On walks avoiding a quadrant

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Abstract

Two-dimensional (random) walks in cones are very natural both in combinatorics and probability theory: they are interesting for themselves and also because they are strongly related to other discrete structures. While walks restricted to the first quadrant have been studied a lot, the case of planar, non-convex cones—equivalent to the three-quarter plane after a linear transform—has been approached only recently. In this article we develop an analytic approach to the case of walks in three quadrants. The advantage of this method is to provide uniform treatment in the study of models corresponding to different step sets. After splitting the three quadrants in two symmetric convex cones, the method is composed of three main steps: write a system of functional equations satisfied by the counting generating function, which may be simplified into one single equation under symmetry conditions; transform the functional equation into a boundary value problem; and finally solve this problem, using a concept of anti-Tutte's invariant. The result is a contour-integral expression for the generating function. Such systems of functional equations also appear in queueing theory with the famous Join-the-Shortest-Queue model, which is still an open problem in the non-symmetric case.

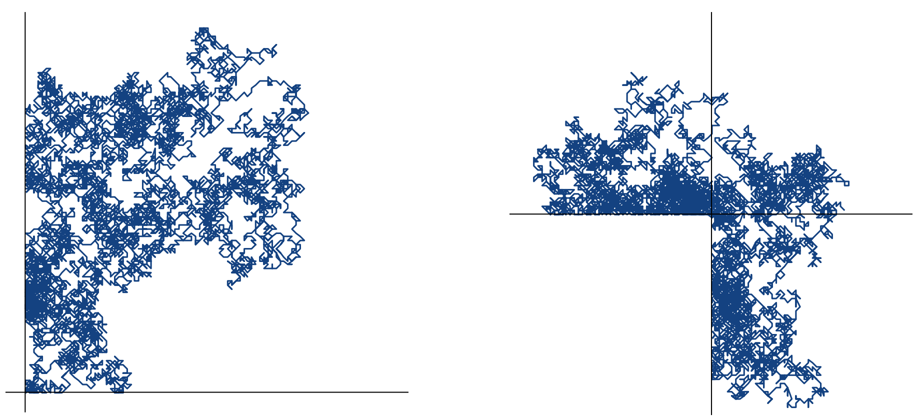

Figure 1: Walks in the quarter plane and in three-quarter plane

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1 Introduction

Context. Two-dimensional (random) walks in cones are very natural both in combinatorics and probability theory: they are interesting for themselves and also because they are strongly related to other discrete structures, see [10] and references therein. Most of the attention has been devoted to the case of convex cones (equivalent to the quarter plane, after a linear transform), see Figure 1, left. Thanks to an appealing variation of techniques, which complement and enrich each other (from combinatorics [31, 10], complex analysis [20, 34], probability theory [13], computer algebra [6, 5], Galois difference equations [14]), one now has a very good understanding of these quadrant models, most of the time via their generating function, which counts the number of walks of length n, starting from a fixed point, ending at an arbitrary point (i, j) and remaining in the cone (see (9) below). Throughout the present work, all walk models will be assumed to have small steps, i.e., jumps in $\{-1, 0, 1\}^2$, see Figure 2 for a few examples. Let us recall a few remarkable results:

- *Exact expressions* exist for the generating function (to illustrate the variety of techniques, remark that the generating functions are infinite series in [31], positive part extractions of diagonals in [10], contour integrals on quartics in [20, 34], integrals of hypergeometric functions in [5], etc.);

- The *algebraic nature of the generating function* is known: it is D-finite (that is, satisfies a linear differential equation with polynomial coefficients) if and only if a certain group of birational transformations is finite [10, 6, 27]. More recently, the differential algebraicity (existence of non-linear differential equations) of the generating function has also been studied [3, 14];

- The *asymptotics of the number of excursions* (an excursion is a path joining two given points and remaining in the cone) [10, 20, 13, 7] is known. Although the full picture is still incomplete, the asymptotics of the total number of walks is also obtained in several cases [10, 20, 13, 15, 5].

Almost systematically, the starting point to solve the above questions is to use a functional equation that satisfies the generating function—it corresponds to the intuitive step-by-step construction of a walk, and will be stated later on, see (8) and (10).

Given the vivid interest in combinatorics of walks confined to a quadrant, it is very natural to consider next the non-equivalent case of non-convex cones, that can be taken without loss of generality as the union of three quadrants

$$\mathcal{C} = \{(i, j) \in \mathbb{Z}^2 : i \geqslant 0 \text{ or } j \geqslant 0\},$$

see Figure 1. After [9], one will also speak about walks avoiding a quadrant. Although walks avoiding a quarter plane have many common features with walks in a quarter plane, the first cited model is definitely much more complicated. To illustrate this fact, let us remind from [9] that the simple walk (usually the simplest model, see Figure 2) in three quadrants has the same level of complexity as the notoriously difficult Gessel's model [6, 8] in the quadrant!

Three-quadrant walks have been approached only recently. In [9], Bousquet-Mélou solves the simple walk and the diagonal walk (see Figure 2 for a representation of these step sets) starting at various points. She obtains an exact expression of the generating function and derives several interesting combinatorial identities, among which a new proof of Gessel's conjecture via the reflection principle. Mustapha [33] computes the asymptotics of the number of excursions, following [13, 7] (interestingly and in contrast with combinatorics, probabilistic results on random walks in cones

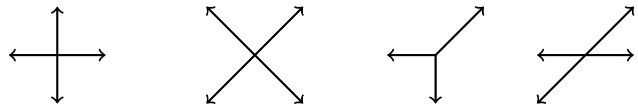

Figure 2: Some famous models of planar walks. The first two ones are solved in the three-quadrant in [9]

do not really depend on convexity). Using a deep connection with planar maps, Budd [11] obtains various enumerating formulas for planar walks, keeping track of the winding angle. These formulas can be used to enumerate simple walks in the three-quarter plane [11, 32]. As recalled in [9], the problem of diagonal walks on the square lattice was also raised in 2001 by David W. Wilson in entry A060898 of the OEIS [25].

In this article we develop the analytic approach of [18, 19, 34] to walks in three quadrants, thereby partially answering to a question of Bousquet-Mélou in [9, Sec. 7.2].

Functional equations for three-quadrant walks and JSQ-like problems. Once a step set $\mathcal{S}$ is fixed, our starting point is a functional equation satisfied by the generating function

$$C(x,y) = \sum_{n\geqslant 0} \sum_{(i,j)\in\mathcal{C}} c_{i,j}(n) x^i y^j t^n, \tag{1}$$

where $c_{i,j}(n)$ counts n-step $\mathcal{S}$-walks going from $(0,0)$ to (i,j) and remaining in $\mathcal{C}$. Stated in (8), this functional equation translates the step-by-step construction of three-quadrant walks and takes into account the forbidden moves which would lead the walk into the forbidden negative quadrant. At first sight, this equation is very similar to its one-quadrant analogue (we will compare the equations (8) to (10) in Section 2.1), the only difference is that negative powers of x and y arise: this can be seen in the definition of the generating function (1) and on the functional equation (8) as well, since the right-hand side of the latter involves some generating functions in the variables $\frac{1}{x}$ and $\frac{1}{y}$. This difference is fundamental and all the methodology of [10, 34] (namely, performing algebraic substitutions or evaluating the functional equation at well-chosen complex points) breaks down, as the series are not convergent anymore.

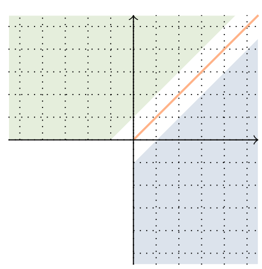

Figure 3: Splitting of the three-quadrant cone in two wedges of opening angle $\frac{3\pi}{4}$

The idea in [9] is to see $\mathcal{C}$ as the union of three quarter planes, and to state for each quadrant a new equation, which is more complicated but (by construction) may be evaluated. Our strategy follows the same line: we split the three-quadrant in two convex cones (of opening angle $\frac{3\pi}{4}$, see Figure 3) and write a system of two functional equations, one for each domain. The drawbacks of this decomposition is that it increases the complexity:

- There are two functional equations instead of one;
- The functional equations involve more unknowns (corresponding to the diagonal and close-to-diagonal terms) in their right-hand sides, see Appendix C.

On the other hand:

- The fundamental advantage is that the new equations may be evaluated—and ultimately will be solved;
- Unexpectedly, this splitting of the cone allows us to link the combinatorial model of walks avoiding a quadrant to a well-known problem in queueing theory: the Join-the-Shortest-Queue (JSQ) model, see Figure 4.

  This is a model with (say) two queues, in which (as its name suggests) the arriving customers choose the shortest queue; if the two queues happen to have the same length, then a queue is chosen according to an a priori fixed probability law. From a random walk viewpoint, this means splitting the quarter plane in two octants (cones of opening angle $\frac{\pi}{4}$) as on Figure 4. In general, the service times depend on the servers, and thus the transition probabilities are different in the upper and lower octants (one speaks about spatially inhomogeneous random walks, and of the general asymmetric JSQ). On the other hand, when the probability laws are symmetric the one of the other, the model is said symmetric. Classical references are [17, 24, 1, 22, 28] and [19, Chap. 10]. Surprisingly, the non-symmetric JSQ is still an open problem: a typical interesting problem in queueing theory would be to compute a closed-form expression for the stationary distribution.

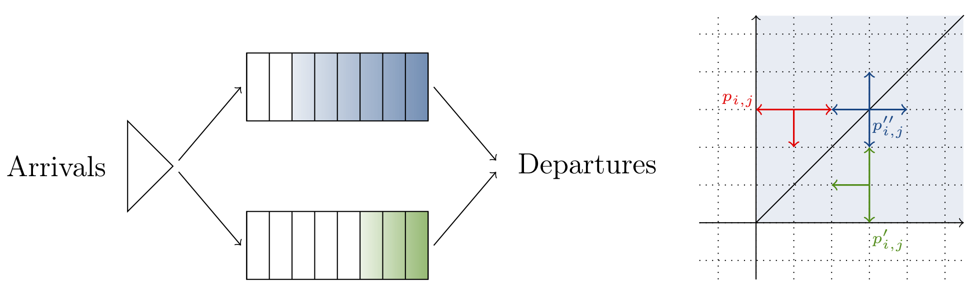

Figure 4: Left: the JSQ model can be represented as a system of two queues, in which the customers choose the shortest one (the green one, on the picture). Right: representation of the JSQ model as an inhomogeneous random walk in the quadrant

Let us briefly notice that quadrant walks could also be treated with a JSQ approach, by decomposing the quarter plane into two octants, see e.g. [28] for asymptotic results.

