arXiv:1808.01416v1 [math.CO] 4 Aug 2018

ON A POSITIVITY CONJECTURE IN THE CHARACTER TABLE OF S_n

SHEILA SUNDARAM

Abstract. In previous work of this author it was conjectured that the sum of power sums p_λ, for partitions λ ranging over an interval $[(1^n), \mu]$ in reverse lexicographic order, is Schur-positive. Here we investigate this conjecture and establish its truth in the following special cases: for $\mu \in [(n-4, 1^4), (n)]$ or $\mu \in [(1^n), (3, 1^{n-3})]$, or $\mu = (3, 2^k, 1^r)$ when $k \geq 1$ and $0 \leq r \leq 2$. Many new Schur positivity questions are presented.
Keywords: conjugacy action, character table, Schur positivity, power sum symmetric functions

1. Introduction and Preliminaries

In this paper we consider Schur positivity questions related to the reverse lexicographic order on integer partitions. Recall that this total order is defined as follows [2, p. 6]. For partitions λ, μ of the same integer n, we say a partition λ is preceded by a partition μ in reverse lexicographic order if $\lambda_1 > \mu_1$ or there is an index $j \geq 2$ such that $\lambda_i = \mu_i$ for $i < j$ and $\lambda_j > \mu_j$. Thus for $n = 4$ we have the total order $(1^4) < (2, 1^2) < (2^2) < (3, 1) < (4)$. In particular our convention is that the minimal and maximal elements in this total order are (1^n) and (n) respectively. Our primary goal is to address the following conjecture:

Conjecture 1. [6, Conjecture 1] *Let L_n denote the reverse lexicographic ordering on the set of partitions of n. Then the sum of power sum symmetric functions $\sum p_\lambda$, taken over any initial segment of the total order L_n, i.e. any interval of the form $[(1^n), \mu]$ for fixed μ, (and thus necessarily including the partition (1^n)), is Schur-positive.*

In general, for arbitrary subsets T of partitions of n with $(1^n) \in T$, the sums $\sum_{\mu \in T} p_\mu$ define (possibly virtual) representations of the symmetric group S_n, of dimension $n!$ There are many instances where Schur-positivity fails; see the remarks following Example 1.4. Proposition 4.1 in Section 4 gives a lower bound for the number of failures.

Conjecture 1 has an equivalent formulation in terms of the character table of S_n. If the columns of the table are indexed by the integer partitions of n corresponding to the conjugacy classes, in reverse lexicographic order, left to right, and the rows by the irreducible characters (hence also corresponding to partitions), then the conjecture states that, for each row, indexed by some fixed partition λ of n, the sum of the entries in the first k consecutive columns, beginning with the column indexed by (1^n), is a nonnegative integer. If the kth column corresponds to the conjugacy class indexed by the partition μ, this row sum is the multiplicity of the Schur function s_λ in the sum $\sum_{\nu \in [(1^n), \mu]} p_\nu$.

Date: August 7, 2018.
2010 *Mathematics Subject Classification.* 20C05, 20C15, 20C30, 05E18, 06A07.

Let ψ_n denote the Frobenius characteristic of the conjugacy action on S_n. The orbits of this action are the conjugacy classes. Let f_n denote the Frobenius characteristic of the conjugacy action of S_n on the class of n-cycles. Let h_n, e_n denote respectively the homogeneous and elementary symmetric functions of degree n, and let $[\quad]$ denote plethysm. By a general observation of Solomon [4] for finite groups (see also [5, Exercise 7.71], [6, Corollary 4.3]), we have the following facts. In view of Part (2) of the theorem below, Conjecture 1 may be seen as a generalisation of the Schur positivity of the sum of all power sums $\sum_{\lambda\vdash n} p_\lambda$.

Theorem 1.1. *The Frobenius characteristic ψ_n of the conjugacy action of S_n admits the following decompositions:*

(1) $\psi_n = \sum_{\lambda\vdash n}\prod_i h_{m_i}[f_i]$, where the partition λ has m_i parts equal to i.
(2) $\psi_n = \sum_{\lambda\vdash n} p_\lambda$, and hence the latter sum is Schur-positive.

Definition 1.2. If μ is a partition of n, we write ψ_μ for the sum of power sums

$$\sum_{\lambda\in[(1^n),\mu]} p_\lambda.$$

More generally if T is any subset of partitions of n, define ψ_T to be the sum $\sum_{\mu\in T} p_\mu$.

Thus $\psi_{(n)} = \psi_n$, and the multiplicity of the Schur function s_λ in ψ_μ is the sum of the values of the irreducible character χ^λ on the conjugacy classes in the interval $[(1^n), \mu]$.

Clearly $\psi_{(1^n)}$ is just the characteristic of the regular representation. Also since $\psi_2 = 2s_{(2)}$ is twice the trivial representation, $\psi_{(2,1^{n-2})} = p_1^{n-2}\psi_{(2)} = 2s_{(2)}p_1^{n-2}$. The Schur function expansion of ψ_n for $n\le 10$ appears in [6, Table 1]. We have verified Conjecture 1 in Maple up to $n = 20$.

The main result of this paper gives an affirmative answer to Conjecture 1 in the following cases:

Theorem 1.3. *The symmetric function $\psi_\mu = \sum_{(1^n)\le\lambda\le\mu} p_\lambda$ is Schur-positive if $\mu \le (3, 1^{n-3})$ or $\mu \ge (n-4, 1^4)$ in reverse lexicographic order.*

Our approach to Conjecture 1 proceeds in two directions. One can start at the bottom of the chain, with p_1^n (which contains all irreducibles), and add successive p_λ's going up the chain. The arguments in this case are subtle, and give an interesting decomposition of the corresponding representation. See Theorem 2.11. Alternatively, one can start at the top of the chain, with the known Schur positive function ψ_n, which is also known to contain all irreducibles (see Section 2), and examine what happens to the irreducibles upon subtracting successive p_λ's going down the chain, from ψ_n. This is done in Theorem 2.16, and requires a careful analysis (Lemmas 2.12 to 2.14) of the Schur functions appearing in products of power sums. The technical difficulty here is in ensuring that the resulting expressions (Proposition 2.15) are *reduced*, i.e. each term corresponds to a unique Schur function. The argument now hinges on the following fact: no irreducible in the partial sum of power sums appears with multiplicity exceeding the lower bound, established in Lemma 2.6, for the multiplicity of each irreducible in ψ_n.

The proof of Theorem 2.11 hints at interesting properties of the representations $\psi_{(2^k)}$. In Section 3 we present conjectures suggested by that proof, and establish more Schur positivity results (the case $\mu = (3, 2^k, 1^r)$ for $0 \le r \le 2$, Proposition 3.7), as well as generalisations of Theorem 2.16 to the twisted conjugacy action as defined in [6]. Section 4 concludes the paper with an analysis of the number of subsets of partitions whose associated sum of power sums is *not* Schur-positive. Tables of the Schur expansion of ψ_μ appear in Section 5.

An interesting implication of Conjecture 1 is obtained by examining the occurrence of the sign representation in ψ_μ. The multiplicity of $s_{(1^n)}$ in ψ_μ is clearly

$$\sum_{(1^n)\leq\lambda\leq\mu}(-1)^{n-\ell(\lambda)}, \tag{1.1}$$

where $\ell(\lambda)$ denotes the number of parts of λ, because the value of the sign character on the conjugacy class indexed by λ is $(-1)^{n-\ell(\lambda)}$. In our reverse lexicographic ordering, these values are the partial sums (computed left to right, with the left-most column indexing the class of the identity (1^n)) in the first row of the character table. Hence Conjecture 1 implies that the sum (1.1) is nonnegative for all $\mu \vdash n$. When $\psi_\mu = \psi_n$ is the characteristic of the full conjugacy action of S_n, the expression (1.1) can be shown to equal the number of partitions of n all of whose parts are odd and distinct, or equivalently, the number of self-conjugate partitions of n [6, Proposition 4.21]. In general the character values form a sequence of 1's and (-1)'s, with the partitions written in reverse lexicographic order; it is not obvious why the resulting partial sums should be nonnegative. The nonnegativity is easily checked for $n \leq 7$. The example below contains data for $8 \leq n \leq 13$. As observed above, the last partial sum in each case is the number of partitions with all parts odd and distinct. Theorem 2.16 will imply (see Corollary 2.17) that at least the last five partial sums are necessarily positive.

Example 1.4. (See the discussion following Corollary 2.17 for the meaning of the italics and the underlined runs.) The 22 values of the sign character for S_8, on the conjugacy classes taken in reverse lexicographic order, beginning with (1^n), are

$$\mathbf{1,-1,1,-1,1,1,-1,1,1,-1,-1,1,-1,-1,1,1,-1,1,-1,1,1,-1},$$

with partial sums: $\mathbf{1,0,1,\underline{0,1,2},\underline{1,2,\mathit{3}},\mathit{2,1,2,1},\underline{\mathit{0},1,2},1,2,\underline{1,2,3},2}$;

the 30 values for S_9 are

$\mathbf{1,-1,1,-1,1,1,-1,1,-1,1,-1,1,-1,1,-1,-1,\underline{1,1,1},-1,1,1,-1,-1,1,-1,1,-1,-1,1,1}$,

with partial sums: $\mathbf{1,0,1,\underline{0,1,2},1,2,1,2,1,2,1,\mathit{2,1},\underline{\mathit{0},1,2,3},2,3,4,3,2,3,2,3,2,1,2}$;

and the 42 values for S_{10} are

$\mathbf{1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,\underline{1,1},\mathit{-1,-1},1,-1,1,-1,-1,\underline{1,1},-1,1,-1,\underline{1,1},-1}$,

with sequence of partial sums:

$\mathbf{1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \underline{0, 1, 2}, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \underline{1, 2, \mathit{3}}, \mathit{2, 1}, 2, 1, 2, 1, 0, 1, 2, 1, 2, \underline{1, 2, 3}, 2.}$

The partial sums for S_{11} are:

$\mathbf{1, 0, 1, 0, 1, 0, 1, 0, 1, \underline{0, 1, 2}, \underline{1, 2, \mathit{3}}, \mathit{2, 1}, 2, 1, 2, 1, \mathit{2, 1, 0}, 1, \underline{0, 1, 2}, 1, 2, 1, 2, 1, 2, \underline{1, 2, 3}, 2, 3, 2, \underline{1, 2, 3}, 2, 3, \underline{2, 3, \mathit{4}}, \mathit{3, 2}, 3, 2, \mathit{3, 2, 1}, 2}$

The partial sums for S_{12} are:

$\mathbf{1, 0, 1, 0, 1, \underline{0, 1, 2}, 1, 2, \underline{1, 2, 3}, 2, 3, 2, 3, 2, 3, 2, 3, 2, \mathit{3, 2, 1, 2, 1, 0}, 1, 2, \underline{1, 2, 3}, 2, 3, 2, 3, 2, 3, \underline{2, 3, 4}, 3, 4, 3, \mathit{4, 3, 2}, 3, 2, 3, 2, 3, 2, \mathit{3, 2, 1}, 2, 3, \underline{2, 3, \mathit{4}}, \mathit{3, 2}, 3, 2, \mathit{3, 2}, \underline{\mathit{1}, 2, 3}, 2, 3, \underline{2, 3, 4}, 3}$

The partial sums for S_{13} are:

$\mathbf{1, 0, 1, 0, 1, \underline{0, 1, 2}, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \underline{1, 2, 3}, 2, 3, 2, \mathit{3, 2, 1}, 2, 1, 2, 1, 2, 1, 2, \underline{1, 2, \mathit{3}}, \mathit{2, 1}, 2, 1, 2, \underline{1, 2, 3}, \underline{2, 3, \mathit{4}}, \mathit{3, 2, 3, 2}, \underline{\mathit{1}, 2, 3}, 2, 3, 2, 3, 2, 3, 2, \mathit{3, 2, 1}, 2, 1, 2, \underline{1, 2, 3, 4}, 3, 4, 3, 4, 3, 4, \underline{3, 4, \mathit{5}}, \mathit{4, 3}, 4, 3, \underline{2, 3, 4}, 3, 4, \underline{3, 4, \mathit{5}}, \mathit{4, 3}, 4, 3, \mathit{4, 3, 2}, 3}$

In addition to the sign, it is also interesting to examine the multiplicity of the representation $(2,1^{n-2})$.

Example 1.5. The values of the irreducible character indexed by $(2,1^{n-2})$ on the conjugacy classes in reverse lexicographic order are as follows.

For $n = 7$:

$$\mathbf{6}, -\mathbf{4}, \mathbf{2}, \mathbf{0}, \mathbf{3}, -\mathbf{1}, -\mathbf{1}, \mathbf{0}, -\mathbf{2}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{0}, -\mathbf{1},$$

with partial sums: $\mathbf{6}, \mathbf{2}, \mathbf{4}, \mathbf{4}, \mathbf{7}, \mathbf{6}, \mathbf{5}, \mathbf{5}, \mathbf{3}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{6}, \mathbf{5}$.

For $n = 8$:

$$\mathbf{7}, -\mathbf{5}, \mathbf{3}, -\mathbf{1}, -\mathbf{1}, \mathbf{4}, -\mathbf{2}, \mathbf{0}, \mathbf{1}, \mathbf{1}, -\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{0}, -\mathbf{1}, \mathbf{2}, \mathbf{0}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{0}, \mathbf{1}$$

with partial sums: $\mathbf{7}, \mathbf{2}, \mathbf{5}, \mathbf{4}, \mathbf{3}, \mathbf{7}, \mathbf{5}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{6}, \mathbf{5}, \mathbf{7}, \mathbf{7}, \mathbf{6}, \mathbf{5}, \mathbf{4}, \mathbf{4}, \mathbf{5}$.

These examples also highlight the fact that there are many ways of reordering the conjugacy classes so that the resulting partial (row) sums in the character table may be negative, and the corresponding sum of power sums will thus fail to be Schur-positive. We will return to this observation in Section 4.

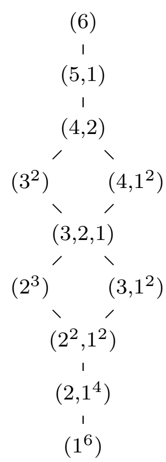

Figure 1: Dominance order for partitions of 6

We remark that Conjecture 1 is false if one considers dominance order instead of reverse lexicographic order. It fails for the first case in which dominance departs from reverse lexicographic order, $n = 6$, as the following example shows:

Example 1.6. The seven partitions (weakly) dominated by $(4,1^2)$ are (see Figure 1 above)

$$\{(4,1^2), (3,2,1), (2^3), (3,1^3), (2^2,1^2), (2,1^4), (1^6)\}$$

The sum of power sums is thus $p_4p_1^2 + p_3p_2p_1 + p_2^3 + p_3p_1^3 + p_2^2p_1^2 + p_2p_1^4 + p_1^6$; in the corresponding S_n-module, the sign appears with negative multiplicity (all other irreducibles occur with positive coefficient):

$$7s_{(6)} + 11s_{(5,1)} + 15s_{(4,2)} + 8s_{(4,1^2)} + 3s_{(3^2)} + 14s_{(3,2,1)}$$
$$+ 10s_{(3,1^3)} + 7s_{(2^3)} + 5s_{(2^2,1^2)} + 5s_{(2,1^4)} - s_{(1^6)}.$$

This is the only instance that fails for $n = 6$. For $\mu = (4,1^{n-4})$, (the hook with one part equal to 4), similarly, up to $n = 12$, the only irreducible with negative coefficient is the sign, appearing with coefficient (-1).

2. Intervals in Reverse Lexicographic Order

The following fact about the representation ψ_n was first proved by Avital Frumkin. See [5, Solution to Exercise 7.71] for more references.

Theorem 2.1. [1] *If $n \neq 2$, the representation ψ_n contains all irreducibles.*

We will need the following stronger result of [7] characterising the conjugacy classes containing all irreducibles. Recall (see [7] for references to the literature) that a conjugacy class in a finite group G is called *global* if the orbit of the conjugacy action corresponding to that class contains all irreducibles of G.

Theorem 2.2. [7, Theorem 5.1] *Let $n \neq 4, 8$. Then the conjugacy class indexed by a partition λ contains all irreducibles, i.e. it is a global class, if and only if λ has at least two parts, and all its parts are distinct and odd. If $n = 8$, the conjugacy class indexed by $(7, 1)$ is global, while the class of the partiition $(5, 3)$ contains all irreducibles except those indexed by (4^2) and (2^4).*

We also require some information on the irreducibles appearing in f_n, the S_n-action by conjugation on the class of n-cycles. Since this is a permutation representation with one orbit, the trivial representation appears exactly once. It is also easy to see that the sign representation appears only if n is odd. We will make use of the following definitive result of Joshua Swanson:

Theorem 2.3. *(*[9], [6, Lemma 4.2]*) Let $n \geq 1$. If n is odd, the representation f_n contains all irreducibles except those indexed by $(n-1, 1)$ and $(2, 1^{n-2})$. If n is even, f_n contains all irreducibles except $(n-1, 1)$ and (1^n).*

The result below was stated without proof in [6]; we sketch a proof here.

