 36 & 36 & 36 & 36 \\ 81 & 65 & 50 & 36 & 6 & 6 & 6 & 6 \\ 6 & 22 & 36 & 48 & 36 & 36 & 36 & 36 \\ 21 & 21 & 22 & 24 & 42 & 42 & 42 & 42 \\ 36 & 36 & 36 & 36 & 28 & 28 & 28 & 28 \\ 6 & 6 & 6 & 6 & 81 & 65 & 50 & 36 \\ 36 & 36 & 36 & 36 & 6 & 22 & 36 & 48 \\ 42 & 42 & 42 & 42 & 21 & 21 & 22 & 24 \end{pmatrix}.$$

For the sequence \mathcal{A}_{++--} the substitution rules are given on the figure 17. The substitution matrix is

$$M_{++--} = \begin{pmatrix} 28 & 28 & 28 & 28 & 36 & 36 & 36 & 36 \\ 57 & 43 & 30 & 18 & 27 & 27 & 27 & 27 \\ 30 & 42 & 52 & 60 & 18 & 18 & 18 & 18 \\ 21 & 23 & 26 & 30 & 39 & 39 & 39 & 39 \\ 36 & 36 & 36 & 36 & 28 & 28 & 28 & 28 \\ 27 & 27 & 27 & 27 & 57 & 43 & 30 & 18 \\ 18 & 18 & 18 & 18 & 30 & 42 & 52 & 60 \\ 39 & 39 & 39 & 39 & 21 & 23 & 26 & 30 \end{pmatrix}.$$

For the sequence \mathcal{A}_{-+++} the substitution rules are given on the figure 18. The substitution

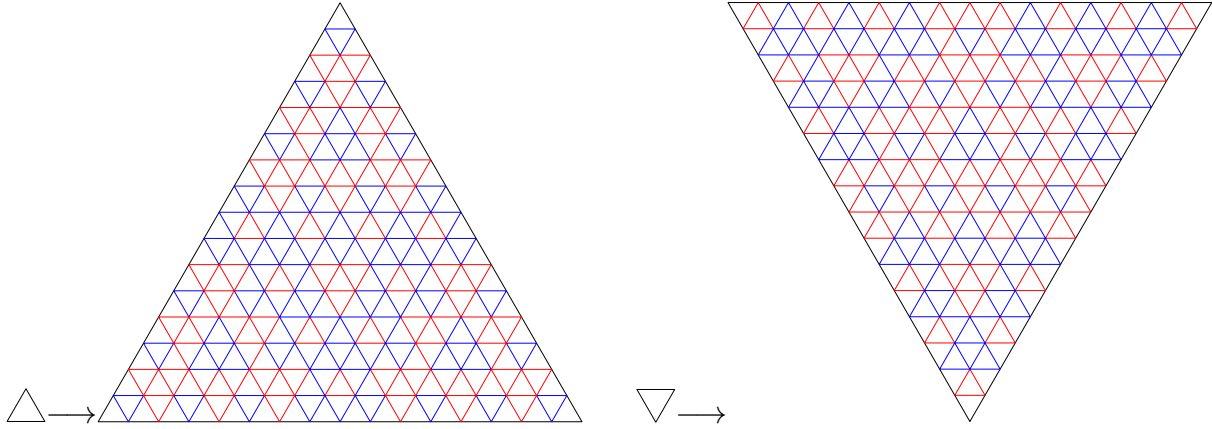


FIGURE 17. Substitution for ++-- sequence.

