NON-CROSSING TREES, QUADRANGULAR DISSECTIONS, TERNARY TREES, AND DUALITY PRESERVING BIJECTIONS

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ABSTRACT. Using the theory developed in [1] we define an involutory duality for non-crossing trees and provide a bijection between the set of non-crossing trees with n vertices and quadrangular dissections of a $2n$-gon by $n-1$ non-crossing diagonals that transforms that duality to reflection across an axis connecting the midpoints of two diametrically opposite sides of the $2n$-gon. We also show that this bijection fits well with well known bijections involving the set of ternary trees with $n-1$ internal vertices and the set of Flagged Perfectly Chain Decomposed Binary Ditrees.

Further by analyzing the natural dihedral group action on the set of quadrangular dissections of a $2n$-gon we provide closed formulae for the number of quadrangular dissections up to rotations and up to rotations and reflections, the set of non-crossing trees up to rotations and up to rotations and reflections, the number of self-dual non-crossing trees, and the number of oriented and unoriented unlabeled self-dual non-crossing trees. With the exception of the formula giving the number of unoriented unlabeled non-crossing trees, these formulae are new.

1. THE BIJECTIONS

In [1] we introduced a notion of duality (called mind-body duality) for factorizations in a symmetric group $\mathcal{S}_n$, and interpreted it in terms of e-v-graphs (that is graphs with ordered edges and vertices) and pegs (that is graphs properly embedded in surfaces with boundary). In this paper we focus on vertex-labeled trees pegged on a disk, or as they are more commonly known, non-crossing trees. We start by fixing conventions and definitions and recalling some basic facts from [1], and refer the reader there for more details.

By a *non-crossing tree* we mean a labeled tree t pegged in $\mathbb{D}^2$ the 2-dimensional disk endowed with the counterclockwise orientation, and we denote the set of non-crossing trees with n vertices by $\mathcal{N}_n$. We assume that the vertices of t form the vertices of a regular n-gon and that the order induced by their labels is compatible with the cyclic order of the boundary circle induced by the orientation of the disk, and to be concrete for each n we fix the vertices of a regular n-gon with a standard labeling and we assume that all non-crossing trees have those vertices and that all edges are embedded as chords of the circle. We emphasize that the orientation of the disk is part of the definition and we denote by $\mathcal{N}_n^{\intercal}$ the set of trees with n vertices pegged in $\left(\mathbb{D}^2\right)^{\intercal}$, the disk endowed with the clockwise orientation. We assume that the elements of $\mathcal{N}_n^{\intercal}$ have the same vertices as the elements of $\mathcal{N}_n$ but with their labels reflected across the diameter that passes through the vertex labeled 1. For a $t \in \mathcal{N}_n$ we denote by $t^{\intercal}$ the element of $\mathcal{N}_n^{\intercal}$ that has the same underlying vertex labeled tree, see the left and middle of Figure 1 for an example. On the other hand the element of $\mathcal{N}_n$ that is obtained from t by reflecting the edges of t across the diameter passing through 1 will be denoted $\bar{t}$, in other words $\bar{t}$ has an edge $(n+2-i, n+2-j)$ (addition is taken $\pmod n$) for every edge (i,j) of t. We will sometimes denote by $s\colon \mathcal{N}_n \to \mathcal{N}_n$ the map $t \mapsto \bar{t}$. Figure 1 illustrates these conventions and definitions.

The set $\mathcal{F}_{n-1}$ of factorizations of the n-cycle $\zeta_n := (n\, n-1\, \ldots, 1) \in \mathcal{S}_n$ as a product of minimal number of transpositions, is in bijection with the set $\mathcal{E}_n^*$ of rooted e-trees with n vertices (that is rooted trees with their edges labeled with labels from the set $[n-1]$). An e-tree t can be pegged in an oriented disk, in such a way that the local order of the edges around every vertex is consistent with the cyclic order, and declaring the root of t to have label 1 while labeling the other vertices so that going around the boundary circle in the counterclockwise direction we get the order $1, 2, \ldots, n$, gives a non-crossing tree, with labeled edges. Forgetting the e-labels defines a surjection $\mathcal{E}_n^* \to \mathcal{N}_n$. Composing this surjection with the bijection $\mathcal{F}_n \to \mathcal{E}_n^*$ gives a surjection $\pi_n\colon \mathcal{F}_n \to \mathcal{N}_n$.

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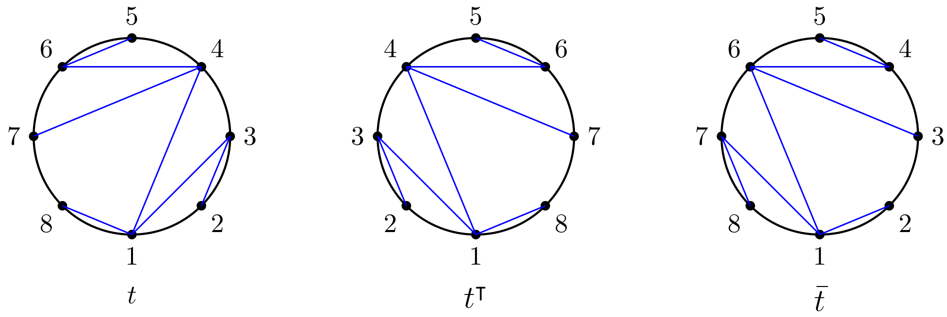


FIGURE 1. A non-crossing tree and its reverses

The braid group on $n-1$ strands B_{n-1} acts from the right on $\mathcal{E}_n^*$ and a duality (the *mind-body duality*) was defined in [1] that can be expressed via the this action as

$$t^* = (t\Delta)^{\intercal}$$

where Δ is the Garside element of B_{n-1}, and $t^{\intercal}$ stands for the reverse of t, that is the e-tree obtained by relabeling the edges of t via $i \to n-i$. A related notion of duality was also defined via

$$t^{\bar{*}} = (t\Delta^{-1})^{\intercal}$$

It is known (see [11] and [6]) that two factorizations belong to the same fiber of π_n if and only if they differ by a sequence of interchanges of consecutive commuting factors. A single such interchange can be effected by the action of a braid generator σ_i, and since $\Delta\sigma_i = \sigma_{n-1-i}\Delta$ it follows that the action of Δ descends to a map $\delta\colon \mathcal{N}_n \to \mathcal{N}_n$, and therefore so do the dualities $*$ and $\bar{*}$; since no confusion is likely we will continue to use $*$ and $\bar{*}$ to denote these dualities even at the level of $\mathcal{N}_n$[1]. In what follows we will also use the notations $r(t)$ and $\bar{r}(t))$ to denote the image of a non-crossing tree under $*$ and $\bar{*}$ respectively.

It is easily seen that δ coincides with the dual defined in [8], and was used in [4] to explain enumerative coincidences. It is also easy to see that

$$t^* = s\left(\delta(t)\right), \quad t^{\bar{*}} = s\left(\delta^{-1}(t)\right) \tag{1}$$

See Figure 2 for an example.

It was shown in [1] that the action of Δ on $\mathcal{E}_n^*$ has order $2n$, and so it follows that δ also has order $2n$; actually as was also observed in [4] δ^2 coincides with the map induced in $\mathcal{N}_n$ by rotation by $2\pi/n$ radians. From this and Equation 1 it follows that the group generated by the involutions r and s is isomorphic to D_{2n} the dihedral group of order $4n$.