Main results: a contour-integral expression for the generating function. Throughout this paper we will do the following assumption:

(H) The step set $\mathcal{S}$ is symmetric (i.e., if $(i, j) \in \mathcal{S}$ then $(j, i) \in \mathcal{S}$) and does not contain the jumps $(-1, 1)$ and $(1, -1)$.

An exhaustive list of which of step sets obeying to (H) is given on Figures 5 and 6. Indeed, we are not able to deal with asymmetric walks (as we are unable to solve the asymmetric JSQ model, see above), because of the complexity of the functional equations. The jumps $(-1,1)$ and $(1,-1)$ are discarded for similar reasons: they would lead to additional terms in the functional equation.

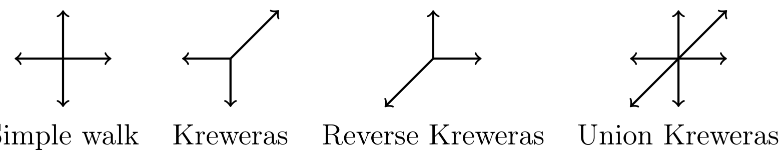

Figure 5: Symmetric models with a finite group and no jumps $(-1,1)$ *and* $(1,-1)$*. The notion of group associated to a model will be properly introduced in Section 2*

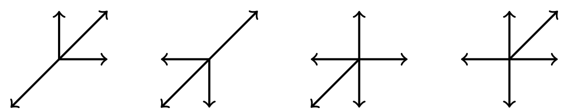

Figure 6: Symmetric models with infinite group and no jumps $(-1,1)$ *and* $(1,-1)$

Our main result is a contour-integral expression for the diagonal section

$$D(x) = \sum_{n\geqslant 0, i\geqslant 0} c(i,i;n)x^i t^n.$$

We shall see later that knowing $D(x)$ actually suffices to give a complete solution to the problem (i.e., to find an expression for $C(x,y)$ in (1)). Let us postpone to Theorem 6 the very precise statement, and instead let us give now the main idea and the shape of the solution. We will show that

$$D(x) = w'(x)f(w(x)) \int g(u,w(u)) \frac{w'(u)}{w(x)-w(u)}\,\mathrm{d}u, \tag{2}$$

where f and g are algebraic functions. The integral in (2) is taken over a quartic curve, constructed from the step set of the model. The function w is interpreted as a conformal mapping for the domain bounded by the quartic, and its algebraic nature heavily depends on the model under consideration: it can be algebraic (finite group case) or non-D-finite (otherwise).

Five consequences of our main results. Our first contribution is about methodology: we show that under the symmetry condition (H), three-quadrant walk models are exactly soluble, in the sense that their generating function admits an explicit (contour-integral) expression (2).

The second point is that our techniques allow to compare walks in a quadrant and walks in three quadrants. More precisely, it is proved in [34] that the generating function counting quadrant walks ending on the horizontal axis can typically be expressed as

$$\widetilde{f}(x) \int \widetilde{g}(u) \frac{w'(u)}{w(x)-w(u)}\,\mathrm{d}u, \tag{3}$$

with the same function w as in (2) but different functions $\widetilde{f}$ (rational) and $\widetilde{g}$ (algebraic). Though simpler, Equation (3) is quite similar to (2). This similarity opens the way to prove combinatorial formulas relating the two models.

Our third corollary is a partial answer to two questions raised by Bousquet-Mélou in [9], that we briefly recall: first, could it be that for any step set associated with a finite group, the generating function $C(x, y)$ is D-finite? Second, could it be that for the four step sets [Kreweras, reverse Kreweras, union Kreweras (see Figure 5) and Gessel (Figure 2)], for which [the quadrant generating function] is known to be algebraic, $C(x, y)$ is also algebraic?

The expression (2) rather easily entails that if w is algebraic (which will correspond to the finite group case, see Section 2), the generating function $D(x)$ is D-finite, being the Cauchy integral of an algebraic function. On the other hand, when the group is infinite the function w is non-D-finite by [34, Thm. 2], and the expression (2) uses non-D-finite functions (note, this does not a priori imply that $D(x)$ itself is non-D-finite).

Fourthly, although we do not solve them, the expression (2) provides a way to attack the following questions:

- Starting from the integral (3), various asymptotic questions concerning quadrant models are solved in [20] (asymptotics of the excursions, of the number of walks returning to one axis, etc.). Similar arguments should lead to the asymptotics of walks in three quadrants. Let us remind, however, that the asymptotics of the excursion sequence is already found in [33].

- A further natural question (still unsolved in the quadrant case) is to find, in the finite group case, a concrete differential equation (or minimal polynomial in case of algebraicity) for the generating function, starting from the contour integrals (2) or (3). It seems that the technique of creative telescoping could be applied here.

- Several interesting (and sometimes surprising) combinatorial identities relating quadrant walks to three-quadrant walks are proved in [9] (in particular, a proof of the former Gessel's conjecture by means of simple walks in $\mathcal{C}$ and the reflection principle). Moreover, Bousquet-Mélou asks in [9] whether $C(x, y)$ could differ from (a simple D-finite series related to) the quadrant generating function by an algebraic series? Taking advantage of the similarity between (2) and (3) provides a starting point for this question.

Finally, along the way of proving our results, we encounter a noteworthy concept of anti-Tutte's invariant, namely a function g such that ($\overline{y}$ denoting the complex conjugate number of $y \in \mathbb{C}$)

$$g(y) = \frac{1}{g(\overline{y})} \tag{4}$$

when y lies on the contour of (2). The terminology comes from [3], where a function g satisfying to $g(y) = g(\overline{y})$ is interpreted as a Tutte's invariant and is strongly used in solving the models. Originally, Tutte introduced the notion of invariant to solve a functional equation counting colored planar triangulations, see [35]. Tutte's equation is rather close to functional equations arising in two-dimensional counting problems. Interestingly, a function g as in (4) appears in the book [12], which proposes an analytic approach to quadrant walk problems (the latter is more general than [19] in the sense that it works for arbitrarily large positive jumps, i.e., not only small steps). In [12] it is further assumed that $g(\overline{y}) = \overline{g(y)}$, so that with (4) one has $|g(y)| = 1$, and g may be interpreted as a conformal mapping from the domain bounded by contour of (2) onto the unit disc.

Equations with (too) many unknowns. What about non-symmetric models? From a functional equation viewpoint, the latter are close to random walks with big jumps [21, 4] or random walks with catastrophes [2], in the sense that the functional equation has more than two unknowns in its right-hand side. One idea to get rid of these extra terms is to transform the initial functional equation, as in [9], where Bousquet-Mélou solves the simple and diagonal models, starting from non-symmetric points ($(-1,0)$, for instance). Another idea, present in [4], is to extend the kernel method by computing weighted sums of several functional equations, each of them being an algebraic substitution of the initial equation. However, finding such combinations is very difficult in general.

From the complex analysis counterpart [19, 34, 21], equations with many unknowns become systems of boundary value problems, which seem not to have a solution in the literature. It is also shown in [19, Chap. 10] that the asymmetric JSQ is equivalent to solving an integral Fredholm equation for the generating function, but again, no closed-form expression seems to exist.

A conjecture. Although it is not directly inspired by our work, let us state the following. Consider an arbitrary finite group step set $\mathcal{S}$ (not necessarily satisfying to (H) but with small steps). We conjecture that the generating function $C(x,y)$ is algebraic as soon as the starting point $(i_0,j_0)\in\mathcal{C}$ is such that $i_0=-1$ or $j_0=-1$.

Structure of the paper. • Section 2: statement of various functional equations satisfied by the generating functions (in particular Lemma 1), definition of the group of the model, study of the zero-set of the kernel

• Section 3: statement of a boundary value problem (BVP) satisfied by the diagonal generating function (Lemma 5), resolution of the BVP (Theorems 6 and 7)

• Appendix A: list and properties of conformal mappings used in Theorems 6 and 7

• Appendix B: important statements from the theory of BVP

• Appendix C: proof of the main functional equation stated in Lemma 1

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2 Preliminaries

2.1 Kernel functional equations

The starting point is to write a functional equation satisfied by the generating function (1), which, as explained in the introduction, translates the step-by-step construction of a walk. Before dealing with those functional equations, let us define some important objects.

First of all, a step set $\mathcal{S}\subset\{-1,0,1\}^2$ is characterized by its inventory (or jump polynomial) $\sum_{(i,j)\in\mathcal{S}}x^iy^j$ as well as by the associated kernel

$$K(x,y)=xy\left(t\sum_{(i,j)\in\mathcal{S}}x^iy^j-1\right). \tag{5}$$

The kernel is a polynomial of degree 2 in x and y, which we can write as

$$K(x,y) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y) = a(x)y^2 + b(x)y + c(x), \tag{6}$$

where

$$\begin{cases} a(x) = tx\sum_{(i,1)\in\mathcal{S}} x^i, & b(x) = tx\sum_{(i,0)\in\mathcal{S}} x^i - x, & c(x) = tx\sum_{(i,-1)\in\mathcal{S}} x^i, \\ \widetilde{a}(y) = ty\sum_{(1,j)\in\mathcal{S}} y^j, & \widetilde{b}(y) = ty\sum_{(0,j)\in\mathcal{S}} y^j - y, & \widetilde{c}(y) = ty\sum_{(-1,j)\in\mathcal{S}} y^j. \end{cases} \tag{7}$$

Define further $\delta_{-1,-1} = 1$ if $(-1,-1) \in \mathcal{S}$ and $\delta_{-1,-1} = 0$ otherwise.

