

(k, λ) -ANTI-POWERS AND OTHER PATTERNS IN WORDS

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ABSTRACT. Given a word, we are interested in the structure of its contiguous subwords split into k blocks of equal length, especially in the homogeneous and anti-homogeneous cases. We introduce the notion of (μ_1, \dots, μ_k) -block-patterns, words of the form $w = w_1 \cdots w_k$ where, when $\{w_1, \dots, w_k\}$ is partitioned via equality, there are μ_s sets of size s for each $s \in \{1, \dots, k\}$. This is a generalization of the well-studied k -powers and the k -anti-powers recently introduced by Fici, Restivo, Silva, and Zamboni, as well as a refinement of the (k, λ) -anti-powers introduced by Defant. We generalize the anti-Ramsey-type results of Fici et al. to (μ_1, \dots, μ_k) -block-patterns and improve their bounds on $N_\alpha(k, k)$, the minimum length such that every word of length $N_\alpha(k, k)$ on an alphabet of size α contains a k -power or k -anti-power. We also generalize their results on infinite words avoiding k -anti-powers to the case of (k, λ) -anti-powers. We provide a few results on the relation between α and $N_\alpha(k, k)$ and find the expected number of (μ_1, \dots, μ_k) -block-patterns in a word of length n .

1. INTRODUCTION

In 1975 Erdős, Simonivits, and Sós [4] introduced anti-Ramsey theory, the idea that sufficiently large partitioned structures cannot avoid anti-homogeneous substructures. Their investigation was initially graph-theoretic, but with time anti-Ramsey-type results have permeated many areas of combinatorics, including the studies of Sidon sets, canonical Ramsey theory, and the spectra of colorings [1, 10, 12]. The study of homogeneous and anti-homogeneous substructures can also be extended to *words*, finite or infinite (to the right) sequences of letters from a fixed alphabet. The substructures of interest are contiguous subwords, known as *factors*. A well-studied type of regularity in words concerns k -powers, that is, words of the form $u^k = uu \cdots u$ (concatenated k times) for some nonempty word u (see, for example, [7]). Recently Fici et al. [6] introduced a notion of anti-regularity in words through their definition of k -anti-powers.

Definition 1.1. Let $|u|$ denote the length of a word u . A k -anti-power is a word w of the form

$$w = w_1 w_2 \cdots w_k$$

such that $|w_1| = \cdots = |w_k|$ and w_1, \dots, w_k are distinct.

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Fici et al. [6] were able to show several properties of anti-powers in words, including anti-Ramsey results concerning the existence of ℓ -powers or k -anti-powers. Defant [3] and Narayanan [9] showed that $\text{ap}(t, k)$, the minimum $m > 0$ for which the factor of length km beginning at the first index of the famous Thue-Morse word t is a k -anti-power, grows linearly in k . Defant also introduced the notion of (k, λ) -anti-powers, which is a generalization of k -anti-powers.

Definition 1.2. A (k, λ) -anti-power is a word w of the form

$$w = w_1 w_2 \cdots w_k$$

such that $|w_1| = \cdots = |w_k|$ and $|\{i : w_i = w_j\}| \leq \lambda$ for each fixed $j \in \{1, \dots, k\}$.

Note that when $\lambda = 1$, this is precisely the definition of a k -anti-power. Whenever such a generalization is nontrivial, we prove that the results of Fici et al. in [6] concerning k -anti-powers generalize to the case of (k, λ) -anti-powers. In fact, many of these results can be strengthened by enforcing a particular structure on the partition of the blocks by equality. We generalize the notions of k -powers and k -anti-powers while refining the (k, λ) -anti-powers through the introduction of (μ_1, \dots, μ_k) -block-patterns.

Definition 1.3. Let μ_1, \dots, μ_k be nonnegative integers satisfying $\sum_{s=1}^k s\mu_s = k$. A (μ_1, \dots, μ_k) -block-pattern is a word of the form $w = w_1 \cdots w_k$ where, if the set $\{1, \dots, k\}$ is partitioned via the rule $i \sim j \iff w_i = w_j$, there are μ_s parts of size s for all $1 \leq s \leq k$.

For example, 10 01 00 01 10 is a $(1, 2, 0, 0, 0)$ -block-pattern. Let $P_{k, \leq \lambda}$ denote the set of k -tuples of natural numbers (μ_1, \dots, μ_k) such that $\sum_{s=1}^k s\mu_s = k$ and $\mu_s = 0$ for $s > \lambda$. These correspond to the partitions of k such that each part has size at most λ . We can relate (μ_1, \dots, μ_k) -block-patterns to (k, λ) -anti-powers via the following observation.

Remark 1.4. Let $\mathcal{AP}_{\mathbb{A}}(k, \lambda)$ be the set of (k, λ) -anti-powers on an alphabet \mathbb{A} . Let $\mathcal{BP}_{\mathbb{A}}(\mu_1, \dots, \mu_k)$ be the set of (μ_1, \dots, μ_k) -block-patterns on \mathbb{A} . Then

$$\mathcal{AP}_{\mathbb{A}}(k, \lambda) = \bigcup_{(\mu_1, \dots, \mu_k) \in P_{k, \leq \lambda}} \mathcal{BP}_{\mathbb{A}}(\mu_1, \dots, \mu_k).$$

In particular, the k -anti-powers are precisely the $(k, 0, \dots, 0)$ -block-patterns, and moreover the k -powers are precisely the $(0, \dots, 0, 1)$ -block-patterns.

The generalizations of the anti-Ramsey results of Fici et al. in [6] to the case of (μ_1, \dots, μ_k) -block-patterns are the focus of Section 3. In particular, we obtain bounds on the sizes of words avoiding powers or block-patterns with at most σ pairs of equal blocks. In Section 4, we generalize the results of [6] on avoiding k -anti-powers in infinite words to (k, λ) -anti-powers. We also observe that Sturmian words have anti-powers of every order starting at each index.

A slight strengthening of the arguments of Fici et al. in [6] also provide better bounds for $N_{\alpha}(k, k)$, the smallest positive integer such that every word of length $N_{\alpha}(k, k)$ over an

alphabet of size α contains a k -power or k -anti-power. Namely, it is shown in [6] that for $k > 2$, $k^2 - 1 \leq N_\alpha(k, k) \leq k^3 \binom{k}{2}$. In Section 5, we improve both the lower and upper bounds according to the following theorem.

Theorem 5.1. For any $k > 3$,

$$2k^2 - 2k \leq N_\alpha(k, k) \leq (k^3 - k^2 + k) \binom{k}{2}.$$

In Section 5 we also investigate how the size of the alphabet affects $N_\alpha(k, k)$. In Section 6, we return to the more general setting and compute the expected number of (μ_1, \dots, μ_k) -block-patterns in a word of length n .

2. PRELIMINARIES

Let $\mathbb{N} = \{1, 2, 3, \dots\}$. The i^{th} letter of a word x is denoted $x[i]$, and for $i < j$ the contiguous substring beginning at the i^{th} letter and ending with the j^{th} is denoted $x[i..j]$. A word v is a *factor* of x if $x = uvw$ for words u and w . In the case that u is empty, v is a *prefix* of x , and if w is empty, then v is a *suffix* of x . The suffix of x beginning at the j^{th} index of x is denoted $x_{(j)}$. If w is both a prefix and suffix of x , then w is a *border* of x .