It is known that $\mathcal{N}_n$ is counted by the generalized Catalan numbers for $p=3$ that among many other objects also count the set $\mathcal{Q}_{2n}$ of quadrangular dissections of a labeled $2n$-gon by $n-1$ dissecting diagonals (see for example [9]). We use ν_n to denote the generalized Catalan numbers, so that

$$|\mathcal{N}_n| = \nu_n := \frac{1}{2n-1}\binom{3(n-1)}{n-1}$$

There is an obvious D_{2n}-action on $\mathcal{Q}_{2n}$ induced by the defining action of D_{2n} on the circle, and it turns out that there is a simple bijection $\phi\colon \mathcal{Q}_{2n} \to \mathcal{N}_n$ which explains the D_{2n}-action on $\mathcal{N}_n$. Indeed we can prove that:

Theorem 1.1. *There is a bijection $\phi\colon \mathcal{Q}_{2n} \to \mathcal{N}_n$ such that δ is the push forward under ϕ of rotation by π/n radians, and r is the push forward under ϕ of the reflection $r_{1\,2}$ across the axis through the midpoints of the edges $1\ 2$ and $n+1\ n+2$. Furthermore s is the push forward of the reflection across the axis through 1.*

Proof. Let q be a quadrangular dissection of a $2n$-gon. Then q has $n-1$ quadrangular cells, and $n-2$ dissecting diagonals. Since there are $2n$ vertices and $n-1$ cells, there is at least one cell with boundary containing three edges of the polygon. By inductively removing such extremal cells one

[1] We emphasize that we refer to the mind-body duality at the level of rooted e-trees, and not mind-body duality at the level of factorizations which maps factorizations of ζ_n to factorizations of ζ_n^{-1}. Mind-body duality at the level of factorizations descends to a duality $\mathcal{N}_n \to \mathcal{N}_n^{\intercal}$ given by by $t \mapsto \delta(t)^{\intercal}$, see the discussion in Section 5 of [1].

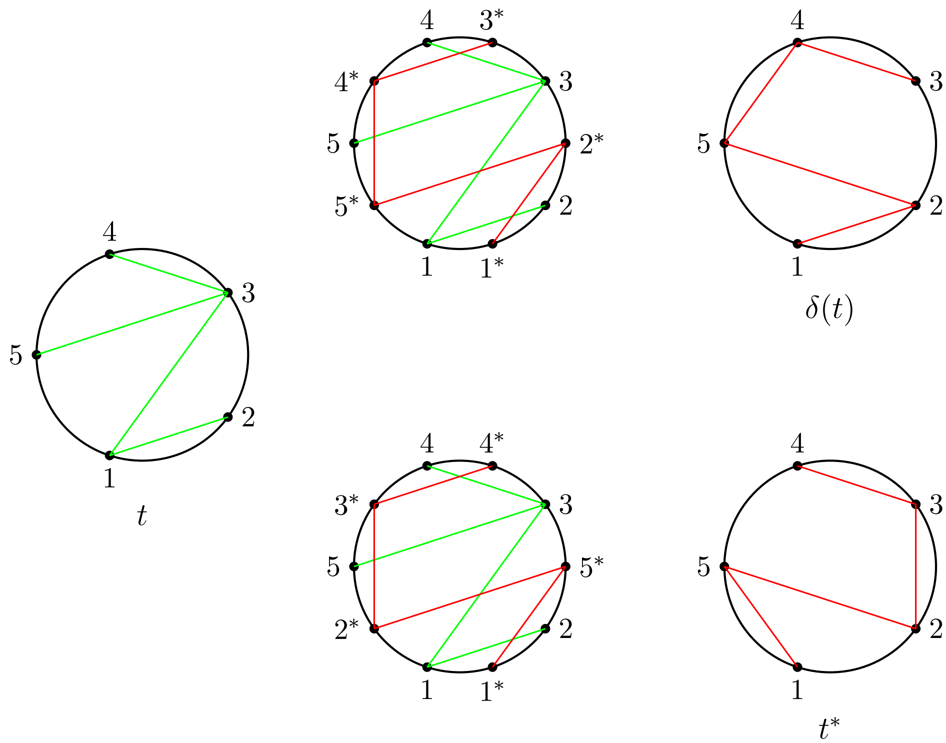


FIGURE 2. A non-crossing tree t on the left, $\delta(t)$ on top right, and t^* on bottom right

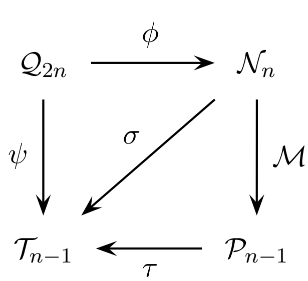
can see that each dissecting diagonal connects two vertices of opposite parity, and so each cell has a diagonal that connects two odd vertices and a diagonal that connects two even vertices. The non-crossing tree $\phi(q)$ is the tree obtained by taking the "odd" diagonals of the cells, deleting the even vertices, and relabeling the odd vertices via $2i-1 \mapsto i$. Notice that this construction with the even diagonals will give $\delta(\phi(q))$, thus showing that δ is the push-forward of rotation by π/n.

To obtain $\phi^{-1}(t)$, for a non-crossing tree t, start by pegging t on the disk with vertices labeled $1, 3, \ldots, 2n-1$, and construct $\delta(t)$ with vertices labeled $2, 4, \ldots, 2n$. We get a decomposition of the $2n$-gon into $2n$ triangles and $n-2$ quadrangulars, and each of the quadrangulars has two opposite vertices on the boundary and two opposite internal vertices. The dissecting diagonals of $\phi(t)$ are the diagonals of the quadrangulars that connect the boundary vertices. See Figure 3, for an example of this construction.

To see that r is the push forward of that reflection $r_{1\,2}$, notice that $r_{1\,2}$ interchanges "even" and "odd" diagonals, and maps the vertex labeled i to the vertex labeled $2n+3-i \pmod{2n}$, so that the $2i-1$ (the label of the i-th vertex of t) is mapped to $2(n+2-1)$ (the label of the i^*-th label of t^*). Similarly we see that s is the push forward of the reflection across the diameter through 1. □

It is well known that the generalized Catalan numbers ν_n also count the set $\mathcal{T}_{n-1}$ of ternary trees with $n-1$ internal vertices. They also count $\mathcal{P}_{n-1}$ the set of *Flagged Perfectly Chain Decomposed Binary Ditrees* with $n-1$ vertices defined in [1] and reviewed bellow. We will show that there is a commutative diagram of bijections:

(2)



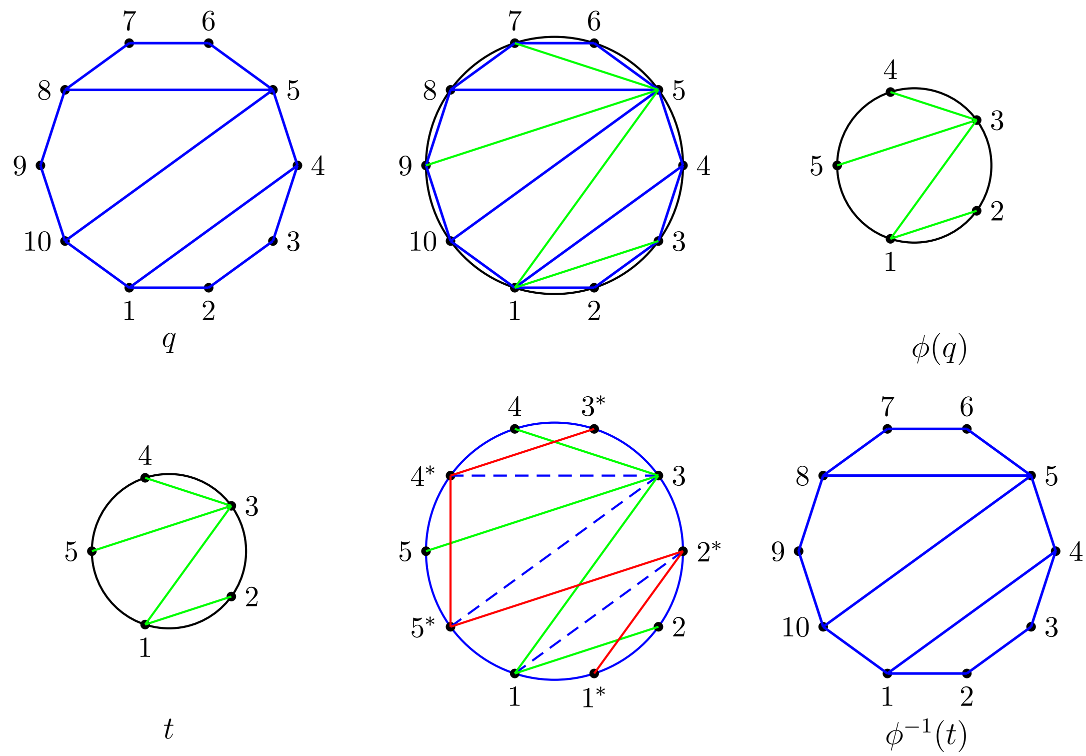


FIGURE 3. The construction of $\phi\colon\ \mathcal{Q}_{2n} \to \mathcal{N}_n$ (top) and its inverse (bottom)