Proposition 2.4. [6, Proposition 4.21] *The multiplicity in ψ_n of the irreducible indexed by the partition*

(1) (n) is $p(n)$, the number of partitions of n.
(2) (1^n) is the number of partitions of n into parts that are distinct and odd, which is also the number of self-conjugate partitions of n. This multiplicity is nonzero for $n \neq 2$.
(3) $(n-1, 1)$ is $\sum_{\lambda \vdash n} (|\{i : m_i(\lambda) \geq 1\}| - 1)$, which in turn equals the number of distinct parts in all the partitions of n, minus the number of partitions of n. In particular this multiplicity is at least the number of non-rectangular partitions of n, and hence at least $(n-1)$.
(4) $(2, 1^{n-2})$ is $\sum_{\lambda \vdash n} (\ell(\lambda) - 1) + \sum(|\{i : m_i(\lambda) \geq 1\}|)$, where the first sum runs over all partitions λ with parts that are distinct and odd and the second sum runs over the set of partitions λ such that $m_i(\lambda) \leq 2, m_j(\lambda) = 2$ for exactly one part j.

Proof. Part (1) is clear since $p(n)$ is the number of conjugacy classes of n. As alluded to in the Introduction, Part (2) is a computation of the sum $\sum_{\mu \vdash n} (-1)^{n-\ell(\mu)}$, and follows from the standard generating function identity for integer partitions by number of parts. Since the sign representation always occurs in f_n if n is odd, it also occurs in $f_1 f_{n-1}$ if n is even, i.e. in the conjugacy class $(n-1, 1)$ for $n \neq 2$. The second statement of Part (2) follows.

For Part (3), we use Frobenius reciprocity, and the fact that $s_{(n-1,1)} = h_{n-1}h_1 - h_n$. Hence the required multiplicity is $p(n)$ less than the multiplicity of $s_{(n-1)}$ in the restriction of ψ_n to S_{n-1}, which can be computed using the partial derivative with respect to p_1 (see e.g. [2]) and Theorem 1.1 (1).

The last statement follows because for $n \geq 3$ there is always a partition with two unequal parts. □

Lemma 2.5. *(1) Let $n \neq 3$. Then $f_n f_1$ contains all irreducibles except (1^{n+1}) if n is even.*

(2) Let $n \geq 5$, and let n be odd. Then the product $f_n f_2$ contains all irreducibles except the sign.

(3) Let $n \geq 5$. If n is odd, every irreducible except the sign appears in each of the conjugacy classes $(n-2, 2)$ and $(n-2, 1, 1)$.

(4) Let $n \geq 6$. If $n = 2k$ is even, every irreducible except the sign appears in $f_{n-3}f_2f_1$.

(5) Let $n \geq 6$ be even. Then every irreducible appears in $f_n f_2$ except for the sign and the one indexed by $(2, 1^n)$, which does however appear in $f_{n-2}f_3f_1$. In particular all the irreducibles except for the sign appear among the two conjugacy classes $(n, 2)$ and $(n-2, 3, 1)$.

Proof. For **Part (1)**: The result is clear for $n = 1, 2$ so assume $n \geq 4$.

If n is odd this is immediate from Theorem 2.2. If n is even, then by Theorem 2.3, f_n contains all irreducibles except (1^n) and $(n-1, 1)$. Now $s_{(n-1,1)} \cdot f_1 = s_{(n+1)} + s_{(n,1)} + s_{(n-1,2)}$. But each of these summands appears in the product $g_n \cdot f_1$ for $g_n = s_{(n)}, s_{(n)}, s_{(n-2,2)}$ respectively, and each g_n appears in f_n. The only irreducible that does not appear is (1^{n+1}).

For **Part (2)**: It is easy to compute, since $f_2 = h_2$ and $f_3 = h_3 + e_3$, $f_3 f_2 = s_{(5)} + s_{(4,1)} + s_{(3,2)} + s_{(3,1,1)} + s_{(2,1,1,1)}$, so the product does not contain $s_{(2,2,1)}$.

So let $n \geq 5$ be odd. By Theorem 2.3, f_n contains all irreducibles except those indexed by $(n-1, 1)$ and $(2, 1^{n-2})$. We have

$$s_{(n-1,1)} \cdot f_2 = s_{(n-1,1)} \cdot h_2 = s_{(n+1,1)} + s_{(n,2)} + s_{(n,1,1)} + s_{(n-1,3)} + s_{(n-1,2,1)}. \quad (A)$$

The first two summands appear in $g_n \cdot f_2$ for $g_n = s_{(n)} \cdot f_2$. The last two summands appear in $g_n \cdot f_2$ for $g_n = s_{(n-1,2)}$. Finally $s_{(n-1,1,1)}$ appears in the product $g_n \cdot f_2$ for $g_n = s_{(n-2,1,1)}$, and this appears in f_n for $n \geq 5$. Thus in all cases g_n appears in f_n.

Next consider the other missing irreducible, $(2, 1^{n-2})$. We have

$$s_{(2,1^{n-2})} \cdot f_2 = s_{(2,1^{n-2})} \cdot h_2 = s_{(4,1^{n-2})} + s_{(3,2,1^{n-3})} + s_{(3,1^{n-1})} + s_{(2^2,1^{n-2})}. \quad (B)$$

The first three appear in the product $g'_n \cdot f_2$ for $g'_n = s_{(3,1^{n-3})}$, which is a constituent of $f_n, n \geq 5$. The last one appears in the product $g'_n \cdot f_2$ for $g'_n = s_{(2^2,1^{n-4})}$, which again is a constituent of $f_n, n \geq 5$. This completes the argument.

For **Part (3)**, observe that the conjugacy classes indexed by $(n, 2)$ and $(n, 1, 1)$ both afford the same representation, namely $f_n \cdot h_2 = f_n \cdot f_2$. The result now follows from Part (2).

Part (4) follows by applying Part (2) to $f_{n-3}f_2$, since $n-3$ is odd.

For **Part (5)**: Since n is even, again Theorem 2.3 tells us that f_n contains all irreducibles except those indexed by $(n-1, 1)$ and (1^n). From (A) above we see that the product $f_n f_2$ may miss the irreducibles indexed by

$$(n+1, 1), (n, 2), (n, 1^2), (n-1, 3) \text{ and } (n-1, 2, 1),$$

but all these appear in the set of products $\{s_{(n)}f_2, s_{(n-2,1^2)}f_2\}$, and $s_{(n)}, s_{(n-2,1^2)}$ appear in f_n. The only other irreducibles possibly missed by the product f_nf_2 are those in the product $s_{(1^n)}f_2$, namely

$$(2, 1^{n-2}), (3, 1^{n-3}) \text{ and } (1^{n+2}).$$

Clearly $s_{(3,1^{n-3})}$ occurs in $s_{(2,1^{n-4})} \cdot f_2$, and f_n contains $s_{(2,1^{n-4})}$ since n is even.

To establish the claim, we now need only show that the irreducible $(2, 1^{n-2})$ appears in $f_{n-2}f_3f_1$. But $f_3 = h_3 + e_3$, so $f_{n-2}f_3$ contains all the irreducibles in the product $f_{n-2}e_3$. Since $n - 2$ is even, it contains $s_{(2,1^{n-4})}$ and this finishes the argument. □

Lemma 2.6. *Let $n \geq 5$. Let do_n denote the number of partitions of n with at least two parts and with all parts odd and distinct. In the conjugacy representation ψ_n, every irreducible except possibly the sign occurs with multiplicity at least* $\begin{cases} 4 + do_n, & n \text{ odd;} \\ 3 + do_n, & n \text{ even.} \end{cases}$ *This number is at least 5 for odd $n \geq 7$, and at least 4 for even $n \geq 6$.*

Proof. First let n be odd. By Lemma 2.5, the following conjugacy classes contain all irreducibles except the sign: $(n-1, 1), (n-2, 2), (n-2, 1^2)$. Also by Theorem 2.3, the conjugacy class (n) (or equivalently the symmetric function f_n) contains all irreducibles except for the one indexed by $(n-1, 1)$ and (1^n). But the irreducible $(n-1, 1)$ appears at least $n - 1$ times in ψ_n, by Proposition 2.4. Thus we have multiplicity at least 4 for each irreducible. Since none of the four conjugacy classes listed above is global by Theorem 2.1, we have a multiplicity of at least 4 plus the number of global classes.

Now let n be even, $n \geq 8$. (The case $n = 6$ can be checked by direct computation. See, e.g. [6, Table 1]. Then by Theorem 2.3, the conjugacy class (n) has all irreducibles except for $(n-1, 1)$ and (1^n). Also by Lemma 2.5, $f_{n-3}f_2$ has all irreducibles except for (1^n). Hence so does the conjugacy class $(n-3, 2, 1)$. Finally this is also true by Lemma 2.5 again, for the sum $(f_{n-2}f_2 + f_{n-4}f_3f_1)$. We have accounted for a multiplicity of at least 3 for every irreducible except the sign, in addition to the global classes.

We now show that the number of global classes is at least $\lfloor \frac{k}{2} \rfloor$ if $n = 2k, 2k + 1$.

First let $n = 2k \geq 6$ be even. In this case, applying Theorem 2.2, we have at least $\lfloor \frac{k}{2} \rfloor \geq 1$ global conjugacy classes: $\{(2k-r, r) : r = 1, 3, \ldots, 2\lfloor \frac{k}{2} \rfloor - 1\}$. If $n = 2k+1 \geq 9$, then again there at least $\lfloor \frac{k}{2} \rfloor - 1 \geq 1$ global conjugacy classes in the set $\{(2k - r, r, 1) : r = 3, 5, \ldots, 2\lfloor \frac{k}{2} \rfloor - 1\}$ and one more: $(2k - 7, 5, 3)$. □

Remark 2.7. Tables of the decomposition into irreducibles for $\psi_n, n \leq 10$, are given in [6]. We point out a misprint in Table 1 of [6] for $n = 7$: the fifth entry from the bottom, for the multiplicity of $(3, 1^4)$ in ψ_7, should be 13, not 7. From this data, the truth of the lemma follows for $n \leq 10$. It is worth noting that the tables indicate far greater lower bounds than we have just established, for the multiplicity of the irreducible indexed by μ when $\mu \neq (1^n), (2, 1^{n-2})$.

We begin our analysis by directing our attention to the bottom of the chain, to examine the representations ψ_μ for $\mu > (1^n)$. Our argument in this case is somewhat mysterious. One interesting aspect is the role played by the following calculation.

Lemma 2.8. *The symmetric function $p_2^2 + h_2e_2 = h_2^2 - e_2h_2 + e_2^2$ is Schur-positive.*

Proof. Note that $p_2 = h_2 - e_2$. It is straightforward to compute, using Pieri rules (see e.g. [2]), that $h_2^2 - e_2h_2 + e_2^2 = s_{(4)} + s_{(1^4)} + 2s_{(2,2)}$. □

Definition 2.9. Let T be any subset of integer partitions. Denote by $p_{n,T}$ the sum of power sums $\sum_{\lambda\vdash n:\lambda_i\in T \text{ for all } i} p_\lambda$.

Theorem 2.10. [6, Theorem 4.23] *If $T = \{\lambda \vdash n : \lambda_i = 1, 2$ for all $i\}$ then $p_{n,T}$ is Schur-positive. We have $p_{2m+1,T} = p_1 p_{2m,T}$ and*

$$p_{2m,T} = \sum_{\substack{j=1\\ j \text{ odd}}}^{m+1} \binom{m+1}{j} h_2^{m+1-j} e_2^{j-1}$$

Theorem 2.11. *Let μ be a partition in the interval $[(1^n), (3, 1^{n-3})]$. Then ψ_μ is Schur-positive. Equivalently, the following are Schur-positive:*

(1) $\psi_{(2^k,1^{n-2k})} = \sum_{i=0}^{k} p_2^i p_1^{n-2i}$, for $k \le n/2$; one has the recurrence

$$\psi_{(2^{k+1},1^{n-2(k+1)})} = \psi_{(2^k,1^{n-2k})} + p_2^{k+1} p_1^{n-2(k+1)}, 0 \le k < n/2.$$

(2) $\psi_{(3,1^{n-3})} = p_3 p_1^{n-3} + \psi_{(2^k,1^{n-2k})}$, where $k = \lfloor \frac{n}{2} \rfloor$.

Proof. For Part (1):

$$\psi_{(2^k,1^{n-2k})} = \sum_{i=0}^{k} p_2^i p_1^{n-2i} = p_1^{n-2k} \sum_{i=0}^{k} p_2^i p_1^{2k-2i} = p_1^{n-2k} p_{2k,T},$$

where $T = \{\lambda \vdash 2k : \lambda_i = 1, 2 \text{ for all } i\}$. But by Theorem 2.10 we know that $p_{2k,T}$ is Schur-positive as a representation of S_{2k}.

For Part (2): Writing $m = \lfloor \frac{n}{2} \rfloor$, since in reverse lexicographic order, $(3, 1^{n-3})$ covers the partition with at most one part equal to 1 and all other parts equal to 2, we have,

$$\psi_{(3,1^{n-3})} = \psi_{(2^m,1^{n-2m})} + p_3 p_1^{n-3}.$$

Note that $p_3 = h_3 + e_3 - (h_2 h_1 - h_3) = 2s_{(3)} + s_{(1^3)} - h_2 p_1$. Hence

$$\psi_{(3,1^{n-3})} = \psi_{(2^m,1^{n-2m})} - h_2 p_1^{n-2} + (2s_{(3)} + s_{(1^3)}) p_1^{n-3}.$$

We will establish the stronger claim that

$$\psi_{(2^m,1^{n-2m})} - h_2 p_1^{n-2} \tag{2.1}$$

is Schur-positive. From Theorem 2.10, it suffices to assume that $n = 2m$. In this case, with T being the set of partitions of $2m$ with parts 1,2, we have

$$\psi_{(2^m)} = p_{2m,T} = \sum_{\substack{j=1\\ j \text{ odd}}}^{m+1} \binom{m+1}{j} h_2^{m+1-j} e_2^{j-1} = h_2 V_{2m-2} + e_2^m \,\mathrm{Odd}(m+1), \tag{2.2}$$

where the notation $\mathrm{Odd}(n)$ is used to signify 1 if n is odd, and 0 otherwise, and we have set

$$V_{2m-2} = \sum_{\substack{j=1\\ j \text{ odd}}}^{m} \binom{m+1}{j} h_2^{m-j} e_2^{j-1}. \tag{2.3}$$

From (2.2) and (2.3), we need to establish the Schur positivity of

$$h_2(V_{2m-2} - p_1^{2m-2}) + e_2^m \,\mathrm{Odd}(m+1). \tag{2.4}$$

In fact when m is odd, we will show that $V_{2m-2} - p_1^{2m-2}$ itself is Schur-positive, whereas when m is even, we will need to multiply this by h_2 and examine the entire expression (2.4) in order to obtain Schur positivity.

Since $p_1^2 = h_2 + e_2$, we can write

$$p_1^{2m-2} = \sum_{t=0}^{m-1} \binom{m-1}{t} e_2^t h_2^{m-t-1}. \tag{2.5}$$

Also

$$\begin{aligned} V_{2m-2} &= \sum_{\substack{j=1\\ j \text{ odd}}}^{m} (\binom{m}{j} + \binom{m}{j-1}) h_2^{m-j} e_2^{j-1} \\ &= \sum_{\substack{t=0\\ t \text{ even}}}^{m-1} (\binom{m}{t+1} + \binom{m}{t}) h_2^{m-t-1} e_2^t, \quad (\text{setting } t = j-1) \\ &= \sum_{\substack{t=0\\ t \text{ even}}}^{m-2} (\binom{m}{t+1} + \binom{m}{t}) h_2^{m-t-1} e_2^t + (m+1) e_2^{m-1} \operatorname{Odd}(m). \end{aligned}$$

Combining this with (2.5), we obtain

$$\begin{aligned} V_{2m-2} - p_1^{2m-2} = \sum_{\substack{t=0\\ t \text{ EVEN}}}^{m-2} e_2^t h_2^{m-1-t} \left(\binom{m-1}{t+1} + \binom{m-1}{t} + \binom{m-1}{t-1} \right) \\ + (m+1) e_2^{m-1} \operatorname{Odd}(m) - \sum_{\substack{t=0\\ t \text{ ODD}}}^{m-1} e_2^t h_2^{m-1-t} \binom{m-1}{t}, \end{aligned} \tag{2.6}$$

where by convention $\binom{m-1}{t-1}$ is zero if $t < 1$. We will split the first sum in (2.6) (over even t) into three sums as follows:

$$\sum_{\substack{t=0\\ t \text{ EVEN}}}^{m-2} P_{(t+)} + \sum_{\substack{t=0\\ t \text{ EVEN}}}^{m-2} Q_{(t)} + \sum_{\substack{t=2\\ t \text{ EVEN}}}^{m-2} R_{(t-)},$$

where, for $0 \le t \le m-2$,

$$P_{(t+)} = e_2^t h_2^{m-1-t} \binom{m-1}{t+1},$$

$$Q_{(t)} = e_2^t h_2^{m-1-t} \binom{m-1}{t},$$

and for $2 \le t \le m-2$,

$$R_{(t-)} = e_2^t h_2^{m-1-t} \binom{m-1}{t-1}.$$

Next consider the negated terms in (2.6). For these we write, for odd k, $1 \le k \le m-1$,

$$N_k = e_2^k h_2^{m-1-k} \binom{m-1}{k}.$$

Our goal is to absorb every negated term N_k into a Schur positive term. We now describe a judicious grouping which will allow us to accomplish this.

Collect the terms in $V_{2m-2} - p_1^{2m-2}$ as follows:

$$(P_{(0+)} - N_1 + R_{(2-)} + (P_{(2+)} - N_3 + R_{(4-)}) + \ldots + (P_{(t+)} - N_{t+1} + R_{(t+2)-}) + \ldots,$$

where t is even.