A word is called *recurrent* if every finite factor appears infinitely many times in the word. A word x is called *eventually periodic* if there exists an index $j \geq 0$ and a finite word u such that $x_{(j)} = u^\omega$; otherwise x is called *aperiodic*. A word is called ω -*power-free* if for every finite factor u , there exists an $\ell \in \mathbb{N}$ such that u^ℓ is not a factor. Note that a word that avoids k -powers for some $k \in \mathbb{N}$ is ω -power-free, but the converse is not necessarily true.

Let $[\alpha] = \{1, \dots, \alpha\}$. The *lower density* and *upper density* of a subset S of \mathbb{N} are given respectively by

$$\underline{d}(S) = \liminf_{n \rightarrow \infty} \frac{|S \cap [n]|}{n} \quad \text{and} \quad \bar{d}(S) = \limsup_{n \rightarrow \infty} \frac{|S \cap [n]|}{n}.$$

3. GENERALIZATION OF AN ANTI-RAMSEY RESULT TO (μ_1, \dots, μ_k) -BLOCK-PATTERNS

A main result of Fici et al. [6] is that every infinite word contains either powers of all orders or anti-powers of all orders. Since powers are homogeneous substructures whereas the anti-powers are anti-homogeneous, one may wonder if similar results can be demonstrated for substructures between these extremes. We will generalize their result to the case of (μ_1, \dots, μ_k) -block-patterns in infinite words. The density bounds rely on the number of pairs of equal blocks that are forced in the prefixes of length $km, \dots, k(m + \beta)$ for some m, β . The following definition is created to account for these pairs.

Definition 3.1. Let $D(x, k, \sigma)$ be the set of $m \in \mathbb{N}$ such that the prefix of the word x of length km is a (μ_1, \dots, μ_k) -block-pattern satisfying $\sum_{s=1}^k \mu_s \binom{s}{2} \leq \sigma$.

Note that $D(x, k, \sigma)$ is closed downward with respect to the dominance order. That is, if $m \in D(x, k, \sigma)$ and $m' \in \mathbb{N}$ are such that $x[1..km]$ is a (μ_1, \dots, μ_k) -block-pattern and $x[1..km']$ is a (μ'_1, \dots, μ'_k) -block-pattern satisfying $\sum_{s=1}^\ell \mu_s \geq \sum_{s=1}^\ell \mu'_s$ for each $\ell \in \{1, \dots, k\}$, then $m' \in D(x, k, \sigma)$.

For the proof of Theorem 3.3, we make use of the following lemma of Fici, Restivo, Silva, and Zamboni.

Lemma 3.2. ([6], Lemma 3) *Let v be a border of a word w and let u be the word such that $w = uv$. If ℓ is an integer such that $|w| \geq \ell|u|$, then u^ℓ is a prefix of w .*

Theorem 3.3. *Let x be an infinite word such that*

$$\bar{d}(D(x, k, \sigma)) \geq \left(1 + \left\lfloor \frac{1}{\sigma} \binom{k}{2} \right\rfloor\right)^{-1}$$

for some $k, \sigma \in \mathbb{N}$. For every ℓ , there is a word u with $|u| \leq (k-1) \left\lfloor \frac{1}{\sigma} \binom{k}{2} \right\rfloor$ such that u^ℓ is a factor of x .

Proof. Fix such a k and σ . Fix an arbitrary $\ell \geq 1$, and let $\beta = \left\lfloor \frac{1}{\sigma} \binom{k}{2} \right\rfloor$. By the condition on the upper density of $D(x, k, \sigma)$, there exists some integer $m > \ell(k-1)\beta$ such that $\{m, m+1, \dots, m+\beta\} \subset D(x, k, \sigma)$. Following [6], for every $j \in \{0, \dots, k-1\}$ and $r \in \{m, \dots, m+\beta\}$, set

$$U_{j,r} = x[jr + 1..(j+1)r].$$

That is, $U_{0,r} \cdots U_{k-1,r} = x[1..kr]$. Since $\{m, m+1, \dots, m+\beta\} \subset D(x, k, \sigma)$, we are guaranteed at least $(\beta+1)\sigma > \binom{k}{2}$ triples (i, j, r) such that $i < j$ and $U_{i,r} = U_{j,r}$. By the Pigeonhole Principle, there exist i, j, r, s such that $0 \leq i < j \leq k-1$, $m \leq r < s \leq \beta+1$, $U_{i,r} = U_{j,r}$, and $U_{i,s} = U_{j,s}$.

Setting $w = x[is + 1..(i+1)r]$ and $v = x[js + 1..(j+1)r]$, we have

$$|v| = (j+1)r - js < (i+1)r - is = |w|,$$

so v is a border of w . Writing $w = uv$, we have

$$1 \leq |u| = |w| - |v| = (j-i)(s-r) \leq (k-1)\beta$$

while

$$|v| = r - j(s-r) \geq m - (k-1)\beta \geq \ell(k-1)\beta \geq (\ell-1)|u|.$$

Hence, $|w| = |u| + |v| \geq \ell|u|$. By Lemma 3.2, u^ℓ is a factor of x . \square

Theorem 3.3 can be applied to the special case of (k, λ) -anti-powers. The definition of (k, λ) -anti-powers suggests the following generalization of $\text{AP}(x, k)$, the set of integers m such that the prefix of x of length km is a k -anti-power.

Definition 3.4. Let $\text{AP}(x, k, \lambda)$ be the set of $m \in \mathbb{N}$ such that the prefix of the word x of length km is a (k, λ) -anti-power.

Note that $\text{AP}(x, k, 1) = \text{AP}(x, k)$.

Corollary 3.5. *Let x be an infinite word such that*

$$\underline{d}(\text{AP}(x, k, \lambda)) < \left(1 + \left\lfloor \frac{k^2 - k}{\lambda^2 + \lambda} \right\rfloor\right)^{-1}$$

for some $k, \lambda \in \mathbb{N}$. For every ℓ , there is a word u with $|u| \leq (k-1) \left\lfloor \frac{k^2 - k}{\lambda^2 + \lambda} \right\rfloor$ such that u^ℓ is a factor of x .

Proof. Fix k and λ as above. Note that $\mathbb{N} \setminus \text{AP}(x, k, \lambda) \subseteq D\left(x, k, \binom{\lambda+1}{2}\right)$. Hence,

$$\underline{d}(\text{AP}(x, k, \lambda)) < \left(1 + \left\lfloor \frac{k^2 - k}{\lambda^2 + \lambda} \right\rfloor\right)^{-1}$$

implies

$$\bar{d}\left(D\left(x, k, \binom{\lambda+1}{2}\right)\right) \geq \frac{\left\lfloor \frac{k^2 - k}{\lambda^2 + \lambda} \right\rfloor}{1 + \left\lfloor \frac{k^2 - k}{\lambda^2 + \lambda} \right\rfloor} \geq \frac{1}{1 + \left[\binom{\lambda+1}{2}^{-1} \binom{k}{2} \right]}.$$

This shows that x satisfies the conditions of Theorem 3.3 for the same k and $\sigma = \binom{\lambda+1}{2}$. \square

In the case that our alphabet is finite, there are finitely many factors of length at most $(k-1) \left\lfloor \frac{k^2 - k}{\lambda^2 + \lambda} \right\rfloor$. Thus, the Pigeonhole Principle allows us to choose a word u that works for every ℓ in Theorem 3.5.