In the three-quarter plane, we can generalize Equation (12) in [9, Sec. 2.1] and deduce the following equation satisfied by $C(x,y)$ defined in (1):

$$K(x,y)C(x,y) = c(x)C_{-0}(x^{-1}) + \widetilde{c}(y)C_{0-}(y^{-1}) - t\delta_{-1,-1}C_{0,0} - xy, \tag{8}$$

where

$$C_{-0}(x^{-1}) = \sum_{i\leqslant 0, n\geqslant 0} c_{i,0}(n)x^i t^n, \quad C_{0-}(y^{-1}) = \sum_{j\leqslant 0, n\geqslant 0} c_{0,j}(n)y^j t^n \quad \text{and} \quad C_{0,0} = \sum_{n\geqslant 0} c_{0,0}(n)t^n.$$

In comparison, let us recall the standard functional equation in the case of the quarter plane

$$\mathcal{Q} = \{(i,j) \in \mathbb{Z}^2 : i \geqslant 0 \text{ and } j \geqslant 0\}.$$

By [10, Lem. 4] and using similar notation as above, the generating function

$$Q(x,y) = \sum_{n\geqslant 0} \sum_{(i,j)\in\mathcal{Q}} q_{i,j}(n)x^i y^j t^n \tag{9}$$

satisfies the equation

$$K(x,y)Q(x,y) = c(x)Q_{-0}(x) + \widetilde{c}(y)Q_{0-}(y) - t\delta_{-1,-1}Q_{0,0} - xy, \tag{10}$$

where

$$Q_{-0}(x) = \sum_{i\geqslant 0, n\geqslant 0} q_{i,0}(n)x^i t^n, \quad Q_{0-}(y) = \sum_{j\geqslant 0, n\geqslant 0} q_{0,j}(n)y^j t^n \quad \text{and} \quad Q_{0,0} = \sum_{n\geqslant 0} q_{0,0}(n)t^n. \tag{11}$$

At first sight, the two functional equations (8) and (10) are very similar. However, due to the presence of infinitely many terms with positive and negative valuations in x or y, the first one is much more complicated, and almost all the methodology of [10, 34] (namely, performing algebraic substitutions or evaluating the functional equation at well-chosen complex points) breaks down, as the series are not convergent anymore.

The idea in [9] is to see $\mathcal{C}$ as the union of three quarter planes, and to state for each quadrant a new equation, which is more complicated but (by construction) may be evaluated. Our strategy follows the same line: we split the three-quadrant cone in two domains (two cones of opening angle $\frac{3\pi}{4}$, see Figure 3) and write two functional equations, one for each domain.

2.2 Functional equations on the $\frac{3\pi}{4}$-cones

We start by splitting the domain of possible ends of the walks into three parts: the diagonal, the lower part $\{i \geqslant 0, j \leqslant i-1\}$ and the upper part $\{i \geqslant 0, j \geqslant i+1\}$, see Figure 3. We may write

$$C(x,y) = \widehat{L}(x,y) + \widehat{D}(x,y) + \widehat{U}(x,y), \tag{12}$$

where

$$\widehat{L}(x,y) = \sum_{\substack{i\geqslant 0\\ j\leqslant i-1\\ n\geqslant 0}} c_{i,j}(n)x^iy^jt^n, \quad \widehat{D}(x,y) = \sum_{\substack{i\geqslant 0\\ n\geqslant 0}} c_{i,j}(n)x^iy^it^n \quad \text{and} \quad \widehat{U}(x,y) = \sum_{\substack{i\leqslant 0\\ j\geqslant i+1\\ n\geqslant 0}} c_{i,j}(n)x^iy^jt^n.$$

(We will promptly change the variables (x,y) and remove the hats from our notation.) Let $\delta_{i,j} = 1$ if $(i,j) \in \mathcal{S}$ and 0 otherwise.

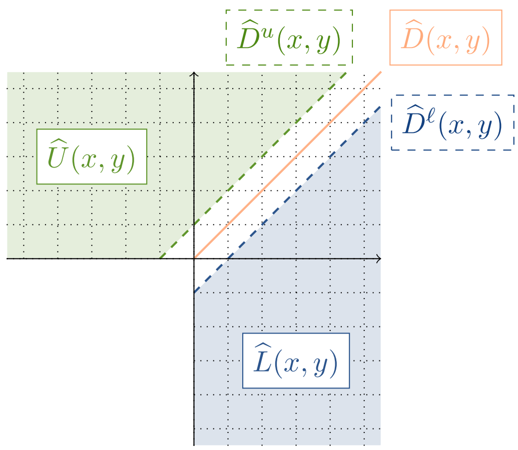


Figure 7: Decomposition of the three-quarter plane and associated generating functions

Lemma 1. *For any step set which satisfies* (H) *and starts at* (i_0, i_0), $i_0 \geqslant 0$, *one has*

$$\begin{aligned}\widehat{K}(x,y)\widehat{L}(x,y) = &-\frac{1}{2}x^{i_0+1}y^{i_0+1} + txy\left(\delta_{-1,-1}x^{-1}y^{-1} + \delta_{-1,0}x^{-1}\right)\widehat{L}_{0-}(y^{-1})\\ &- xy\left(-\frac{1}{2} + t\Big(\frac{1}{2}\big(\delta_{1,1}xy + \delta_{-1,-1}x^{-1}y^{-1}\big) + \delta_{0,-1}y^{-1} + \delta_{1,0}x\Big)\right)\widehat{D}(x,y),\end{aligned} \tag{13}$$

with $\widehat{L}_{-0}(y) = \sum_{n\geqslant 0, j<0} c_{0,j}(n)y^jt^n$.

The proof of Lemma 1 is postponed to Appendix C. The functional equation for non-symmetric models (as well as for symmetric models with non-diagonal starting points) is commented in Appendix C. From now, we only consider symmetric models starting at $(0,0)$; our study can be easily generalized to arbitrary starting points $(i_0, i_0) \in \mathcal{C}$.

In order to simplify the functional equation (13), we perform the change of variable

$$\varphi(x,y) = (xy, x^{-1}).$$

Then (13) becomes

$$K(x,y)L(x,y) = c(x)L_{-0}(x) - x(x\widetilde{a}(y) + \frac{\widetilde{b}(y)}{2})D(y) - \frac{1}{2}xy, \tag{14}$$

where $K(x,y) = xy(t\sum_{(i,j)\in\mathcal{S}} x^{i-j}y^i - 1) = x\widehat{K}(\varphi(x,y))$, $L_{-0}(x) = \sum_{n\geqslant 0, j\geqslant 1} c_{0,-j}x^j t^n$ and similarly

$$L(x,y) = \widehat{L}(\varphi(x,y)) = \sum_{\substack{i\geqslant 1\\ j\geqslant 0\\ n\geqslant 0}} c_{j,j-i}(n)x^i y^j t^n \quad \text{and} \quad D(y) = \widehat{D}(\varphi(x,y)) = \sum_{\substack{i\geqslant 0\\ n\geqslant 0}} c_{i,i}(n)y^i t^n. \tag{15}$$

The change of coordinates φ simplifies the resolution of the problem, as the functional equation (14) is closer to a (solvable) quadrant equation, compare with (10). Throughout the manuscript, functions with (resp. without) a hat will be associated to the step set $\mathcal{S}$ (resp. to the step set after change of variables φ).

For the reader's convenience, we have represented on Figure 1 the effect of φ on the symmetric models of Figures 5 and 6. We also remark on Figure 8 that the presence of anti-diagonal jumps $(-1,1)$ or $(1,-1)$ would lead to the bigger steps $(-2,-1)$ or $(2,1)$: this is the reason why they are discarded.

2.3 Group of the model

With our notation (7), the group of the walk is the dihedral group of bi-rational transformations $\langle\Phi,\Psi\rangle$ generated by the involutions

$$\Phi(x,y) = \left(\frac{\widetilde{c}(y)}{\widetilde{a}(y)}\frac{1}{x}, y\right) \qquad \text{and} \qquad \Psi(x,y) = \left(x, \frac{c(x)}{a(x)}\frac{1}{y}\right).$$

It was introduced in [30] in a probabilistic context and further used in [19, 10]. The group $\langle\Phi,\Psi\rangle$ may be finite (of order $2n \geqslant 4$) or infinite. The order of the group for the 79 quadrant models is computed in [10]: there are 23 models with a finite group (16 of order 4, 5 of order 6 and 2 of order 8) and 56 models with infinite order.

For instance, the simple walk has a group of order 4, while the three right models on Figure 5 have a group of order 6. Indeed, taking Kreweras model as an example, we have $\Phi(x,y) = (\frac{1}{xy}, y)$ and $\Psi(x,y) = (x, \frac{1}{xy})$, and the orbit of (x,y) under the action of Φ and Ψ is

$$(x,y) \stackrel{\Phi}{\to} (\tfrac{1}{xy}, y) \stackrel{\Psi}{\to} (\tfrac{1}{xy}, x) \stackrel{\Phi}{\to} (y,x) \stackrel{\Psi}{\to} (y, \tfrac{1}{xy}) \stackrel{\Phi}{\to} (x, \tfrac{1}{xy}) \stackrel{\Psi}{\to} (x,y).$$

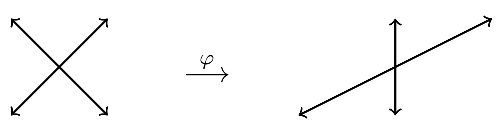


Figure 8: The diagonal model is transformed by φ into a model with bigger steps

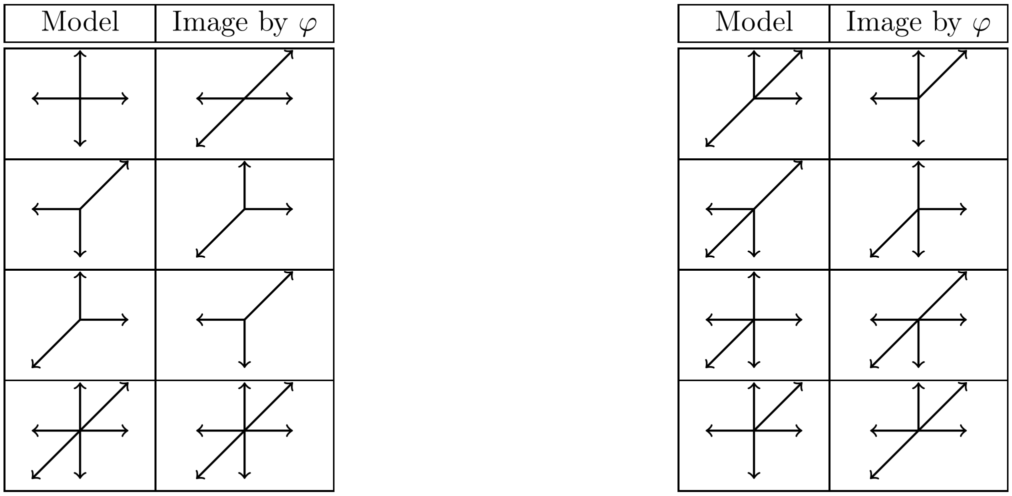


Table 1: Transformation φ on the eight symmetric models (with finite group on the left and infinite group on the right) without the steps $(-1,1)$ and $(1,-1)$. In particular, the simple walk is related by φ to Gessel's model. After [9], this is another illustration that counting simple walks in three-quarter plane is related to counting Gessel walks in a quadrant

2.4 Roots and curves defined by the kernel

We define the discriminants in x and y of the kernel (6):

$$\widetilde{d}(y) = \widetilde{b}(y)^2 - 4\widetilde{a}(y)\widetilde{c}(y) \qquad \text{and} \qquad d(x) = b(x)^2 - 4a(x)c(x). \tag{16}$$

The discriminant $d(x)$ (resp. $\widetilde{d}(y)$) in (16) is a polynomial of degree three or four. Hence it admits four roots (also called branch points) x_1, x_2, x_3, x_4 (resp. y_1, y_2, y_3, y_4), with $x_4 = \infty$ (resp. $y_4 = \infty$) when $d(x)$ (resp. $\widetilde{d}(y)$) is of degree 3.