All of the sets in (2) carry a duality $*$: for $\mathcal{N}_n$ and $\mathcal{P}_{n-1}$ this is the mind-body duality, for $\mathcal{Q}_{2n}$ it is the reflection $r_{1\,2}$ across the diameter through the midpoints of $1\ 2$ and $n\ n+1$, and for $\mathcal{T}_{n-1}$ we define t^* to be obtained from t recursively by interchanging left and right children at every node[2] We've already seen that ϕ preserves duality, and we will actually see that all the maps in (2) are duality preserving.

The bijection $\psi\colon\ \mathcal{Q}_{2n} \to \mathcal{T}_{n-1}$ is defined in [9]. Essentially $\psi(q)$ is a sort of dual of q viewed as a graph embedded in the disk with all its vertices mapped on the boundary circle: the disk is divided into $n-1$ quadrangulars (the cells of the dissection q) and $2n$ bigons formed by the edges of the polygons and the arcs of the boundary circle. Let T be the 4-valent plane tree that has a vertex for each of these regions, and an edge between two vertices if the corresponding regions share an edge. See Figure 4, where, in the middle, a vertex that correspond to a cell is drawn in the interior of that cell, and a vertex that corresponds to a bigon is drawn in the boundary arc of that bigon. Clearly bigons give leaves of T and cells give internal vertices. The ternary tree $\psi(t)$ is obtained from T by removing the leaf that comes from the bigon that contains the edge $1\,2$, declaring the vertex it was attached to be the root of the remaining tree, and using the orientation of the disk to order the children of any internal vertex. See Figure 4 for an example of this construction.

Clearly the process of obtaining $\psi(q)$ can be reversed: starting with a ternary tree t construct a 4-valent plane tree T by attaching a new leaf labeled $1\ 2$ bellow the root[3]. Then list the leaves of T in preorder (starting at $1\ 2$) and label them by the edges of the $2n$-gon in the order $1\ 2, 2\ 3, \ldots, 2n\ 1$, and label the corresponding pendant edges by the same label. Since t has $2n-1$ leaves and only $n-1$ internal vertices there is at least one internal vertex with all its children being leaves; if such a vertex has children (from right to left) $i\ i+1, i+1\ i+2, i+2\ i+3$, label it $i\ i+1\ i+2\ i+3$ and the the edge connecting it to its parent $i\ i+3$. Proceeding recursively we can label all internal vertices with (the vertices of) a quadrangular, and all non-pendant edges of T with a diagonal of the polygon. From this decorated tree we can reconstruct the quadrangular dissection, for an example see Figure 5, where $\psi^{-1}(t)$ is shown as a 4-cluster in the sense of [7].

It is clear that ψ is duality preserving.

[2]This definition was given in [3]. See also the remark following Theorem 2.2.

[3]We consider ternary trees growing upwards, as in the right side of Figure 4

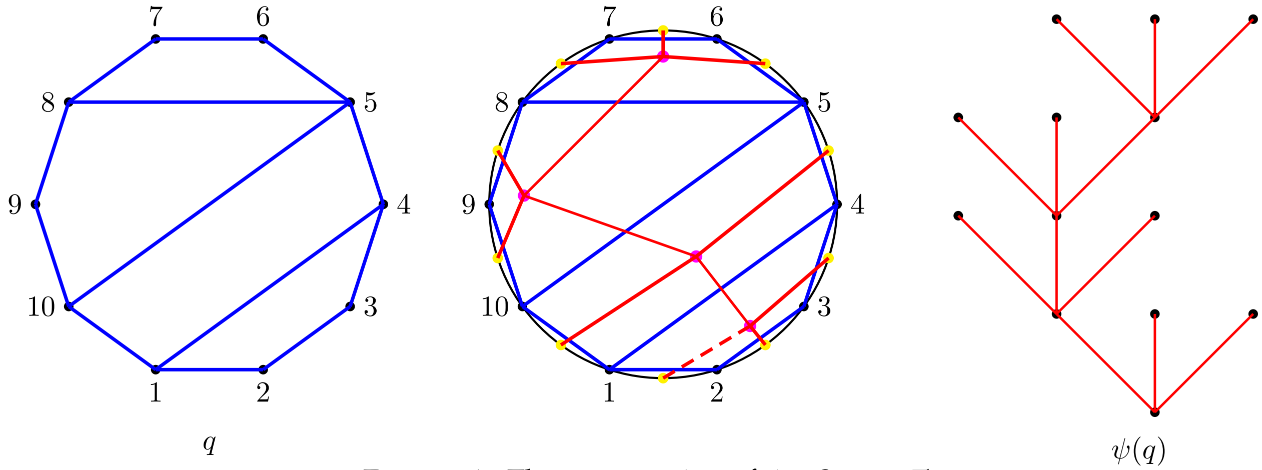


FIGURE 4. The construction of $\psi\colon\ \mathcal{Q}_{2n} \to \mathcal{T}_{n-1}$

The bijection $\sigma\colon\ \mathcal{N}_n \to \mathcal{T}_{n-1}$ is (modulo obvious modifications due to different conventions) the bijection defined in [5]. The definition of σ there is recursive for $n \geq 3$ and it depends on an arbitrary choice of one of the six bijections $\mathcal{N}_3 \to \mathcal{T}_2$. For the rest of this section we recast this definition as a composition $\tau \circ \mathcal{M}$ of two bijections defined for all $n \geq 1$, thus eliminating that arbitrary choice, and then prove that $\sigma = \psi \circ \phi^{-1}$.

We note that the construction of ψ^{-1} can be expressed in terms of ternary algebra: the label of an internal vertex v of the intermediate tree T is obtained by applying the ternary operator $\mathcal{S}_n^3 \to \mathcal{S}_n\colon\ (a,b,c) \mapsto abc$ to the labels of the outgoing edges of v viewed as transpositions, while the label of an edge to an internal vertex is obtained by applying the ternary operator $\mathcal{S}_n^3 \to \mathcal{S}_n\colon\ (a,b,c) \mapsto c^{ba}$.

