arXiv:1808.01594v2 [math.RT] 7 Sep 2018

DEGENERATE SCHUBERT VARIETIES IN TYPE A

ROCCO CHIRIVÌ, XIN FANG, AND GHISLAIN FOURIER

Abstract. We introduce rectangular elements in the symmetric group. In the framework of PBW degenerations, we show that in type A the degenerate Schubert variety associated to a rectangular element is indeed a Schubert variety in a partial flag variety of the same type with larger rank. Moreover, the degenerate Demazure module associated to a rectangular element is isomorphic to the Demazure module for this particular Schubert variety of larger rank. This generalises previous results by Cerulli Irelli, Lanini and Littelmann for the PBW degenerate flag variety.

1. Introduction

1.1. **Motivation.** In the framework of PBW filtrations on simple Lie algebras and PBW degenerations of their cyclic modules, Feigin [11] introduced in type A the PBW degenerate flag variety $\mathcal{F}^a$. These projective varieties can be defined by a variation of the containing relations in the definition of the complete flag variety. Similar to the complete flag varieties, the degenerate flag varieties can be realised either in [11] using PBW degenerated modules as a highest weight orbit of a degenerated algebraic group, or in [7] as a quiver Grassmannian associated to the type A equi-oriented quiver and a fixed dimension vector.

It turned out, that these projective varieties have rich geometric and combinatorial structures. For example, they are normal locally complete intersections and an analogous of the Borel-Weil-Bott theorem holds [13]; their torus fixed points are counted by the median Genocchi numbers [12], *etc.* Moreover, a monomial basis for the homogeneous coordinate ring of $\mathcal{F}^a$, parametrised by lattice points in a normal polyhedron, has been provided in [15], again influencing a whole new chapter on toric degenerations of the flag variety and beyond.

Due to the abelianization, it is quite surprising, that Cerulli Irelli and Lanini [8], later also together with Littelmann [9], recovered the PBW degenerate flag variety (resp. the degenerate module) as a Schubert variety (resp. a Demazure module) in a partial flag variety of double rank. This certainly implies all the aforementioned geometric and combinatorial properties. One has actually all the tools from Schubert calculus on hand to compute singularities, cohomologies *etc.*

The concept of PBW degenerations can be applied to spherical varieties in general: one partially abelianizes the corresponding Lie algebra and algebraic group (see [15, 11]) and considers its orbit in the associated graded module. A natural class of spherical

2010 *Mathematics Subject Classification.* 17B10, 16S30, 14M15, 06A07.

Key words and phrases. PBW filtration, Schubert varieties, Demazure modules.

varieties is that of Schubert varieties and a first step towards the analysis of PBW degenerate Schubert varieties has been carried out in [17], where for a triangular element in the Weyl group, a monomial basis of the homogeneous coordinate ring has been provided. From a combinatorial point of view, this basis can be identified with lattice points on a face of the previous polyhedron. Our main motivation for this paper is to figure out whether one can identify these degenerate Schubert varieties again as Schubert varieties in a partial flag variety of type A.

1.2. **Main result.** The following main result of this paper, as a combination of Theorem 2.4 and Proposition 5.6, answers this question for a large class of Weyl group elements which will be introduced after the statement.

Theorem. *Let $\tau \in \mathfrak{S}_{n+1}$ be a rectangular element and λ be a dominant integral weight for $\mathfrak{sl}_{n+1}$.*

(1) The PBW degenerate Demazure module $V_\tau^{-,a}(\lambda)$ (see Section 2.3 for the definition) is isomorphic to the Demazure module $V_{\widetilde{w}}(\widetilde{\lambda})$, where $\widetilde{\tau} \in \mathfrak{S}_{2n}$ and $\widetilde{\lambda}$ is a dominant integral weight for $\mathfrak{sl}_{2n}$.

(2) The degenerate Schubert variety $X_w^a(\lambda)$ (again see Section 2.3) is isomorphic to a Schubert variety $X_{\widetilde{w}}(\widetilde{\lambda})$ in a partial flag variety of $\mathfrak{sl}_{2n}$.

We explain the terminology *rectangular* in the theorem. Rectangular elements may be characterized in various way: via a property of the set of positive roots sent to negative roots as in Definition 3.6 using rectangular subsets, or via certain inequalities as in Proposition 3.10, or, finally, as permutations avoiding the four patterns 2413, 2431, 4213 and 4231 as proved in Corollary 3.12. For example, in $\mathfrak{S}_4$, there are 20 rectangular elements. We give an example in Remark 6.3 of a non-rectangular element, showing that the above theorem does not hold by following the construction in this paper.

Since the longest element in the Weyl group of type A is a rectangular element, the results in [8] and [9] in type A are special cases of the above theorem (Example 6.2). The proof provided here is different from that in [8], while partially follows the strategy of [9], indeed this last paper was the main inspiration for our combinatorial construction.

Parallel to the results in type A, the theory of PBW filtration has been well understood for the symplectic Lie algebras [16, 14], we expect results parallel to this paper in the symplectic case, depending on the presence of symplectic triangular and rectangular elements and corresponding useful bases for the degenerate Demazure modules.