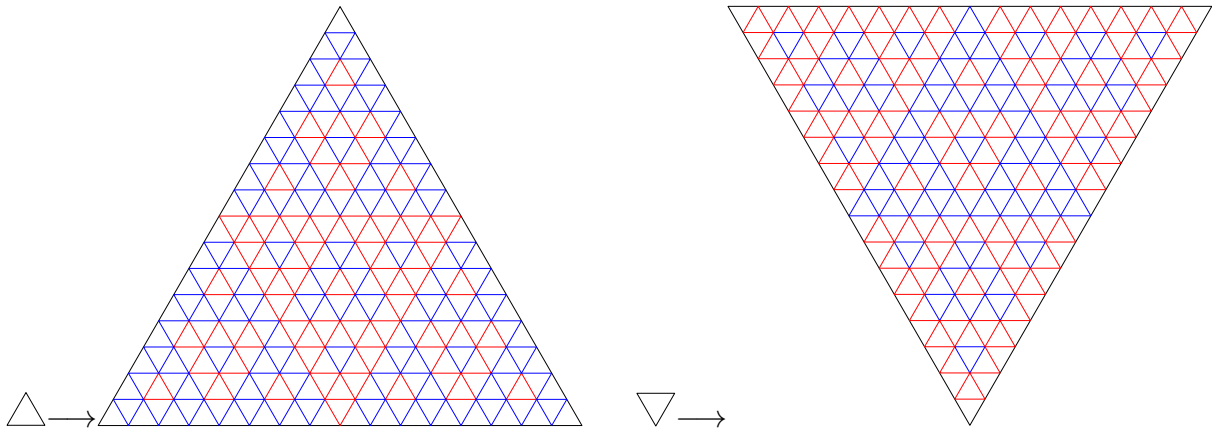
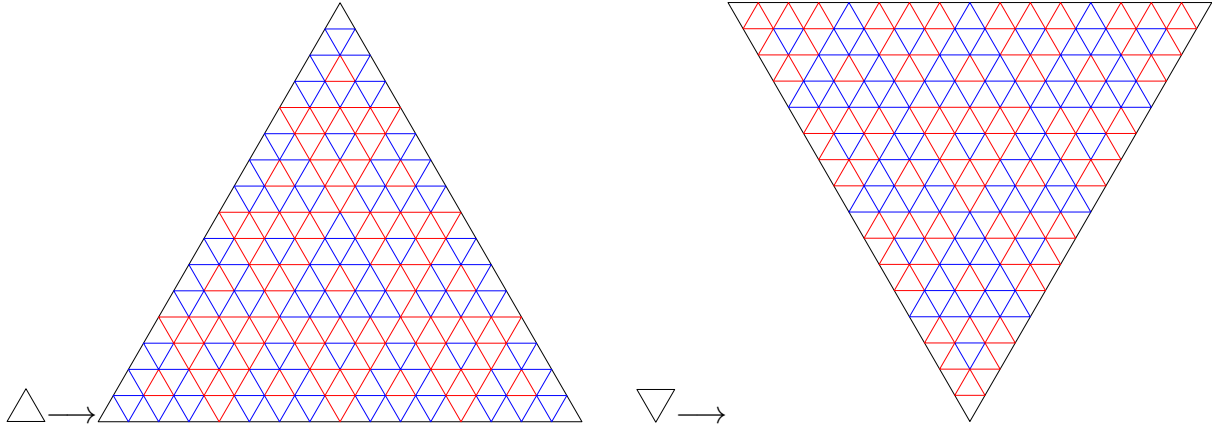


FIGURE 18. Substitution for +-++ sequence.

matrix is

$$M_{+-++} = \begin{pmatrix} 28 & 28 & 28 & 28 & 36 & 36 & 36 & 36 \\ 30 & 26 & 23 & 21 & 39 & 39 & 39 & 39 \\ 60 & 52 & 42 & 30 & 18 & 18 & 18 & 18 \\ 18 & 30 & 43 & 57 & 27 & 27 & 27 & 27 \\ 36 & 36 & 36 & 36 & 28 & 28 & 28 & 28 \\ 39 & 39 & 39 & 39 & 30 & 26 & 23 & 21 \\ 18 & 18 & 18 & 18 & 60 & 52 & 42 & 30 \\ 27 & 27 & 27 & 27 & 18 & 30 & 43 & 57 \end{pmatrix}.$$