Let $\mathcal{A}$ be an algebra with one ternary operator $\Upsilon\colon\ \mathcal{A}^3 \to \mathcal{A}$ freely generated by one element λ, and let $\mathcal{A}_m$ stand for the set of elements of $\mathcal{A}$ with level m, that is those elements of $\mathcal{A}$ that are obtained by m applications of Υ. There is an obvious and well known (see for example [10], or any "Discrete Mathematics for Computer Science" textbook) bijection $\mathcal{T}_m \to \mathcal{A}_m$ defined by labeling all the leaves of a ternary tree t by λ and then recursively labeling each internal vertex by the result of Υ applied to its children; the image of t is then just the label of the root. In the literature (e.g. [9]) the elements of $\mathcal{A}_{n-1}$ are often referred to as ways of parenthesizing $n-1$ applications of a ternary operation. Actually the standard recursive definition of $\mathcal{T} := \bigcup_{m\in\mathbb{N}} \mathcal{T}_m$ can be construed as defining a (tautological) ternary algebra freely generated by the ternary tree λ with one vertex and no edges. The duality of ternary trees is then defined by $\lambda^* = \lambda$ and

$$\Upsilon\left(t_l, t_m, t_r\right)^* = \Upsilon\left(t_r^*, t_m^*, t_l^*\right) \tag{3}$$

We recall from [1] that a *binary ditree* is a digraph whose underlying graph is a tree and the in and out degree of every vertex is at most 2. A vertex of binary ditree is called *internal* if both its in and out degrees are at least 1. A *Perfect Chain Decomposition (PCD)* of a binary ditree t is a decomposition of the edges of t into chains where trivial chains consisting of only one vertex are allowed, with the property that every edge of t belongs to exactly one chain and every vertex to exactly two. A *Perfectly Chain Decomposed Binary Ditree* is a pair $(t,\mathcal{C})$ consisting of a binary ditree endowed with a PCD. A *flagged Perfectly Chain Decomposed Binary Ditree* is a triple $(t,\mathcal{C},f)$ where $(t,\mathcal{C})$ is a Perfectly Chain Decomposed Binary Ditree and $f \in \mathcal{C}$ a distinguished chain called its *flag*. We will use the abbreviation *PCDD* to stand for a Flagged Perfectly Chain Decomposed Binary Ditree, and we will use the same notation (typically t) to denote the PCDD and its underlying binary ditree, and in that case the flag will be denoted by $f(t)$. For a chain $c \in \mathcal{C}$ we use the notation $\alpha(c)$ (resp. $\omega(c)$) to stand for the first (resp. last) vertex of c, and for a PCDD t we use the notation $\alpha(t)$ and $\omega(t)$ to stand for $\alpha(f)$ and $\omega(f)$ respectively, where f is the flag of t.

The PCD of the PCDD is determined by making a binary choice at every internal vertex, one has to chose which incoming edge is connected to which outgoing edge. The dual of a PCD $\mathcal{C}$ is the PCD $\mathcal{C}^*$ obtained by making the opposite choice at every internal vertex. The mind-body dual t^*

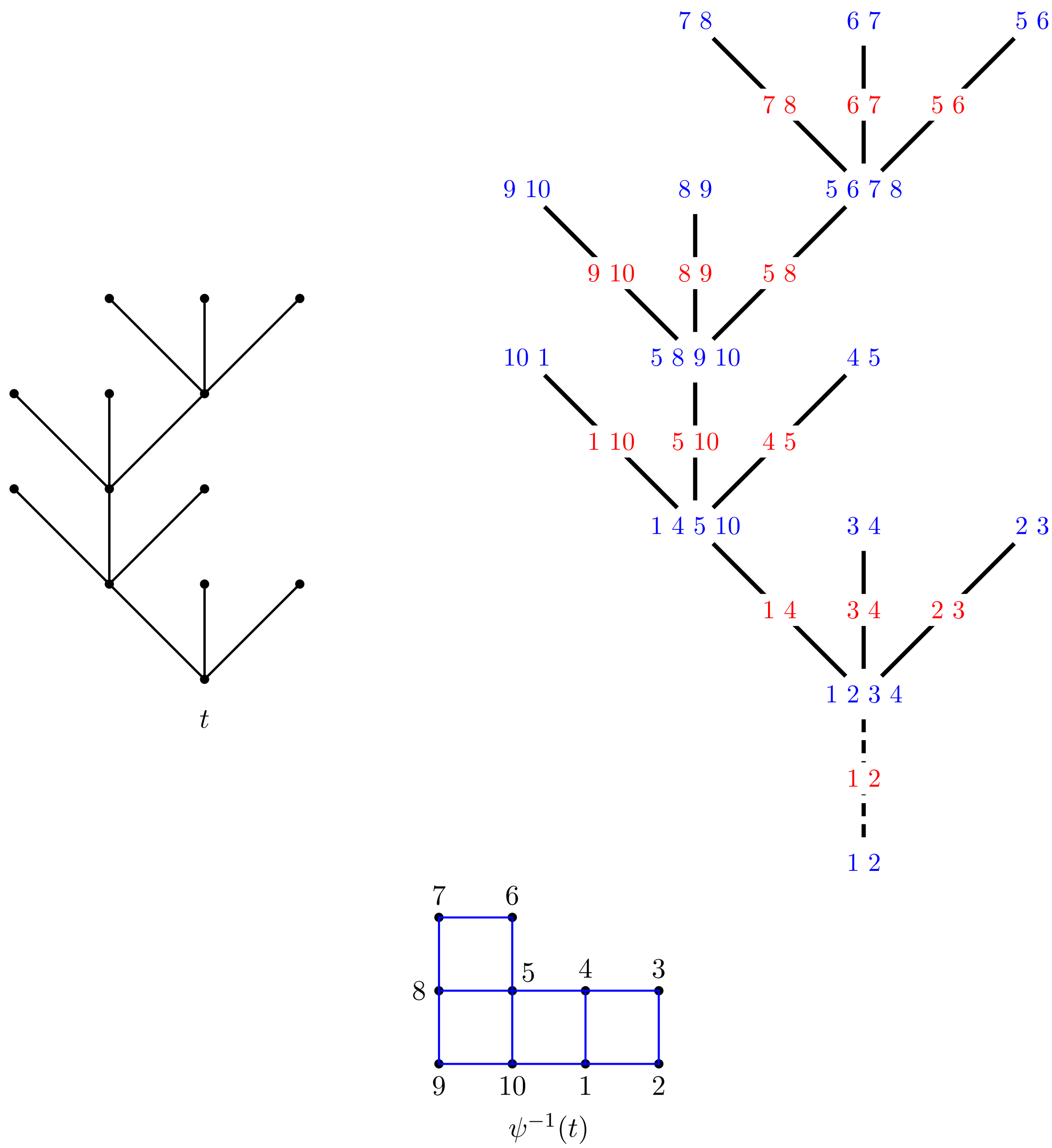


FIGURE 5. The construction of $\psi^{-1}\colon \mathcal{T}_{n-1} \to \mathcal{Q}_{2n}$

of $(t, \mathcal{C}, f) \in \mathcal{P}_m$ is $(t, \mathcal{C}^*, f^*)$ where f^* is the chain of $\mathcal{C}^*$ determined as follows: $\alpha(f^*) = \alpha(f)$ and if f is the only chain that starts at $\alpha(f)$ then f^* is the only chain of t^* that starts at $\alpha(f)$, otherwise the first edge of f^* is the outgoing edge incident at $\alpha(f)$ that does not belong to f, if no such edge exist then f^* is a trivial chain. Similarly, $t^{\bar{*}}$ is $(t, \mathcal{C}, f^{\bar{*}})$, where the definition of $f^{\bar{*}}$ is obtained by the definition of f^* by replacing "start" with "end" and α with ω.

The duality-preserving bijection $\mathcal{M}\colon \mathcal{N}_n \to \mathcal{P}_{n-1}$ was defined in [1]: given a non-crossing tree t, the star of each vertex inherits a linear order from the orientation of the disk, and the *medial ditree* $\mathcal{M}(t)$ is defined as the binary ditree with vertices the edges of t, and there is a dart $e_1 \to e_2$ if and only if, e_1 and e_2 are consecutive edges in the star of a vertex of t. The chains of $\mathcal{M}(t)$ are the stars of the vertices of t, and the flag is the star of 1. For a PCDD d, $\mathcal{M}^{-1}(d)$ has vertices the chains of d, two chains being connected by an edge if they have a vertex of d in common, and the local edge orders inherited by the chains give $\mathcal{M}^{-1}(t)$ a *leo* structure which pegs it into a disk. The vertex corresponding to the flag of d is labeled 1 and the orientation of the disk gives us a labeling or the remaining vertices.