For each odd $k = 2j-1, 1 \le k \le m-3$, we have

$$P_{((2j-2)+)} - N_{2j-1} + R_{((2j)-)} = P_{((k-1)+)} - N_k + R_{((k+1)-}$$

$$= \binom{m-1}{k} e_2^{k-1} h_2^{m-k-2}(h_2^2 - e_2h_2 + e_2^2), \quad 1 \le k \le m-3. \tag{2.7}$$

But this is Schur-positive by Lemma 2.8. This absorbs the negative terms N_k for k odd, $k \le m-3$, into Schur-positive expressions. Looking at (2.6), there is only one more negated term to investigate.

Suppose m is **odd**, so that $k = m-2$ is odd, and thus N_{m-2} is the last negated summand in (2.6), the only one not taken care of in (2.7). Group the terms as before, and noting that $P_{((m-3)+)}$ was NOT used in the groupings of (2.7), we have

$$\begin{aligned} &P_{((m-3)+))} - N_{m-2} + e_2^{m-1}(m+1) = P_{((m-3)+} - N_{m-2} + e_2^{m-1}(m+1) \\ &= \binom{m-1}{m-2} e_2^{m-3} h_2^2 - \binom{m-1}{m-2} e_2^{m-2} h_2 + (m+1) e_2^{m-1} \\ &= (m-1) e_2^{m-3}(h_2^2 - e_2h_2 + e_2^2) + 2e_2^{m-1} \end{aligned}$$

and this is again Schur-positive by Lemma 2.8. (Admittedly this is something of a miracle.)

Next suppose m is **even**, so that $k = m-1$ is odd; then N_{m-1} is the last negated summand in (2.6), and the only one not absorbed in (2.7). Now we have (since $P_{((m-2)+)}$ was not used in any of the groupings in (2.8)):

$$P_{((m-2)+)} - N_{m-1} = \binom{m-1}{m-1} e_2^{m-2} h_2 - \binom{m-1}{m-1} e_2^{m-1} = e_2^{m-2}(h_2 - e_2).$$

While this is not itself Schur-positive, from (2.4) we can multiply by h_2, and then we have (again somewhat fortuitously), since $m+1$ is odd,

$$h_2(P_{((m-2)+} - N_{m-1}) + e_2^m \operatorname{Odd}(m+1) = e_2^{m-2}(h_2^2 - h_2e_2 + e_2^2),$$

and this is Schur-positive as before.

By (2.4), this completes the argument, in which Lemma 2.8 clearly played a crucial role. □

Next we examine the chain from the top down. Here the arguments are more direct, but also computationally technical. Our strategy for establishing Schur positivity will be to show that in the Schur expansion of the sum $\sum_{\nu \vdash n: \mu < \nu \le (n)} p_\nu$, i.e. starting from the partition (n) and moving down the chain, the positive multiplicities of the irreducibles never exceed those in ψ_n. For the cases we consider, we are able to show that, with the exception of the trivial representation, which clearly occurs as many times as the number of partitions in the interval $[\mu, (n)]$, this sum has multiplicities at most 4. This allows us to apply the lower bounds for the multiplicities in ψ_n developed earlier in

Lemma 2.6. The Schur function expansion of the product $p_n p_m$ figures prominently in this analysis.

In the proofs that follow, for simplicity and clarity we write λ for the Schur function s_λ indexed by λ. The context should make clear when $(n-2,2)$ indicates the Schur function $s_{(n-2,2)}$ rather than the partition itself. Also for any statement S, δ_S denotes the value 1 if and only if S is true, and is zero otherwise.

Lemma 2.12. *Let $n \geq m$ and $4 \geq m \geq 1$; if $n = m$ assume $m \neq 2$. Then the Schur function expansion of the product $p_n p_m$ has only the coefficients $0, \pm 1$. More precisely, one has the following expansion into distinct irreducibles:*

$$\begin{aligned}&D_{n,m}\\&+[\alpha(n,m)+\delta_{m\geq 2}\beta(n,m)+\delta_{m=3}\gamma(n,3)+\delta_{m=4}\gamma(n,4)]\\&+(-1)^{n+m}\omega\left([\alpha(n,m)+\delta_{m\geq 2}\beta(n,m)+\delta_{m=3}\gamma(n,3)+\delta_{m=4}\gamma(n,4)]\right);\end{aligned}$$

where

$$D_{n,m} = \sum_{s=0}^{m-1}\sum_{t=0}^{n-m-2}(-1)^{t+1}(n-t-s-1, m-s+1, 2^s, 1^t),$$

$$\alpha(n,m) = \sum_{r=0}^{m-1}(-1)^r(n-r+m, 1^r), \quad \beta(n,m) = \sum_{s=0}^{\mathbf{m-2}}(-1)^s(n, m-s, 1^s),$$

$$\gamma(n,3) = (n-1, 2^2), \text{ and } \gamma(n,4) = (n-1,3,2) - (n-1,2,2,1) + (n-2,2^3).$$

The number of irreducibles appearing in the expansion is

$$m(n-m+1) + 2(\delta_{m\geq 2}(m-1) + \delta_{m=3} + 3\delta_{m=4}).$$

Proof. Note that these definitions imply that

$$(-1)^{n+m}\omega(\alpha(n,m)) = \sum_{t=n}^{n+m-1}(-1)^{t-1}(n-t+m, 1^t)$$

and

$$(-1)^{n+m}\omega(\beta(n,m)) = \sum_{s=\mathbf{1}}^{m-1}(-1)^{n-s-1}(m-s+1, 2^s, 1^{n-s-1}).$$

We begin with the well-known expansion of p_n into Schur functions of hook shape (a special case of the Murnaghan-Nakayama rule):

$$p_n = \sum_{r=0}^{n-1}(-1)^r(n-r, 1^r), n \geq 2.$$

The Murnaghan-Nakayama rule says the Schur functions in the product $p_n p_m$ are indexed by partitions obtained by attaching border strips (or rim hooks) of size m to each of the above hooks, with sign $(-1)^s$ where s is one less than the number of rows occupied by the border strip. (See [2] or [5].) We enumerate the disjoint possibilities in the figures below. Note that we have excluded the partition (1^m) (respectively, (m)) from Figure 1a because it is counted in Figure 2b (respectively, Figure 2a). In particular Figures 1a-1b need to be considered separately only if $m \geq 2$. (When $m = 1$, Figure 1a is included as the special case of Figure 2b for $r = 0$, and similarly Figure 1b is the

$r = n-1$ case of Figure 2a.) Figures 2-3 are possible configurations for all $m \geq 1$, and Figures 4a-4b, 5a-5d can occur only if $m = 3, 4$ respectively.

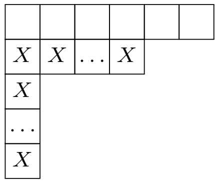

Figure 1a: $\mu = (n)$, with hook $(m-s, 1^s), 0 \leq s \leq \mathbf{m-2}$, in row 2.

The contribution to $p_n p_m$ here is $(-1)^s(n, m-s, 1^s)$, if $m \geq 2$.

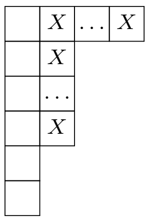

Figure 1b: $\mu = (1^n)$, with hook $(m-s, 1^s), \mathbf{1} \leq s \leq m-1$, in column 2.

The contribution to $p_n p_m$ is $(-1)^{n-1+s}(m-s+1, 2^s, 1^{n-s-1})$, if $m \geq 2$.

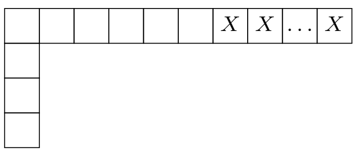

Figure 2a: $\mu = (n-r, 1^r), 0 \leq r \leq n-1$, with horizontal strip of size m.

The contribution to $p_n p_m$ here is $(-1)^r(n-r+m, 1^r), 0 \leq r \leq n-1$.

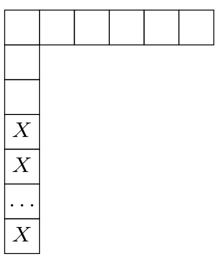

Figure 2b: $\mu = (n-r, 1^r), 0 \leq r \leq n-1$, with vertical strip of size m.

The contribution to $p_n p_m$ here is $(-1)^{r+m-1}(n-r, 1^{r+m}), 0 \leq r \leq n-1$.

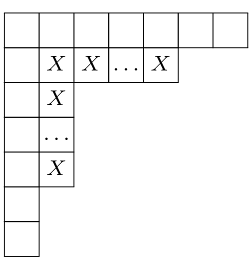

Figure 3: $\mu = (n-r, 1^r), r \geq 1, n-r-1 \geq 1$ with hook $(m-s, 1^s), 0 \leq s \leq m-1$, attached.

The contribution to $p_n p_m$ is
$$(-1)^{r+s}(n-r, m-s+1, 2^s, 1^{r-1-s}), 0 \leq s \leq m-1, 1 \leq r \leq n-2.$$

If $m = 3$ we have a conjugate pair of additional configurations:

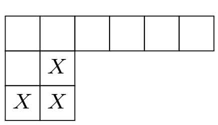

Figure 4a: $\mu = (n-1, 1)$, with rim hook of size $m = 3$.

The contribution to $p_n p_3$ is $(-1)^2(n-1, 2, 2)$.

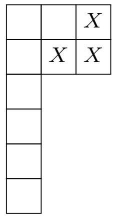

Figure 4b: $\mu = (2, 1^{n-2})$, with rim hook of size $m = 3$.

The contribution to $p_n p_3$ is $(-1)^{n-2} \cdot (-1)(3, 3, 1^{n-3})$.

If $m = 4$ we have six additional configurations, which we list in successive conjugate pairs:

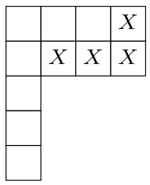

Figure 5a: $\mu = (3, 1^{n-3})$, with rim hook of size $m = 4$.

The contribution to $p_n p_4$ is $(-1)^{n-3} \cdot (-1)(4, 4, 1^{n-4})$.

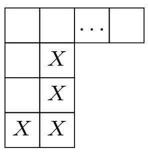

Figure 5b: $\mu = (n-2, 1^2)$, with rim hook of size $m = 4$.

The contribution to $p_n p_4$ is $(-1)^2 \cdot (-1)^2(n-2, 2, 2, 2)$.

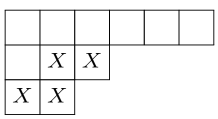

Figure 5c: $\mu = (n-1, 1)$, with rim hook of size $m = 4$.

The contribution to $p_n p_4$ is $(-1)^2(n-1, 3, 2)$.

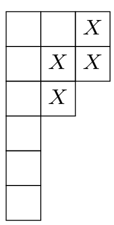

Figure 5d: $\mu = (2, 1^{n-2})$, with rim hook of size $m = 4$.

The contribution to $p_n p_4$ is $(-1)^{n-2} \cdot (-1)^2(3, 3, 2, 1^{n-4})$.

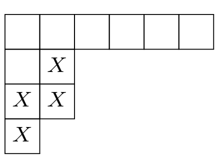

Figure 5e: $\mu = (n-1, 1)$, with rim hook of size $m = 4$.

The contribution to $p_n p_4$ is $(-1) \cdot (-1)^2(n-1, 2, 2, 1)$.

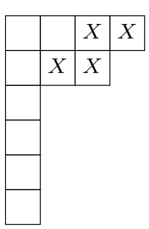

Figure 5f: $\mu = (2, 1^{n-2})$, with rim hook of size $m = 4$.

The contribution to $p_n p_4$ is $(-1)^{n-2} \cdot (-1)(4, 3, 1^{n-4})$.

The sum of hooks in Figures 2a-2b collapses as follows:

$$\sum_{r=0}^{n-1}(-1)^r(n-r+m,1^r)+\sum_{r=0}^{n-1}(-1)^{r+m-1}(n-r,1^{r+m}),$$

and since $n-1\geq m,$ we can split the first sum:

$$= \sum_{r=0}^{m-1}(-1)^r(n-r+m,1^r)+\sum_{r=m}^{n-1}(-1)^r(n-r+m,1^r)+\sum_{t=0}^{n+m-1}(-1)^{t-1}(n+m-t,1^t)$$

$$= \sum_{r=0}^{m-1}(-1)^r(n-r+m,1^r)+\sum_{r=m}^{n-1}(-1)^r(n-r+m,1^r)$$

$$+ \sum_{t=0}^{m-1}(-1)^{t-1}(n+m-t,1^t)+\sum_{t=m}^{n+m-1}(-1)^{t-1}(n+m-t,1^t)$$

$$= \sum_{r=m}^{n-1}(-1)^r(n-r+m,1^r)+\sum_{t=m}^{n+m-1}(-1)^{t-1}(n+m-t,1^t).$$

The first two lines in the statement clearly come from Figures 4 and 5 respectively. The first two summations come from Figures 1a-1b, and the third and fourth summations are the result of the collapse in between Figures 2a and 2b.

Note that in the double hook of Figure 3, we must have $n-r\geq m-s+1\geq 1$ and similarly $r>s\geq 0$. This gives $0\leq s\leq m-1$ and $s<r<n-m+s$. Hence Figure 3 contributes the sum of double hooks

$$D_{n,m} = \sum_{s=0}^{m-1}\sum_{\substack{r\\ s<r<n-m+s}}(-1)^{r+s}(n-r,m-s+1,2^s,1^{r-1-s})$$

$$= \sum_{s=0}^{m-1}\sum_{t=0}^{n-m-2}(-1)^{t+1}(n-t-s-1,m-s+1,2^s,1^t),$$

where we have put $t=r-1-s$.

Figures 1a-1b contribute the sum

$$\delta_{m\geq 2}\left(\sum_{s=0}^{\mathbf{m-2}}(-1)^s(n,m-s,1^s)+\sum_{s=\mathbf{1}}^{m-1}(-1)^{n-s-1}(m-s+1,2^s,1^{n-s-1})\right);$$

putting $\beta(n,m)$ for the first sum above, we see that this can by rewritten as

$$\delta_{m\geq 2}\left(\beta(n,m)+(-1)^{n-m}\omega(\beta(n,m)\right).$$

(Check that $\omega(n,m-s,1^s)=(s+2,2^{m-s-1},1^{n-m+s})$, and make the substitution $t=m-s-1$ to get the second sum multiplied by $(-1)^{n-m}$.)

Figures 2a-2b contribute

$$+\sum_{r=0}^{m-1}(-1)^r(n-r+m,1^r)+\sum_{t=n}^{n+m-1}(-1)^{t-1}(n-t+m,1^t);$$

putting $\alpha(n,m)$ for the first sum of m hooks above, we that this can be rewritten as

$$\alpha(n,m)+(-1)^{n+m}\omega(\alpha(n,m)).$$

Putting $\gamma(n,3) = (n-1, 2^2)$, the contribution of Figures 4a-4b is seen to be

$$\delta_{m=3}\left(\gamma(n,3) + (-1)^{n+m}\omega(\gamma(n,3))\right), \text{ since, when } m=3, (-1)^{n-1} = (-1)^{n+m}$$

and similarly setting $\gamma(n,4) = (n-1,3,2) - (n-2,3,2,1) + (n-2,2^3)$, the contribution of Figures 5a-5f is

$$\delta_{m=4}\left(\gamma(n,4) + (-1)^{n+m}\omega(\gamma(n,4))\right) \text{ again since, when } m=4, (-1)^{n} = (-1)^{n+m}$$

□

We now specialise this lemma to the values $m \le 4$. In what follows it will be convenient to write partitions of n as $(*, \mu)$, where $*$ will indicate a single part equal to $n - |\mu|$, and μ is a partition whose largest part does not exceed the part indicated by $*$. For example, $(*, 2, 2, 1^t)$ means the partition $(n-4-t, 2, 2, 1^t)$.