Lemma 2 (Sec. 3.2 in [34])**.** *Let $t \in (0, 1/|\mathcal{S}|)$. The branch points x_i's are real and distinct. Two of them (say x_1 and x_2) are in the open unit disc, with $x_1 < x_2$ and $x_2 > 0$. The other two (say x_3 and x_4) are outside the closed unit disc, with $x_3 > 0$ and $x_3 < x_4$ if $x_4 > 0$. The discriminant $d(x)$ is negative on (x_1, x_2) and (x_3, x_4), where if $x_4 < 0$, the set (x_3, x_4) stands for the union of intervals $(x_3, \infty) \cup (-\infty, x_4)$. Symmetric results hold for the branch points y_i.*

Let $Y(x)$ (resp. $X(y)$) be the algebraic function defined by the relation $K(x, Y(x)) = 0$ (resp. $K(X(y), y) = 0$). Obviously we have

$$Y(x) = \frac{-b(x) \pm \sqrt{d(x)}}{2a(x)} \qquad \text{and} \qquad X(y) = \frac{-\widetilde{b}(y) \pm \sqrt{\widetilde{d}(y)}}{2\widetilde{a}(y)}. \tag{17}$$

The function Y has two branches Y_0 and Y_1, which are meromorphic on the cut plane $\mathbb{C} \setminus ([x_1, x_2] \cup [x_3, x_4])$. On the cuts $[x_1, x_2]$ and $[x_3, x_4]$, the two branches still exist and are complex conjugate (but possibly infinite at $x_1 = 0$, as discussed in Lemma 3). At the branch points x_i, we have $Y_0(x_i) = Y_1(x_i)$ (when finite), and we denote this common value by $Y(x_i)$.

Fix the notation of the branches by choosing $Y_0 = Y_-$ and $Y_1 = Y_+$ in (17). We further fix the determination of the logarithm so as to have $\sqrt{d(x)} > 0$ on (x_2, x_3). Then clearly with (17) we have

$$|Y_0| \leqslant |Y_1| \tag{18}$$

on (x_2, x_3), and as proved in [19, Thm. 5.3.3] the latter inequality (18) holds true on the whole complex plane and is strict, except on the cuts, where Y_0 and Y_1 are complex conjugate.

A key object is the curve $\mathcal{L}$ defined by

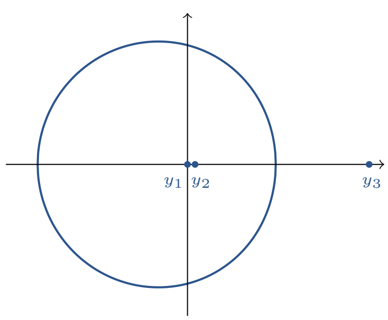
$$\mathcal{L} = Y_0([x_1, x_2]) \cup Y_1([x_1, x_2]) = \{y \in \mathbb{C} : K(x, y) = 0 \text{ and } x \in [x_1, x_2]\}. \tag{19}$$

By construction, it is symmetric with respect to the real axis. We denote by $\mathcal{G}_\mathcal{L}$ the domain delimited by $\mathcal{L}$ and avoiding the real point at $+\infty$. See Figure 9 for two examples. Furthermore, let $\mathcal{L}_0$ (resp. $\mathcal{L}_1$) be the upper (resp. lower) half of $\mathcal{L}$, i.e., the part of $\mathcal{L}$ with non-negative (resp. non-positive) imaginary part, see Figure 11. Likewise, we define $\mathcal{M} = X_0([y_1, y_2]) \cup X_1([y_1, y_2])$.

Lemma 3 (Lem. 18 in [3])**.** *The curve $\mathcal{L}$ in (19) is symmetric in the real axis. It intersects this axis at $Y(x_2) > 0$.*

If $\mathcal{L}$ is unbounded, $Y(x_2)$ is the only intersection point. This occurs if and only if neither $(-1, 1)$ nor $(-1, 0)$ belong to $\mathcal{S}$. In this case, $x_1 = 0$ and the only point of $[x_1, x_2]$ where at least one branch $Y_i(x)$ is infinite is x_1 (and then both branches are infinite there). Otherwise, the curve $\mathcal{L}$ goes through a second real point, namely $Y(x_1) \leqslant 0$.

Consequently, the point 0 is either in the domain $\mathcal{G}_\mathcal{L}$ or on the curve $\mathcal{L}$. The domain $\mathcal{G}_\mathcal{L}$ also contains the (real) branch points y_1 and y_2, of modulus less than 1. The other two branch points, y_3 and y_4, are in the complement of $\mathcal{G}_\mathcal{L} \cup \mathcal{L}$.



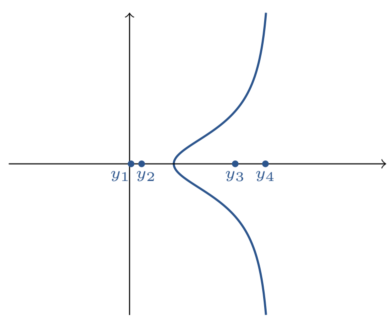


Figure 9: The curve $\mathcal{L}$ for Gessel's model (on the left) and for the model with jumps $\{\mathsf{E}, \mathsf{N}, \mathsf{SW}, \mathsf{S}\}$ *(on the right), for $t = 1/8$*

The step sets with jumps $\{\mathsf{E}, \mathsf{N}, \mathsf{SW}\}$, $\{\mathsf{E}, \mathsf{NE}, \mathsf{N}, \mathsf{S}\}$ and $\{\mathsf{E}, \mathsf{N}, \mathsf{SW}, \mathsf{S}\}$ have an unbounded contour, whereas the other models in Table 1 have a bounded contour.

We close this section by introducing a particular conformal mapping for $\mathcal{G}_\mathcal{L}$, which will happen to be very useful for our study.

Definition 4 (Conformal gluing functions)**.** A function w is said to be a conformal gluing function for the set $\mathcal{G}_\mathcal{L}$ if:

- w is meromorphic in $\mathcal{G}_\mathcal{L}$ and admits finite limits on $\mathcal{L}$;
- w is one-to-one on $\mathcal{G}_\mathcal{L}$;
- for all y on $\mathcal{L}$, $w(y) = w(\overline{y})$.

For example, $w(y) = \frac{1}{2}(y + \frac{1}{y})$ is a conformal gluing function for the unit disc. See Appendix A for a list of conformal gluing functions for the models we are considering.

3 Expression for the generating functions

3.1 Main results and discussion

The first and crucial point is to prove that the diagonal $D(y)$ in (15) satisfies a BVP, in the sense of the lemma below, the proof of which is postponed to Section 3.2. Let $\mathcal{D}$ denote the open unit disc and let $\widetilde{d}$ be the discriminant (16).

Lemma 5. *The function $D(y)$ can be analytically continued from the unit disc to the domain $\mathcal{D}\cup\mathcal{G}_{\mathcal{L}}$ and admits finite limits on $\mathcal{L}$. Moreover, $D(y)$ satisfies the following boundary condition, for $y \in \mathcal{L}$:*

$$\sqrt{\widetilde{d}(y)}D(y) - \sqrt{\widetilde{d}(\overline{y})}D(\overline{y}) = y - \overline{y}. \tag{20}$$

In the remainder of the paper, we solve Lemma 5 in two different ways, leading to the contour-integral expressions of $D(y)$ given in Theorem 6 and Theorem 7 below. Let us first remark that contrary to the usual quadrant case [34], the prefactor $\sqrt{\widetilde{d}(y)}$ in front of the unknown $D(y)$ is not meromorphic in $\mathcal{G}_{\mathcal{L}}$, simply because it is the square root of a polynomial, two roots of which being located in $\mathcal{G}_{\mathcal{L}}$ (see Section 2.4). This innocent-looking difference has strong consequences on the resolution:

- Due to the presence of a non-meromorphic prefactor in (20), solving the BVP of Lemma 5 requires the computation of an index (in the sense of Section 3.3 and Appendix B). This index is an integer and will be non-zero in our case, which will increase the complexity of the solutions. In Theorem 6 we solve the BVP, by taking into account this non-zero index.
- A second, alternative idea is to reduce to the case of a meromorphic boundary condition, and thereby to an index equal to 0. To do so, we will find an analytic function f with the property that

  $$\frac{\sqrt{\widetilde{d}(\overline{y})}}{\sqrt{\widetilde{d}(y)}} = \frac{f(\overline{y})}{f(y)} \tag{21}$$

  for $y \in \mathcal{L}$, see Section 3.4 for more details. Such a function f allows us to rewrite (20) as

  $$f(y)D(y) - f(\overline{y})D(\overline{y}) = \frac{f(y)}{\sqrt{\widetilde{d}(y)}}(y - \overline{y}), \tag{22}$$

  which by construction admits a meromorphic prefactor $f(y)$. In Theorem 7 we solve this zero-index BVP by this technique.

Although they represent the same function $D(y)$ (and so should be equal!), it will be apparent that the expressions obtained in Theorems 6 and 7 are quite different, and that the second one is simpler. However, we decided to present the two resolutions, as we think that they offer different insights on this boundary value method, and also because it is not obvious at all to be able to solve an equation of the form (21) and thereby to reduce to the zero-index case.

Recall (Section 2.4) that $\mathcal{L}_0$ is the upper half of the curve $\mathcal{L}$.