We extend the definition of PCDD to include the following two degenerate[4] cases:

[4]The first one may even be called pointless.

Definition 1.2. The *empty PCDD* λ is the triple $(\emptyset, \{\emptyset\}, \emptyset)$ consisting of the empty ditree, the perfect chain decomposition consisting of the empty chain, and the empty chain as flag. The functions α and ω are not defined for the empty flag, and therefore not for λ either.

The *point PCDD* $\mathfrak{p}$ is the triple $(p, \{p, p\}, p)$, consisting of a ditree with one vertex and no edges, a chain decomposition consisting of two identical trivial chains, and the unique chain as a flag.

The non-crossing trees that corresponds to λ and $\mathfrak{p}$ are the unique non-crossing trees with one and two vertices respectively.

If c is a chain and v a vertex not in c we will use the notation $v \to c$ to denote the trivial chain $\{v\}$ if c is empty, and the chain consisting of the edges of c together with an edge $v \to \alpha(c)$. Similarly $c \to v$ is the trivial chain $\{v\}$ if c is empty, and c augmented by edge $\omega(c) \to v$ if c is non-empty.

Definition 1.3. Let t_l, t_m, t_r be PCDDs. Their *fusion* is defined to be the PCDD $\Upsilon(t_l, t_m, t_r)$ where:

- The underlying ditree has vertices the (disjoint) union of the vertices of t_l, t_m, t_r, plus a new vertex v_0. The edges are the edges of t_m, t_l, t_r plus, provided that the corresponding flags are not empty, edges connecting v_0 to $\alpha(t_1)$ and $\alpha(t_r)$ and an edge connecting $\omega(t_m)$ to v_0.
- The chains are the non-flag chains of t_l, t_m and t_r, and two additional chains: $f_m \to v_0 \to f_r$, and $v_0 \to f_l$
- The flag is $v_0 \to f_l$.

Notice that with this definition $\Upsilon(\lambda, \lambda, \lambda) = \mathfrak{p}$. An other example is shown in Figure 6: ditrees are drawn with the convention that the edges are directed upwards, and the flag of each PCDD is drawn in red.

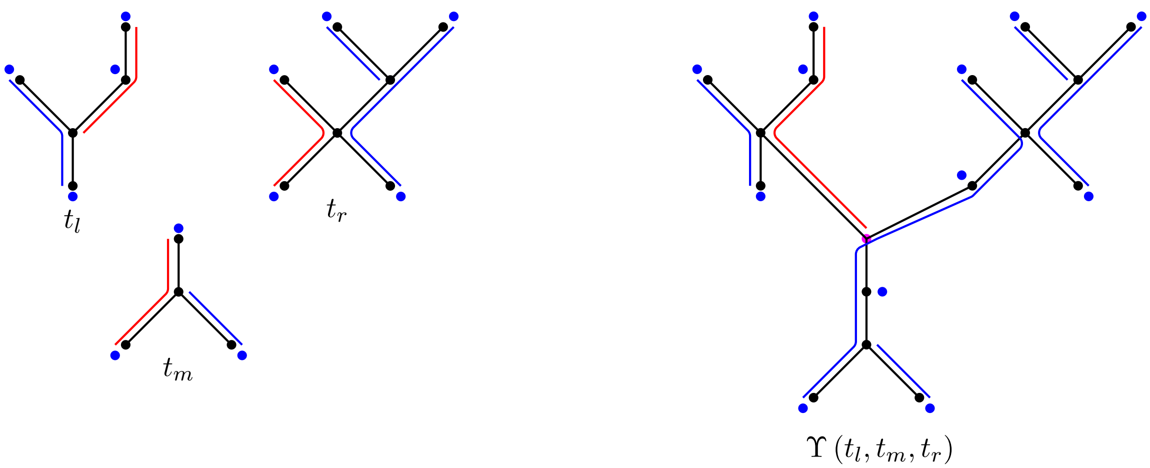


FIGURE 6. An example of the fusion of PCDDs

Starting with a non-empty PCDD t and removing $\alpha(t)$ we obtain three PCDDs: t_m induced by the vertices of t that are bellow $\alpha(t)$, t_l induced by those vertices of t that are above $\alpha(t)$ and were connected to $\alpha(t)$ by the first edge of f, and t_r induced by the remaining vertices. Clearly $t = \Upsilon(t_l, t_m, t_r)$, and since t is finite it's clear that by recursively continuing this process we will eventually find an expression for t that consists of applications of Υ and λ, and that such expression is unique. So we have:

Theorem 1.4. *The set* $\mathcal{P} := \bigcup_{m \in \mathbb{N}} \mathcal{P}_m$ *endowed with the ternary operator* Υ *is isomorphic to* $\mathcal{A}$. *In particular, every non-empty PCDD* t *is* $\Upsilon(t_l, t_m, t_r)$ *for some unique* t_l, t_m, t_r.

Theorem 1.4 provides a bijection $\mathcal{P}_m \to \mathcal{T}_m$ but that bijection is not duality-preserving, since Equation (3) does not quite hold in $\mathcal{P}$, instead we have:

Lemma 1.5. *For PCDDs* t_l, t_m, *and* t_r *we have:*

$$\Upsilon(t_l, t_m, t_r)^* = \Upsilon(t_r^*, t_m^{\bar{*}}, t_l^*)$$

Proof. By the local nature of $*$, for vertices different than v_0, $\alpha(t_l)$, $\omega(t_m)$, and $\alpha(t_r)$ switching the connections of edges at every vertex can be done either before or after fusing the PCDDs. Switching the connections at $\omega(t_m)$ means that we connect $f\left(t^{\bar{*}}\right)$ to v_0, and by the switching at v_0 the resulting chain continues by connecting v_0 to $f(t_l^{\bar{*}})$. The other chain in $\Upsilon(t_l, t_m, t_r)^*$ containing v_0, which by definition has to be the flag, connects v_0 to $f\left(t_r^{\bar{*}}\right)$. □

Definition 1.6. We define $\tau\colon \mathcal{P}_m \to \mathcal{T}_m$ recursively: $\tau(\lambda) = \lambda$ and

$$\tau\left(\Upsilon(t_l, t_m, t_4)\right) = \Upsilon(\tau\left(t_l\right), \tau\left(\overline{t_m}\right), \tau\left(t_r\right))$$

Now using the fact that $t^{\bar{*}} = \overline{\overline{t^*}}$ we can inductively prove that $\tau(t)^* = \tau\left(t^*\right)$. Indeed:

$$\begin{aligned}
\tau\left(\Upsilon\left(t_l, t_m, t_r\right)^*\right) &= \tau\left(\Upsilon\left(t_r^*, t_m^{\bar{*}}, t_m^*\right)\right) \\
&= \Upsilon\left(\tau\left(t_r^*\right), \tau\left(\overline{t_m}^*\right), \tau\left(t_l^*\right)\right) \\
&= \Upsilon\left(\tau\left(t_r\right)^*, \tau\left(\overline{t_m}\right)^*, \tau\left(t_l\right)^*\right) \\
&= \Upsilon\left(\tau\left(t_r\right), \tau\left(\overline{t_m}\right), \tau\left(t_l\right)\right)^* \\
&= \tau\left(\Upsilon\left(t_l, t_m, t_r\right)\right)^*
\end{aligned}$$

In terms of the non-crossing tree $T = \mathcal{M}^{-1}(t)$, v_0 corresponds to the right-most edge $1\ k$ incident to 1, $T_L = \mathcal{M}^{-1}(t_l)$ to the tree to the left of 1, $T_m = \mathcal{M}^{-1}(t_m)$ to the tree to the left of k, and $T_r = \mathcal{M}^{-1}(t_r)$ to the tree to the right of k, see the left side of Figure 7. To see that $\sigma = \psi \circ \phi^{-1}$, notice that the edge of T^* dual to $1\ k$, is mapped to a diagonal $2\ l$ of the $2n$-gon, and therefore the cell of $\phi^{-1}\left(T\right)$ determined by those two dual edges has the edge $1\ 2$ of the $2n$-gon in its boundary, and therefore it corresponds to the root of $\psi\left(\phi^{-1}\left(T\right)\right)$. Clearly the left child of the root is the root of $\psi\left(\phi\left(T_l\right)\right)$, the middle child is the root of $\psi\left(\phi\left(T_m\right)\right)$, and the right child is the root of $\psi\left(\phi\left(T_r\right)\right)$; see the right side of Figure 7.