Lemma 2.13. *One has the following Schur function expansions:*

(1) $p_n = (n) + (-1)^{n-1}(1^n) + \sum_{r=1}^{n-2}(-1)^r(*, 1^r)$ *for* $n \ge 2$ *(the summation is nonzero if and only if* $n \ge 3$*);*

(2) $p_1 p_{n-1} = (n) + (-1)^n(1^n) + \sum_{r=0}^{n-4}(-1)^{r-1}(*, 2, 1^r)$ *for* $n \ge 3$ *(note the summation is nonzero if and only if* $n \ge 4$*);*

(3) $p_{n-2}p_2$ *(for* $n \ge 5$*)*

$$= (n) - (n-1,1) + (n-2,2) + (-1)^n[(1^n) - (2, 1^{n-2}) + (2^2, 1^{n-4})] + \sum_{t=0}^{n-6}(-1)^{t-1}(*, 3, 1^t) + \sum_{t=0}^{n-6}(-1)^{t-1}(*, 2, 2, 1^t).$$

(Note that the summations are nonzero if and only if $n \ge 6$*).*

(4) $p_{n-2}h_2$ *(for* $n \ge 5$*)*

$$= (n) + (-1)^n(2^2, 1^{n-4}) + (-1)^{n-1}(2, 1^{n-2}) + \sum_{r=0}^{n-6}(-1)^{r+1}(*, 3, 1^r).$$

(5) $p_{n-2}p_1^2$ *(for* $n \ge 5$*)*

$$= (n) + (n-1,1) - (n-2,2) - (-1)^n(1^n) + (-1)^n(2^2, 1^{n-4}) + (-1)^{n-1}(2, 1^{n-2}) + \sum_{r=0}^{n-6}(-1)^{r+1}(*, 3, 1^r) + \sum_{t=0}^{n-6}(-1)^t(*, 2, 2, 1^t)$$

(6) $p_{n-3}p_3$ *(for* $n \ge 6$*)*

$$= [(n) - (n-1,1) + (n-2,1^2) + (n-3,3) - (n-3,2,1)] + (n-4, 2^2) + (-1)^n[(1^n) - (2, 1^{n-2}) + (3, 1^{n-3}) + (2^3, 1^{n-6}) - (3, 2, 1^{n-5})] + (-1)^{n-4}(3^2, 1^{n-6}) + \sum_{t=0}^{n-8}(-1)^{t-1}(*, 4, 1^t) + \sum_{t=0}^{n-8}(-1)^{t-1}(*, 3, 2, 1^t) + \sum_{t=0}^{n-8}(-1)^{t-1}(*, 2, 2, 2, 1^t)$$

(7) $p_{n-3}p_2p_1$

$$= (n) - (n-2, 1^2) + (n-3, 2, 1) - (n-4, 2^2) + (-1)^{n-1}[(1^n) - (3, 1^{n-3}) + (3, 2, 1^{n-5}) - (3^2, 1^{n-6})] + \sum_{t=0}^{n-8}(-1)^{t-1}(*, 4, 1^t) + \sum_{r=0}^{n-8}(-1)^r(*, 2, 2, 2, 1^r)$$

(8) $p_{n-3}h_2p_1$

$$= (n) + (n-1,1) - (n-3,3) + (-1)^{n-2}(2,1^{n-2}) + (-1)^{n-1}(2^3,1^{n-6})$$
$$+ (-1)^{n-1}(3,2,1^{n-5}) + (-1)^{n}(3,1^{n-3}) + (-1)^{n-6}(3^2,1^{n-6})$$
$$+ \sum_{r=0}^{n-8}(-1)^{r+1}(*,4,1^r) + \sum_{t=0}^{n-8}(-1)^{t}(*,3,2,1^t)$$

(9) $p_{n-4}p_4$

$$= (n) - (n-1,1) + (n-2,1^2) - (n-3,1^3)$$
$$+ (*,4) - (*,3,1) + (*,2,1^2) + (*,3,2) - (*,2^2,1) + (*,2^3)$$
$$+ (-1)^n[(1^n) - (2,1^{n-2}) + (3,1^{n-3}) - (4,1^{n-4}) + (2^4,1^{n-8})$$
$$- (3,2^2,1^{n-7}) + (4,2,1^{n-6}) + (3^2,2,1^{n-8}) - (4,3,1^{n-7}) + (4^2,1^{n-8})]$$
$$+ \sum_{t=0}^{n-10}(-1)^{t+1}\left[(*,5,1^t) + (*,4,2,1^t) + (*,3,2^2,1^t) + (*,2,2^3,1^t)\right]$$

(10) $p_{n-4}p_3p_1$

$$= (n) - (n-2,2) + (n-3,3) + (n-3,1^3) - (n-4,2,1^2) + (n-5,2^2,1)$$
$$+ (-1)^{n-1}[(1^n) - (2^2,1^{n-4}) + (2^3,1^{n-6}) + (4,3,1^{n-7}) - (4,2,1^{n-6}) + (4,1^{n-4})]$$
$$+ (-1)^n(4^2,1^{n-8}) - (n-6,2^3)$$
$$+ \sum_{t=0}^{n-10}(-1)^{t-1}(*,5,1^t) + \sum_{t=0}^{n-9}(-1)^{t-1}(*,3^2,1^t) + \sum_{t=1}^{n-9}(-1)^{t-1}(*,2^4,1^{t-1})$$

(11) $p_{n-4}p_2p_1^2$

$$= (n) + (n-1,1) - (n-2,1^2) - (n-3,1^3) - (n-4,4)\delta_{n\geq 9} + (n-4,3,1)$$
$$+ (n-4,2,1^2) - (n-5,3,2) - (n-5,2^2,1) + (n-6,2^3)$$
$$+ (-1)^n[(1^n) + (2,1^{n-2}) - (2^4,1^{n-8}) - (3,1^{n-3}) + (3,2^2,1^{n-7}) - (3^2,2,1^{n-8})]$$
$$+ (-1)^n[(-(4,1^{n-4}) + (4,2,1^{n-6}) - (4,3,1^{n-7}) + (4^2,1^{n-8})]$$
$$+ \delta_{n\geq 10}\sum_{t=0}^{n-10}(-1)^{t-1}\{(*,5,1^t) - (*,3,2^2,1^t)\} + \delta_{n\geq 10}\sum_{r=0}^{n-10}(-1)^{r}\{(*4,2,1^r) - (*,2^4,1^r)\}$$

Proof. **Parts (2)-(3)** and **Part (6)**, **Part (9)** follow by putting $m = 1,2,3,4$ in Lemma 2.7, and replacing n with $n-1, n-2, n-3, n-4$ respectively. In **Part (6)**, note that $\gamma(n-3,3) = (n-4,2,2)$ and hence the corresponding term is $(n-4,2,2)+(-1)^{n-4}(3,3,1^{n-6})$.

The expansion of $p_{n-2}p_1^2$ in **Part (5)** follows by subtracting twice the equation in **Part (3)** from **Part (4)**, by virtue of the identity $p_1^2 = 2h_2 - p_2$.

For **Part (4)**:

Write $p_{n-2} = (n-2) + \sum_{r=1}^{n-5}(-1)^r(n-2-r,1^r) + (-1)^{n-4}(2,1^{n-4}) + (-1)^{n-3}(1^{n-2})$.

Using the Pieri rule, we have

$$\begin{aligned}
p_{n-2}h_2 &= (n) + (n-1,1) + (n-2,2) \\
&+ (-1)^{n-4}\left((4,1^{n-4}) + (3,2,1^{n-5}) + (3,1^{n-3}) + (2^2,1^{n-4})\right) \\
&+ (-1)^{n-3}\left((3,1^{n-3}) + (2,1^{n-2})\right) + \sum_{r=1}^{n-5}(-1)^r(n-2-r,3,1^{r-1}) \\
&+ \sum_{r=1}^{n-5}(-1)^r(n-1-r,2,1^{r-1}) + \sum_{r=1}^{n-5}(-1)^r(n-2-r,2,1^r) \quad (F) \\
&+ \sum_{r=1}^{n-5}(-1)^r(n-r,1^r) + \sum_{r=1}^{n-5}(-1)^r(n-1-r,1^{r+1}). \quad (G)
\end{aligned}$$

Line (F) is a telescoping sum which collapses into

$$\sum_{t=0}^{n-6}(-1)^{t+1}(n-2-t,2,1^t) + \sum_{r=1}^{n-5}(-1)^r(n-2-r,2,1^r) = -(n-2,2) + (-1)^{n-5}(3,2,1^{n-5}).$$

Similarly line (G) collapses into

$$\sum_{r=1}^{n-5}(-1)^r(n-r,1^r) + \sum_{t=0}^{n-4}(-1)^{t-1}(n-t,1^t) = -(n-1,1) + (-1)^{n-5}(4,1^{n-4}).$$

Hence $p_{n-2}h_2$ reduces to

$$(n) + \sum_{r=1}^{n-5}(-1)^r(n-2-r,3,1^{r-1}) + (-1)^n(2^2,1^{n-4}) + (-1)^{n-1}(2,1^{n-2}).$$

For **Part (7):** We start with the expression for $p_{n-3}p_2$ and multiply by p_1. One checks that this gives

$$\begin{aligned}
&(n) - (n-2,1^2) + (n-3,3) + (n-3,2,1) \\
&+ (-1)^{n-1}(1^n) + (-1)^{n-1}(2^3,1^{n-6}) + (-1)^{n-2}(3,1^{n-3}) + (-1)^{n-1}(3,2,1^{n-5}) \\
&\sum_{t=0}^{n-7}(-1)^{t+1}[(*,3,1^t) + (*,4,1^t) + (*,3,2,1^{t-1}) + (*,3,1^{t+1})] \quad (A1) \\
&\sum_{t=0}^{n-7}(-1)^{t+1}[(*,2^2,1^t) + (*,3,2,1^t) + (*,2^3,1^{t-1}) + (*,2^2,1^{t+1})] \quad (A2)
\end{aligned}$$

Note that the sums in (A1) and (A2) vanish identically unless $n \geq 7$. The first and last summands in (A1) collapse to $(-(n-3,3) + (-1)^{n-6}(3^2,1^{n-6})$, and similarly the first and last summands in (A2) collapse to $-(n-4,2^2) + (-1)^{n-6}(2^3,1^{n-6})$. Also note that where we have written 1^{t-1} for part 1 with multiplicity $t-1$, there is no contribution unless $t \geq 1$. So the third sum in (A1) and the second sum in (A2) cancel each other.

Likewise, **Parts (8) and (10)** follow respectively from **Parts (4) and (6)**. Finally **Part (11)** follows from **Part (7)**. □

Lemma 2.14. *The expansion of $p_2^2 p_{n-4}$ is as follows:*

(1) The irreducibles appearing with multiplicity ± 2 *are indexed by the following partitions* λ *:*

(a) $(n-2,2)$ *with coefficient 2 and* $(n-3,3)$ *with coefficient* -2;
(b) $(2^2, 1^{n-4})$ *with coefficient* $2(-1)^{n-1}$;
(c) $(2^3, 1^{n-6})$ *with coefficient* $2(-1)^n$;
(d) $(*, 3^2, 1^t)$ *with coefficient* $2(-1)^t$;

The remaining irreducibles (with multiplicity) are:

(2) $(n), (-1)^n(4^2, 1^{n-8}), (-1)^{n-1}(4,3,1^{n-7}), (-1)^n(4,2,1^{n-6}), (-1)^{n-1}(4,1^{n-4}),$ $(-1)^{t-1}(*,5,1^t)$;
(3) $(-1)^{n-1}(1^n), (n-3,1^3), (-1)(n-6,2^3), (-1)(n-4,2,1^2), (n-5,2^2,1), (-1)^t(*,2^4,1^t)$;
(4) $(-1)^n(3,1^{n-3}), (-1)(n-1,1), (-1)^t(3,2^2,1^t), (-1)^t(3^2,2,1^t), (-1)^{t-1}(*,4,2,1^t), (n-4,4)$.
(5) $(-1)(n-2,1^2), (-1)^n(2,1^{n-2}), (-1)^{n-1}(2^4,1^{n-8}), (n-4,3,1), (-1)(n-5,3,2),$ $(-1)^t(*,3,2^2,1^t)$.

Proof. Using the Murnaghan-Nakayama rule, we list the different configurations for the irreducibles appearing in the expansion of $p_2^2 p_{n-4}$. The list below is organised by considering border strips of size $(n-4)$ attached to the shapes appearing in p_2^2 :

(1)

1	1	X	X	...	X
2	2	X			
X	X	X			
...					
...					
X					

or

1	2	X	X	...	X
1	2	X			
X	X	X			
...					
...					
X					

Each of these contributes exactly the same set of shapes λ containing the shape $(2,2)$. We therefore obtain the following possibilities with (signed) multiplicity 2:

$\lambda_2 = 2$ **and** $\lambda_1 = 2$**:** $(-1)^{t-1}(2,2,1^t)$ or $(-1)^t(2,2,2,1^t)$
$\lambda_2 = 2$ **and** $\lambda_1 \geq 3$**:** $(n-2,2)$
$\lambda_2 = 3$ **and** $\lambda_3 = 3$**:** $(-1)^{t+2}(*,3,3,1^t)$
$\lambda_2 = 3$ **and** $\lambda_3 = 0$**:** $(-1)(*,3)$

(2)

1	1	2	2	X	X	...	X
X	X	X	X	X			
X							
...							
X							

$\lambda \not\supset (2,2)$**:** Then λ is a hook, so (n) or $(-1)^{n-1}(4,1^{n-4})$.
$\lambda \supset (2,2)$**:** If $\lambda_2 = 4$ we obtain $(-1)^t(4,4,1^t)$, $(-1)^t(4,3,1^t)$, $(-1)^t(4,2,1^t)$; finally if $\lambda_2 = 5$ we obtain $(-1)^{t+1}(*,5,1^t)$.

(3)

1	X	X	...	X
1	X			
2	X			
2	X			
X				
...				
X				

$\lambda \not\supset (2,2)$**:** Then λ is a hook, so $(n-3,1^3)$ or $(-1)^{n-5}(1^n)$.
$\lambda \supset (2,2)$**:** The possibilities are: $(-1)(*,2,1^2), (-1)^2(*,2^2,1), (-1)^3(*,2^3)$ and $(-1)^t(*,2^4,1^t)$.

(4)
$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & X & \dots & X \\ \hline 1 & X & X & X \\ \cline{1-4} X & X \\ \cline{1-2} X \\ \cline{1-1} \dots \\ \cline{1-1} X \\ \cline{1-1} \end{array}$$

$\lambda \not\supset (2,2)$: Then $\lambda_2 = 1$, so λ is a hook, $(-1)\cdot(-1)^{n-3}(3, 1^{n-3})$ or $(-1)\cdot(n-1, 1)$.

$\lambda \supset (2,2)$: Then $2 \le \lambda_2 \le 4$. If $\lambda_2 = 2$ then $(-1) \cdot (-1)^{t+1}(3, 2^2, 1^t)$; if $\lambda_2 = 3$ then $(-1) \cdot (-1)^{t+1}(3, 3, 2, 1^t)$; if $\lambda_2 = 4$ then $(-1) \cdot (-1)^{t+2}(*, 4, 2, 1^t)$ or $(-1) \cdot (-1)(n-4, 4)$.

(5)
$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & X & X & \dots & X \\ \hline 2 & X & X \\ \cline{1-3} 2 & X \\ \cline{1-2} X \\ \cline{1-1} \dots \\ \cline{1-1} X \\ \cline{1-1} \end{array}$$

$\lambda \not\supset (2,2)$: Thus $\lambda_2 = 1$, yielding $(-1)\cdot(n-2, 1^2)$ or $(-1)\cdot(-1)^{n-5}(2, 1, 1, 1^{n-4})$.

$\lambda \supset (2,2)$: If $\lambda_2 = 2$ we have $(-1) \cdot (-1)^{n-6}(2, 2, 2, 2, 1^{n-8})$.
If $\lambda_2 = 3$ we have $(-1) \cdot (-1)(*, 3, 1), (-1) \cdot (-1)^2(*, 3, 2)$, and $(-1) \cdot (-1)^{t+3}(*, 3, 2^2, 1^t)$.

□

Using Lemmas 2.13-2.14, we can compute partial sums in the reverse lexicographic order, starting from the top element (n).

Proposition 2.15. *The partial sums beginning with the top element (n) are:*

(1) $p_n = (n) + (-1)^{n-1}(1^n) - (n-1, 1) + (-1)^{n-2}(2, 1^{n-2}) + \sum_{r=2}^{n-3}(-1)^r(*, 1^r)$ *for* $n \ge 2$

(2) $\sum_{(n)\ge\mu\ge(n-1,1)} p_\mu = p_n + p_{n-1}p_1 = 2s_{(n)} + \sum_{r=1}^{n-2}(-1)^r(*, 1^r) + \sum_{r=0}^{n-4}(-1)^{r-1}(*, 2, 1^r)$.