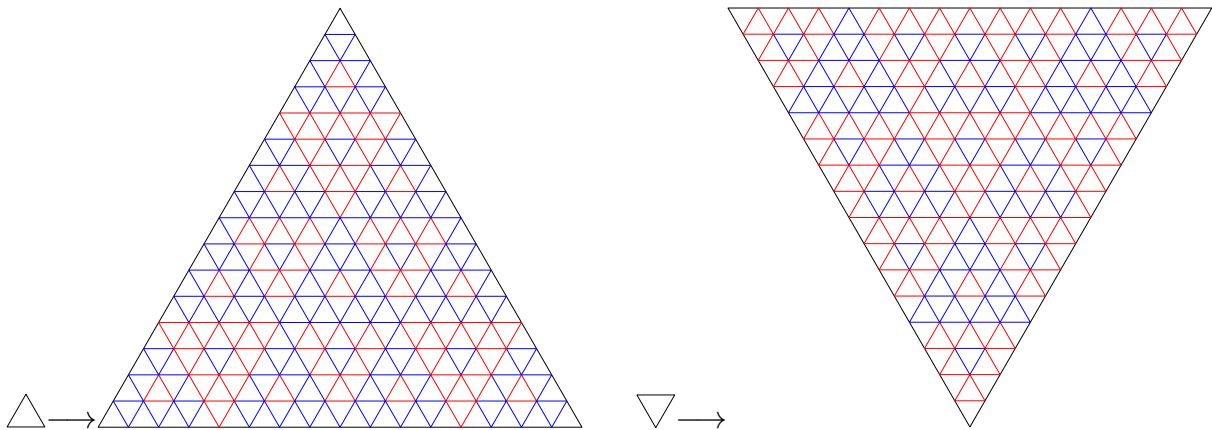
For the sequence \mathcal{A}_{+-++} the substitution rules are given on the figure 19. The substitution

FIGURE 19. Substitution for $+ - - +$ sequence.

matrix is

$$M_{+---} = \begin{pmatrix} 28 & 28 & 28 & 28 & 36 & 36 & 36 & 36 \\ 45 & 37 & 30 & 24 & 30 & 30 & 30 & 30 \\ 54 & 54 & 52 & 48 & 12 & 12 & 12 & 12 \\ 9 & 17 & 26 & 36 & 42 & 42 & 42 & 42 \\ 36 & 36 & 36 & 36 & 28 & 28 & 28 & 28 \\ 30 & 30 & 30 & 30 & 45 & 37 & 30 & 24 \\ 12 & 12 & 12 & 12 & 54 & 54 & 52 & 48 \\ 42 & 42 & 42 & 42 & 9 & 17 & 26 & 36 \end{pmatrix}.$$

Finally, for the sequence \mathcal{A}_{+---} the substitution rules are given on the figure 20. The

FIGURE 20. Substitution for $+ - - -$ sequence.

substitution matrix is

$$M_{+---} = \begin{pmatrix} 28 & 28 & 28 & 28 & 36 & 36 & 36 & 36 \\ 33 & 27 & 22 & 18 & 39 & 39 & 39 & 39 \\ 54 & 50 & 44 & 36 & 18 & 18 & 18 & 18 \\ 21 & 31 & 42 & 54 & 27 & 27 & 27 & 27 \\ 36 & 36 & 36 & 36 & 28 & 28 & 28 & 28 \\ 39 & 39 & 39 & 39 & 33 & 27 & 22 & 18 \\ 18 & 18 & 18 & 18 & 54 & 50 & 44 & 36 \\ 27 & 27 & 27 & 27 & 21 & 31 & 42 & 54 \end{pmatrix}.$$

All six matrices above as well as matrices M_{++}^2 and M_{+-}^2 have eigenvalues 256, 16, 16, 16, 1, 1, 0, 0. Moreover the right eigenvector corresponding to Perron-Frobenius eigenvalue 256 is $(1, 1, 1, 1, 1, 1, 1, 1)^T$ meaning that in each pattern (or tiling) generated by a 4-periodic sequence of elementary foldings densities of all triangles are equal. In section 6 we show that densities are equal not only for periodic, but for any sequence of elementary foldings. In section 7 we study eigenvalues and eigenvectors of substitution matrices for all periodic sequences of elementary foldings.

6. DENSITIES FOR ARBITRARY SEQUENCE

The goal of this section is to prove that for every sequence \mathcal{A} of elementary foldings the densities of all types of unit triangles (four types of positive colorings, and four types of negative colorings) in $P_{\mathcal{A}}$ are equal to $\frac{1}{8}$.