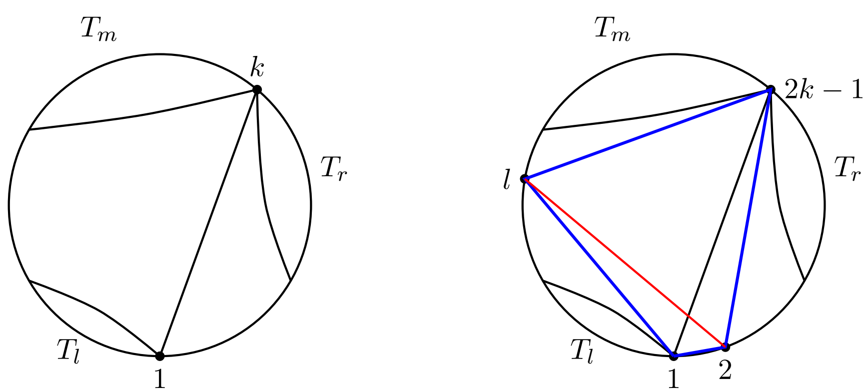


FIGURE 7. $\sigma = \psi \circ \phi^{-1}$

So we have proved:

Theorem 1.7. *Diagram (2) commutes and all arrows are duality preserving.*

We close this section by remarking that ϕ can be generalized to bijections between graphs pegged on an arbitrary surface with boundary and quadrangular dissections of that surface. One such generalization, for graphs pegged in an annulus will be explored in a future work (see [2]).

2. ENUMERATIONS

We start by enumerating the set of self-dual non-crossing trees, i.e. non-crossing trees satisfying $t^* = t$. Since σ is duality preserving we can work with self-dual ternary trees instead. Since $\mathcal{T}$ is a free ternary algebra, Equation (3) implies that a ternary tree t is self-dual if and only if it has the form

$$t = \Upsilon(t_0, t_1, t_0^*) \tag{4}$$

where t_1 is self-dual. It follows that

Theorem 2.1. *The set S_m of self-dual ternary trees with m internal vertices is in bijection with the set $\mathcal{T}_{\frac{m}{2}}$ for m even, and with the set of ordered pairs of ternary trees with a total number of $\frac{m-1}{2}$ internal vertices when m is odd.*

Proof. We will recursively define a bijection β that sends a ternary tree t with m vertices to an ordered triple of ternary trees when m is even and an ordered pair of ternary trees when m is odd. For $m = 0, 1$ all relevant sets have one element so β is defined. Assume then that β has been defined for all values less than m and let t be a self-dual ternary tree with m internal vertices. Express $t = \Upsilon(t_0, t_1, t_0^*)$, where t_1 is self-dual, and notice that if m is even then t_1 has an odd number of internal vertices so we can define $\beta(t) = (t_0, \beta(t_1))$, while if m is odd then t_1 has an even number of internal vertices and so we can define $\beta(t) = (t_0, \Upsilon(\beta(t_1)))$. □

The number of pairs of ternary trees is given by sequence A006013 in the Online Encyclopedia of Integer Sequences, see also [10]. So as a corollary have the following explicit formula for the number of self-dual non-crossing trees.

Theorem 2.2. *The number of self-dual labeled non-crossing trees (i.e. non-crossing trees t such that $t^* = t$) is*

$$s_n = \begin{cases} \dfrac{1}{n+1}\dbinom{3n/2}{n/2} & \textit{if } n \textit{ is even} \\[2ex] \dfrac{2}{n+1}\dbinom{(3n-1)/2}{(n-1)/2} & \textit{if } n \textit{ is odd} \end{cases}$$

Obviously s_n is also the number of non-crossing trees on $[n]$ such that $t^{\bar{*}} = t$.

We remark that Equation (4) was used in [3] to deduce the formula of Theorem 2.2 using a generating function argument. In that paper the authors prove that s_n is the number of self-dual ternary trees with $n-1$ internal vertices.

Next we take a closer look at the dihedral group action on the set of quadrangular dissections of a $2n$-gon. If δ stands for the rotation by $\frac{\pi}{n}$ radians, and r for the reflection $r_{1\,2}$ then $D_{2n} = \{\delta^i r^j : i = 0, \ldots, 2n-1,\ \ j = 0, 1\}$. Note that if $s = \delta\, r$, then the subgroup $\langle \delta^2, s \rangle$ is isomorphic to D_n, and the restriction of the D_{2n}-action on $\mathcal{Q}_{2n}$ on that subgroup coincides with the standard action of D_n to $\mathcal{N}_n$, where δ^2 is rotation by $\frac{2\pi}{n}$ radians and s is reflection across the diameter that passes through 1.

Theorem 2.3. *Every reflection in D_{2n} fixes s_n elements of $\mathcal{Q}_{2n}$. Rotation by π radians has ns_n fixed points if n is odd, and $\frac{ns_n}{2}$ if n is even. When $n \equiv 2 \pmod 4$, rotations by $\pm\frac{\pi}{2}$ have $\frac{n}{2} s_{\frac{n}{2}}$ fixed points. No other rotation has fixed points.*

Proof. The basic observation is that the center of the polygon is fixed by all rotations and reflections, and for a quadrangular dissection q of a $2n$-gon fixed by an element of D_{2n} we have two cases: the center is in the interior of a cell or it's the midpoint of a dissecting diagonal (which has then to be a diameter of the circle) of q, and that cell or dissecting diagonal has then to be invariant.

We first examine rotations. If the center is on a dissecting diameter, then since all dissecting diagonals connect vertices of opposite parity, this can happen if and only if n is odd. This diameter has to be invariant under the rotation and it follows that the rotation is by π radians. Then q consists of two dissections (one a rotation by π of the other) of the $(n+1)$-gon, glued together along an edge. See Figure 8 for an example of a rotation invariant dissection of a octadecagon: the diameter 1 10 is a dissecting diagonal, and q consists of a dissection of a decagon, glued along an edge to its rotation.

There are n diameters that could be dissecting diagonals, and there are $\nu_{n+1} = s_{2n}$ dissections of the $(n+1)$-gon. It follows that the central rotation by π has ns_n fixed points, and no other rotation has fixed points.

If on the other hand the center belongs to an invariant cell, then the two diagonal of the cell are diameters and the rotation either fixes them or rotates one into the other. In the first case we have rotation by π and in the second by $\frac{\pi}{2}$. The number of cells that are to the south or east of the invariant cell equals the number of cells to the west or north, and thus there is is an odd number

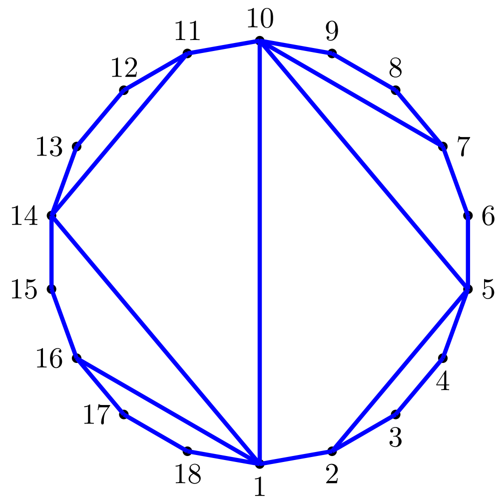


FIGURE 8. A rotation invariant quadrangular dissection of the octadecagon

of total cells. It follows that this case occurs only when n is even. For a dissection invariant under rotation by π radians one can see that it consists of a pair of smaller dissections (not necessarily both non-empty), one to the south which rotates to the one in the north, and one to the east that rotates to the one on the west. See Figure 9 for two examples in the case $n = 8$.