(3) $\sum_{(n)\ge\mu\ge(n-2,2)} p_\mu = p_n + p_{n-1}p_1 + p_{n-2}p_2$
$$= 3(n) - 2(n-1, 1) + (-1)^n(1^n) + \sum_{r=2}^{n-3}(-1)^r(*, 1^r) + \sum_{r=1}^{n-5}(-1)^{r-1}(*, 2, 1^r)$$
$$+ \sum_{t=0}^{n-6}(-1)^{t+1}(*, 3, 1^t) + \sum_{t=0}^{n-6}(-1)^{t+1}(*, 2, 2, 1^t)$$

(4) $\sum_{(n)\ge\mu\ge(n-2,1^2)} p_\mu = p_n + p_{n-1}p_1 + p_{n-2}p_2 + p_{n-2}p_1^2$ *($n \ge 6$)*
$$= 4(n) + (-1)^{n-1}(2, 1^{n-2}) + (-1)^n(2^2, 1^{n-4}) + \sum_{r=1}^{n-3}(-1)^r(*, 1^r)$$
$$+ \sum_{r=0}^{n-5}(-1)^{r-1}(*, 2, 1^r) + 2 \cdot \sum_{r=0}^{n-6}(-1)^{r+1}(*, 3, 1^r)$$

(5) $\sum_{(n)\ge\mu\ge(n-3,3)} p_\mu = p_n + p_{n-1}p_1 + p_{n-2}p_2 + p_{n-2}p_1^2 + p_{n-3}p_3$

$$= 5(n) - 2(n-1, 1) - (n-2, 2) + 2(n-2, 1^2) - (n-3, 3) + (n-4, 2^2)$$
$$+ (-1)^n(1^n) + 2(-1)^{n-1}(2, 1^{n-2}) + (-1)^n(2^2, 1^{n-4}) + (-1)^n(2^3, 1^{n-6}) + (-1)^{n-1}(3^2, 1^{n-6})$$
$$+ \sum_{r=\mathbf{3}}^{n-4}(-1)^r(*, 1^r) + \sum_{r=\mathbf{2}}^{\mathbf{n-6}}(-1)^{r-1}(*, 2, 1^r) + 2\sum_{r=\mathbf{1}}^{n-7}(-1)^{r+1}(*, 3, 1^r) + \mathbf{2}\delta_{\mathbf{n=6}}(*, \mathbf{3})$$
$$+ \sum_{t=0}^{n-8}(-1)^{t+1}(*, 4, 1^t) + \sum_{t=0}^{n-8}(-1)^{t+1}(*, 3, 2, 1^t) + \sum_{t=0}^{n-8}(-1)^{t+1}(*, 2^3, 1^t)$$

(6) $\sum_{(n)\geq\mu\geq(n-3,2,1)} p_\mu$

$$\begin{aligned}
&= 6(n) - 2(n-1,1) - (n-2,2) + (n-2,1^2) - (n-3,3) + (n-3,2,1)\\
&+ 2(-1)^{n-1}(2,1^{n-2}) + (-1)^n(2^2,1^{n-4}) + (-1)^n(2^3,1^{n-6})\\
&+ (-1)^n(3,1^{n-3}) + (-1)^{n-1}(3,2,1^{n-5})\\
&+ \sum_{r=3}^{n-4}(-1)^r(*,1^r) + \sum_{r=2}^{n-6}(-1)^{r-1}(*,2,1^r) + 2\sum_{r=1}^{n-7}(-1)^{r+1}(*,3,1^r) + \mathbf{2}\delta_{\mathbf{n=6}}(*,\mathbf{3})\\
&+ 2\sum_{t=0}^{n-8}(-1)^{t-1}(*,4,1^t) + \sum_{t=0}^{n-8}(-1)^{t+1}(*,3,2,1^t)
\end{aligned}$$

(7) $\sum_{(n)\geq\mu\geq(n-3,1^3)} p_\mu$

$$\begin{aligned}
&= 7(n) - (n-2,2) - 3(n-3,3) + 2(n-2,1^2) + (n-4,2^2)\\
&+ (-1)^n(1^n) + (-1)^n(2^2,1^{n-4}) + (-1)^{n-1}(2^3,1^{n-6})\\
&+ 2(-1)^{n-1}(3,2,1^{n-5}) + 2(-1)^n(3,1^{n-3}) + (-1)^n(3^2,1^{n-6})\\
&+ \sum_{r=3}^{n-4}(-1)^r(*,1^r) + \sum_{r=2}^{n-6}(-1)^{r-1}(*,2,1^r) + 2\sum_{r=1}^{n-7}(-1)^{r+1}(*,3,1^r)\\
&+ 3\sum_{t=0}^{n-8}(-1)^{t+1}(*,4,1^t) + \sum_{t=0}^{n-8}(-1)^t(*,3,2,1^t) + \sum_{t=0}^{n-8}(-1)^{t+1}(*,2^3,1^t).
\end{aligned}$$

(8) $\sum_{(n)\geq\mu\geq(n-4,4)} p_\mu$

$$\begin{aligned}
&= 8(n) - (n-1,1) - (n-2,2) + 3(n-2,1^2) - 3(n-3,3) - 2(n-3,1^3)\\
&- 2(n-4,4) + (n-4,3,1) + (n-4,2^2) + 2(n-5,3,2) - (n-5,2^2,1)\\
&+ (-1)^n 2(1^n) + (-1)^{n-1}(2,1^{n-2}) + (-1)^n(2^2,1^{n-4}) + (-1)^{n-1}(2^3,1^{n-6}) + (-1)^n(2^4,1^{n-8})\\
&+ 2(-1)^{n-1}(3,2,1^{n-5}) + 3(-1)^n(3,1^{n-3}) + (-1)^n(3^2,1^{n-6}) + (-1)^n(3^2,2,1^{n-8})\\
&+ (-1)^{n-1}(3,2^2,1^{n-7}) + (-1)^{n-1}(4,3,1^{n-7}) + (-1)^n(4^2,1^{n-8})\\
&+ \sum_{r=\mathbf{4}}^{\mathbf{n-5}}(-1)^r(*,1^r) + \sum_{r=\mathbf{3}}^{\mathbf{n-7}}(-1)^{r-1}(*,2,1^r)\\
&+ 2\sum_{r=\mathbf{2}}^{n-7}(-1)^{r+1}(*,3,1^r) + 3\sum_{t=\mathbf{1}}^{n-8}(-1)^{t+1}(*,4,1^t) + \sum_{t=\mathbf{1}}^{n-8}(-1)^t(*,3,2,1^t) + \sum_{t=\mathbf{1}}^{n-8}(-1)^{t+1}(*,2^3,1^t)\\
&+ \sum_{t=0}^{n-10}(-1)^{t+1}\left[(*,5,1^t) + (*,4,2,1^t) + (*,3,2^2,1^t) + (*,2^4,1^t)\right]
\end{aligned}$$

(9) $\sum_{(n)\geq\mu\geq(n-4,3,1)} p_\mu$

$$= 9(n) - (n-1,1) - 2(n-2,2) + 3(n-2,1^2) - 2(n-3,3) - (n-3,1^3)$$
$$- 2(n-4,4) + (n-4,3,1) + (n-4,2^2) - (n-4,2,1^2) + 2(n-5,3,2) - (n-6,2^3)$$
$$+ (-1)^n(1^n) + (-1)^{n-1}(2,1^{n-2}) + 2(-1)^n(2^2,1^{n-4}) + 2(-1)^{n-1}(2^3,1^{n-6})$$
$$+ (-1)^n(2^4,1^{n-8})$$
$$+ 2(-1)^{n-1}(3,2,1^{n-5}) + 3(-1)^n(3,1^{n-3}) + (-1)^n(3^2,1^{n-6}) + (-1)^n(3^2,2,1^{n-8})$$
$$+ (-1)^{n-1}(3,2^2,1^{n-7})$$
$$+ 2(-1)^{n-1}(4,3,1^{n-7}) + 2(-1)^n(4^2,1^{n-8}) + (-1)^n(4,2,1^{n-6}) + (-1)^{n-1}(4,1^{n-4})$$
$$+ \sum_{r=\mathbf{4}}^{\mathbf{n-5}}(-1)^r(*,1^r) + \sum_{r=\mathbf{3}}^{\mathbf{n-7}}(-1)^{r-1}(*,2,1^r)$$
$$+ 2\sum_{r=\mathbf{2}}^{n-7}(-1)^{r+1}(*,3,1^r) + 3\sum_{t=\mathbf{1}}^{n-8}(-1)^{t+1}(*,4,1^t) + \sum_{t=\mathbf{1}}^{n-8}(-1)^t(*,3,2,1^t)$$
$$+ \sum_{t=\mathbf{1}}^{n-8}(-1)^{t+1}(*,2^3,1^t)$$
$$+ \sum_{t=0}^{n-10}(-1)^{t+1}\left[2(*,5,1^t) + (*,4,2,1^t) + (*,3,2^2,1^t)\right] + \sum_{t=0}^{n-9}(-1)^{t-1}(*,3^2,1^t)$$

(10) $\sum_{(n)\geq\mu\geq(n-4,2^2)} p_\mu$

$$= 10(n) - 2(n-1,1) + 2(n-2,1^2) - 4(n-3,3)$$
$$- (n-4,4) + 2(n-4,3,1) + (n-4,2^2) - 2(n-4,2,1^2)$$
$$+ (n-5,3,2) + (n-5,2^2,1) - 2(n-6,2^3) + (-1)^n 4(3,1^{n-3})$$
$$+ 2(-1)^{n-1}(3,2,1^{n-5}) + (-1)^n(3^2,1^{n-6}) + \mathbf{3}(-1)^n(3^2,2,1^{n-8}) + 2(-1)^{n-1}(3,2^2,1^{n-7})$$
$$+ (-1)^{n-1}(4,3,1^{n-7}) + \mathbf{0}(-1)^n(4^2,1^{n-8}) + (-1)^n\mathbf{2}(4,2,1^{n-6}) + 2(-1)^{n-1}(4,1^{n-4})$$
$$+ \sum_{r=\mathbf{4}}^{\mathbf{n-5}}(-1)^r(*,1^r) + \sum_{r=\mathbf{3}}^{\mathbf{n-7}}(-1)^{r-1}(*,2,1^r) + 2\sum_{r=\mathbf{2}}^{\mathbf{n-8}}(-1)^{r+1}(*,3,1^r)$$
$$+ 3\sum_{t=\mathbf{1}}^{\mathbf{n-9}}(-1)^{t+1}(*,4,1^t)$$
$$+ \sum_{t=\mathbf{1}}^{\mathbf{n-9}}(-1)^t(*,3,2,1^t) + \sum_{t=\mathbf{1}}^{n-8}(-1)^{t+1}(*,2^3,1^t) + \sum_{t=0}^{n-10}(-1)^t(*,2^4,1^t)$$
$$+ \sum_{t=0}^{n-10}(-1)^{t-1}\left[\mathbf{3}(*,5,1^t) + \mathbf{2}(*,4,2,1^t)\right] + \sum_{t=0}^{n-9}(-1)^t(*,3^2,1^t)$$

(11) $\sum_{(n)\geq\mu\geq(n-4,2,1^2)} p_\mu$

$$= 11(n) - (n-1,1) + (n-2,1^2) - 4(n-3,3) - (n-3,1^3)$$
$$- (1+\delta_{n\geq 9})(n-4,4) + 3(n-4,3,1) + (n-4,2^2) - (n-4,2,1^2)$$
$$- (n-6,2^3) + (-1)^n(1^n) + (-1)^n(2,1^{n-2}) + \mathbf{2}(-1)^{n-1}(2^4,1^{n-8})$$
$$+ (-1)^n 3(3,1^{n-3}) + 2(-1)^{n-1}(3,2,1^{n-5}) + (-1)^n(3^2,1^{n-6}) + \mathbf{2}(-1)^n(3^2,2,1^{n-8})$$
$$+ (-1)^{n-1}(3,2^2,1^{n-7})$$
$$+ \mathbf{2}(-1)^{n-1}(4,3,1^{n-7}) + (-1)^n(4^2,1^{n-8}) + (-1)^n\mathbf{3}(4,2,1^{n-6}) + (-1)^{n-1}\mathbf{3}(4,1^{n-4})$$
$$+ \sum_{\mathbf{r=4}}^{\mathbf{n-5}}(-1)^r(*,1^r) + \sum_{\mathbf{r=3}}^{\mathbf{n-7}}(-1)^{r-1}(*,2,1^r) + 2\sum_{\mathbf{r=2}}^{\mathbf{n-8}}(-1)^{r+1}(*,3,1^r) + 3\sum_{\mathbf{t=1}}^{\mathbf{n-9}}(-1)^{t+1}(*,4,1^t)$$
$$+ \sum_{\mathbf{t=1}}^{\mathbf{n-9}}(-1)^t(*,3,2,1^t) + \sum_{\mathbf{t=1}}^{\mathbf{n-9}}(-1)^{t+1}(*,2^3,1^t)$$
$$+ \sum_{t=0}^{n-10}(-1)^{t-1}\left[4(*,5,1^t) + \mathbf{1}(*,4,2,1^t) - (*,3,2^2,1^t)\right] + \sum_{t=0}^{n-9}(-1)^t(*,3^2,1^t)$$

Proof. We sketch the proof, since the details are routine and tedious. In general, each partial sum is obtained from the preceding one by adding the expansion of the power sum indexed by the appropriate partition. More precisely, if μ^+ covers μ in reverse lexicographic order, then $\psi_{\mu^+} = \psi_\mu + p_{\mu^+}$.

Thus the first two partial sums follow by adding the first two sums in Lemma 2.13. A similar procedure is applied for the remaining sums, with the following exceptions.

For (3), we compute the partial sum by using $p_2 + p_1^2 = 2h_2$ and thus it suffices to add the expansion of $2h_2p_{n-2}$ from Lemma 2.13 to the preceding partial sum.

Similarly for (7), we compute the sum $\sum_{(n)\geq\mu\geq(n-3,1^3)} p_\mu$ by adding $p_{n-3}(p_2p_1 + p_1^3) = 2p_{n-3}h_2p_1$ (using the expansion (8) of Lemma 2.13), to $\sum_{(n)\geq\mu\geq(n-3,3)} p_\mu$, which is given in (5) above.

In general, in all cases we use the relevant computations of Lemma 2.13, the chief exception being the expression in (10), which requires the expansion of $p_{n-4}p_2^2$ computed in Lemma 2.14. The expression (11) of this proposition then follows cumulatively using **Part (11)** of Lemma 2.13.

It is important to note that the sums have been carefully rewritten so that there is no "collapsing": as an example, we give here an analysis of what happens in computing the partial sum (10). This sum is obtained by adding to the sum in (9) the expansion for $p_{n-4}p_2^2$, which from Lemma 2.14 is

$$(n)-(n-1,1)+2(n-2,2)-(n-2,1^2)-2(n-3,3)+(n-3,1^3)$$
$$+(n-4,4)+(n-4,3,1)-(n-4,2,1^2)-(n-5,3,2)+(n-5,2^2,1)-(n-6,2^3)$$
$$+(-1)^{n-1}[(1^n)-(2,1^{n-2})+2(2^2,1^{n-4})-2(2^3,1^{n-6})+(2^4,1^{n-8})]$$
$$+(-1)^n(3,1^{n-3})+(-1)^{n-1}(3,2^2,1^{n-7})+(-1)^n(3^2,2,1^{n-8})$$
$$+(-1)^n(4^2,1^{n-8})+(-1)^{n-1}(4,3,1^{n-7})+(-1)^n(4,2,1^{n-6})+(-1)^{n-1}(4,1^{n-4})$$
$$+\textstyle\sum_{t=0}^{n-10}(-1)^t[(*,2^4,1^t)+(*,3,2^2,1^t)]+\sum_{t=0}^{n-10}(-1)^{t-1}[(*,5,1^t)+(*,4,2,1^t)]$$
$$+2\textstyle\sum_{t=0}^{n-9}(-1)^t(*,3^2,1^t)$$

Adding this to (9) of Proposition 2.15 produces

$\sum_{(n)\geq\mu\geq(n-4,2^2)} p_\mu$

$=10(n)-2(n-1,1)+2(n-2,1^2)-4(n-3,3)$

$-(n-4,4)+2(n-4,3,1)+(n-4,2^2)-2(n-4,2,1^2)$

$+(n-5,3,2)+(n-5,2^2,1)-2(n-6,2^3)+(-1)^n 4(3,1^{n-3})$

$+2(-1)^{n-1}(3,2,1^{n-5})+(-1)^n(3^2,1^{n-6})+(A1)\ 2(-1)^n(3^2,2,1^{n-8})+2(-1)^{n-1}(3,2^2,1^{n-7})$

$+(A2)\ \mathbf{3}(-1)^{n-1}(4,3,1^{n-7})+(A3)\ \mathbf{3(-1)^n(4^2,1^{n-8})}+(-1)^n\mathbf{2}(4,2,1^{n-6})+2(-1)^{n-1}(4,1^{n-4})$

$+\sum_{r=\mathbf{4}}^{\mathbf{n-5}}(-1)^r(*,1^r)+\sum_{r=\mathbf{3}}^{\mathbf{n-7}}(-1)^{r-1}(*,2,1^r)+(A2)\ 2\sum_{r=\mathbf{2}}^{n-7}(-1)^{r+1}(*,3,1^r)+(A3)\ \mathbf{3}\sum_{\mathbf{t=1}}^{\mathbf{n-8}}(\mathbf{-1})^{\mathbf{t+1}}(*,\mathbf{4},\mathbf{1^t})$

$+(A1)\ \sum_{t=\mathbf{1}}^{n-8}(-1)^t(*,3,2,1^t)+\sum_{t=\mathbf{1}}^{n-8}(-1)^{t+1}(*,2^3,1^t)+\sum_{t=0}^{n-10}(-1)^t(*,2^4,1^t)$

$+\sum_{t=0}^{n-10}(-1)^{t-1}\left[\mathbf{3}(*,5,1^t)+\mathbf{2}(*,4,2,1^t)\right]+\sum_{t=0}^{n-9}(-1)^t(*,3^2,1^t)$

The items of matching colour (or matching labels, in the absence of colour) can be combined as follows:

$(A2)\ \mathbf{3}(-1)^{n-1}(4,3,1^{n-7})+2\sum_{r=\mathbf{2}}^{n-7}(-1)^{r+1}(*,3,1^r)=(-1)^{n-1}(4,31^{n-7})+2\sum_{r=\mathbf{2}}^{n-8}(-1)^{r+1}(*,3,1^r)$;

$(\mathbf{A3})\ \mathbf{3(-1)^n(4^2,1^{n-8})}+\mathbf{3}\sum_{\mathbf{t=1}}^{\mathbf{n-8}}(\mathbf{-1})^{\mathbf{t+1}}(*,\mathbf{4},\mathbf{1^t})=\mathbf{3}\sum_{\mathbf{t=1}}^{\mathbf{n-9}}(\mathbf{-1})^{\mathbf{t+1}}(*,\mathbf{4},\mathbf{1^t})$;

$(A1)\ 2(-1)^n(3^2,2,1^{n-8})+\sum_{t=\mathbf{1}}^{n-8}(-1)^t(*,3,2,1^t)=3(-1)^n(3^2,2,1^{n-8})+\sum_{t=\mathbf{1}}^{n-9}(-1)^t(*,3,2,1^t)$

Making these replacements finally yields the completely reduced expression (10).