1.3. **Outline of proof.** We briefly explain the idea of the proof. For $\sigma \in \mathfrak{S}_{n+1}$, we denote $N(\sigma)$ to be the image of the negative roots in the positive roots under σ^{-1}. Let $\tau \in \mathfrak{S}_{n+1}$ be a rectangular element. We construct an element $\widetilde{\tau} \in \mathfrak{S}_{2n}$ such that for any $\alpha, \beta \in N(\widetilde{\tau})$, $\alpha + \beta \notin N(\widetilde{\tau})$, which implies that the Lie subalgebra of $\mathfrak{sl}_{2n}$ spanned by roots in $N(\widetilde{\tau})$ is abelian. Moreover, for a given dominant integral weight λ for $\mathfrak{sl}_{n+1}$, we provide a dominant integral weight $\widetilde{\lambda}$ for $\mathfrak{sl}_{2n}$ such that the scalar products of λ with the roots of $N(\tau)$ coincide with the scalar products of $\widetilde{\lambda}$ with $N(\widetilde{\tau})$.

In this sense, we have identified the abelianized Lie subalgebras of $\mathfrak{sl}_{n+1}$ generating the Demazure modules, with abelian Lie subalgebras of $\mathfrak{sl}_{2n}$. We then show that this induces an isomorphism between the PBW degenerated Demazure modules $V_\tau^{-,a}(\lambda)$ for $\mathfrak{sl}_{n+1}$ and the Demazure modules $V_{\widetilde{\tau}}(\widetilde{\lambda})$ of $\mathfrak{sl}_{2n}$, terminating the proof of our main theorem.

1.4. **Organisation of paper.** The paper is organised as follows: we start by giving a few more details on the main theorem in Section 2, then we introduce the combinatorics related to rectangular subsets in Section 3. In Section 4, we build the bridge from $\mathfrak{sl}_{n+1}$ to $\mathfrak{sl}_{2n}$, followed by the proof of the main theorem in Section 5. We conclude the paper with a few consequences and remarks in Section 6.

1.5. **Acknowledgements.** Part of the work was carried out during a research visit of X.F. to University of Hannover. He would like to thank Leibniz University of Hannover for the hospitality.

2. Statement of main result

2.1. **Notations in Lie theory.** Let $G = \mathsf{SL}_{n+1}(\mathbb{C})$, B be the Borel subgroup of G consisting of upper triangular matrices, T be the maximal torus contained in B with normalizer $N(T)$ and let $W = N(T)/T$ be the Weyl group.

Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ be the corresponding Lie algebra and $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the triangular decomposition with $\mathfrak{h}$ the Lie algebra of T and $\mathfrak{b}^+ := \mathfrak{n}^+ \oplus \mathfrak{h}$ the Lie algebra of B. We denote by $U(\mathfrak{g})$, $U(\mathfrak{n}^-)$ and $U(\mathfrak{b}^+)$ the corresponding universal enveloping algebras.

The root system of $\mathfrak{g}$ is denoted by Φ; Φ^+ (resp. Φ^-) stands for the set of positive (resp. negative) roots. We choose the usual numbering of the simple roots $\alpha_1, \cdots, \alpha_n$ in $\mathfrak{h}^*$ so that

$$\Phi^+ = \{\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mid 1 \leq i \leq j \leq n\}.$$

The reflection associated to a root $\alpha \in \Phi$ will be denoted by s_α; if $\alpha = \alpha_i$, we will abbreviate $s_i := s_{\alpha_i}$. Having fixed an enumeration of the simple roots, we may identify W with $\mathfrak{S}_n$.

The *support* $\mathrm{supp}(\alpha_{i,j})$ of the root $\alpha_{i,j}$ is defined to be the set $\{i, i+1, \cdots, j\}$ of $[n] := \{1, 2, \ldots, n\}$. A subset $I \subseteq [n]$ is called connected, if it is an interval. For a subset I, we let Φ_I^+ denote the set of positive roots in Φ^+ supported in I.

For $\alpha \in \Phi^+$ we fix e_α and f_α to be basis elements in the root spaces of $\mathfrak{g}$ corresponding to α and $-\alpha$. We choose $h_\alpha \in \mathfrak{h}$ such that $e_\alpha, f_\alpha, h_\alpha$ form a $\mathfrak{sl}_2$-triplet in $\mathfrak{g}$. When $\alpha = \alpha_{i,j} \in \Phi^+$, we will denote $e_{i,j} := e_{\alpha_{i,j}}$ and $f_{i,j} := f_{\alpha_{i,j}}$.

Let Λ (resp. Λ^+) denote the weight lattice (resp. the monoid of the dominant weights in Λ) and $\varpi_1, \cdots, \varpi_n$ denote the fundamental weights.