It follows that for a given invariant cell, there are as many invariant dissections as pairs of invariant dissections with total number of cells equal to $2n - 2$, which is counted by s_{2n}. Now an invariant cell is determined by a pair of invariant diagonals (the two dual edges of the pair of dual non-crossing trees) and there are $\frac{n}{2}$ such pairs of dual edges.

Thus rotation by π has $\frac{ns_{2n}}{2}$ invariant dissections.

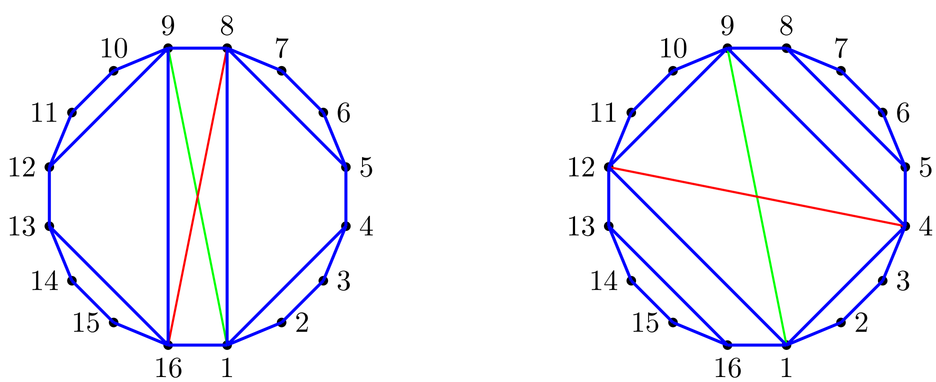


FIGURE 9. Quadrangular dissections of the decahehaxon invariant under rotation

Notice that if $n = 2k$ with k-odd, those pairs that consist of two equal dissections, are also invariant under rotation by $\pm\frac{\pi}{2}$, see for example Figure 10 for a dissection of a dodecagon invariant under $\frac{\pi}{2}$ rotation.

The analysis for reflection invariant dissections is analogous. There are two conjugacy classes of reflections in D_{2n}: those whose axis passes trough two diametrically opposite vertices, and those whose axis passes through the midpoints of two diametrically opposite edges. The second conjugacy class is represented by $r_{1\,2}$ and it has been dealt with in Theorem 2.2.

For a reflection whose axis passes through two diametrically opposite vertices, we observe that if n is odd, there can be no invariant cell (because there is an even number of them) and so the axis of reflection is a dissecting diagonal. The whole dissection then consists of a dissection of a $(n + 1)$-gon glued to its reflection along an edge. So there are $\nu_{n+1} = s_n$ of invariant dissections, for each of the n diameters. For example, Figure 11 displays the three dissections of a decagon that are invariant under reflection across the axis $1\ 6$.

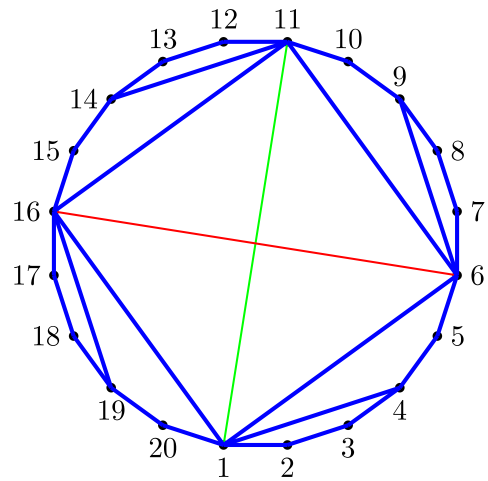


FIGURE 10. A quadrangular dissection of the icosagon invariant under rotation by $\frac{\pi}{2}$ radians

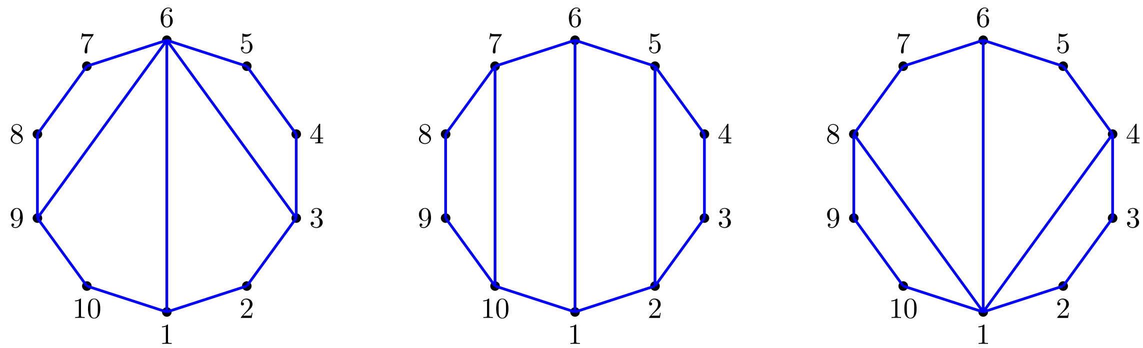


FIGURE 11. The three dissections of the decagon invariant under under reflection across $1\,6$

If n is even, because we have an odd number of cells, the axis of symmetry cannot be one of the dissecting diagonals, and there has to be an invariant cell, one of whose diagonals is the axis of symmetry. That means that at in one of the fixed vertices (say 1) we have two (reflections of each other) dissecting chords, and the invariant cell is completed by an other pair of reflected dissecting chords meeting at the other vertex. An invariant dissection is then determined by an ordered pair of dissections, one to the left of the chord $1\,j$ and the other to the left of the chord $j\,n+1$. So there are s_n such invariant dissections of each of the n vertex axes. For example Figure 12 shows all the invariant quadrangular dissections of a dodecagon invariant under reflection across the axis $1\ 7$. □

Let $\mathcal{Q}'_{2n}$ be the set of *unlabeled oriented* quadrangular dissections of the $2n$-gon, that is quadrangular dissections up to rotation, and $\widetilde{\mathcal{Q}}_{2n}$ the set of *unlabeled unoriented* quadrangular dissections that is quadrangular dissections up to rotations and reflections. The number of such dissections q'_{2n} and $\tilde{q}_{2n}$ respectively, are sequences A005034 and A005036 respectively, in the Online Encyclopedia of Integer Sequences [14]. Using Theorem 2.3 and Burnside's lemma we obtain the following explicit formulas:

Theorem 2.4. *The number quadrilateral dissections of a $2n$-gon up to rotations is*

$$q'_{2n} = \begin{cases} \frac{1}{2n}\left(\nu_n + n s_n\right) & \text{if } n \equiv 1 \pmod 2 \\ \frac{1}{4n}\left(\nu_n + \frac{n}{2} s_n\right) & \text{if } n \equiv 0 \pmod 4 \\ \frac{1}{4n}\left(\nu_n + \frac{n}{2} s_n + n s_{\frac{n}{2}}\right) & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