Likewise, the reduced expression (11) is obtained by adding to (10) the expansion of $p_{n-4}p_2p_1^2$. There is only one pair that recombines into one term here, namely

$(-1)^{n-1}(2^4,1^{n-8})+\sum_{t=1}^{n-8}(-1)^{t+1}(*,2^3,1^t)=2(-1)^{n-1}(2^4,1^{n-8})+\sum_{t=1}^{n-9}(-1)^{t+1}(*,2^3,1^t)$.

□

We can now deduce the positivity of the functions ψ_μ for $\mu\geq(n-4,1^4)$.

Theorem 2.16. *Let $n\geq 6$. Let μ be a partition in the interval $[(n-4,1^{n-4}),(n)]$ in the reverse lexicographic order on partitions of n. Then ψ_μ is Schur-positive.*

Proof. It is clear from the definition that

$$\psi_\mu=\psi_n-\sum_{\nu\vdash n:\mu<\nu\leq(n)} p_\nu. \tag{2.8}$$

We use the partial sum computations in Proposition 2.15. Observe that, in each of those expansions, no Schur function appears with multiplicity greater than +4, except for the trivial representation, which appears with multiplicity equal to the number of partitions in the interval $(\mu,(n)]$.

For example, for $\psi_{(n-4,3,1)}$ we see from (8) that we need to subtract from ψ_n a virtual representation in which no multiplicity in the sum exceeds +3, other than the multiplicity of the trivial representation which is now 8.

Similarly to obtain $\psi_{(n-4,2^2)}$, from (9) it follows that we subtract from ψ_n a representation in which no multiplicity in the sum exceeds +3, other than the multiplicity of the trivial representation which is now 9.

In fact the largest multiplicity (in absolute value) of +4 is obtained for the first time in the penultimate sum (10) of Proposition 2.15, $\sum_{(n)\geq\mu\geq(n-4,2^2)} p_\mu$ (for the two irreducibles $(n-3,3),(3,1^{n-3})$). In the last sum of Proposition 2.15, viz. $\sum_{(n)\geq\mu\geq(n-4,2,1^2)} p_\mu$, again the largest multiplicity in absolute value is 4, and this multiplicity occurs several times.

But Lemma 2.6 guarantees that ψ_n has multiplicity at least 4 for each irreducible except the trivial module, and hence, examining the partial sums in Proposition 2.15, it is clear that the right-hand side is Schur-positive in all the cases enumerated. The fact that all the expressions are reduced (no further simplification occurs) is important in this argument. The multiplicity of the trivial representation is the partition number $p(n)$, which is certainly at least the length of the interval $(\mu,(n)]$. The theorem is proved. □

Together Theorems 2.11 and 2.16 complete the proof of Theorem 1.3.

It is difficult to see how to generalise this argument. Already for $n=8,9,10$, computation with Maple shows that in the Schur function expansion of the sum $\sum_{\lambda\geq(n-4,1^4)} p_\lambda$, $s_{(n-3,3)}$ occurs with multiplicity -6; $s_{(n-4,4)}$ occurs with multiplicity -5 when $n=9,10$, and $s_{(n-4,3,1)}$ occurs with multiplicity -5 for $n=8$. The lower bound that we were able to establish in Lemma 2.6 is therefore insufficient to guarantee Schur positivity of ψ_μ by these arguments, in the case when μ is strictly below $(n-4,1^4)$ in reverse lexicographic order.

From Theorem 2.11 and Proposition 2.4 it is also easy to derive the following information about the multiplicity of the sign representation. Clearly if μ and ν are consecutive partitions in reverse lexicographic order, this multiplicity differs by 1 in absolute value, since $\psi_\mu - \psi_\nu = \pm p_\nu$. It is also clear that the multiplicity of the trivial representation decreases by one as we descend the chain from (n) to (1^n).

Corollary 2.17. *We have*

(1) $\langle \psi_n, s_{(1^n)}\rangle = \langle \psi_{(n-2,2)}, s_{(1^n)}\rangle =$ *the number of partitions of self-conjugate partitions of* n.
(2) $\langle \psi_{(n-1,1)}, s_{(1^n)}\rangle = \langle \psi_n, s_{(1^n)}\rangle + (-1)^n$;
(3) $\langle \psi_{(n-2,1^2)}, s_{(1^n)}\rangle = \langle \psi_n, s_{(1^n)}\rangle - (-1)^n$;
(4) $\langle \psi_{(n-3,3)}, s_{(1^n)}\rangle = \langle \psi_n, s_{(1^n)}\rangle$.
(5) $\langle \psi_{(2^k,1^{n-2k})}, s_{(1^n)}\rangle = (-1)^k + \langle \psi_{(2^{k-1},1^{n-2k+2})}, s_{(1^n)}\rangle, \quad 1\leq k\leq \lfloor \frac{n}{2}\rfloor$.
(6) $\langle \psi_{(3,1^{n-3})}, s_{(1^n)}\rangle = 1 + \langle \psi_{(2^k,1^{n-2k})}, s_{(1^n)}\rangle$, *where* $k=\lfloor \frac{n}{2}\rfloor$

In Example 1.4, we have underlined the increasing runs, and italicised the decreasing runs, of length 3 or more, in the multiplicity $\langle \psi_\mu, s_{(1^n)}\rangle$. It is unclear how to predict the runs of 1's and (-1)'s, i.e. the increasing and decreasing sequences in the partial sums. The longest such run in the examples occurs in S_{13}, namely $1,2,3,4$. The corresponding partitions are

$$[6,5,2] < [6,6,1] < [7,1^6],$$

all with sign $+1$.

Schur positivity also holds for the following (unsaturated) chain of partitions in reverse lexicographic order.

Proposition 2.18. *Let* $T_n = \{\lambda \vdash n : \lambda = (n-r,1^r), 0\leq r\leq n-1\}$. *Then* $Hk_n = \sum_{\mu\in T_n} p_\mu$ *is Schur-positive. In fact* Hk_n *contains all irreducibles unless* n *is even, in which case only the irreducible indexed by* (1^n) *does not appear.*

Proof. Clearly $Hk_n = p_n + p_1 Hk_{n-1}$. Since $p_n = \sum_{r=0}^n (-1)^r s_{(n-r,1^r)}$, by Frobenius reciprocity, denoting by $\langle,\rangle$ the inner product on the ring of symmetric functions for which the Schur functions form an orthonormal basis, we have

$$\langle Hk_n, s_\lambda\rangle = \begin{cases} \langle Hk_{n-1}, s_{\lambda/(1)}\rangle, & \text{if } \lambda \text{ is not a hook;} \quad (A)\\ \langle Hk_{n-1}, s_{(1^{n-1})}\rangle + (-1)^{n-1}, & \lambda=(1^n); \quad (B)\\ \langle Hk_{n-1}, s_{(n-1)}\rangle + 1, & \lambda=(n); \quad (C)\\ \langle Hk_{n-1}, s_{(n-r-1,1^r)}\rangle & \\ +\langle Hk_{n-1}, s_{(n-r,1^{r-1})}\rangle + (-1)^r, & \lambda=(n-r,1^r), 1\le r\le n-2. \quad (D)\end{cases}$$

We verify that $Hk_1 = s_{(1)}$, $Hk_2 = p_2 + p_1^2 = 2s_{(2)}$, $Hk_3 = \psi_3^C = 3s_{(3)} + s_{(2,1)} + s_{(1^3)}$, $Hk_4 = 4s_{(4)} + 3s_{(3,1)} + 3s_{(2,1^2)} + s_{(2,2)}$, $Hk_5 = 5s_{(5)} + 6s_{(4,1)} + 7s_{(3,1^2)} + 2s_{(2,1^3)} + s_{(1^5)} + 4s_{(3,2)} + 4s_{(2^2,1)}$.

First we claim that

(1) $\langle Hk_n, s_{(1^n)}\rangle = \begin{cases} 1, & n \text{ odd;}\\ 0, & \text{otherwise.}\end{cases}$

(2) $\langle Hk_n, s_{(n)}\rangle = n$ for all n.

(3) $\langle Hk_{n-1}, s_{(n-r,1^r)}\rangle \ge 1$ for all r.

Claims (1) and (2) are immediate by an easy induction from (A) and (B) above.

We will show that claim (3) also follows by induction. It is clearly true for $r=0$. For $r=1$, we have, using (2), the recurrence

$$\langle s_{(n-1,1)}, Hk_n\rangle - \langle s_{(n-2,1)}, Hk_{n-1}\rangle = (n-2),$$

and hence, since $\langle s_{(2,1)}, Hk_3\rangle = 1$,

$$\langle s_{(n-1,1)}, Hk_n\rangle = \textstyle\sum_{r=2}^{n-1}(n-r) = \binom{n-1}{2} \ge 3 \text{ if } n\ge 4.$$

Similarly we have, for $r=2$,

$$\langle s_{(n-2,1^2)}, Hk_n\rangle - \langle s_{(n-3,1^2)}, Hk_{n-1}\rangle = (-1)^2 + \langle s_{(n-2,1)}, Hk_{n-1}\rangle = 1 + \binom{n-2}{2} \text{ if } n\ge 4.$$

Taking this recurrence down to the last line, namely

$$\langle s_{(2,1^2)}, Hk_4\rangle - \langle s_{(1,1^2)}, Hk_3\rangle = (-1)^2 + \langle s_{(2,1)}, Hk_3\rangle,$$

we have,

$$\langle s_{(n-2,1^2)}, Hk_n\rangle = \langle s_{(1,1^2)}, Hk_3\rangle + (n-3) + \textstyle\sum_{j=2}^{n-2}\binom{j}{2} = n-2+\binom{n-1}{3} \ge 3, \text{ if } n\ge 4.$$

Our induction hypothesis will be that $\langle s_{(n-r,1^r)}, Hk_n\rangle \ge 3$ for some r such that $r \le n-2$. This has now been verified for $r=0,1,2$. Then it follows from (D) above, using the same telescoping sum, that

$$\langle s_{(n-r,1^r)}, Hk_n\rangle - \langle s_{(n-(r-2),1^{r-(r-2)})}, Hk_{n-r+2}\rangle \ge \\ \langle s_{(n-(r-1),1^{r-(r-1)})}, Hk_{n-r+1}\rangle + \textstyle\sum_{i=2}^{r}(-1)^i \ge 0 \text{ since } n\ge r+2,$$

and hence

$$\langle s_{(n-r,1^r)}, Hk_n\rangle \ge \langle s_{(n-(r-2),1^2)}, Hk_{n-r+2}\rangle \ge 3$$

by induction hypothesis.

This establishes the induction step and hence claim (3). In view of (A) above, the positivity of the multiplicities of hooks in Hk_n implies that Hk_n is Schur-positive for all n. The last statement is clear. □

The representations Hk_n give rise to an interesting family of nonnegative integers. Let $a_{n,r} = \langle s_{(n-r,1^r)}, Hk_n \rangle, 0 \le r \le n-1$.

Using $Hk_n = p_n + p_1 Hk_{n-1}$, from (D) in Proposition 2.18 we have the recurrence

$$a_{n,r} = (-1)^r + a_{n-1,r} + a_{n-1,r-1}, 1 \le r \le n-2, n \ge 2$$

and $a_{n,0} = n, a_{n,n-1} = \delta_{n \text{ odd}}$ for all n. Thus $a_{n,1} - a_{n-1,1} = n-2$, giving $a_{n,1} = \binom{n-1}{2}$ for $n \ge 2$.

Table 1: $a_{n,r}$, row $n \ge 1$, column $r \ge 0$

1											
2	0										
3	1	1									
4	3	3	0								
5	6	7	2	1							
6	10	14	8	4	0						
7	15	25	21	13	3	1					
8	21	41	45	35	15	5	0				
9	28	63	85	81	49	21	4	1			
10	36	92	147	167	129	71	24	6	0		
11	45	129	238	315	295	201	94	31	5	1	
12	55	175	366	554	609	497	294	126	35	7	0

Define $b_{n,r} = \langle s_{n-r,r}, Hk_n \rangle, 0 \le r \le \frac{n}{2}$. Then (A) gives

$$b_{n,r} = b_{n-1,r} + b_{n-1,r-1}, r \ge 2, \quad b_{n,1} = a_{n,1} = \binom{n-1}{2}.$$

Let Lie_n denote the S_n-module obtained by inducing a primitive nth root of unity from the cyclic subgroup generated by an n-cycle up to S_n. It is a well-known fact that Lie_n is also the representation of the symmetric group acting on the multilinear component of the free Lie algebra ([5, Ex. 7.88-89]). Write Lie for $\sum_{n\ge 1} \mathrm{ch}\, Lie_n$. Recall from the Introduction that we denote by f_n the conjugacy action on the n-cycles, and that $f[g]$ denotes plethysm. The functions Hk_n satisfy an interesting plethystic identity. In order to establish this, we need the following connection between Lie and the conjugacy action. In keeping with the notation of [8], we will write $Conj_n$ for f_n in the remainder of this section.

Proposition 2.19. [8, Proposition 6.6]

$$\sum_{m\ge 1} p_m[Lie] = \sum_{n\ge 1} Conj_n.$$

Proposition 2.20. *Let W_n be the representation with characteristic Hk_n. Then W_n satisfies the following properties:*

(1) The restriction $W_{n+1} \downarrow_{S_n}$ from S_{n+1} to S_n is isomorphic to the direct sum of W_n and the induced module $(W_n \downarrow_{S_{n-1}}) \uparrow^{S_n}$.

(2) $Conj_n$ is the degree n term in

$$(1 - Lie) \cdot \sum_{n\ge 1} Hk_n[Lie].$$

Proof. The following symmetric function identity is immediate from the definition of Hk_n :

$$\sum_{n\geq 1} p_n = (1-p_1)\sum_{n\geq 1} Hk_n. \tag{2.9}$$

Taking partial derivatives with respect to p_1 gives

$$\frac{\partial}{\partial p_1} Hk_{n+1} = Hk_n + p_1 \frac{\partial}{\partial p_1} Hk_n,$$

which is (1).

Taking the plethysm of both sides of equation (2.9) with Lie, and invoking Proposition 2.19, now gives (2). □

3. The representations ψ_{2^k} and the twisted conjugacy action

The functions ψ_{2^k} of Theorem 2.11 appear to have interesting properties. We state separately the following consequence of the proof of Theorem 2.11:

Corollary 3.1. *$\psi_{2m} - h_2 p_1^{2m-2}$ is Schur-positive, and hence so is $\psi_{2m} - h_2^r p_1^{2m-2r}$, $1 \leq r \leq m$.*

Proof. The second statement follows by induction from the Schur positivity of expression (2.1), upon writing

$$\psi_{2m} - h_2^r p_1^{2m-2r} = \psi_{2m} - h_2^{r-1}(p_1^2 - e_2) p_1^{2m-2r} = (\psi_{2m} - h_2^{r-1} p_1^{2m-2(r-1)}) + e_2 p_1^{2m-2r}.$$ □

In fact the following stronger statement appears to be true.

Conjecture 2. *For $k \geq 2$, $\psi_{2^k} - 2h_2^2 p_1^{2k-4}$ is Schur-positive. This has been verified for $k \leq 16$.*

Remark 3.2. Note however that $\psi_{2^k} - 2h_2 p_1^{2k-2}$ is NOT Schur-positive. This can be easily verified by computation, using Theorem 2.11, for $k = 2, 3, 4$.

However, we do have the following:

Lemma 3.3. *The symmetric function $\psi_{2^m} - h_2 \psi_{2^{m-1}}$ is Schur-positive. More generally, for $k \leq m$, the function $\psi_{2^m} - h_2^k \psi_{2^{m-k}}$ is Schur-positive.*

Proof. Equation (2.2) of Theorem 2.11 gives

$$\begin{aligned}
\psi_{2^m} - h_2 \psi_{2^{m-1}} &= \sum_{\substack{j=1 \\ j \text{ odd}}}^{m+1} \binom{m+1}{j} h_2^{m+1-j} e_2^{j-1} - h_2 \cdot \sum_{\substack{k=1 \\ k \text{ odd}}}^{m} \binom{m}{k} h_2^{m-k} e_2^{k-1} \\
&= \sum_{\substack{j=1 \\ j \text{ odd}}}^{m+1} \binom{m+1}{j} h_2^{m+1-j} e_2^{j-1} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{m} \binom{m}{k} h_2^{m-k+1} e_2^{k-1} \\
&= \sum_{\substack{j=1 \\ j \text{ odd}}}^{m} \left[\binom{m+1}{j} - \binom{m}{j} \right] h_2^{m+1-j} e_2^{j-1} + e_2^m \text{ Odd}(m+1) \\
&= \sum_{\substack{j=1 \\ j \text{ odd}}}^{m} \binom{m}{j-1} h_2^{m+1-j} e_2^{j-1} + e_2^m \text{ Odd}(m+1),
\end{aligned}$$

and this is clearly Schur-positive. (As in the proof of Theorem 2.11, $\mathrm{Odd}(m+1)$ is 1 if $m+1$ is odd and zero otherwise.)