Corollary 3.6. *Let x be an infinite word on a finite alphabet such that*

$$\underline{d}(\text{AP}(x, k, \lambda)) < \left(1 + \left\lfloor \frac{k^2 - k}{\lambda^2 + \lambda} \right\rfloor\right)^{-1}$$

for some $k, \lambda \in \mathbb{N}$. There is a word u with $|u| \leq (k-1) \left\lfloor \frac{k^2 - k}{\lambda^2 + \lambda} \right\rfloor$ such that u^ℓ is a factor of x for every $\ell > 0$. In particular, x is not ω -power-free.

There is a $\lambda = 1$ analogue to Corollary 3.5 in [6] (their Theorem 4), which claims under the same density condition that x is not ω -power-free. Though the condition that the alphabet is finite is not explicitly stated, their result is false for infinite alphabets. In fact, there exist ω -power-free words which avoid k -anti-power prefixes for some fixed $k \in \mathbb{N}$. These words also show

that Theorem 6 of [6], which states that ω -power-free words have anti-powers of every order beginning at each index, is false when infinite alphabets are allowed. Theorem 3.7 provides a counterexample to Theorems 4 and 6 in [6] when infinite alphabets are permitted.

Theorem 3.7. *There exists an ω -power-free word x on an infinite alphabet such that $\text{AP}(x, k)$ is empty for some $k \in \mathbb{N}$.*

Proof. Let $y = \prod_{i=1}^{\infty} (a_i)^{2^i}$. Since there are finitely many appearances of each letter a_i , y is clearly ω -power-free. Note that if $2^{i+1} - 2^i = 2^i \geq 4m$ for some block length m and some i satisfying $2^{i+1} < km$, then two blocks of the prefix of length km must equal a_i^m . Hence, $m \notin \text{AP}(x, k)$. For $k \geq 17$, such an i always exists. $\text{AP}(x, k)$ is empty for $k \geq 17$, despite x being ω -power-free. \square

We return to a modified version of the proof of Theorem 3.3 in order to find bounds on the length of words avoiding k -powers and k -anti-powers.

Theorem 3.8. *For all integers $\ell > 1, k > 1, \sigma \geq 1$ there exists $N'_\alpha(\ell, k, \sigma)$ such that every word of length $N'_\alpha(\ell, k, \sigma)$ on $[\alpha]$ contains an ℓ -power or (μ_1, \dots, μ_k) -block-pattern satisfying $\sum_{s=1}^k \mu_s \binom{s}{2} \leq \sigma$. Moreover,*

$$k \left(k - \left\lfloor \frac{1}{2}(\sqrt{8\sigma + 1} + 1) \right\rfloor \right) \leq N'_\alpha(k, k, \lambda) \leq \left\lfloor \frac{1}{\sigma} \binom{k}{2} \right\rfloor (k^3 - k^2 + k).$$

Proof. As in [6], the upper bound follows from the proof of the infinite case in Theorem 3.3. Let $\beta = \left\lfloor \frac{1}{\sigma} \binom{k}{2} \right\rfloor$. Let x be any word of length $\beta(k^3 - k^2 + k)$. For each $r \in \{(k^2 - k)\beta, \dots, (k^2 - k + 1)\beta\}$, consider the first k consecutive blocks of length r in x , denoted by $U_{0,r}, U_{1,r}, \dots, U_{k-1,r}$. If x does not contain any element of $D(x, k, \sigma)$, then there exist i, j, r, s such that $0 \leq i < j \leq k - 1$, $m \leq r < s \leq \beta + 1$, $U_{i,r} = U_{j,r}$ and $U_{i,s} = U_{j,s}$. Setting $w = x[is + 1..(i + 1)r]$ and $v = x[js + 1..(j + 1)r]$, we have that v is a border of w . Writing $w = uv$, we have $|u| \leq (k - 1)\beta$ and

$$|w| = |u| + |v| \geq |u| + r - j(s - r) \geq |u| + (k - 1)^2\beta \geq k|u|.$$

By Lemma 3.2, we get that u^k is a factor of x , i.e., x contains a k -power. The length of x is chosen to accommodate k blocks of size at most $(k^2 - k + 1)\beta$.

The lower bound is proven via a construction; we will show that the word

$$x = 0^{k-1} (10^{k-1})^{k - \lfloor \frac{1}{2}(\sqrt{8\sigma + 1} + 1) \rfloor - 1}$$

avoids k -powers and (μ_1, \dots, μ_k) -block-patterns with $\sum_{s=1}^k \mu_s \binom{s}{2} \leq \sigma$. Since $\sigma \geq 1$, we have

$$k - \left\lfloor \frac{1}{2}(\sqrt{8\sigma + 1} + 1) \right\rfloor - 1 \leq k - 1.$$

If u^k were a factor of x , either u would contain the letter 1, contradicting the fact that x has at most $k - 1$ copies of the letter 1, or $u = 0^m$ for some $m \geq 1$, contradicting the fact that x

has no factor equal to 0^k . Hence, x avoids k -powers. We can see that for every factor v of length km , at least $\lfloor \frac{1}{2}(\sqrt{8\sigma+1}+1) \rfloor + 1$ blocks of v are equal to 0^m . v is a (μ_1, \dots, μ_k) -block-pattern with

$$\sum_{s=1}^k \mu_s \binom{s}{2} \geq \binom{\lfloor \frac{1}{2}(\sqrt{8\sigma+1}+1) \rfloor + 1}{2} > \frac{1}{8}(\sqrt{8\sigma+1}+1)(\sqrt{8\sigma+1}) \geq \sigma.$$

□

We can specialize Theorem 3.8 to the case of (k, λ) -anti-powers.

Corollary 3.9. *For all integers $\ell > 1, k > 1, \lambda \geq 1$, there exists $N_\alpha(\ell, k, \lambda)$ such that every word of length $N_\alpha(\ell, k, \lambda)$ on $[\alpha]$ contains an ℓ -power or (k, λ) -anti-power. Moreover,*

$$k(k-\lambda) \leq N_\alpha(k, k, \lambda) \leq \left\lfloor \frac{k^2 - k}{\lambda^2 + \lambda} \right\rfloor (k^3 - k^2 + k).$$

Proof. Suppose a word avoids (k, λ) -anti-powers. Then it avoids (μ_1, \dots, μ_k) -block-patterns with $\sum_{s=1}^k \mu_s \binom{s}{2} \leq \binom{\lambda+1}{2}$. Applying Theorem 3.8 with $\sigma = \binom{\lambda+1}{2}$ yields the corresponding bounds. □

In particular, this improves upon the upper bound for $N_\alpha(k, k)$ (in their notation, $N(k, k)$) in [6].