Lemma 6.1. *Let $\mathcal{A} = \{a_i\}_{i=1}^{\infty}$ be a sequence of elementary foldings with a_1 being a folding up. Then any positive unit triangle colored with red and any negative unit triangle colored with blue has all three sides from layer \mathcal{L}_1 .*

Proof. From the coloring of layer \mathcal{L}_1 on figure 9 we see that each positive unit triangle either coincides with a positive triangle of \mathcal{L}_1 colored with red, or has a blue side from a hexagon of \mathcal{L}_1 . This proves the statement for positive triangles. The arguments for negative triangles are similar. \square

Corollary 6.2. *If \mathcal{A} is a sequence of elementary foldings from the previous lemma, then the densities of positive unit triangles colored with red and of negative triangles colored with blue are equal to $\frac{1}{8}$.*

Proof. It is enough to notice that densities of the corresponding triangles in \mathcal{L}_1 are equal to $\frac{1}{8}$. \square

Definition 6.3. Let H be a hexagon of layer \mathcal{L}_1 (see definition 3.2). According to the figure 8, the central point h of H is a vertex of two triangles of some layer \mathcal{L}_i with $i > 1$. Also h belongs to a side of a triangle from some layer \mathcal{L}_j with $j > i$.

In that case we call H a *hexagon of type (i, j)* or an *(i, j) -hexagon*.

If $1 < i < j$, then each side of a (positive or negative) triangle of \mathcal{L}_j contains exactly 2^{j-i-1} centers of (i, j) -hexagons. Namely, if a side is represented by a segment $[0, 1]$, then the centers of (i, j) -hexagons are at the points $\frac{a}{2^{j-i}}$ for odd a .

Let $\mathcal{H}_{i,j}$ denote the set of all (i, j) -hexagons. This set is periodic because the layer \mathcal{L}_j is periodic. Thus the total density $D_{i,j}$ of the set covered by all hexagons from $\mathcal{H}_{i,j}$ is defined.

Lemma 6.4. *Let $1 < i < j$ be integers. Let \mathcal{A} be a sequence of elementary foldings with a_1 being a folding up. Then all unit triangles except positive triangle colored red and negative triangle colored blue have equal densities in the set $\mathcal{H}_{i,j}$.*

Proof. First we notice that if H is a hexagon from $\mathcal{H}_{i,j}$, then the coloring of the sides of H is known, and the coloring of inside segments is determined by the direction and color of the diagonal of H from \mathcal{L}_j because this diagonal determines which pair of segments at the center of H belongs to a positive triangle from \mathcal{L}_i and which pair of segments belongs to a negative triangle from \mathcal{L}_i . And the coloring of these two triangles are determined by a_i . We will call the hexagons from $\mathcal{H}_{i,j}$ red or blue depending on the color of the diagonal from \mathcal{L}_j .

If a_i is a folding up, then the colorings of the red and blue hexagons from $\mathcal{H}_{i,j}$ are as shown on figure 21 or their rotation by $\pm \frac{2\pi}{3}$.



FIGURE 21. Coloring inside hexagons for $a_i = +$.

If a_i is a folding down, then the colorings of the red and blue hexagons from $\mathcal{H}_{i,j}$ are as shown on figure 22 or their rotation by $\pm \frac{2\pi}{3}$.



FIGURE 22. Coloring inside hexagons for $a_i = -$.

Also it is easy to notice that red and blue hexagons have the same density within $\mathcal{H}_{i,j}$, thus the density of each type of unit triangle exists and can be found by computing the density of the same type of triangle on figure 21 or 22 depending on a_i . However each type of triangle (except positive red triangle and negative blue triangle) has two representatives in figure 21 and two on figure 22, therefore all the densities are equal. \square

Theorem 6.5. *Let \mathcal{A} be a sequence of elementary foldings. Then the densities of all types of unit triangles are equal in $P_{\mathcal{A}}$.*

Proof. Without loss of generality we can assume that a_1 is a folding down. Then corollary 6.2 implies that positive red triangles and negative blue triangles have density $\frac{1}{8}$.

For $n \geq 3$ let

$$\mathcal{H}_n := \bigsqcup_{j=3}^n \bigsqcup_{i=2}^{j-1} \mathcal{H}_{i,j}$$

denote the set of all hexagons of \mathcal{L}_1 with all segments in the union $\bigsqcup_{i=1}^n \mathcal{L}_i$. Then densities of six remaining types of triangles are equal in \mathcal{H}_n because they are equal in each $\mathcal{H}_{i,j}$ due to lemma 6.4.