Theorem 6. *Let w be a conformal gluing function with a pole at y_2. For any step set $\widehat{\mathcal{S}}$ satisfying to* (H)*, the diagonal section* (15) *can be written, for $y \in \mathcal{G}_{\mathcal{L}}$,*

$$D(y) = \frac{\Psi(w(y))}{2i\pi} \int_{\mathcal{L}_0} \frac{z - \overline{z}}{\sqrt{\widetilde{d}(z)}} \frac{w'(z)}{\Psi^+(w(z))(w(z) - w(y))} \mathrm{d}z,$$

with

$$\begin{cases} \Psi(y) &= (y - Y(x_1)) \exp \Gamma(y), \\ \Psi^+(y) &= (y - Y(x_1)) \exp \Gamma^+(y), \\ \Gamma(w(y)) &= \dfrac{1}{2i\pi} \displaystyle\int_{\mathcal{L}_0} \log \left(\frac{\sqrt{\widetilde{d}(\overline{z})}}{\sqrt{\widetilde{d}(z)}} \right) \frac{w'(z)}{w(z) - w(y)} \mathrm{d}z. \end{cases}$$

All quantities are computed according to the step set $\mathcal{S} = \varphi(\widehat{\mathcal{S}})$ after the change of coordinates.

The left limit Γ^+ (and thereby Ψ^+) appearing in Theorem 6 can be computed with the help of Sokhotski-Plemelj formulas, that we have recalled in Proposition 14 of Appendix B.

We now turn to our second main result.

Theorem 7. *Let w be a conformal gluing function with a pole at y_2. For any step set $\widehat{\mathcal{S}}$ satisfying to* (H)*, the diagonal section* (15) *can be written, for $y \in \mathcal{G}_{\mathcal{L}}$,*

$$D(y) = \frac{-w'(y)}{\sqrt{(w(y) - w(Y(x_1)))(w(y) - w(Y(x_2)))}} \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{z w'(z)}{\sqrt{w(z) - w(y_1)}(w(z) - w(y))} \mathrm{d}z.$$

All quantities are computed according to the step set $\mathcal{S} = \varphi(\widehat{\mathcal{S}})$.

Let us propose some comments around our results.

• First, it is important to notice that having an expression for $D(y)$ is sufficient for characterizing the complete generating function $C(x, y)$. Indeed, looking at Figure 7 one is easily convinced that

$$C(x, y) = L(\varphi^{-1}(x, y)) + D(\varphi^{-1}(x, y)) + L(\varphi^{-1}(y, x)),$$

with

$$\begin{cases} L(x, y) &= \dfrac{1}{K(x, y)} c(x) L_{-0}(x) - x(x\widetilde{a}(y) + \dfrac{1}{2}\widetilde{b}(y)) D(y) - \dfrac{1}{2} xy, \\ L_{-0}(x) &= \dfrac{x}{c(x)} \left(\dfrac{1}{2} Y_0(x) + (x\widetilde{a}(Y_0(x)) + \dfrac{1}{2}\widetilde{b}(Y_0(x))) D(Y_0(x)) \right), \\ \varphi^{-1}(x, y) &= (y^{-1}, xy). \end{cases}$$

• Regarding the question of determining the algebraic nature of the diagonal series $D(y)$, the second expression is much simpler. Indeed, the integrand as well as the prefactor of the integral of Theorem 7 are algebraic functions of y, z, t and w (and its derivative) evaluated at various points. In addition, let us recall from [34, Thm. 2] that w is algebraic if and only if the group is finite, and non-D-finite in the infinite group case. See Table 2 for some implications. On the contrary, based on the exponential of a D-finite function, the integrand in Theorem 6 is a priori non-algebraic.

• Lemma 5 entails that the function $D(y)$ can be analytically continued to the domain $\mathcal{D} \cup \mathcal{G}_{\mathcal{L}}$. This is apparent on the first statement (using properties of contour integrals). This is a little bit less explicit on Theorem 7, because of the prefactor.

• Theorem 6 (resp. Theorem 7) will be proved in Section 3.3 (resp. Sections 3.4 and 3.5).

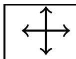
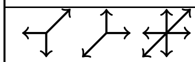
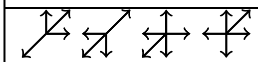
Model	Nature of w	Nature of $Q(x,y)$	Nature of $C(x,y)$
	rational [34]	D-finite [10]	D-finite by [9] and Thm. 7
	algebraic [34]	algebraic [10]	D-finite by Thm. 7 (algebraic?)
	non-D-finite [34]	non-D-finite [27, 7, 14]	(non-D-finite?)

Table 2: Algebraic nature of the conformal mapping w, the quadrant generating function $Q(x,y)$ and the three-quarter plane counting function $C(x,y)$

3.2 Proof of Lemma 5

Assuming that $D(y)$ may be continued as in the statement of Lemma 5, it is easy to prove the boundary condition (20). We evaluate the functional equation (14) at $Y_0(x)$ for x close to $[x_1,x_2]$:

$$-\frac{1}{2}xY_0(x)+c(x)L_{-0}(x)-x(x\widetilde{a}(Y_0(x))+\frac{1}{2}\widetilde{b}(Y_0(x)))D(Y_0(x))=0. \tag{23}$$

We obtain two new equations by letting x go to any point of $[x_1,x_2]$ with a positive (resp. negative) imaginary part. We do the subtraction of the two equations and obtain (20).

We now prove the analytic continuation. Note that similar results are obtained in [19, Thm. 3.2.3], [34, Thm. 5] and [3, Prop. 19]. We follow the same idea as in [34, Thm. 5]. Starting from (14) we can prove that

$$c(X_0(y))L_{-0}(X_0(y))+X_0(y)\sqrt{\widetilde{d}(y)}D(y)-X_0(y)y=0$$

for $y\in\{y\in\mathbb{C}:|X_0(y)|<1\}\cap\mathcal{D}$, and then

$$c(X_0(y))\sum_{n\geqslant 0,j\geqslant 0}c_{0,-j-1}(n)X_0(y)^jt^n+\sqrt{\widetilde{d}(y)}D(y)-y=0$$

for $y\in\{y\in\mathbb{C}:|X_0(y)|<1 \text{ and } X_0(y)\neq 0\}\cap\mathcal{D}$ which can be continued in $\mathcal{G}_{\mathcal{L}}\cup\mathcal{D}$. Being a power series, $D(y)$ is analytic on $\mathcal{D}$ and on $(\mathcal{G}_{\mathcal{L}}\cup\mathcal{D})\setminus\mathcal{D}$, $D(y)$ may have the same singularities as X_0 and $\sqrt{\widetilde{d}(y)}$, namely the branch cuts $[y_1,y_2]$ and $[y_3,y_4]$. But none of these segments belong to $(\mathcal{G}_{\mathcal{L}}\cup\mathcal{D})\setminus\mathcal{D}$ (see Lemma 3). Then $D(y)$ can be analytically continued to the domain $\mathcal{G}_{\mathcal{L}}\cup\mathcal{D}$. Using the same idea, we can prove that $D(y)$ has finite limits on $\mathcal{L}$. From (23), it is enough to study the zeros of $(x\widetilde{a}(Y_0(x))+\frac{1}{2}\widetilde{b}(Y_0(x))$ for x in $[x_1,x_2]$. Using the relation $X_0(Y_0(x))=x$ valid in $\mathcal{G}_{\mathcal{M}}$ (see [19, Cor. 5.3.5]) shows that it recurs to study the zeros of $\widetilde{d}(y)$ for $y\in(\mathcal{G}_{\mathcal{L}}\cup\mathcal{D})\setminus\mathcal{D}$. None of these roots (y_1,y_2,y_3,y_4) belong to the last set, then D has finite limits on $\mathcal{L}$.

3.3 Proof of Theorem 6

The function $\sqrt{\widetilde{d}(y)}D(y)$ satisfies a BVP of Riemann-Carleman type on $\mathcal{L}$, see Lemma 5. Following the literature [19, 34], we use a conformal mapping to transform the latter into a more classical Riemann-Hilbert BVP. Throughout this section, we shall use notation and results of Appendix B.

More precisely, let w be a conformal gluing function for the set $\mathcal{G}_{\mathcal{L}}$ in the sense of Definition 4, and let $\mathcal{U}$ denote the real segment

$$\mathcal{U}=w(\mathcal{L}).$$

(With this notation, w is a conformal mapping from $\mathcal{G}_\mathcal{L}$ onto the cut plane $\mathbb{C}\setminus\mathcal{U}$.) The segment $\mathcal{U}$ is oriented such that the positive direction is from $w(Y(x_2))$ to $w(Y(x_1))$, see Figure 11.

Define v as the inverse function of w. The latter is meromorphic on $\mathbb{C}\setminus\mathcal{U}$. Following the notation of Appendix B and [19], we denote by v^+ and v^- the left and right limits of v on $\mathcal{U}$. The quantities v^+ and v^- are complex conjugate on $\mathcal{U}$, and more precisely, since w preserves angles, we have for $u\in\mathcal{U}$ and $y\in\mathcal{L}_0$

$$\begin{cases} v^+(u) &= v^+(w(y)) &= y, \\ v^-(u) &= v^-(w(y)) &= \overline{y}, \end{cases}$$

see Figure 11 for an illustration of the above properties.

Then (20) may be rephrased as the following new boundary condition on $\mathcal{U}$:

$$D(v^+(u)) = \frac{\sqrt{\widetilde{d}(v^-(u))}}{\sqrt{\widetilde{d}(v^+(u))}} D(v^-(u)) + \frac{v^+(u)-v^-(u)}{\sqrt{\widetilde{d}(v^+(u))}}. \tag{24}$$

As explained in Appendix B (see in particular Definition 15), the first step in the way of solving the Riemann-Hilbert problem with boundary condition (24) is to compute the index of the BVP.

Proposition 8. *The index of* $\frac{\sqrt{\widetilde{d}(v^-(u))}}{\sqrt{\widetilde{d}(v^+(u))}}$ *along the curve* $\mathcal{U}$ *is* -1.

Proof. First of all, let us recall that when $\mathcal{L}$ is a closed curve of interior $\mathcal{G}_\mathcal{L}$ and G is a non-constant, meromorphic function without zeros or poles on $\mathcal{L}$, then

$$\operatorname{ind}_\mathcal{L} G = \frac{1}{2i\pi}\int_\mathcal{L} \frac{G'(z)}{G(z)}\,\mathrm{d}z = Z - P,$$

where Z and P are respectively the numbers of zeros and poles of G in $\mathcal{G}_\mathcal{L}$, counted with multiplicity.

Applying this result to the function $d(y)$, which in $\mathcal{G}_\mathcal{L}$ has no pole and exactly two zeros (at y_1 and y_2—remember that y_3 and y_4 are also roots of $d(y)$ but are not in $\mathcal{G}_\mathcal{L}$), we have $\operatorname{ind}_\mathcal{L}\widetilde{d}(y) = 2$, see Figure 10 for an illustration.