Corollary 3.10. *For all $k > 1$,*

$$N_\alpha(k, k) \leq (k^3 - k^2 + k) \binom{k}{2}.$$

4. AVOIDING ANTI-POWERS

This section is devoted to generalizing the results of Fici et al. [6] on infinite words avoiding k -anti-powers to the case of (k, λ) -anti-powers. Many of these generalizations can be achieved using proofs similar to those in [6]. We also provide a condensed proof of the fact that the Sturmian words contain anti-powers of every order beginning at every index.

We begin with a straightforward lemma.

Lemma 4.1. *Suppose $k > \lambda > j > 1$. If a word avoids (k, λ) -anti-powers, then it avoids $(k-j, \lambda-j)$ -anti-powers.*

Proof. It is enough to show that if a word avoids (k, λ) -anti-powers, then it avoids $(k-1, \lambda-1)$ -anti-powers. Suppose that a word x contains a $(k-1, \lambda-1)$ -anti-power w of length km . If we extend to the right by m letters, we obtain a (k, λ) -anti-power, since we increase the number of equal blocks, $|\{i : w_i = w_j\}|$ for any j , by at most 1. □

Definition 4.2. We call an infinite word *constant* if it is of the form a^ω for some $a \in \mathbb{A}$.

In order to classify the words avoiding $(k, k-2)$ -anti-powers, we will use two results of Fici, Restivo, Silva, and Zamboni.

Lemma 4.3. ([6], Lemma 9) *Let x be an infinite word. If x avoids 3-anti-powers, then x is a binary word.*

Proposition 4.4. ([6], Proposition 10) *Let x be an infinite word. If x avoids 3-anti-powers, then it cannot contain a factor of the form $10^n 1$ or $01^n 0$ with $n > 1$.*

Theorem 4.5.

- (1) *For $k > 1$, the infinite words avoiding $(k, k-1)$ -anti-powers are precisely the constant words.*
- (2) *For $k > 2$, infinite words avoiding $(k, k-2)$ -anti-powers are the words that differ from a constant word in at most one position.*
- (3) *For $k > 3$, there exist infinite aperiodic words avoiding $(k, k-3)$ -anti-powers.*

Proof. The first claim is trivial; merely note that the avoidance of $(k, k-1)$ -anti-powers implies that every factor whose length is a multiple of k is a k -power.

For the second claim, let x be a word avoiding $(k, k-2)$ -anti-powers. By Lemma 4.1, x avoids 3-anti-powers. By Lemma 4.3, x is a binary word. Suppose, seeking a contradiction, that x has at least 2 instances of 1 and at least 2 instances of 0, i.e., x differs from a constant word in more than one position. Then x has a factor of the form $10^a 1^b 0$ or $01^a 0^b 1$ for some $a \geq 1, b \geq 1$; without loss of generality assume it is the first. By Proposition 4.4, $a = b = 1$. However, under these conditions, x has a factor of the form 1010, which is itself a $(4, 2)$ -anti-power.

For the third claim, we exhibit a family of infinite aperiodic words avoiding $(k, k-3)$ -anti-powers. Let $\{\gamma_i\}_{i=1}^n$ be an increasing sequence such that $\gamma_{i+1} \geq (k+1)\gamma_i$ for all $i \in \mathbb{N}$. Define a word x as follows:

$$x[j] = \begin{cases} 1 & \text{if } j = \gamma_i \text{ for some } i; \\ 0 & \text{otherwise.} \end{cases}$$

We will show that x avoids $(k, k-3)$ -anti-powers. Note that if $x[\ell+1..\ell+n]$ has at least two nonzero entries, then for some i we have

$$\ell + 1 \leq \gamma_i < (k+1)\gamma_i \leq \gamma_{i+1} \leq \ell + n.$$

This implies that $n > k\gamma_i \geq k(\ell+1)$, so $\ell+1 \leq \frac{n}{k}$. Suppose, seeking a contradiction, that the k consecutive blocks $x[j+1..j+m], \dots, x[j+(k-1)m+1..j+km]$ form a $(k, k-3)$ -anti-power. At most $k-3$ of these blocks can be 0^m , so the word $x[j+m+1..j+km]$ has at least two nonzero entries. Thus, $j+m+1 \leq \frac{(k-1)m}{k}$. It follows that $j+1 < 0$, a contradiction. \square

Theorem 4.6. *For all $k \geq 6$, there exist aperiodic recurrent words avoiding $(k, k - 5)$ -anti-powers.*

Proof. Let w be the limit of the sequence $w_0 = 0$, $w_{n+1} = w_n 1^{(k-3)|w_n|} w_n$. Note that each occurrence of w_n except the first is preceded and followed by $1^{(k-3)|w_n|}$. Let $v = v_1 v_2 \cdots v_k$ be a factor of w , where $|v_i| = \ell > 0$ for all $i \in \{1, \dots, k\}$. Let n be the largest integer such that

$$|w_n| = (k - 1)^n < 2\ell < (k - 1)^{n+1} = |w_{n+1}|.$$

Since w is recurrent, we can assume v appears after the first appearance of w_n .

We claim that at most four blocks of v can intersect an occurrence of w_n . Each occurrence of w_n intersects at most two blocks of v by the condition $2\ell > |w_n|$. Moreover, any three occurrences of w_n are separated by factors of $1^{(k-3)|w_n|}$ and $1^{(k-3)|w_{n+1}|}$. As

$$|v| = k\ell < \frac{k}{2}|w_{n+1}| \leq (k - 3)|w_{n+1}|,$$

v can intersect at most 2 occurrences of w_n . We can conclude that at most four blocks of v are not equal to 1^ℓ . \square

We now restrict ourselves to the setting of k -anti-powers. In [6], Fici et al. question under what conditions aperiodic recurrent words can avoid k -anti-powers. It is known this is possible for $k \geq 6$ and impossible for $k \leq 3$, but nothing has been shown for $k = 4$ or 5 . One class of aperiodic recurrent words that we can exclude from this search are the *Sturmian words*.

Definition 4.7. A *Sturmian word* is an infinite word x such that for all $n \in \mathbb{N}$, x has exactly $n + 1$ distinct factors of length n .

Note that Sturmian words are necessarily binary. An alternate characterization of the Sturmian words in terms of *irrationally mechanical words* was given by Morse and Hedlund [8] in 1938.

Definition 4.8. The *upper mechanical word* $s_{\theta, x}$ and the *lower mechanical word* $s'_{\theta, x}$ with angle θ and initial position x are defined, respectively, by

$$s_{\theta, x}[n] = \begin{cases} 1 & \text{if } \theta(n - 1) + x \in [1 - \theta, 1) \pmod{1} \\ 0 & \text{if } \theta(n - 1) + x \in [0, 1 - \theta) \pmod{1} \end{cases}$$

$$s'_{\theta, x}[n] = \begin{cases} 1 & \text{if } \theta(n - 1) + x \in (1 - \theta, 1) \pmod{1} \\ 0 & \text{if } \theta(n - 1) + x \in [0, 1 - \theta] \pmod{1} \end{cases}$$

for some $\theta, x \in \mathbb{R}$. A word w is called *irrationally mechanical* if $w = s_{\theta, x}$ or $w = s'_{\theta, x}$ for some $x \in \mathbb{R}$ and irrational $\theta \in \mathbb{R}$.