The density of all hexagons of \mathcal{L}_1 that are not in \mathcal{H}_n goes to 0 as n goes to ∞ because the centers of these hexagons are located on sparser layers \mathcal{L}_k with $k > n$. Therefore the overall

densities of remaining types of triangles exist and these densities are equal in $P_{\mathcal{A}}$. Moreover, the densities are equal to $\frac{1}{6} \cdot (1 - 2 \cdot \frac{1}{8}) = \frac{1}{8}$. \square

7. EIGENVALUES AND EIGENVECTORS OF TRIANGULAR FOLDINGS SUBSTITUTION

The goal of this section is to study spectral properties of the substitution rules described in theorem 5.1. For this we fix a $2n$ -periodic sequence \mathcal{A} of elementary foldings starting from a folding up (the second case is similar), and we fix the substitution rules F_P described in the proof of theorem 5.1 that generate the pattern $P_{\mathcal{A}}$. Let $M_{\mathcal{A}}$ be the corresponding 8×8 substitution matrix. Our goal is to describe the eigenvalues and the (right) eigenvectors of $M_{\mathcal{A}}$.

The Perron-Frobenius eigenvalue of $M_{\mathcal{A}}$ is the square of the inflation factor which is $\lambda_{PF} = 2^{4n}$. The right eigenvector of $M_{\mathcal{A}}$ corresponding to λ_{PF} is the vector of densities of prototiles. According to theorem 6.5 all densities are equal, so the eigenvector is $(1, 1, 1, 1, 1, 1, 1, 1)^T$. Also the left eigenvector corresponding to λ_{PF} represents the areas of prototiles, and since the areas of all unit triangles are equal, this eigenvector is $(1, 1, 1, 1, 1, 1, 1, 1)$.

First we describe the general structure of $M_{\mathcal{A}}$.

Lemma 7.1. *The matrix $M_{\mathcal{A}}$ can be written as a block matrix*

$$M = \left(\begin{array}{c|c} X & Y \\ \hline Y & X \end{array} \right)$$

where X is a 4×4 matrix with the first row

$$2^{2n-2}(2^{2n-1} - 1) \cdot (1, 1, 1, 1)$$

and Y is a 4×4 matrix with the first row

$$2^{2n-2}(2^{2n-1} + 1) \cdot (1, 1, 1, 1)$$

and all equal columns.

Proof. The block structure of the matrix $M_{\mathcal{A}}$ follows from the description of the substitution rule (see theorem 5.1). If we take the rule for a positive triangle and flip it over a side while swapping the colors as well, then we will get the substitution rule for a negative triangle.

This gives us the following correspondence between columns of $M_{\mathcal{A}}$

$$\begin{aligned} 1 &\rightarrow 5 \\ 2 &\rightarrow 6 \\ 3 &\rightarrow 7 \\ 4 &\rightarrow 8 \end{aligned}$$

based on the order of the prototiles. The flipping changes each positive unit triangle into a negative unit triangle and vice versa, and this forces a similar interchange between rows of $M_{\mathcal{A}}$. As a result we get that the upper left 4×4 corner of $M_{\mathcal{A}}$ is equal to the lower right 4×4 corner of $M_{\mathcal{A}}$, and the lower left 4×4 corner of $M_{\mathcal{A}}$ is equal to the upper right 4×4 corner of $M_{\mathcal{A}}$. We denoted these blocks as X and Y in the statement of the theorem.

The first row of the block X and the first row of the block Y show the numbers of positive red triangles and negative blue triangles in the substitution rule related to positive unit triangles. According to lemma 6.1 these triangles come only from the layer \mathcal{L}_1 . The number of positive red triangles in such a substitution is equal to

$$1 + 2 + \dots + (2^{2n-1} - 1) = 2^{2n-2}(2^{2n-1} - 1),$$

and the number of negative blue triangles in such substitution is equal to

$$1 + 2 + \dots + 2^{2n-1} = 2^{2n-2}(2^{2n-1} + 1).$$

Finally, the columns of the block Y are equal because these columns represent numbers of negative triangles in the substitution rules related to positive unit triangles, and these rules differ only on the boundaries of inflated triangles with side 2^{2n} . No negative unit triangle has a side on that boundary, so numbers of negative triangles will be equal for different prototiles. \square

Next we are going to obtain more information about each block X and Y . Let a, b, c be the number of positive triangles \triangle , \triangle , \triangle respectively in the interior of the positive triangle T with side 2^{2n} obtained after applying the substitution rule F_P to a positive unit triangle with red sides. Similarly, let x and y be the number of positive triangles \triangle , \triangle adjacent to one fixed side of the same triangle T with side 2^{2n} .