The more general statement follows from the telescoping sum

$$\psi_{2^m} - h_2^k\psi_{2^{m-k}} = \sum_{i=1}^{k} h_2^{i-1}(\psi_{2^{m+1-i}} - h_2 \cdot \psi_{2^{m-i}}).$$

□

Proposition 3.4. *Let $k, r \geq 1$, and let $m = \lceil \frac{r}{2} \rceil$. (So $m = \frac{r+1}{2}$ if r is odd, and $m = \frac{r}{2}$ if r is even.) We have*

$$\psi_{(3,2^k,1^r)} = \begin{cases} [\psi_{2^{m+k+1}} - h_2 p_1^{r+1}\psi_{2^k}] + p_1^r(2h_3+e_3)\psi_{2^k}, & r \text{ odd;} \\ p_1[\psi_{(2^{m+k+1})} - h_2 p_1^r\psi_{2^k}] + p_1^r(2h_3+e_3)\psi_{2^k}, & r \text{ even.} \end{cases}$$

Proof. We have the recurrence

$$\psi_{(3,2^k,1^r)} - \psi_{(3,2^{k-1},1^{r+3})} = p_3 p_1^r p_2^k.$$

Iterating this, the last two lines of this recurrence are

$$\psi_{(3,2,1^{r+2k-2})} - \psi_{(3,1^{r+2k})} = p_3 p_1^{r+2k-2} p_2,$$

and

$$\psi_{(3,1^{r+2k})} - \psi_{(2^m,1^a)} = p_3 p_1^{r+2k},$$

where in the last line $a = 0$ if $r + 2k + 3 = 2m$ is even, i.e. if r is odd, and $a = 1$ if $r + 2k + 3 = 2m + 1$ is odd, i.e. if r is even.

This telescoping sum collapses to give

$$\psi_{(3,2^k,1^r)} - \psi_{(2^m,1^a)} = p_3 p_1^r \cdot (p_2^k + {p_2}^{k-1} p_1^2 + \ldots + p_1^{2k}) = p_3 p_1^r \cdot \psi_{2^k}.$$

But $p_3 p_1 = h_3 - s_{(2,1)} + e_3 = 2h_3 + e_3 - h_2 h_1$. Hence we have

$$\psi_{(3,2^k,1^r)} - \psi_{(2^m,1^a)} = (2h_3 + e_3) p_1^r \psi_{2^k} - h_2 p_1^{r+1} \psi_{2^k}.$$

The proposition follows. □

This leads us to make the following conjecture, which has been shown to be true in Lemma 3.3 for $m = 1$:

Conjecture 3. *Let $k, m \geq 1$. Then $\psi_{2^{k+m}} - h_2 p_1^{2m-2}\psi_{2^k}$ is Schur-positive, and hence so is $\psi_{2^{k+m}} - h_2^r p_1^{2m-2r}\psi_{2^k}$, $1 \leq r \leq m$. We have verified this for $1 \leq k, m \leq 5$.*

In view of Proposition 3.4, the truth of this conjecture would immediately imply Schur positivity of $\psi_{(3,2^k,1^r)}$ for all $r, k \geq 1$.

Remark 3.5. In contrast to Conjecture 2, computations show that $\psi_{2^{k+m}} - 2h_2 p_1^{2m-2}\psi_{2^k}$ is NOT Schur-positive.

Lemma 3.6. *The function $g_4 = p_3 p_1 + h_2^2$ is Schur-positive.*

Proof. It is easily verified, using the expansion $p_3 = h_3 - s_{(2,1)} + e_3, = h_3 + e_3 - (h_2 h_1 - h_3)$, that

$$p_3 p_1 + h_2^2 = p_1(2h_3 + e_3) - h_2 p_1^2 + h_2^2 = p_1(2h_3 + e_3) - h_2 e_2 = 2s_{(4)} + s_{(3,1)} + s_{(1^4)}.$$

□

We are able to settle the following special cases:

Proposition 3.7. *Let $r = 0, 1, 2$. Then $\psi_{(3,2^k,1^r)}$ is Schur-positive.*

Proof. We use Proposition 3.4. First let $r = 1$. Then we have

$$\psi_{(3,2^k,1)} = [\psi_{2^{k+2}} - h_2 p_1^2 \psi_{2^k}] + p_1(2h_3 + e_3)\psi_{2^k} = [\psi_{2^{k+2}} - h_2^2 \psi_{2^k}] + g_4 \psi_{2^k},$$

where $g_4 = p_1(2h_3 + e_3) - h_2 e_2$. and is thus Schur-positive by Lemma 3.6. But the last expression in brackets is also Schur-positive by Lemma 3.3.

If $r = 0$, Propostion 3.4 reduces to

$$\psi_{(3,2^k)} = p_1[\psi_{2^{k+1}} - h_2 \psi_{2^k}] + \psi_{2^k}(2h_3 + e_3),$$

and again this is Schur-positive by Lemma 3.3.

Finally if $r = 2$, Proposition 3.4 gives

$$\psi_{(3,2^k,1^2)} = p_1[\psi_{2^{k+2}} - h_2 p_1^2 \psi_{2^k}] + p_1^2(2h_3 + e_3)\psi_{2^k} = p_1[\psi_{2^{k+2}} - h_2^2 \psi_{2^k}] + p_1 g_4 \psi_{2^k};$$

invoking Lemmas 3.3 and 3.6, this is Schur-positive as before. □

This argument fails for $r = 3$. Proposition 3.4 then gives

$$\psi_{(3,2^k,1^3)} = [\psi_{2^{k+3}} - h_2 p_1^4 \psi_{2^k}] + p_1^3(2h_3 + e_3)\psi_{2^k} = [\psi_{2^{k+3}} - h_2^3 \psi_{2^k}] + g_6 \psi_{2^k},$$

but the function $g_6 = p_1^3(2h_3 + e_3) - h_2 e_2(h_2 + p_1^2)$ is no longer Schur-positive.

In previous work of this author, a sign-twisted conjugacy action of S_n was defined in terms of the exterior powers of the conjugacy action, and the following analogue of Theorem 1.1 was established (recall that f_n is the characteristic of the conjugacy action on the class of n-cycles):

Theorem 3.8. [6, Theorem 4.2] *The twisted conjugacy action has Frobenius characteristic ε_n satisfying*

(1) $\varepsilon_n = \sum_{\lambda \vdash n} \prod_i e_{m_i}[f_i]$, where λ has m_i parts equal to i;
(2) $\varepsilon_n = \sum_{\substack{\lambda \vdash n \\ \text{all parts odd}}} p_\lambda$; hence the latter sum is Schur-positive.

Note that ε_n is self-conjugate, so in particular the multiplicities of the trivial and sign representations coincide (and are equal to the number of partitions of n with all parts odd). Based on character tables up to $n = 10$, we were led to make a conjecture in the spirit of Conjecture 1, which we have subsequently verified for $n \le 28$.

Let $\mu \vdash n$ be a partition with all parts odd. Define

$$\varepsilon_\mu = \sum_{\substack{(1^n) \le \lambda \le \mu \\ \text{all parts odd}}} p_\lambda.$$

Conjecture 4. *Let $\mu \vdash n$ be a partition with all parts odd. The symmetric function ε_μ is Schur-positive.*

Note that ε_μ is necessarily self-conjugate.

Theorem 3.8 says that $\varepsilon_{(n)}$ for n odd and $\varepsilon_{(n-1,1)}$ for n even are Schur-positive, since we now have

$$\varepsilon_n = \begin{cases} \varepsilon_{(n)}, & n \text{ odd}, \\ \varepsilon_{(n-1,1)}, & n \text{ even}. \end{cases}$$

The chains in reverse lexicographic order are now as follows:

(1) If n is odd:

$(n) > (n-2, 1^2) > (n-4, 3, 1) > (n-4, 1^4) > (n-6, 3^2) > (n-6, 3, 1^3) > (n-6, 1^6) > \dots$

(2) If n is even:

$$(n-1,1) > (n-3,3) > (n-3,1^3) > (n-5,3,1^2) > (n-5,1^5) > \dots$$

(3) At the bottom of the chain we always have

$$(1^n) < (3,1^{n-3}) < (3^3,1^{n-6}) < \dots < (3^{\lfloor \frac{n}{3} \rfloor},1^r) < (5,1^{n-5}) < \dots$$

where $r \equiv n \mod 3$, $0 \le r \le 2$.

Some cases of Conjecture 4 are easy to establish, e.g. for $\mu = (3,1^{n-3})$, $p_1^n + p_3 p_1^{n-3}$ is clearly Schur-positive. More generally we have

Proposition 3.9. *If* $\mu = (3^r, 1^{n-3r})$ *then* $\varepsilon_\mu = \sum_{\substack{(1^n)\le\lambda\le\mu \\ \textit{all parts odd}}} p_\lambda$ *is Schur-positive.*

Proof. Note that $\varepsilon_{(3^r,1^{n-3r})} = p_1^{n-3r}\varepsilon_\nu$ where $\nu = (3^r)$. But ε_ν is the sum of power sums for λ in the set T_{3r} consisting of all partitions with parts equal to 1 or 3. By [6, Theorem 4.23], this is Schur-positive. □

An analogue of Theorem 2.1 holds here as well. It is also a consequence of Theorem 2.2, since the global classes defined there are also conjugacy classes appearing in ε_n.

Theorem 3.10. [6, Theorem 4.9, Proposition 4.22] *The representation* ε_n *contains all irreducibles. The multiplicity of the trivial representation (and hence also the sign) is the number of partitions of* n *into odd parts. In particular this multiplicity is at least* $\lceil \frac{n}{2} \rceil \ge 3$ *for* $n \ge 5$.

The last statement in theorem is simply a consequence of the observation that if n is odd, the partitions $(n-2r, 1^{2r}), 0 \le r \le \frac{n-1}{2}$ all have odd parts, while if n is even, the partitions $(n-1-2r, 1^{2r}), 0 \le r \le \frac{n-2}{2}$ all have odd parts.

Proposition 3.11. *Let* n *be odd. Then* $\varepsilon_{(n-2,1^2)}, \varepsilon_{(n-4,3,1)}$ *and* $\varepsilon_{(n-4,1^3)}$ *are all Schur-positive. If* n *is even, then* $\varepsilon_{(n-3,3)}$ *and* $\varepsilon_{(n-3,1^3)}$ *are Schur-positive.*

Proof. We use Theorem 3.10. Since ε_n contains all irreducibles, from Lemma 2.13 (1), $\varepsilon_{(n-2,1^2)} = \varepsilon_n - p_n$ must be Schur-positive. Similarly, $\varepsilon_{(n-4,3,1)} = \varepsilon_n - p_n - p_{n-2}p_1^2$. From Lemma 2.13 (1) and (9), $p_n + p_{n-2}p_1^2$ is multiplicity-free except for the occurrence of $2(n) + 2(-1)^n(1^n) = 2(n) - 2(1^n)$. But the trivial representation occurs in ε_n with multiplicity equal to the number of partitions of n with all parts odd, and this is at least 2 for any odd n. Since these representations are all self-conjugate, the proof is complete.

Finally observe that from (1), (9) and (10) of Lemma 2.13, $p_n + p_{n-2}p_1^2 + p_{n-4}p_3p_1$ is multiplicity-free except for the occurrence of $3(n) + 3(-1)^n(1^n) = 3(n) - 3(1^n)$. The result follows as before from Theorem 3.10. □

.

Proposition 3.12. *Let* n *be even. Then* $\varepsilon_{(n-3,3)}$ *and* $\varepsilon_{(n-3,1^3)}$ *are Schur-positive.*

Proof. We have $\varepsilon_{(n-3,3)} = \varepsilon_n - p_1p_{n-1}$, so the result follows again from Theorem 3.10 and Lemma 2.13 (2).

Next we have $\varepsilon_{(n-3,1^3)} = \varepsilon_n - p_1p_{n-1} - p_3p_{n-3}$, which from Lemma 2.13 is multiplicity-free except for the term $2((n) + (1^n))$. But for n even, $n \ge 6$, there are at least three partitions with odd parts, namely $(n-1,1)$, $(3,1^{n-3})$ and $(n-3,3)$. This ensures the trivial representation (and hence the sign, since the representations are self-conjugate) occurs with positive multiplicity, completing the argument. □

Let ω denote the involution on the ring of symmetric functions which sends h_n to e_n. Another result of [6] states that

Theorem 3.13. [6, Theorem 4.11] *The sum* $\sum_{\substack{\lambda\vdash n\\ n-\ell(\lambda)\ \text{even}}} p_\lambda$ *equals* $\frac{1}{2}(\psi_n+\omega(\psi_n))$ *and is Schur positive.*

Similarly we have, for any partition μ of n, $\frac{1}{2}(\psi_\mu+\omega(\psi_\mu)) = \sum_{\substack{(1^n)\leq\lambda\leq\mu\\ n-\ell(\lambda)\ \text{even}}} p_\lambda$. This leads us to make the following conjecture, which has been verified for $n\leq 20$:

Conjecture 5. *Let* $\mu\vdash n$. *The sum* $\sum_{\substack{(1^n)\leq\lambda\leq\mu\\ n-\ell(\lambda)\ \text{even}}} p_\lambda$ *is Schur-positive.*

Clearly Conjecture 1 implies Conjecture 5. Maple computations with the character table of S_n show that the sum $\sum_{\lambda\in T} p_\lambda$ is NOT Schur-positive for arbitrary subsets T containing (1^n) and consisting of all partitions λ with $n-\ell(\lambda)$ even. The first counterexample occurs only for $n=14$, and there are then at least 2^{11} such subsets for which Schur positivity fails.

Note that if we require that $n-\ell(\lambda)$ be odd, but also include the regular representation in the sum, the preceding conjecture is false:

the sum $p_1^n+\sum_{\substack{(1^n)\leq\lambda\leq\mu\\ n-\ell(\lambda)\ \text{odd}}} p_\lambda$ is not Schur-positive.

Question 3.14. In [6] and [8], S_n-modules are constructed whose characteristics are multiplicity-free sums of power sums, thereby settling the Schur positivity question in these cases. Is there a representation-theoretic context for the sums ψ_μ?

4. Arbitrary subsets of conjugacy classes

In this section we examine the following more general question: Let $f(n)$ be the number of subsets of $\{p_\lambda : \lambda\vdash n\}$ containing p_1^n, and having the property that the sum of their elements is NOT Schur-positive. What can be said about $f(n)$? Richard Stanley computed the values of $f(n)$ for $n\leq 7$ after seeing a preprint of [6]. Table 1 extends these values up to $n=10$.

Recall from Section 1 that ψ_T denotes the Schur function $\sum_{\mu\in T} p_\mu$. The analysis of the multiplicity of the sign representation in Example 1.4 suggests a way to obtain a lower bound for the numbers $f(n)$. Indeed, let $A(n)=\{\mu\vdash n : n-\ell(\mu)\text{ is even}\}$, and let $B(n)=\{\mu\vdash n : n-\ell(\mu)\text{ is odd}\}$. Let $\alpha(n),\beta(n)$ respectively be the cardinalities of $A(n),B(n)$. Clearly $\alpha(n)+\beta(n)=p(n)$. As in [6, Proposition 4.21] (see also equation (1.1)),

$$\alpha(n)-\beta(n)=\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)} \tag{4.1}$$

is the number of self-conjugate partitions of n, and hence $\alpha(n)\geq\beta(n)$.

By manipulating generating functions it can be seen that $\alpha(n)$ is also the number of partitions of n with an even number of even parts, and an arbitrary number of odd parts. The sequence appears in [3, A046682].

Proposition 4.1. *Let T be a subset of the set of partitions of n not containing the partition (1^n). The Schur function indexed by (1^n) appears with negative multiplicity in the Schur expansion of $\psi_{T\cup\{(1^n)\}}$ if and only if $|T\cap B(n)|\geq 2+|T\cap A(n)|$. Hence the number of such subsets gives the following lower bound for $f(n)$:*

$$\ell b(n) = \sum_{i=0}^{p(n)-\alpha(n)-2} \binom{p(n)-1}{i}.$$

In particular $f(n)$ is positive for all $n \geq 4$.

Proof. Immediate from the fact that each p_μ contributes $(-1)^{n-\ell(\mu)}$ to the multiplicity of (1^n) in ψ_T. A simple count then tells us that this multiplicity is negative for exactly as many subsets T as given by the following sum:

$$\sum_{a=0}^{\alpha(n)-1} \binom{\alpha(n)-1}{a} \sum_{b=a+2}^{\beta(n)} \binom{\beta(n)}{b}.$$

This is precisely the sum of the coefficients of the powers of $x^j, j \geq 2$, in the Laurent series expansion of

$$(1+x^{-1})^{\alpha(n)-1}(1+x)^{\beta(n)} = x^{-(\alpha(n)-1)}(1+x)^{\alpha(n)+\beta(n)-1}.$$

Since $\alpha(n)+\beta(n) = p(n)$, this in turn is the sum of the coefficients of the terms x^j in $(1+x)^{p(n)-1}$, for $j \geq \alpha(n)+1$, i.e:

$$\sum_{j=\alpha(n)+1}^{p(n)-1} \binom{p(n)-1}{j}.$$

Now replace j with $p(n)-1-i$. The last claim follows because $\alpha(n)$ is the number of partitions with an even number of even parts and thus $\alpha(n) \leq p(n)-2$. (If $n \geq 4$, exclude the partitions $(2, 1^{n-2})$ and (n) if n is even, $(n-1, 1)$ if n is odd.) ☐

Table 1 includes data up to $n = 10$, and the resulting lower bound $\ell b(n)$ on the number $f(n)$ of non-Schur-positive functions ψ_T, omitting the trivial values $f(n) = 0$ for $n \leq 3$.