Theorem 4.9. ([8]) *A word is Sturmian if and only if it is irrationally mechanical.*

Irrationally mechanical words can be interpreted through the lens of mathematical billiards. Consider the unit circle centered at the origin, parameterized by $g(t) = (\cos(2\pi t), \sin(2\pi t))$ for $t \in \mathbb{R}$. Place an (infinitesimal) ball at point $g(x)$ on the circle and shoot it in a straight trajectory toward $g(x + \pi)$. At each moment the ball "bounces off" the circle, it generates a 0 if it hits the point $g(x)$ for $x \in [0, 1 - \theta)$ and a 1 otherwise. The sequence generated by the trajectory of such a ball is precisely the word $s_{\theta, x}$. For example, a trajectory generating the famous Fibonacci word is shown below.

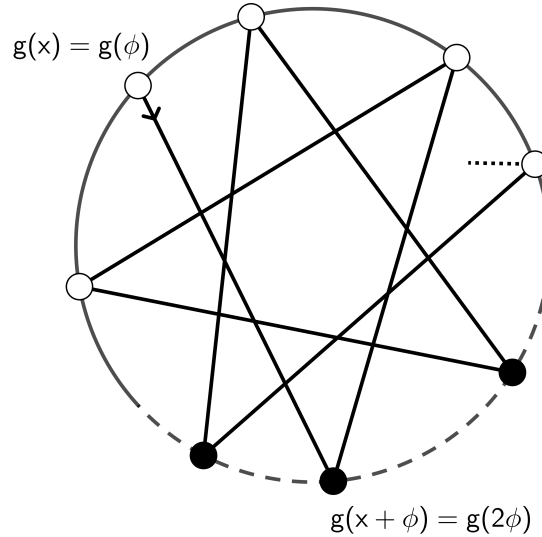


FIGURE 1. The trajectory associated with the Fibonacci word $s_{\phi, \phi} = 01001010\dots$, where ϕ is the golden ratio $1.6180339\dots$. A white point indicates that the letter 0 is generated, and a black point indicates that the letter 1 is generated. The Fibonacci word can also be generated as the limit of the sequence $\{S_n\}_{n=1}^{\infty}$, where $S_1 = 0$, $S_2 = 01$, and $S_n = S_{n-1}S_{n-2}$ for $n \geq 3$.

The Sturmian words comprise a well-studied class of aperiodic recurrent words. We will show that for any Sturmian word x , $j \geq 0$, and $k \geq 1$, x contains a k -anti-power beginning at $x[j]$. Hence, the Sturmian words cannot avoid k -anti-powers for any $k \geq 1$. It is enough to show that the Sturmian words are ω -power-free by the following theorem from [6].

Theorem 4.10. ([6], Theorem 6) *Let x be an ω -power-free word on a finite alphabet. For every $k > 1$ there is an occurrence of a k -anti-power starting at every position of x .*

In fact, for every Sturmian word x there exists an $M \in \mathbb{N}$ such that x avoids M powers. This follows from the work of Fici, Langiu, Lecroq, Lefebvre, Mignosi, Peltomäki, and Prieur-Gaston in [5]. They prove a stronger but somewhat lengthy result about a generalized notion of powers known as *abelian powers*. We provide a condensed proof of our weaker claim.

Theorem 4.11. *Let w be a Sturmian word with angle θ . Let $M = \lceil (\min\{\theta, 1 - \theta\})^{-1} \rceil + 1$. Then s_θ avoids M -powers.*

Proof. Suppose w is an upper mechanical word; the case for lower mechanical words follows analogously. Let $w = s_{\theta,x}$ for $x \in \mathbb{R}$. Suppose, seeking a contradiction, that there exists a factor u of length m such that u^M is also a factor. Let $\{z\}$ denote the fractional part of $z \in \mathbb{R}$. There is some nonnegative integer r such that

$$\{x + r\theta\}, \{x + (r + m)\theta\}, \{x + (r + 2m)\theta\}, \dots, \{x + (r + (M - 1)m)\theta\}$$

either all lie in the interval $[0, 1 - \theta)$ or all lie in the interval $[1 - \theta, 1)$.

In the first case, we divide $[0, 1)$ up into $\lceil \theta^{-1} \rceil$ intervals of uniform size, starting at 0. By the Pigeonhole Principal, at least two of the M points lie in the same interval. In other words, there exist $0 \leq q, i, j \leq \lceil \theta^{-1} \rceil$ such that

$$\{(x + (r + im)\theta), \{x + (r + jm)\theta\} \in \left[\frac{q}{\lceil \theta^{-1} \rceil}, \frac{q+1}{\lceil \theta^{-1} \rceil} \right).$$

Hence, $\{x + (r + |i - j|m)\theta\} \in [1 - \lceil \theta^{-1} \rceil^{-1}, 1) \subseteq [1 - \theta, 1)$, a contradiction.

In the second case, we divide $[0, 1)$ up into $\lceil (1 - \theta)^{-1} \rceil$ intervals of uniform size, starting at 0. By the Pigeonhole Principal, at least two of the M points lie in the same interval. In other words, there exist $0 \leq q, i, j \leq \lceil (1 - \theta)^{-1} \rceil$ such that

$$\{(x + (r + im)\theta), \{x + (r + jm)\theta\} \in \left[\frac{q}{\lceil (1 - \theta)^{-1} \rceil}, \frac{q+1}{\lceil (1 - \theta)^{-1} \rceil} \right).$$

Hence, $\{x + (r + |i - j|m)\theta\} \in [0, \lceil (1 - \theta)^{-1} \rceil^{-1}) \subseteq [0, 1 - \theta)$, a contradiction. \square

Corollary 4.12. *Let x be a Sturmian word. For every $k > 1$, there is an occurrence of a k -anti-power starting at every position of x .*

Definition 4.13. Given a sequence $\{v_n\}_{n=1}^\infty$ of finite words, define words w_n by $w_1 = v_1$ and $w_{n+1} = w_n v_n w_n$. The limit of the sequence of words $\{w_n\}_{n=1}^\infty$ is called the *sesquipower induced by the sequence* $\{v_n\}_{n=1}^\infty$.

It is well-known that an infinite word is recurrent if and only if it is a sesquipower (see, for example, [7]). We will show that if an aperiodic recurrent word avoids k -anti-powers, then we can deduce some properties about the sequence $\{v_n\}_{n=1}^\infty$.