We get then

$$\begin{aligned}\operatorname{ind}_\mathcal{U} \frac{\sqrt{\widetilde{d}(v^-(u))}}{\sqrt{\widetilde{d}(v^+(u))}} &= \operatorname{ind}_\mathcal{U}\sqrt{\widetilde{d}(v^-(u))} - \operatorname{ind}_\mathcal{U}\sqrt{\widetilde{d}(v^+(u))} = -\operatorname{ind}_{\mathcal{L}_1}\sqrt{\widetilde{d}(y)} - \operatorname{ind}_{\mathcal{L}_0}\sqrt{\widetilde{d}(y)} \\ &= -\operatorname{ind}_\mathcal{L}\sqrt{\widetilde{d}(y)} = -\frac{1}{2}\operatorname{ind}_\mathcal{L}\widetilde{d}(y) = -1.\end{aligned}$$

The proof is complete. ∎

With Theorem 16, we deduce a contour-integral expression for the function $D(v(u))$, namely

$$D(v(u)) = \frac{\Psi(u)}{2i\pi}\int_\mathcal{U} \frac{v^+(s)-v^-(s)}{\sqrt{\widetilde{d}\,(v^+(s))}}\frac{1}{\Psi^+(s)(s-u)}\,\mathrm{d}s.$$

With the change of variable $s = w(y)$ we easily have the result of Theorem 6.

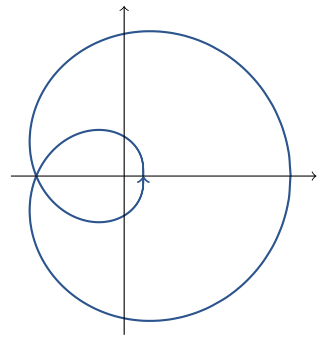


Figure 10: Plot of $\widetilde{d}(y)$ when y lies on $\mathcal{L}$, in the case of Gessel's step set

3.4 Anti-Tutte's invariant

Our aim here is to find a function f satisfying to the decoupling condition (21), namely

$$\frac{\sqrt{\widetilde{d}(\overline{y})}}{\sqrt{\widetilde{d}(y)}} = \frac{f(\overline{y})}{f(y)}, \qquad \forall y \in \mathcal{L}.$$

Indeed, such a function is used in a crucial way in Theorem 7.

Before giving a systematic construction of a function f as above, we start by an example. For Gessel's model, we easily prove that the function

$$g(y) = \frac{y}{t(y+1)^2}$$

satisfies $g(Y_0)g(Y_1) = 1$, and so for $x \in [x_1, x_2]$ the condition (4) announced in the introduction. By the same reasoning as in the proof of Theorem 9 below, we deduce that

$$f(y) = \frac{g(y)}{g'(y)} = \frac{y(y+1)}{y-1}$$

satisfies the decoupling condition (21).

However, a simple rational expression of f as above does not exist in general. Instead, our general construction consists in writing f in terms of a conformal mapping. Our main result is the following.

Theorem 9. *Let g be any conformal mapping from $\mathcal{G}_\mathcal{L}$ onto the unit disc $\mathcal{D}$, with the property that $g(\overline{y}) = \overline{g(y)}$. Then the function f defined by*

$$f = \frac{g}{g'}$$

satisfies the decoupling condition (21). Moreover, f is analytic in $\mathcal{G}_\mathcal{L}$ and has finite limits on $\mathcal{L}$.

Finally, defining $h(z) = -z + \sqrt{z^2-1}$ and letting w be a conformal gluing function as in Definition 4, one can choose

$$g(y) = h\left(\frac{2}{w(Y(x_2)) - w(Y(x_1))}\left(w(y) - \frac{w(Y(x_1)) + w(Y(x_2))}{2}\right)\right), \tag{25}$$

see Figure 11.

To obtain the expression of g in (25) for a given model, we refer to the list of conformal mappings w provided in Appendix A.

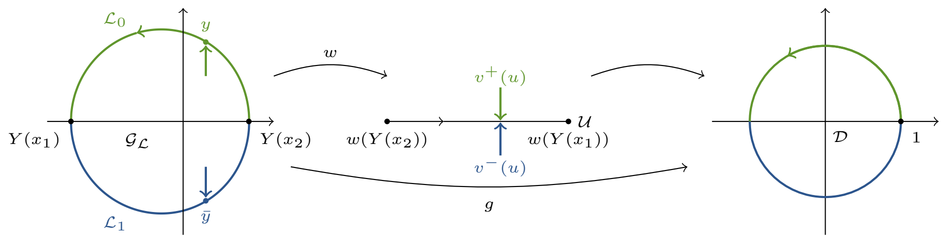


Figure 11: Conformal gluing functions from $\mathcal{G}_\mathcal{L}$ to $\mathbb{C}\setminus\mathcal{U}$ and conformal mappings from $\mathcal{G}_\mathcal{L}$ to the unit disc $\mathcal{D}$

Proof. We first prove that if g is a conformal mapping from $\mathcal{G}_\mathcal{L}$ onto the unit disc $\mathcal{D}$ with the property that $g(\overline{y}) = \overline{g(y)}$, then $f = \frac{g}{g'}$ satisfies the decoupling condition (21). First, for $x \in [x_1, x_2]$ one has

$$g(Y_0(x))g(Y_1(x)) = g(Y_0(x))g(\overline{Y_0(x)}) = g(Y_0(x))\overline{g(Y_0(x))} = |g(Y_0(x))|^2 = 1.$$

Differentiating the identity $g(Y_0(x))g(Y_1(x)) = 1$, one finds on $[x_1, x_2]$

$$\frac{f(Y_0(x))}{f(Y_1(x))} = -\frac{Y_0'(x)}{Y_1'(x)}.$$

To conclude the proof, we show that on $[x_1, x_2]$

$$\frac{\sqrt{\widetilde{d}(Y_0(x))}}{\sqrt{\widetilde{d}(Y_1(x))}} = -\frac{Y_0'(x)}{Y_1'(x)}. \tag{26}$$

To that purpose, let us first consider $x \in \mathcal{G}_\mathcal{M} \setminus [x_1, x_2]$. Differentiating the identity $K(x, Y_0(x)) = 0$ in (6) yields

$$Y_0'(x)(2a(x)Y_0(x) + b(x)) = -(a'(x)Y_0(x)^2 + b'(x)Y_0(x) + c'(x)). \tag{27}$$

First, it follows from Section 2.4 that $2a(x)Y_0(x) + b(x) = -\sqrt{d(x)}$. Moreover, differentiating (6) in x and using the relation $X_0(Y_0(x)) = x$ valid in $\mathcal{G}_\mathcal{M}$ (see [19, Cor. 5.3.5]) shows that the right-hand side of (27) satisfies

$$a'(x)Y_0(x)^2 + b'(x)Y_0(x) + c'(x) = -\sqrt{\widetilde{d}(Y_0(x))}.$$

Then for $x \in \mathcal{G}_\mathcal{M} \setminus [x_1, x_2]$, Equation (27) becomes

$$-\sqrt{d(x)}Y_0'(x) = \sqrt{\widetilde{d}(Y_0(x))}.$$

To complete the proof of (26), we let x converge to a point $x \in [x_1, x_2]$ from above and then from below, and we compute the ratio of the two identities so-obtained. The minus sign in (26) comes from that

$$\lim_{x\downarrow[x_1,x_2]} \sqrt{d(x)} = -\lim_{x\uparrow[x_1,x_2]} \sqrt{d(x)},$$

see Section 2.4.

Our second point is to show that the function g in (25) is a conformal mapping from $\mathcal{G}_\mathcal{L}$ onto the unit disc $\mathcal{D}$, which in addition is such that $g(\overline{y}) = \overline{g(y)}$. This is obvious from our construction (25), since as illustrated on Figure 11, $g = h \circ \widehat{w}$ is the composition of the conformal mapping h from the cut plane $\mathbb{C} \setminus [-1, 1]$ onto the unit disc, by the conformal mapping

$$\widehat{w} = \frac{2}{w(Y(x_1)) - w(Y(x_2))} \left(w - \frac{w(Y(x_1)) + w(Y(x_2))}{2} \right) \tag{28}$$

from $\mathcal{G}_\mathcal{L}$ onto the same cut plane.

The third item is to prove that f has finite limits on $\mathcal{L}$, for any initial choice of conformal mapping g. We may propose two different proofs of this fact. First, we could prove that the function f constructed from the particular function g in (25) has the desired properties (this follows from a direct study). Then as any two suitable conformal mappings g_1 and g_2 can necessarily related by a linear fractional transformation

$$g_1 = \frac{\alpha g_2 + \beta}{\gamma g_2 + \delta},$$

it is easily seen that all functions have indeed the good properties.

The second idea is to use a very general statement on conformal mapping. Namely, any conformal mapping which maps the unit disc onto a Jordan domain (the domain $\mathcal{G}_\mathcal{L}$) with analytic boundary (our curve $\mathcal{L}$) can be extended to a univalent function in a larger disc, see [16, Sec. 1.6]. As the extension is univalent, it becomes obvious that the derivative g' in the denominator of f cannot vanish. □

3.5 Proof of Theorem 7

Our main idea here is to reformulate the initial boundary condition (20) as (22), with the help of a function f which is analytic in $\mathcal{G}_\mathcal{L}$, admits finite limits on $\mathcal{L}$ and satisfies on $\mathcal{L}$ the decoupling condition (21). Using Lemma 5 and Theorem 9, we deduce that $f(y)D(y)$ is analytic in $\mathcal{G}_\mathcal{L}$ and has finite limits on $\mathcal{L}$. As a consequence, $f(y)D(y)$ satisfies a Riemann-Carleman BVP with index zero (in the sense of Definition 15). Similarly to Section 3.3 and using again a conformal gluing function, we transform the latter BVP into a Riemann-Hilbert BVP on an open contour, whose solution is

$$D(y)f(y) = \frac{1}{2i\pi} \int_\mathcal{L} \frac{zf(z)}{\sqrt{\widetilde{d}(z)}} \frac{w'(z)}{w(z) - w(y)} \mathrm{d}z + c, \tag{29}$$

where c is constant in y, but may depend on t (as recalled in Theorem 16 from Appendix B, the solutions to a BVP of index zero are determined up to one constant). Notice that f cancels at y_2 (the unique pole of w) and the integral in the right-hand side of (29) as well, it follows that $c = 0$.