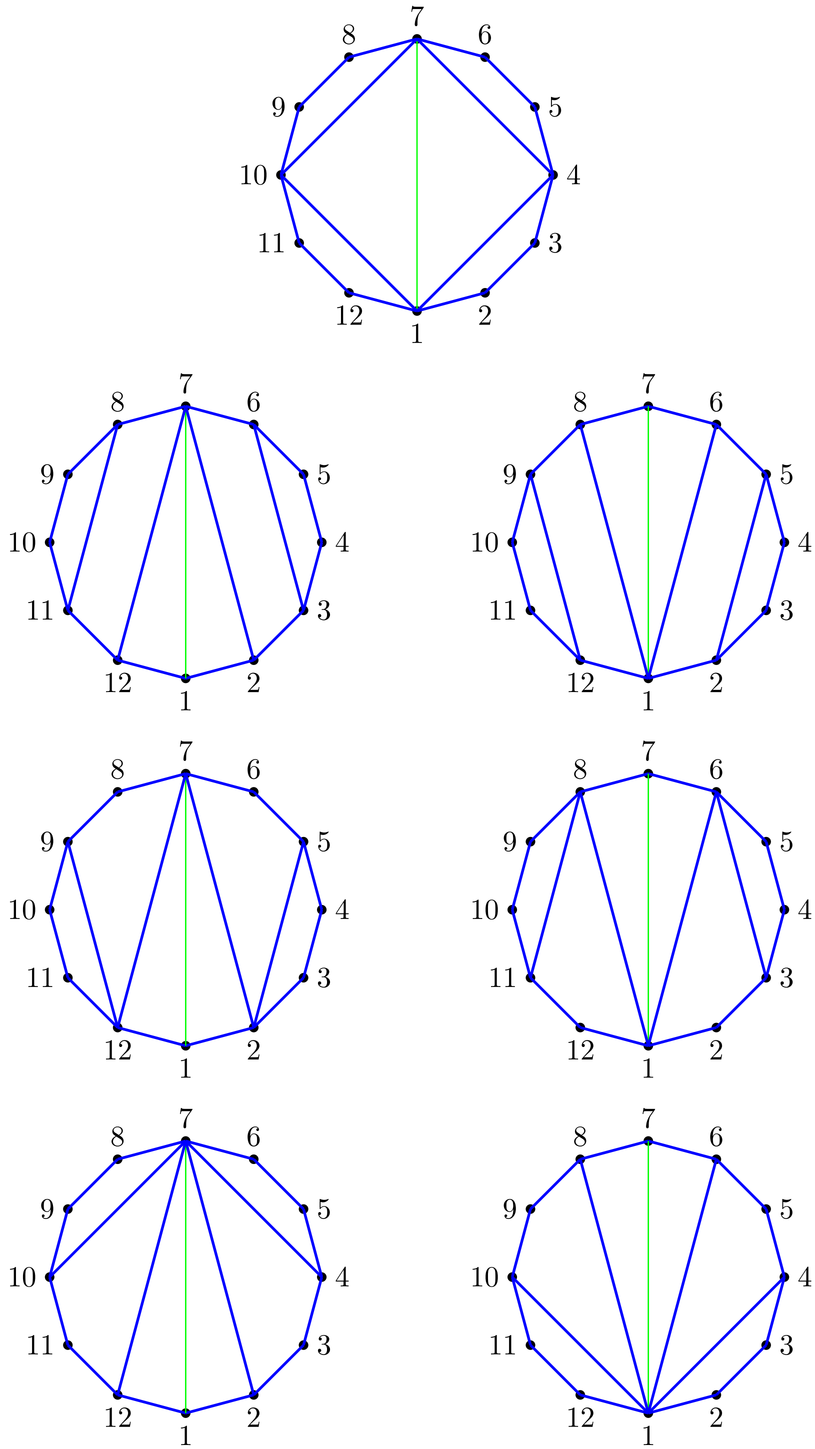


FIGURE 12. Dissections of the dodecagon invariant under reflection across 1 7

The number quadrilateral dissections of a $2n$-gon up to rotations and reflections is

$$\tilde{q}_{2n} = \begin{cases} \frac{1}{4n}\left(\nu_n + 3ns_n\right) & \textit{if } n \equiv 1 \pmod 2 \\ \frac{1}{4n}\left(\nu_n + \frac{5n}{2}s_n\right) & \textit{if } n \equiv 0 \pmod 4 \\ \frac{1}{4n}\left(\nu_n + \frac{5n}{2}s_n + ns_{\frac{n}{2}}\right) & \textit{if } n \equiv 2 \pmod 4 \end{cases}$$

We use the notation $\mathcal{N}_n'$ to denote the set of *unlabeled oriented* non-crossing trees, in other words an element of $\mathcal{N}_n'$ is an orbit of the action of $\langle \delta^2 \rangle \cong \mathbb{Z}/n$, and $\widetilde{\mathcal{N}}_n$ to denote the set of *unlabeled unoriented* non-crossing trees, in other words an element of $\widetilde{\mathcal{N}}_n$ is an orbit of the action of the dihedral group $D_n = \langle \delta^2, s \rangle$. So Theorem 2.3 allow us to calculate the number of unlabeled oriented and unoriented non-crossing trees as well. We note that for n even the central element of D_{2n} (rotation by π) belongs to D_n while for odd n it doesn't, so that for even n there are no rotation invariant non-crossing trees. So we have the following theorem, proved in [12][5].

Theorem 2.5 (Noy [12])**.** *The number of non-crossing trees with n vertices up to rotations is*

$$\nu_n' = \begin{cases} \frac{\nu_n}{2n} & \text{if } n \text{ is odd} \\ \frac{1}{2n}\left(\nu_n + \frac{ns_n}{2}\right) & \text{if } n \text{ is even} \end{cases}$$

The number of unlabeled non-crossing trees with n vertices is

$$\tilde{\nu}_n = \begin{cases} \frac{1}{2n}\left(\nu_n + ns_n\right) & \text{if } n \text{ is odd} \\ \frac{1}{2n}\left(\nu_n + \frac{3n}{2}s_n\right) & \text{if } n \text{ is even} \end{cases}$$

Finally we can use a generalization of Burnside's Lemma, the "Counting Lemma" of [13], to count the number of self-dual unlabeled oriented or unoriented trees. That lemma states that if a group G acts on a set X endowed with a permutation r such that $rG = Gr$, so that r is well defined in the orbits of G, then $N(G,r)$ the number of orbits fixed by r is given by:

$$N(G,r) = \frac{1}{|G|}\sum_{g\in G} |\{x \in G : grx = x\}|$$

Applying this theorem in our case with $G = \langle \delta^2 \rangle \cong \mathbb{Z}/n$ and $G = \langle \delta^2, r \rangle \cong D_n$ we get:

Theorem 2.6. *The number of self-dual unlabeled oriented non-crossing trees with n vertices is*

$$s_n' = s_n$$

The number of self-dual unlabeled unoriented self-dual non-crossing trees with n vertices is

$$\tilde{s}_n = \begin{cases} s_n & \text{if } n \equiv 1 \pmod 2 \\ \frac{s_n}{2} & \text{if } n \equiv 0 \pmod 4 \\ \frac{s_n + s_{\frac{n}{2}}}{2} & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

Proof. For the case of oriented non-crossing trees we need to look at the fixed points of $\delta^{2i}r$ for $i = 0, \ldots, n-1$. Each of these elements is a reflection in D_{2n} and so by Theorem 2.3 has s_n fixed element.

For the case of unoriented unlabeled non-crossing trees, we need in addition to take into account the fixed points of $\delta^{2i}s\,r$ for $i = 0, \ldots, n-1$. Since $s\,r = \delta^{-1}$ this means that we have to take into account all the fixed points of odd powers of δ, so the result follows from Theorem 2.3. □

Call $t \in \mathcal{N}_n'$ *anti-self-dual* if $t^* = \bar{t}$. Clearly an anti-self-dual non-crossing tree is fixed by δ^i for some odd power i. So we have:

Theorem 2.7. *The number of anti-self-dual non-crossing trees with n vertices is*

$$a_n = \begin{cases} s_n & \text{if } n \equiv 1 \pmod 2 \\ 0 & \text{if } n \equiv 0 \pmod 4 \\ s_{\frac{n}{2}} & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

[5] The enumeration of oriented unlabeled trees is not explicitly stated there but a formula can be deduced from the calculations.

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