Theorem 4.14. *Let x be the aperiodic sesquipower on a finite alphabet induced by $\{v_n\}_{n=1}^\infty$, and suppose x avoids k -anti-powers for some $k \geq 2$. There exists a word u of length at most $k - 1$ such that for all $\ell > 0$, there is some $n > 0$ such that u^ℓ is a factor of v_n .*

Proof. Since x avoids k -anti-powers, Corollary 3.6 implies x is not ω -power-free. Thus, there is some factor u of x such that u^ℓ is a factor for every $\ell > 0$. We can assume u is not an m -power

for any $m \geq 2$; otherwise, let $u = u_{(|u|/m)}$. Suppose, seeking a contradiction, that $|u| \geq k$. Note that the prefix of length $k(|u| + 1)$ of u^{k+1} is a k -anti-power, contradicting the fact that u^{k+1} is a factor of x that avoids k -anti-powers. Thus, $|u| \leq k - 1$.

We now know arbitrarily long powers of u occur in x , but, in fact, we can show that arbitrarily long powers of u occur in $\{v_n\}_{n=1}^{\infty}$. Since x is not periodic, there exists a k such that v_k is not a factor of u^ω . Let ℓ_0 be the largest power of u that is a factor of x_k . For sufficiently large ℓ , there is some $m \geq k$ such that u^ℓ is a factor of w_{m+1} but not of w_m . We can conclude $u^{\ell-2\ell_0-2}$ is a factor of v_{m+1} , as w_k is a border of w_m . As ℓ_0 is fixed, this implies that for all $\ell > 0$, u^ℓ is a factor of some v_m . \square

To summarize, if x is an aperiodic recurrent word avoiding k -powers for some $k \geq 2$, then x is non-Sturmian and is the sesquipower induced by a sequence $\{v_n\}_{n=1}^{\infty}$ where the v_n contain arbitrarily long powers of some word u .

5. AVOIDING POWERS AND ANTI-POWERS

In [6], Fici et al. show that for every $\ell, k > 1$, there exists $N_\alpha(\ell, k)$ such that every word of length $N_\alpha(\ell, k)$ on an alphabet of size α contains either an ℓ -power or a k -anti-power. They prove that for $k > 2$, one has $k^2 - 1 \leq N_\alpha(k, k) \leq k^3 \binom{k}{2}$. We improve both these lower and upper bounds.

Theorem 5.1. *For any $k > 3$,*

$$2k^2 - 2k \leq N_\alpha(k, k) \leq (k^3 - k^2 + k) \binom{k}{2}.$$

Proof. The upper bound is precisely the statement of Corollary 3.10.

For the lower bound, consider the word

$$x = 1(0^{k-1}1)^{k-2}0^{k-2}10^{k-2}(10^{k-1})^{k-2}1.$$

We begin by showing that the border $1(0^{k-1}1)^{k-2}0^{k-2}10^{k-2}$ of length $k^2 - 2$ avoids k -powers and k -anti-powers. In their proof that $k^2 - 1 \leq N_\alpha(k, k)$, Fici et al. [6] show that the word $(0^{k-1}1)^{k-2}0^{k-2}10^{k-1}$ avoids k -powers and k -anti-powers, so we need only check this border for k -power or k -anti-power prefixes. We can see immediately that there are no k -power prefixes: as $1 \leq m \leq k - 1$, the first block of length m of the prefix of length km begins with 1 while the second begins with 0. Suppose, seeking a contradiction, that the prefix of length km is a k -anti-power for some m . Since the prefix of length km would need to contain at least $k - 1$ instances of the letter 1 to distinguish the blocks, we require $km \geq 1 + k(k - 2)$. Hence, $m \geq k - 1$. We know the block length m is at most $k - 1$ since $1(0^{k-1}1)^{k-2}0^{k-2}10^{k-2}$ has length $k^2 - 2$. However, as $k(k - 2) + 1 = (k - 1)^2$, the last two blocks must be $0^{k-2}1$. The equality

of these blocks contradicts the assumption that the prefix of length km is a k -anti-power. $1(0^{k-1}1)^{k-2}0^{k-2}10^{k-2}$ avoids both k -powers and k -anti-powers.

Thus, we need to consider only those factors of x intersecting nontrivially with the prefix and suffix of length $k^2 - 2$. Fix such a factor y of length km starting at position j .

Suppose, seeking a contradiction, that y is a k -power. Let $y_\ell = x[j + \ell m..j + (\ell + 1)m - 1]$ be the ℓ^{th} block of length m in y . Choose b such that the central letter 1 of x is contained in y_b . That is, $j + bm \leq k(k-1) \leq j + (b+1)m - 1$. Since $k \geq 4$ and there are exactly two occurrences of the factor $10^{k-2}1$ in x , the block y_b cannot contain $10^{k-2}1$ as a factor. Note $y_\ell \neq 0^m$ for any nonnegative integers ℓ and m . As $k \geq 4$, one of $b-2, b+2 \in \{0, \dots, k-1\}$; without loss of generality assume it is $b+2$. Thus, the two factors $y_{b-1}y_b$ and $y_{b+1}y_{b+2}$ of x each contain an occurrence of the factor $10^{k-2}1$. However, the only two occurrences of $10^{k-2}1$ in x intersect while $y_{b-1}y_b$ and $y_{b+1}y_{b+2}$ are disjoint, so we've reached a contradiction. Therefore, x avoid k -powers.

Now we show that such a factor y is not a k -anti-power. Suppose it were. Since y contains at least $k-1$ occurrences of the letter 1, it follows that $m \geq k-2$. In the case $m = k-2$, each block contains at most one occurrence of the letter 1, but there are only $k-1$ distinct such blocks. One can check $m \neq k-1, k$ by examining the period of the prefix/suffix of length $k(k-2)+1$ or the middle section of length $2k-1$. Thus, taking into consideration the length of x , we have $k+1 \leq m \leq 2k-3$. Consider all blocks except y_b (the block containing the central 1). Note that the letters of each block are determined by the number of leading 0's, which is at most $k-1$. If there are ℓ blocks preceding y_b , and y_{b-1} has z leading zeros, then the numbers of leading zeros for all blocks except y_b are given by the multiset

$$(*) \quad \{z + (\ell - 1)m, \dots, z + m, z, z - 2m - 2, z - 3m - 2, \dots, z - (k - \ell)m - 2\} \pmod k$$

which has a repeated element if and only if the multiset

$$\{(l-1)m, \dots, m, 0, (k-2)m-2, (k-3)m-2, \dots, \ell m-2\} \pmod k$$

has a repeated element. Assuming this has no repeated element, we have

$$\{\ell m - 2, \dots, (k-2)m - 2\} \subseteq \{\ell m, (\ell+1)m, \dots, (k-1)m\}.$$

The left-hand side has size $k - \ell - 1$ while the right-hand side has size at most $k - \ell$, and both are arithmetic progressions with difference m . Thus, either $\ell m - 2 \equiv \ell m \pmod k$ or $\ell m - 2 \equiv (\ell + 1)m \pmod k$. In the former case, $2 \equiv 0 \pmod k$, but this would imply $k = 2$, contradicting the fact that $k \geq 4$. In the latter case, $m \equiv -2 \pmod k$, but this also leads to a contradiction as $k + 1 \leq m \leq 2k - 3$. Therefore, x avoids k -powers and k -anti-powers. \square