It is clear that numbers x and y do not depend on the choice of side of T because T has 3-fold rotational symmetry. Also there is no positive triangle with only blue sides adjacent to a side of T because the sides of T are colored with red, and no positive triangles with only red sides adjacent to a side of T because the triangles adjacent to a side of T are not from \mathcal{L}_1 .

From the length of side of T we get that $x + y = 2^{2n}$, and from the total number of positive triangles inside T we get

$$a + b + c + 2^{2n-2}(2^{2n-1} - 1) = 1 + 2 + \dots + (2^{2n} - 3) = (2^{2n} - 3)(2^{2n-1} - 1)$$

or $a + b + c = 3(2^{2n-2} - 1)(2^{2n-1} - 1)$.

Lemma 7.2. *The block X can be written as*

$$X = \begin{pmatrix} 2^{2n-2}(2^{2n-1} - 1) & 2^{2n-2}(2^{2n-1} - 1) & 2^{2n-2}(2^{2n-1} - 1) & 2^{2n-2}(2^{2n-1} - 1) \\ a + 3x - 3 & a + 2x - 3 & a + x - 2 & a \\ b + 3y & b + x + 2y & b + 2x + y - 2 & b + 3x - 6 \\ c & c + y & c + 2y + 1 & c + 3y + 3 \end{pmatrix}$$

Proof. The first column follows immediately from the definition of the values a, b, c, x, y once we notice that all three sides of T are colored in red, so all three corners of T must have two red sides and one blue side.

In order to get from the first column of X to the second column we need to color one side of T with blue color. This changes x triangles of type \triangle into triangles of type \triangle and y triangles of type \triangle into triangles of type \triangle .

The transformations from the second column to the third one, and from the third one to the fourth one can be described in the same way. \square

The next lemma gives a possibility to describe most of eigenvalues and eigenvectors of M_A .

Lemma 7.3. *If $\alpha + \beta + \gamma + \delta = 0$ and $(\alpha, \beta, \gamma, \delta)$ is a λ -eigenvector of the matrix*

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3x - 3 & 2x - 3 & x - 2 & 0 \\ 3y & x + 2y & 2x + y - 3 & 3x - 6 \\ 0 & y & 2y + 1 & 3y + 3 \end{pmatrix},$$

then the vectors $(\alpha, \beta, \gamma, \delta, 0, 0, 0, 0)^T$ and $(0, 0, 0, 0, \alpha, \beta, \gamma, \delta)^T$ are λ -eigenvectors of M_A .

Proof. We notice that $X = (2^{2n-2}(2^{2n-1} - 1), a, b, c)^T(1, 1, 1, 1) + Q$ from lemma 7.2 and that $Y(\alpha, \beta, \gamma, \delta)^T = (0, 0, 0, 0)^T$ because all columns of Y are equal.

After that remark the statement follows by easy computation using that $(\alpha, \beta, \gamma, \delta)(1, 1, 1, 1)^T = 0$. □

Theorem 7.4. *The eigenvalues (with multiplicities) of M_A are $2^{4n}, 2^{2n}, 2^{2n}, 2^{2n}, 1, 1, 0, 0$.*

The following are (right) eigenvectors of M_A :

- *with eigenvalue 2^{4n} : $(1, 1, 1, 1, 1, 1, 1, 1)^T$;*
- *with eigenvalue 2^{2n} : $(0, 2 - x, x - y - 3, y + 1, 0, 0, 0, 0)^T$ and $(0, 0, 0, 0, 0, 2 - x, x - y - 3, y + 1)^T$;*
- *with eigenvalue 1: $(0, 1, -2, 1, 0, 0, 0, 0)^T$ and $(0, 0, 0, 0, 0, 1, -2, 1)^T$;*
- *with eigenvalue 0: $(1, -3, 3, -1, 0, 0, 0, 0)^T$ and $(0, 0, 0, 0, 1, -3, 3, -1)^T$.*

Remark. Note that we have 8 eigenvalues (counting with multiplicity), but only 7 eigenvectors. The reason for that is because the matrix M_A is not always diagonalizable. Among the examples from section 5 the matrices M_{++} , M_{+---} , and M_{+--+} are diagonalizable (so they have an additional eigenvector with eigenvalue 2^{2n}), and others are not.