We now simplify the integrand in (29). First, noting that h satisfies the simple differential equation $h' = \frac{-h}{\sqrt{z^2-1}}$, we obtain with our notation (28)

$$f = \frac{g}{g'} = \frac{h(\widehat{w})}{\widehat{w}' h'(\widehat{w})} = -\frac{\sqrt{\widehat{w}^2 - 1}}{\widehat{w}'} = -\frac{\sqrt{(w - w(Y(x_1)))(w - w(Y(x_2)))}}{w'}.$$

Furthermore, the conformal gluing function w satisfies the following differential equation

$$\widetilde{d}(z) w'(z)^2 = (w(z) - w(Y(x_1)))(w(z) - w(Y(x_2)))(w(z) - w(y_1)), \tag{30}$$

see [19, Sec. 5.5.2.2]. Taking the square root of (30) in the neighborhood of $[y_2, y_3] \cap \mathcal{G}_{\mathcal{L}}$ gives

$$-\sqrt{\widetilde{d}(z)}w'(z) = \sqrt{(w(z) - w(Y(x_1)))(w(z) - w(Y(x_2)))(w(z) - w(y_1))},$$

as w is decreasing on $[y_2, y_3] \cap \mathcal{G}_{\mathcal{L}}$. It follows that

$$\frac{f(z)}{\sqrt{\widetilde{d}(z)}} = \frac{1}{\sqrt{w(z) - w(y_1)}}.$$

The proof of Theorem 7 is complete.

Remark 10. The differential equation (30) is only true for the conformal gluing function w whose expression is given in (32), with a pole at y_2. If instead we have at hand a function w with a pole at $y_0 \neq y_2$ (for example $y_0 = 0$, as in Lemma 11), we can consider $w_{y_2} = \frac{1}{w - w(y_2)}$, which instead of (30) satisfies the differential equation

$$\widetilde{d}(z)w'_{y_2}(z)^2 = \widetilde{d}'(y_2)w'(y_2)(w_{y_2}(z) - w_{y_2}(Y_0(x_1)))(w_{y_2}(z) - w_{y_2}(Y_0(x_2)))(w_{y_2}(z) - w_{y_2}(y_1)).$$

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A Expression and properties of conformal gluing functions

A crucial ingredient in our main results (see Theorems 6 and 7) is the function $w(y)$, which we interpret as a conformal mapping from the domain $\mathcal{G}_\mathcal{L}$ onto a complex plane cut along an interval, see Section 2.4. In this appendix, we recall from [34, 3] an explicit expression as well as some analytic properties of this function, first in the finite group case, then for infinite group models.

Let us remind that if w is a suitable mapping, then any $\frac{\alpha w+\beta}{\gamma w+\delta}$ is also a suitable mapping, as soon as $\alpha\delta-\beta\gamma\neq 0$. Therefore, all expressions hereafter are given up to such a fractional linear transform.

A.1 Finite group models

We start by giving an expression of the conformal mapping $w(y)$ for the Kreweras trilogy of Figure 5. Let $W = W(t)$ (resp. $Z = Z(t)$) be the unique power series (resp. the unique power series with no constant term) satisfying

$$W = t(2 + W^3) \qquad \text{and} \qquad Z = t\frac{1 - 2Z + 6Z^2 - 2Z^3 + Z^4}{(1 - Z)^2}. \tag{31}$$

Lemma 11. *Let W and Z as in* (31)*. The function*

$$w(y) = \left(\frac{1}{y} - \frac{1}{W}\right)\sqrt{1 - yW^2}$$

is a conformal mapping for Kreweras model. Likewise, a conformal mapping for reverse Kreweras model is given by

$$w(y) = \frac{-ty^3 + y^2 + t}{2yt} - \frac{2y^2 - yW^2 - W}{2yW}\sqrt{1 - yW(W^3 + 4)/4 + y^2W^2/4}.$$

Finally, a conformal mapping for double Kreweras model is

$$w(y) = \sqrt{1 - 2yZ(1 + Z^2)/(1 - Z)^2 + Z^2y^2}\,\frac{(Z(1 - Z) + 2yZ - (1 - Z)y^2)}{2yZ(1 - Z)(1 + y)} + \frac{Z(1 - Z)^2 - Z^2(-1 + 2Z + Z^2)y + (1 - 2Z + 7Z^2 - 4Z^3)y^2 - Z(1 - Z)^2y^3}{2y(1 + y)Z(1 - Z)^2}.$$

Notice that the functions w given in Lemma 11 all have a pole at $y = 0$.

Proof. Expressions for w are given in [34, Thm. 3 (iii)], but some quantities in the latter statement (namely α, β, δ and γ, all depending on t) are not totally explicit. So to derive the above expressions of w, we will rather use a combination of the works [10] and [3]. Indeed, algebraic expressions of $Q(0, y)$ in terms of y and t are obtained in [10] for the three Kreweras models (see Prop. 13, Prop. 14 and Prop. 15 there). On the other hand, an alternative formulation of $Q(0, y)$ as a rational function of $w(y)$, y and t is derived in [3] (see Thm. 23 and Table 8 there). The formulas of Lemma 11 are obtained by equating the two expressions. □

An expression for $w(y)$ for Gessel's model is obtained in [26, Thm. 7].

A.2 Infinite group models

In the infinite group case, the function w is not algebraic anymore (it is even non-D-finite, see [34, Thm. 2]). As $\mathcal{L}$ is a quartic curve [19, Thm. 5.3.3 (i)], w can be expressed in terms of Weierstrass' elliptic functions (see [19, Sec. 5.5.2.1] or [34, Thm. 6]):

Lemma 12 ([19, 34, 3])**.** *The function w defined by*

$$w(y) = \wp_{1,3}\Big(-\frac{\omega_1 + \omega_2}{2} + \wp_{1,2}^{-1}(f(y))\Big) \tag{32}$$

is a conformal mapping for the domain $\mathcal{G}_{\mathcal{L}}$, and has in this domain a unique (and simple) pole, located at y_2. The function w admits a meromorphic continuation on $\mathbb{C} \setminus [y_3, y_4]$. It is D-algebraic in y and in t.

The differential algebraicity is shown in [3, Thm. 33]. The remaining properties stated in Lemma 12 come from [19, 34], see e.g. [34, Thm. 6 and Rem. 7].

Let us now comment on the expression (32), following the discussion in [3, Sec. 5.2]. First, $f(y)$ is a rational function of y whose coefficients are algebraic functions of t:

$$f(y) = \begin{cases} \dfrac{\widetilde{d}''(y_4)}{6} + \dfrac{\widetilde{d}'(y_4)}{y - y_4} & \text{if } y_4 \neq \infty, \\[2ex] \dfrac{\widetilde{d}''(0)}{6} + \dfrac{\widetilde{d}'''(0)y}{6} & \text{if } y_4 = \infty, \end{cases}$$

where $\widetilde{d}(y)$ is the discriminant (16) and y_4 is one of its roots.

The next ingredient in (32) is Weierstrass' elliptic function $\wp$, with periods ω_1 and ω_2:

$$\wp(z) = \wp(z, \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{(i,j)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(z - i\omega_1 - j\omega_2)^2} - \frac{1}{(i\omega_1 + j\omega_2)^2} \right).$$

Then $\wp_{1,2}(z)$ (resp. $\wp_{1,3}(z)$) is the Weierstrass function with periods ω_1 and ω_2 (resp. ω_1 and ω_3) defined by:

$$\omega_1 = i\int_{y_1}^{y_2} \frac{\mathrm{d}y}{\sqrt{-\widetilde{d}(y)}}, \qquad \omega_2 = \int_{y_2}^{y_3} \frac{\mathrm{d}y}{\sqrt{\widetilde{d}(y)}}, \qquad \omega_3 = \int_{Y(x_1)}^{y_1} \frac{\mathrm{d}y}{\sqrt{\widetilde{d}(y)}}.$$

These definitions make sense thanks to the properties of the y_i's and $Y(x_i)$'s (see [3, Sec. 5.1]). If $Y(x_1)$ is infinite (which happens if and only if neither $(-1,0)$ nor $(-1,1)$ are in $\mathcal{S}$), the integral defining ω_3 starts at $-\infty$. Note that $\omega_1 \in i\mathbb{R}_+$ and $\omega_2, \omega_3 \in \mathbb{R}_+$.

Finally, as the Weierstrass function is not injective on $\mathbb{C}$, we need to clarify our definition of $\wp_{1,2}^{-1}$ in (32). The function $\wp_{1,2}$ is two-to-one on the fundamental parallelogram $[0,\omega_1) + [0,\omega_2)$ (because $\wp(z) = \wp(-z+\omega_1+\omega_2)$), but is one-to-one when restricted to a half-parallelogram — more precisely, when restricted to the open rectangle $(0,\omega_1) + (0,\omega_2/2)$ together with the three boundary segments $[0,\omega_1/2]$, $[0,\omega_2/2]$ and $\omega_2/2 + [0,\omega_1/2]$. We choose the determination of $\wp_{1,2}^{-1}$ in this set.

B Riemann-Hilbert BVP

In the way of proving our main results (Theorems 6 and 7), a crucial ingredient is the BVP with shift of Lemma 5. It is solved by reduction to a more classical Riemann BVP (Sections 3.3 and 3.5). In this appendix we present the main formulas used to solve the latter, so as to render our paper self-contained. Our main references are the books of Gakhov [23, Chap. 2] and Lu [29, Chap. 4].

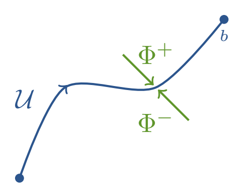

Figure 12: Left and right limits on the open contour $\mathcal{U}$

Suppose that $\mathcal{U}$ is an open, smooth, non-intersecting, oriented curve from a to b, see Figure 12 for an example. Throughout, for $z \in \mathcal{U}$, we will denote by $\Phi^+(z)$ (resp. $\Phi^-(z)$) the limit of a function Φ as $y \to z$ from the left (resp. right) of $\mathcal{U}$, see again Figure 12.

Definition 13 (Riemann BVP)**.** Let $\mathcal{U}$ be as above. A function Φ satisfies a BVP on $\mathcal{U}$ if:

- Φ is sectionally analytic, i.e., analytic in $\mathbb{C} \setminus \mathcal{U}$;
- Φ has finite degree at ∞ (the only singularity at ∞ is a pole of finite order), and Φ is bounded in the vicinity of the extremities a and b;
- Φ has left limits Φ^+ and right limits Φ^- on $\mathcal{U}$;
- Φ satisfies the following boundary condition

$$\Phi^+(z) = G(z)\Phi^-(z) + g(z), \quad z \in \mathcal{U}, \tag{33}$$

  where G and g are Hölder functions on $\mathcal{U}$, and G does not vanish on $\mathcal{U}$.