Note that the above bounds are independent of the alphabet size. This leads to two questions: does $N_\alpha(k, k)$ depend on the size of the alphabet, and if so, in what way? Note that $N_\alpha(k, k)$ is nondecreasing as α increases. The following values of $N_2(k, k)$ were computed by Shallit [11].

$$N_2(1, 1) = 1 \quad N_2(2, 2) = 2 \quad N_2(3, 3) = 9 \quad N_2(4, 4) = 24 \quad N_2(5, 5) = 55$$

For small α and k , $N_\alpha(k, k)$ can be computed by testing all α -ary strings of small length with a computer. In particular, we were able to check that $N_4(3, 3) = N_2(3, 3) = 9$, which implies $N_\alpha(3, 3) = 9$ for all $\alpha \geq 2$. This follows because a word of length 9 avoiding 3-anti-powers would use at most 4 letters. We also computed that $N_{11}(4, 4) = N_2(4, 4) = 24$. It is straightforward to check that a word of length 24 avoiding 4-anti-powers would use at most 11 letters. Thus, $N_\alpha(3, 3)$ and $N_\alpha(4, 4)$ are independent of α . It remains open if this is true for all k .

Another scenario to investigate is under what conditions a word can be extended (in a potentially larger alphabet) and still avoid k -powers and k -anti-powers. We aim to show that for large enough α , no word of length $N_\alpha(k, k) - 1$ can be extended (in a larger alphabet) and avoid k -powers and k -anti-powers. To do so, we require the following lemma.

Lemma 5.2. *If there exists $\alpha \geq 2$ such that $N_\alpha(k, k) < N_{\alpha+1}(k, k)$, then one of the following must hold:*

- (1) *Let $W_{k,\alpha}$ be the set of words on $[\alpha + 1]$ of length $N_\alpha(k, k)$ that avoid k -powers and k -anti-powers. For every word $w \in W_{k,\alpha}$, the two factors of w of length $N_\alpha(k, k) - 1$ each use exactly $\alpha + 1$ letters.*
- (2) *There exists a word on $[\alpha]$ of the form*

$$w = u_1(1u_1)^{k-1}x_1 = u_2(2u_2)^{k-1}x_2 = \cdots = u_\alpha(\alpha u_\alpha)^{k-1}x_\alpha$$

that avoids k -powers and k -anti-powers, where $x_1, \dots, x_\alpha, u_1, \dots, u_\alpha$ are finite words, $|u_1| < \cdots < |u_\alpha|$, and for all $1 \leq i < j \leq \alpha$,

$$\gcd(|u_i| + 1, |u_j| + 1) \leq \frac{|u_j| + 1}{k - 1}.$$

Proof. Suppose that the first case does not hold. There is a word w of length $N_\alpha(k, k) - 1$ on $[\alpha]$ such that $(\alpha + 1)w$, the extension of w by the addition the letter $\alpha + 1$ on the left, avoids k -powers and k -anti-powers. Since $|w|$ is maximal for words on $[\alpha]$ avoiding k -powers and k -anti-powers, the extension aw contains a k -power or k -anti-power for any $a \in [\alpha]$. As $(\alpha + 1)w$ contains no k -anti-powers, neither does aw for any $a \in [\alpha]$. Thus, aw has a prefix that is a k -power for each $a \in [\alpha]$.

Hence,

$$w = u_1(1u_1)^{k-1}x_1 = u_2(2u_2)^{k-1}x_2 = \cdots = u_\alpha(\alpha u_\alpha)^{k-1}x_\alpha,$$

where $x_1, \dots, x_\alpha, u_1, \dots, u_\alpha$ are finite words. Note that this implies $w[m(|u_\ell| + 1)] = \ell$ for all $1 \leq m \leq k - 1$ and $1 \leq \ell \leq k$. Without loss of generality (since the labels of the letters are arbitrary), we can assume $|u_1| < \cdots < |u_\alpha|$. Suppose, seeking a contradiction, that for some $1 \leq i < j \leq \alpha$, we have $\gcd(|u_i| + 1, |u_j| + 1) \geq \frac{|u_j| + 1}{k - 1}$. There is some $1 \leq m \leq k - 1$ such that $m(|u_i| + 1) \equiv 0 \pmod{(|u_j| + 1)}$. Hence, $m(|u_i| + 1) = d(|u_j| + 1)$ for some $1 \leq d \leq k - 1$. However, this implies

$$i = w[m(|u_i| + 1)] = w[d(|u_j| + 1)] = j.$$

Since we assumed $i < j$, we've reached a contradiction. \square

An investigation of the failure of the first case leads to the following corollary.

Corollary 5.3. *Suppose $\alpha > \frac{N_\alpha(k,k)}{k} - k + 3$. If a word w has a factor $u \neq w$ of length $N_\alpha(k,k) - 1$ that uses only α letters, w contains a k -power or k -anti-power.*

Proof. Suppose, seeking a contradiction, that w is as above but contains no k -power or k -anti-power. For all $1 \leq i < j \leq \alpha$, we have by Lemma 5.2 that

$$1 \leq \gcd(|u_i| + 1, |u_j| + 1) \leq \frac{|u_j| + 1}{k - 1}.$$

Thus, $|u_j| \geq k - 2$ for all $j \geq 2$. Since the $|u_j|$'s are strictly increasing, this implies $|u_\alpha| \geq (\alpha - 2) + (k - 2) = \alpha + k - 4$. As $w = u_\alpha(\alpha u_\alpha)^{k-1}x_\alpha$, we have

$$|w| \geq k(\alpha + k - 4) + k - 1 = k\alpha + k^2 - 3k - 1.$$

Since w is a word on $[\alpha]$ avoiding k -powers and k -anti-powers, $k\alpha + k^2 - 3k - 1 \leq N_\alpha(k,k)$. If this inequality is not satisfied, then we can conclude w is as above but contains a k -power or k -anti-power. \square

6. BLOCK PATTERNS AND THEIR EXPECTATION

In this section, we return to the general setting of block-patterns to calculate the expected number of (μ_1, \dots, μ_n) -block-patterns in a word of length n on an alphabet of size α . The special case of this expectation for k -powers was calculated by Christodoulakis, Christou, Crochemore, and Iliopoulos in [2].