Proof. The eigenvector with eigenvalue 2^{4n} was found before and it corresponds to densities of the prototiles. The remaining six eigenvectors follow from eigenvectors $(0, 2 - x, x - y - 3, y + 1)^T$, $(0, 1, -2, 1)^T$ and $(1, -3, 3, -1)^T$ of the matrix Q with corresponding eigenvalues 2^{2n} , 1, and 0 respectively and lemma 7.3.

The last 8th eigenvalue can be found from the trace of the matrix M_A .

$$\text{tr}(M_A) = 2\text{tr}(X) = 2(2^{2n-2}(2^{2n-1} - 1) + a + b + c + 4(x + y) - 2) = 2^{4n-2} - 2^{2n-1} + 2(a + b + c) + 8(x + y) - 4.$$

Using that $a + b + c = 3(2^{2n-2} - 1)(2^{2n-1} - 1)$ and $x + y = 2^{2n}$ we get that $\text{tr}(M_A) = 2^{4n} + 3 \cdot 2^{2n} + 2$ which gives the last eigenvalue 2^{2n} . □

8. ON MIXED FOLDINGS

In general, we are not required to have all three parts of an elementary folding in definition 2.2 to be performed through one half-space of the ambient space. This will give eight options for what we will call *mixed elementary foldings* of a triangle with side $2a$ assuming we can independently choose which half-space is used for every triangle with side a .

Most preliminary results of the current paper are true for mixed foldings as well, but the coloring of layers could be different now. In particular, if a mixed elementary folding does not coincide with elementary folding we introduced before, then the layer \mathcal{L}_1 is colored according to the figure 23 (or its rotation) if the first mixed elementary folding in a sequence defining a pattern has two operations performed through upper half-space, and one through lower half-space.

We can not provide more details on the structure of the resulted pattern. In particular, the densities of the unit triangles are no longer equal as it can be seen from the layer \mathcal{L}_1 on figure 23. For example, all positive triangles from this layer have one red side and two blue sides which gives the density of this type at least $\frac{1}{8}$, however more such triangles could appear in the hexagons of the layer \mathcal{L}_1 .

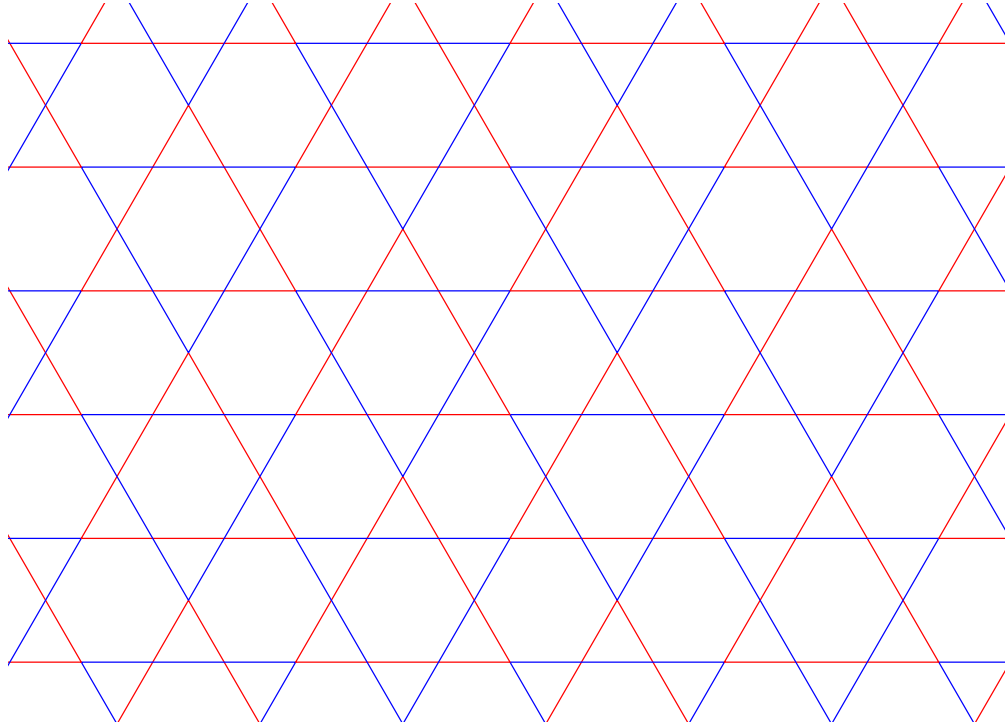


FIGURE 23. Coloring of \mathcal{L}_1 for a mixed folding.

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