Let us remind the so-called Sokhotski-Plemelj formulas, which represent a crucial tool to solve the BVP of Definition 13.

Proposition 14 (Sokhotski-Plemelj formulas)**.** *Let $\mathcal{U}$ be as above, and let f be a Hölder function on $\mathcal{U}$. The contour integral*

$$F(z) = \frac{1}{2i\pi} \int_{\mathcal{U}} \frac{f(u)}{u - z} \, \mathrm{d}u$$

is sectionally analytic on $\mathbb{C} \setminus \mathcal{U}$. Its left and right limit values F^+ and F^- are Hölder functions on $\mathcal{U}$ and satisfy for $z \in \mathcal{U}$

$$F^{\pm}(z) = \pm\frac{1}{2} f(z) + \frac{1}{2i\pi} \int_{\mathcal{U}} \frac{f(u)}{u - z} \, \mathrm{d}u,$$

where the very last integral is understood in the sense of Cauchy-principal value, see [23, Chap. 1, Sect. 12]. This is equivalent to the following equations on $\mathcal{U}$:

$$\begin{cases} F^+(z) - F^-(z) = f(z), \\ F^-(z) + F^-(z) = \dfrac{1}{i\pi} \displaystyle\int_{\mathcal{L}} \frac{f(u)}{u - z} \, \mathrm{d}u. \end{cases} \tag{34}$$

We also define the following important quantity:

Definition 15 (Index)**.** Let $\mathcal{U}$ be as above and let G be the function (continuous on $\mathcal{U}$) as in (33). The index χ of the BVP of Definition 13 is

$$\chi = \mathrm{ind}_{\mathcal{U}}\, G = \frac{1}{\pi}[\arg G]_{\mathcal{U}} = \frac{1}{2i\pi}[\log G]_{\mathcal{U}}.$$

Plainly, χ represents the variation of argument of $G(u)$, when u moves along the contour $\mathcal{U}$ in the positive direction.

The main result is the following, see [29, Chap. 4, Thm. 2.1.2]

Theorem 16 (Solution of Riemann-Hilbert BVP)**.** *Let $\mathcal{U}$ be as above. The solution of the BVP of Definition 13 is given by, for $z \notin \mathcal{U}$,*

$$\Phi(z) = \begin{cases} X(z)\psi(z) + X(z)P_\chi(z) & \text{if } \chi \geqslant 0, \\ X(z)\psi(z) & \text{if } \chi < 0 \text{ and if the conditions (36) are satisfied,} \end{cases} \tag{35}$$

where P_χ is an arbitrary polynomial of degree χ and

$$\begin{cases} X(z) &=& (z-b)^{-\chi} \exp \Gamma(z), \\ X^+(z) &=& (z-b)^{-\chi} \exp \Gamma^+(z), \\ \Gamma(z) &=& \dfrac{1}{2i\pi} \displaystyle\int_{\mathcal{U}} \dfrac{\log G(u)}{u-z} \,\mathrm{d}u, \\ \psi(z) &=& \dfrac{1}{2i\pi} \displaystyle\int_{\mathcal{U}} \dfrac{g(u)}{X^+(u)(u-z)} \,\mathrm{d}u, \end{cases}$$

and the conditions (36) read

$$\frac{1}{2i\pi} \int_{\mathcal{U}} \frac{g(u)t^{k-1}}{X^+(u)} \,\mathrm{d}u = 0, \quad k = 1, 2, \ldots, -\chi - 1. \tag{36}$$

C Proof of Lemma 1

The decomposition in (12) expresses $C(x,y)$ as a sum of three generating functions. Thanks to the symmetry of the step set and the fact that the starting point lies on the diagonal, $\widehat{U}(x,y) = \widehat{L}(y,x)$ and $C(x,y)$ is written as a sum of the two unknowns $\widehat{L}(x,y)$ and $\widehat{D}(x,y)$. We further introduce the sections

$$\widehat{D}^\ell(x,y) = \sum_{n\geqslant 0, i \geqslant 0} c_{i,i-1}(n) x^i y^{i-1} t^n \quad \text{and} \quad \widehat{D}^u(x,y) = \sum_{n\geqslant 0, i \geqslant 0} c_{i+1,i}(n) x^{i+1} y^i t^n,$$

which respectively count walks ending on the lower (resp. upper) diagonal, see Figure 7.

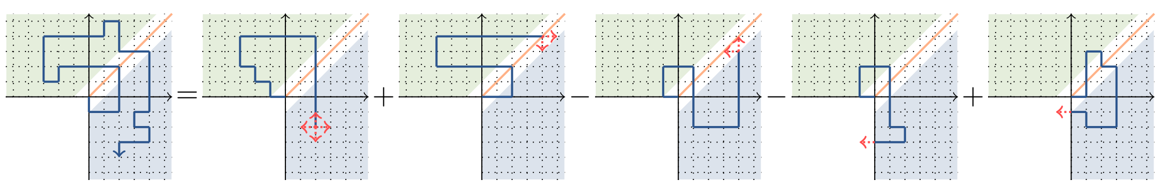

Figure 13: Different ways to end in the lower part (example of the simple walk)

Classically [10], we construct a walk by adding a new step at the end of the walk at each stage. We first derive a functional equation for $\widehat{L}(x,y)$ by taking into account all possibilities of ending in the lower part:

- we may add a step from $\widehat{\mathcal{S}}$ (recall that $\widehat{\mathcal{S}}$ is the step set before the change of variable φ) to walks ending in the lower part, yielding in (37) the term $t(\sum_{(i,j)\in\widehat{\mathcal{S}}} x^i y^j)\widehat{L}(x,y)$, see the second picture on Figure 13 in the particular case of the simple walk;

- walks coming from the diagonal also need to be counted up, giving rise in (37) to the term $t(\delta_{1,0}x + \delta_{0,-1}y^{-1})\widehat{D}(x,y)$ (third picture on Figure 13);
- on the other hand, walks going out of the three-quarter plane need to be removed, yielding the terms $t(\delta_{-1,0}x^{-1}+\delta_{0,1}y)\widehat{D}^\ell(x,y)$ (the lower diagonal) and $t(\delta_{-1,0}x^{-1}+\delta_{-1,-1}x^{-1}y^{-1})\widehat{L}_{0-}(y^{-1})$ (negative y-axis), see the fourth and fifth pictures on Figure 13;
- we finally add the term $t\delta_{-1,0}x^{-1}\sum_{n\geqslant 0} c_{0,-1}(n)y^{-1}t^n$ which was subtracted twice, corresponding to the rightmost picture on Figure 13.

We end up with a first functional equation:

$$\widehat{L}(x,y) = t\sum_{(i,j)\in\widehat{\mathcal{S}}} x^iy^j\widehat{L}(x,y) + t(\delta_{1,0}x+\delta_{0,-1}y^{-1})\widehat{D}(x,y) - t(\delta_{-1,0}x^{-1}+\delta_{0,1}y)\widehat{D}^\ell(x,y) \\ - t(\delta_{-1,0}x^{-1}+\delta_{-1,-1}x^{-1}y^{-1})\widehat{L}_{0-}(y^{-1}) + t(\delta_{-1,0}x^{-1})\sum_{n\geqslant 0} c_{0,-1}(n)y^{-1}t^n. \quad (37)$$

We now prove the second equation

$$\widehat{D}(x,y) = x^{i_0}y^{j_0} + t(\delta_{1,1}xy+\delta_{-1,-1}x^{-1}y^{-1})\widehat{D}(x,y) \\ + 2t(\delta_{-1,0}x^{-1}+\delta_{0,1}y)\widehat{D}^\ell(x,y) - 2t\delta_{-1,0}x^{-1}\sum_{n\geqslant 0} c_{0,-1}(n)y^{-1}t^n, \quad (38)$$

and remark that by plugging in (38) into (37) we get (13), thereby completing the proof of Lemma 1.

This second equation (38) is obtained by writing all possibilities of ending on the diagonal, as illustrated on Figure 14 for simple walks:

- we first count the empty walk, giving the term $x^{i_0}y^{i_0}$;
- we add the walks remaining on the diagonal $t(\delta_{1,1}xy+\delta_{-1,-1}x^{-1}y^{-1})\widehat{D}(x,y)$, the walks ending on the diagonal coming from the upper part $t(\delta_{0,-1}y^{-1}+\delta_{1,0}x)\widehat{D}^u(x,y)$ and those coming from the lower part $t(\delta_{-1,0}x^{-1}+\delta_{0,1}y)\widehat{D}^\ell(x,y)$;
- finally, walks going out of the domain need to be removed, giving $t\delta_{0,-1}y^{-1}\sum_{n\geqslant 0} c_{-1,0}(n)x^{-1}t^n$ and $t\delta_{-1,0}x^{-1}\sum_{n\geqslant 0} c_{0,-1}y^{-1}t^n$.

Thanks to the symmetry of the step set, the number of walks coming from the upper part is the same as the number of walks coming from the lower part.

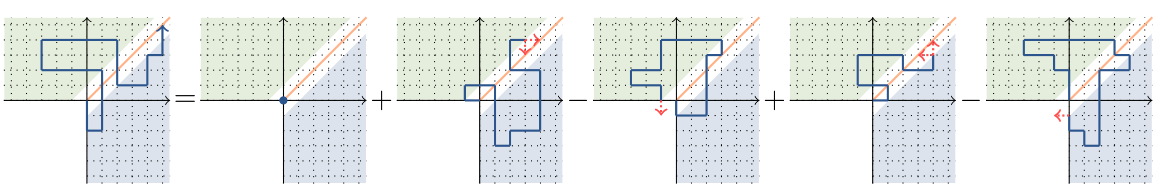

Figure 14: Different ways to end on the diagonal (example of the simple walk)

Remark 17. A step set containing the jumps $(-1,1)$ and $(1,-1)$ would lead to two additional terms in the functional equations, namely

$$\delta_{-1,1}x^{-1}y\sum_{i,n\geqslant 0}c_{i,i-2}(n)x^iy^{i-2}t^n \qquad \text{and} \qquad \delta_{1,-1}xy^{-1}\sum_{j,n\geqslant 0}c_{j-2,j}(n)x^{j-2}y^jt^n,$$

making the resolution much more complicated (not to say impossible, by our techniques!). Likewise, considering an asymmetric step set and/or a starting point out of the diagonal would lead to other terms in the functional equation.