Theorem 6.1. ([2], Theorem 4.1) *On average, a word of length n has $\Theta(n)$ k -powers. More precisely, this number is*

$$(n+1) \frac{\alpha^{1-k}(1 - \alpha^{(1-k)\lfloor \frac{n}{k} \rfloor})}{1 - \alpha^{1-k}} - \frac{k}{\alpha^{k-1}} \left(\frac{1}{1 - \alpha^{1-k}} - \frac{\lfloor \frac{n}{k} \rfloor \alpha^{(1-k)\lfloor \frac{n}{k} \rfloor}}{1 - \alpha^{1-k}} + \frac{\alpha^{1-k}(1 - \alpha^{(1-k)(\lfloor \frac{n}{k} \rfloor - 1)})}{(1 - \alpha^{1-k})^2} \right).$$

Theorem 6.2. *On average, a word of length n has $O(n^2)$ and $\Omega(n)$ (μ_1, \dots, μ_k) -block-patterns. More precisely, the expected number of (μ_1, \dots, μ_k) -block-patterns is*

$$\sum_{m=1}^{\lfloor \frac{n}{k} \rfloor} (n+1 - km) \frac{k!}{\mu_1! \cdots \mu_k!} \frac{1}{\alpha^{km}} \prod_{\ell=1}^{\mu_1 + \cdots + \mu_k} (\alpha^m - (\ell - 1)).$$

Proof. Let x be a word of length n , drawn uniformly at random. Let

$$X_{i,j} = \begin{cases} 1 & \text{if } x[i..j] \text{ is a } (\mu_1, \dots, \mu_k)\text{-block-pattern;} \\ 0 & \text{otherwise.} \end{cases}$$

Let $N = \sum_{i \leq j} X_{i,j}$. That is, N is the number of (μ_1, \dots, μ_k) -block-patterns in x . We have

$$\begin{aligned} \mathbb{E}[N] &= \mathbb{E} \left[\sum_{i=1}^{n-k+1} \sum_{j=i+1}^n X_{i,j} \right] \\ &= \sum_{i=1}^{n-k+1} \sum_{j=i+1}^n \mathbb{E}[X_{i,j}] \\ &= \sum_{i=1}^{n-k+1} \sum_{j=i+1}^n \mathbb{P}(x[i..j] \text{ is a } (\mu_1, \dots, \mu_k)\text{-block-pattern}). \end{aligned}$$

Let us count the number of (μ_1, \dots, μ_k) -block-patterns of length α^{j+1-i} on $[\alpha]$. Partition $[k]$ into unlabeled parts with μ_s parts of size s , and choose $\mu_1 + \dots + \mu_k$ distinct ordered elements from $\alpha^{(j+1-i)/k}$. We can assign elements to parts by order of appearance of the parts, which will yield a (μ_1, \dots, μ_k) -block-pattern. Moreover, the block-pattern is uniquely determined by the choice of an unlabeled partition and ordered m -tuple. Let $[A]$ denote the indicator function of the event A . We have

$$\begin{aligned} \mathbb{E}[N] &= \sum_{i=1}^{n-k+1} \sum_{j=i+1}^n \frac{k!}{\mu_1! \dots \mu_k!} \frac{1}{\alpha^{j+1-i}} \prod_{\ell=1}^{\mu_1 + \dots + \mu_k} \left(\alpha^{(j+1-i)/k} - (\ell - 1) \right) [j+1-i \equiv 0 \pmod{k}] \\ &= \sum_{m=1}^{\lfloor \frac{n}{k} \rfloor} (n+1-km) \frac{k!}{\mu_1! \dots \mu_k!} \frac{1}{\alpha^{km}} \prod_{\ell=1}^{\mu_1 + \dots + \mu_k} (\alpha^m - (\ell - 1)). \end{aligned}$$

Since there are only $\binom{n}{2} + n$ nonempty factors of x , we have $\mathbb{E}[N] = O(n^2)$. Note that the expectation is minimized for k -powers, where $\mu_k = 1$ and $\mu_s = 0$ for all $s < k$. Thus, from Theorem 6.1, we have $\mathbb{E}[N] = \Omega(n)$. \square

Corollary 6.3. *On average, a word of length n has $\Theta(n^2)$ k -anti-powers. More precisely, the expected number of k -anti-powers is*

$$\sum_{m=1}^{\lfloor \frac{n}{k} \rfloor} (n+1-km) \prod_{\ell=0}^{k-1} \left(1 - \frac{\ell}{\alpha^m} \right).$$

Proof. The formula follows Theorem 6.2 in the case $\mu_1 = k$ and $\mu_s = 0$ for $s > 1$. Restricting the sum (of nonnegative terms) to the range $\lfloor \frac{n}{4k} \rfloor \leq m \leq \lfloor \frac{3n}{4k} \rfloor$, we see

$$\mathbb{E}[\#(k\text{-anti-powers in } x)] \geq \left(\frac{n}{2k} - 1 \right) \left(\frac{n}{4} + 1 \right) \left(1 - \frac{k-1}{\alpha^{\lfloor \frac{n}{4k} \rfloor}} \right)^{k-1}.$$

For $n > 4k \left(1 + \log_\alpha \frac{k-1}{1-2^{-k-1}}\right)$, we have $\left(1 - \frac{k-1}{\alpha^{\lfloor \frac{n}{4k} \rfloor}}\right)^{k-1} > \frac{1}{2}$, hence

$$\mathbb{E} [\#(k\text{-anti-powers in } x)] > \frac{n^2}{16k} - \frac{n}{4} - \frac{1}{2} = \Omega(n^2). \quad \square$$

7. FURTHER DIRECTIONS

Recall that Theorem 3.3 shows that having a small enough density of (μ_1, \dots, μ_k) -block-pattern prefixes with few equal blocks implies the existence of arbitrarily long power prefixes. We believe that a strengthening of this argument could yield a lower bound on the density of $P(x, k)$, the set of $m \in \mathbb{N}$ such that the prefix of x of length km is a k -power. In Section 3, we also remark that Theorem 6 of [6], stating that if x is an ω -power-free word then $AP(x_{(j)}, k)$ is nonempty for every j and k , is false if we allow infinite alphabets. Perhaps there is a finer characterization of which ω -power-free words fail this condition.

As in the bounds found by Fici et al. [6], our upper and lower bounds for $N_\alpha(k, k)$ are polynomials in k whose degrees differ by 3. If it is the case that $N_\alpha(k, k)$ depends on α , such a dependence could be used to strengthen the bounds for $N_\alpha(k, k)$. Given the few known values of $N_\alpha(k, k)$, it seems plausible that k always divides $N_\alpha(k, k)$. On the other hand, if $N_\alpha(k, k)$ is independent of α , this alone would be an interesting structural property of the set of words avoiding k -powers and k -anti-powers achieving the length $N_\alpha(k, k)$ for arbitrary α . We believe the second case holds.

Conjecture 7.1. *The quantity $N_\alpha(k, k)$ is independent of α .*

Whether there exist aperiodic recurrent words avoiding 4 or 5 powers remains an open question. One may wish to investigate other large classes of words, such as the morphic words, and their potential to avoid k -anti-powers. A natural generalization is to find the structure of infinite words avoiding (μ_1, \dots, μ_k) -block-patterns other than (k, λ) -anti-powers.

Lastly, let $\mathbb{A} = \{a_1, \dots, a_\alpha\}$ be a finite alphabet. The *Parikh vector* $\mathcal{P}(w) = (e_1, \dots, e_\alpha)$ of a finite word w on \mathbb{A} has entry e_i equal to the number of instances of a_i in w . Define an *abelian (μ_1, \dots, μ_k) -block-pattern* to be a word of the form $w = w_1 \cdots w_k$ where, if the set $\{1, \dots, k\}$ is partitioned via the rule $i \sim j \iff \mathcal{P}(w_i) = \mathcal{P}(w_j)$, there are μ_s parts of size s for all $1 \leq s \leq k$. One may ask questions similar to those addressed in this paper for abelian (μ_1, \dots, μ_k) -block-patterns.

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