Table 2

n	4	5	6	7	8	9	10
$p(n)$	5	7	11	15	22	30	42
f(n)	**1**	**7**	**184**	**3674**	**488,259**	**145,796,658**	**670,141,990,673**
$\ell b(n)$	1	7	176	3473	401,930	123,012,781	585,720,020,356
$\frac{f(n)}{2^{p(n)-1}}$	0.06	0.11	0.18	0.22	0.23	0.272	0.305
$\frac{\ell b(n)}{2^{p(n)-1}}$	0.06	0.11	0.172	0.212	0.192	0.229	0.266

Proposition 4.2. *There exists a subset T of the set of partitions of n, with $(1^n) \in T$, such that*

- *the irreducible $(n-1, 1)$ appears with negative multiplicity in ψ_T, if and only if $n \geq 10$.*
- *the irreducible $(2, 1^{n-2})$ appears with negative multiplicity in ψ_T, if and only if $n \geq 6$.*

Proof. Write χ^μ for the irreducible character indexed by the partition μ. Recall that the value of $\chi^{(n-1,1)}(\lambda)$ is one less than the number $m_1(\lambda)$ of parts of λ which are equal to 1, and is therefore never less than -1. Hence we have

$$\chi^{(n-1,1)}(\lambda) = \begin{cases} -1 & \text{for the } (p(n)-p(n-1)) \text{ partitions } \lambda \text{ with } m_1(\lambda)=0, \\ 0 & \text{for the} (p(n-1)-p(n-2)) \text{ partitions } \lambda \text{ with } m_1(\lambda)=1, \\ \geq 1 & \text{for the } p(n-2) \text{ partitions } \lambda \text{ with } m_1(\lambda)\geq 2. \end{cases}$$

Consider the conjugacy classes indexed by the $p(n)-p(n-1)$ partitions with no part equal to 1, and the partition (1^n). The row sum indexed by $(n-1,1)$ in the character table of S_n will then be $n-1-(p(n)-p(n-1))$. The first claim follows by observing that $p(n)-p(n-1)$ first exceeds $\chi^{(n-1,1)}(1^n)=n-1$ when $n=10$, and the fact that the values $p(n)-p(n-1)$ increase.

Of course we could also append to the set T above any of the $2^{p(n-1)-p(n-2)}$ subsets of conjugacy classes with exactly one fixed point (since these do not contribute to the multiplicity of $(n-1,1)$), to obtain even more non-Schur-positive instances of ψ_T; for the number of subsets with negative multiplicity for $(n-1,1)$ this gives a lower bound of

$$2^{p(n-1)-p(n-2)} \sum_{j=0}^{p(n)-p(n-1)-n} \binom{p(n)-p(n-1)}{n+j}. \tag{4.2}$$

Next we note that $\chi^{(2,1^{n-2})}(\lambda)=(-1)^{n-\ell(\lambda)}\chi^{(n-1,1)}(\lambda)$, since the two irreducibles are conjugate. Thus the number of times that $\chi^{(2,1^{n-2})}(\mu)$ equals (-1) is

$$|\{\mu\vdash n : n-\ell(\mu) \text{ is even and } \mu \text{ has no singleton parts}\}|$$
$$+\,|\{\mu\vdash n : n-\ell(\mu) \text{ is odd and } \mu \text{ has exactly two singleton parts}\}|.$$

Similarly the number of times that $\chi^{(2,1^{n-2})}(\mu)$ equals $(-r), r\geq 2$, is

$$|\{\mu\vdash n : n-\ell(\mu) \text{ is odd and } \mu \text{ has at least three singleton parts}\}|.$$

Combining these two quantities, we have that the number of conjugacy classes for which the value of $\chi^{(2,1^{n-2})}$ is negative is $|C_1|+|C_2|$, where

$$C_1=\{\mu\vdash n : n-\ell(\mu) \text{ is even and } \mu \text{ has no singleton parts}\}$$

and

$$C_2=\{\mu\vdash n : n-\ell(\mu) \text{ is odd, } \mu \text{ has at least } \textit{two} \text{ singleton parts}\}.$$

The set C_2 is in bijection with the set of all odd-signature partitions of $n-2$, so has cardinality $p(n-2)-\alpha(n-2)$. Also, the character values on the classes (1^n) and $(2,1^{n-2})$ together add up to $(n-1)-(n-3)=2$. Hence, by choosing at least 3 additional conjugacy classes in C_2, excluding the partition $(2,1^{n-2})$, we obtain that the multiplicity of $\chi^{(2,1^{n-2})}$ is negative for at least

$$2^{p(n-1)-p(n-2)} \sum_{j\geq 3} \binom{p(n-2)-\alpha(n-2)-1}{j} \tag{4.3}$$

subsets, and this is positive as soon as $n\geq 6$, since then $p(n-2)-\alpha(n-2)\geq p(n-2)-p(n-1)\geq 2$. Likewise the cardinality of C_1 is $p(n-2)-\alpha(n-2)$. □

Note that the lower bound of Proposition 4.1 surpasses the two lower bounds obtained above. In order to test Schur positivity of ψ_T, we need to examine the multiplicity of the irreducible indexed by each $\lambda\vdash n$ in ψ_T. This is given by $a_T(\lambda)=f^\lambda+\sum_{\mu\in T:\mu\neq(1^n)}\chi^\lambda(\mu)$, where $f^\lambda=\chi^\lambda((1^n))$ is the number of standard Young tableaux of shape λ. Now $\chi^\lambda((1^n))$

is larger than any other value of the character χ^λ. Hence one way in which we can see how to make these values negative is to find μ such that $f^\lambda + \chi^\lambda(\mu)$ is small relative to the number $p(n)$ of conjugacy classes. For instance:

Proposition 4.3. *Let $\lambda \vdash n$, and let $\tau = (2, 1^{n-2})$ be the (conjugacy class of) a single transposition. Then $\chi^\lambda((1^n)) + \chi^\lambda(\tau)$ equals*

(1) 2 if $\lambda = (2, 1^{n-2})$,
(2) $2(n-3)$ if $\lambda = (2^2, 1^{n-4})$,
(3) $(n-2)(n-5)$ if $\lambda = (2^3, 1^{n-6})$,
(4) $2(n-2)$ if $\lambda = (3, 1^{n-3})$.

Proof. The first part has already been observed in the proof of Proposition 4.2. For the rest, we use the formula $\chi^\lambda(\tau) = \frac{f^\lambda}{\binom{n}{2}}(b(\lambda') - b(\lambda))$, where $b(\lambda) = \sum_i (i-1)\lambda_i$, as well as the hook length formula for f^λ. (See [5, Ex. 7.51]). When λ' dominates λ we must have $b(\lambda') < b(\lambda)$ and thus $\chi^\lambda(\tau)$ is negative. □

An examination of the character tables of S_n leads to the following observations. The use of character tables eliminates the need for Stembridge's SF package for Maple, by means of which the values $f(n)$ were originally calculated, up to $n = 8$.

- For $n = 6$, of the 184 subsets that fail to be Schur positive, exactly 176 fail to be Schur-positive because of the irreducible (1^6), another 4 fail because the irreducible $(2, 1^4)$ appears with negative coefficient, and the remaining 4 fail because of the irreducible (3^2). From the character table of S_6, it is easy to identify these 8 subsets. (In each of these cases no other irreducibles occur with negative coefficient.)
- For $n = 7$, the number of subsets failing Schur-positivity because of (a negative coefficient for) (1^7) is 3473, and 384 were identified as failing (in part) because of the irreducible $(2, 1^5)$. The count for subsets in which both irreducibles appear with negative coefficient is 183, and this confirms $f(7) = 3674$. From the character table of S_7, it is easy to verify that the number of subsets T resulting in a negative coefficient for $(2, 1^5)$ in ψ_T is exactly 384, and also that no other irreducibles occur with negative coefficient in any subset.
- For $n = 8$, by examining the negative entries in the character table, we see that for any subset T containing (1^n), the only two possibilities for negative coefficients in the Schur expansion of ψ_T are (1^8) and $(2, 1^6)$. There are $\ell b(8) = 401,930$ subsets with negative multiplicity for (1^8), $153,008$ subsets with negative multiplicity for $(2, 1^6)$, and $76,679$ subsets in which *both* irreducibles occur with negative multiplicity. The reader can check that this agrees with the figure for $f(8)$ in the table. This computation took 70 seconds in Maple.
- For $n = 9$, since our lower bound is $\ell b(9) = 123,012,781$, we know that $\frac{f(9)}{2^{p(9)-1}} \geq \frac{123{,}012{,}781}{2^{29}} = 0.22913$. The character table shows that in addition to (1^9) and $(2, 1^7)$, only the irreducibles $(2^2, 1^5)$ and $(3, 1^6)$ will appear with negative coefficient in some subsets. The value of $f(9)$ was calculated by exploiting this fact, and took 6.8 hours in Maple. However, the C code ran in only 36 seconds.

- For $n = 10$ similarly, we have $\frac{f(10)}{2^{p(10)-1}} \geq \frac{585{,}720{,}020{,}356}{2^{41}} = 0.266$. The character tables now show that one or more of only the following five irreducibles will appear with negative coefficient in ψ_T : $(1^{10}), (2, 1^8), (2^2, 1^6), (3, 1^7), (9, 1)$, for some subset T containing (1^{10}). Again, the computation of $f(10)$ exploited this fact. It was coded in C, and took 83 hours to produce the result.
- For $n = 11$ the six irreducibles contributing to negativity in ψ_T are $(1^{11}), (2, 1^9), (2^2, 1^7), (2^3, 1^5), (3, 1^8), (10, 1)$.

Of course this number of irreducibles increases rapidly with n; e.g. for $n = 28$, out of $p(28) = 3,718$ partitions, 89 can occur with negative multiplicity. A far more accurate lower bound than $\ell b(n)$ is obtained by taking the number of subsets T in which *either* of the representations (1^n) or $(2, 1^{n-2})$ appear with negative multiplicity in ψ_T, but a formula for this in the spirit of Proposition 4.1 seems difficult to obtain.

Tables 3a and 3b below contain, for each n, the values of the function $g(n)$, defined to be the number of partitions μ of n such that, for some subset T containing (1^n), the irreducible indexed by μ appears with negative multiplicity in ψ_T.

Table 3a

n	4	5	6	7	8	9	10	11	12
$p(n)$	5	7	11	15	22	30	42	56	77
g(n)	**1**	**1**	**3**	**2**	**2**	**4**	**5**	**6**	**8**

Table 3b

n	13	14	15	16	17	18	19	20	21	22	23	24	25
$p(n)$	101	135	176	231	297	385	490	627	792	1002	1255	1575	1958
g(n)	**9**	**10**	**10**	**15**	**16**	**22**	**23**	**27**	**33**	**36**	**43**	**51**	**56**

Based on our computations, we make the following conjecture:

Conjecture 6. *For $n \geq 6$, the numbers $\frac{f(n)}{2^{p(n)-1}}$ are bounded below by $\frac{1}{16}$, above by $\frac{1}{2}$, and are strictly increasing.*

This would imply an affirmative answer to a question raised by Richard Stanley:

Conjecture 7. *The numbers $\frac{f(n)}{2^{p(n)-1}}$ approach a limit strictly between 0 and 1.*

5. Additional tables

In the tables below we follow our usual convention of writing simply μ to signify the Schur function s_μ.

Table 4: Schur function expansion of ψ_μ, $n \leq 5$

$$\psi_1 = (1), \qquad \psi_2 = 2(2), \qquad \psi_{(1^2)} = (2) + (1^2)$$
$$\psi_3 = 3(3) + (2,1) + (1^3), \qquad \psi_{(2,1)} = 2(3) + 2(2,1), \qquad \psi_{(1^3)} = (3) + 2(2,1) + (1^3)$$

$$\begin{aligned}
\psi_4 &= 5(4) + 2(3,1) + 3(2^2) + 2(2,1^2) + (1^4)\\
\psi_{(3,1)} &= 4(4) + 3(3,1) + 3(2^2) + (2,1^2) + 2(1^4)\\
\psi_{(2^2)} &= 3(4) + 3(3,1) + 4(2^2) + (2,1^2) + (1^4)\\
\psi_{(2,1^2)} &= 2(4) + 4(3,1) + 2(2^2) + 2(2,1^2)\\
\psi_{(1^4)} &= (4) + 3(3,1) + 2(2^2) + 3(2,1^2) + (1^4)\\
\psi_5 &= 7(5) + 5(4,1) + 6(3,2) + 5(3,1^2) + 4(2^2,1) + 3(2,1^3) + (1^5)\\
\psi_{(4,1)} &= 6(5) + 6(4,1) + 6(3,2) + 4(3,1^2) + 4(2^2,1) + 4(2,1^3)\\
\psi_{(3,2)} &= 5(5) + 6(4,1) + 7(3,2) + 4(3,1^2) + 3(2^2,1) + 4(2,1^3) + (1^5)\\
\psi_{(3,1^2)} &= 4(5) + 7(4,1) + 6(3,2) + 4(3,1^2) + 4(2^2,1) + 3(2,1^3) + 2(1^5)\\
\psi_{(2^2,1)} &= 3(5) + 6(4,1) + 7(3,2) + 4(3,1^2) + 5(2^2,1) + 2(2,1^3) + (1^5)\\
\psi_{(2,1^3)} &= 2(5) + 6(4,1) + 6(3,2) + 6(3,1^2) + 4(2^2,1) + 2(2,1^3)\\
\psi_{(1^5)} &= (5) + 4(4,1) + 5(3,2) + 6(3,1^2) + 5(2^2,1) + 4(2,1^3) + (1^5)
\end{aligned}$$

Table 5: Schur function expansion of ψ_μ, $n = 6$

$$\begin{aligned}
\psi_{(6)} &= 11(6)+8(5,1)+15(4,2)+10(4,1^2)+4(3^2)+13(3,2,1)+10(3,1^3)+8(2^3)+5(2^2,1^2)+4(2,1^4)+(1^6)\\
\psi_{(5,1)} &= 10(6)+9(5,1)+15(4,2)+9(4,1^2)+4(3^2)+13(3,2,1)+11(3,1^3)+8(2^3)+5(2^2,1^2)+3(2,1^4)+2(1^6)\\
\psi_{(4,2)} &= 9(6)+9(5,1)+16(4,2)+9(4,1^2)+4(3^2)+12(3,2,1)+11(3,1^3)+8(2^3)+6(2^2,1^2)+3(2,1^4)+(1^6)\\
\psi_{(4,1^2)} &= 8(6)+10(5,1)+15(4,2)+9(4,1^2)+5(3^2)+12(3,2,1)+11(3,1^3)+9(2^3)+5(2^2,1^2)+4(2,1^4)\\
\psi_{(3^2)} &= 7(6)+9(5,1)+16(4,2)+9(4,1^2)+6(3^2)+12(3,2,1)+11(3,1^3)+8(2^3)+4(2^2,1^2)+5(2,1^4)+(1^6)\\
\psi_{(3,2,1)} &= 6(6)+10(5,1)+16(4,2)+8(4,1^2)+4(3^2)+14(3,2,1)+10(3,1^3)+6(2^3)+4(2^2,1^2)+6(2,1^4)\\
\psi_{(3,1^3)} &= 5(6)+10(5,1)+16(4,2)+9(4,1^2)+3(3^2)+14(3,2,1)+9(3,1^3)+7(2^3)+4(2^2,1^2)+6(2,1^4)+(1^6)\\
\psi_{(2^3)} &= 4(6)+8(5,1)+16(4,2)+8(4,1^2)+4(3^2)+16(3,2,1)+8(3,1^3)+8(2^3)+4(2^2,1^2)+4(2,1^4)\\
\psi_{(2^2,1^2)} &= 3(6)+9(5,1)+13(4,2)+10(4,1^2)+7(3^2)+16(3,2,1)+6(3,1^3)+5(2^3)+7(2^2,1^2)+3(2,1^4)+(1^6)\\
\psi_{(2,1^4)} &= 2(6)+8(5,1)+12(4,2)+12(4,1^2)+6(3^2)+16(3,2,1)+8(3,1^3)+4(2^3)+6(2^2,1^2)+2(2,1^4)\\
\psi_{(1^6)} &= (6)+5(5,1)+9(4,2)+10(4,1^2)+5(3^2)+16(3,2,1)+10(3,1^3)+5(2^3)+9(2^2,1^2)+5(2,1^4)+(1^6)
\end{aligned}$$

Acknowledgment: The author is grateful to Erik Altman for invaluable help in computing the values of $f(n)$ in Table 1, particularly for the calculation of the entry for $n = 10$.

References

1. A. Frumkin, *Theorem about the conjugacy representation of S_n*, Israel J. Math. **55** (1986),121–128.
2. I. G. Macdonald, *Symmetric functions and Hall polynomials*, Second Edition, Oxford University Press (1995).
3. The Online Encyclopaedia of Integer Sequences, https://oeis.org/search?q=A046682.
4. L. Solomon, *On the sum of the elements in the character table of a finite group*, Proc. Amer. Math. Soc. **12** No. 6 (1961), 962–963.
5. R. P. Stanley, Enumerative Combinatorics, Vol.2, Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, Cambridge, 1999.
6. S. Sundaram, *The conjugacy action of S_n and modules induced from centralisers*, J. Alg. Comb., to appear. DOI 10.1007/s10801-017-0796-9 (arXiv:160305898v2).

7. S. Sundaram, *On conjugacy classes of S_n containing all irreducibles*, Isr. J. Math. **225(1)** (2018), 321–342. DOI: 10.1007/s11856-018-1659-3. (arXiv.math.GR/1607.08581v3)
8. S. Sundaram, *Variations on the S_n-module Lie_n*, submitted. (arXiv.math.RT/1803.09368).
9. J. Swanson, *On the existence of tableaux with given modular major index*, Algebraic Combinatorics, Volume 1 (2018) no. 1, 3-21.

Pierrepont School, One Sylvan Road North, Westport, CT 06880
E-mail address: shsund@comcast.net