Now, fix a weight $\lambda \in \Lambda^+$, denote by $V(\lambda)$ the irreducible representation of G of highest weight λ and let $v_\lambda \in V(\lambda)$ be a fixed highest weight vector. Let also P_λ be the parabolic subgroup of G containing B and stabilizing the line $\mathbb{C} \cdot v_\lambda$ in $V(\lambda)$. The Schubert variety X_τ^λ associated to an element $\tau \in W$ in the partial flag varieties G/P_λ is the closure $\overline{B.\tau P_\lambda}$ of the B–orbit of any lift of τ to $N(T)$.

The extremal weight space $V(\lambda)_{\tau(\lambda)}$ of weight $\tau(\lambda)$ is one-dimensional; we fix the non-zero basis element $v_{\tau(\lambda)} := \tau \cdot v_\lambda$ in this weight space. The Demazure module $V_\tau(\lambda) := U(\mathfrak{b}^+) \cdot v_{\tau(\lambda)}$ is the cyclic $\mathfrak{b}^+$-module generated by $v_{\tau(\lambda)}$. The orbit through $v_{\tau(\lambda)}$ in the projective space $\mathbb{P}(V_\tau(\lambda))$

$$X_\tau(\lambda) := \overline{B \cdot [v_{\tau(\lambda)}]} \subseteq \mathbb{P}(V_\tau(\lambda))$$

is isomorphic to the Schubert variety X^λ_τ. In particular for $\tau = w_0$ the longest element of W we get an embedding of the partial flag variety G/P_λ.

For $\tau \in W$, we denote $N(\tau) := \Phi^+ \cap \tau^{-1}(\Phi^-)$ the set of positive roots sent to negative roots by τ, and $P(\tau) := \Phi^+ \cap \tau^{-1}(\Phi^+)$ the set of positive roots sent to positive roots by τ. We denote $\mathfrak{n}^-_\tau$ the subspace of $\mathfrak{n}$ consisting of root spaces associated to negative roots in $-N(\tau)$: this is a Lie subalgebra of $\mathfrak{n}$. In this case, we let N^-_τ be the corresponding algebraic group in G having Lie algebra $\mathfrak{n}^-_\tau$: it is generated by root subgroups associated to roots in $-N(\tau)$.

Since $-N(\tau) = \tau^{-1}N(\tau^{-1})$ we have

$$V_\tau(\lambda) = \tau(\tau^{-1}U(\mathfrak{n}^+)\tau) \cdot v_\lambda = \tau U(\mathfrak{n}^-_\tau) \cdot v_\lambda.$$

We define $V^-_\tau(\lambda) := U(\mathfrak{n}^-_\tau) \cdot v_\lambda$ and call it τ^{-1}-*twisted Demazure module*.

Notice that $e_\alpha \cdot v_{\tau(\lambda)} = 0$ unless $\alpha \in N(\tau^{-1})$. We endow $V_\tau(\lambda)$ with an $\mathfrak{n}^-_\tau$-module structure via twisting by τ. Then as $\mathfrak{n}^-_\tau$-modules, $V^-_\tau(\lambda)$ is isomorphic to $V_\tau(\lambda)$.

We define the *twisted Schubert variety* as a highest weight orbit:

$$X^-_\tau(\lambda) := \overline{N^-_\tau \cdot [v_\lambda]} \subseteq \mathbb{P}(V(\lambda)).$$

As projective varieties, $X^-_\tau(\lambda)$ is isomorphic to $X_\tau(\lambda)$. The Schubert varieties which will be studied in this paper are the twisted Schubert varieties.

2.2. **PBW degeneration.** Let $\mathfrak{a}$ be a finite dimensional Lie algebra. The PBW filtration on $U(\mathfrak{a})$ is defined by: for $k \in \mathbb{N}$,

$$U(\mathfrak{a})_{\leq k} := \mathrm{span}\{x_1 \cdots x_s \mid x_1, \cdots, x_s \in \mathfrak{a}, s \leq k\}.$$

It endows $U(\mathfrak{a})$ with a filtered algebra structure. By the Poincaré-Birkhoff-Witt (PBW) theorem, the associated graded algebra is isomorphic to the symmetric algebra $S(\mathfrak{a})$.

Let M be a cyclic $\mathfrak{a}$-module generated by m: that is to say, $M = U(\mathfrak{a}) \cdot m$. The PBW filtration on $U(\mathfrak{a})$ endows M with a filtered $U(\mathfrak{a})$-module structure by defining for $k \in \mathbb{N}$, $M_{\leq k} := U(\mathfrak{a})_{\leq k} \cdot m$. We denote by M^a the associated graded $S(\mathfrak{a})$-module. It is easy to see that M^a is a cyclic $S(\mathfrak{a})$-module generated by m^a, the image of m in M^a. The defining ideal $I^a(M)$ of M^a in $S(\mathfrak{a})$ is defined to be the kernel of the following map

$$S(\mathfrak{a}) \longrightarrow S(\mathfrak{a}) \cdot m = M^a, \quad x \longmapsto x \cdot m^a.$$

When $\mathfrak{a} = \mathfrak{n}^-$ and $M = V(\lambda)$ for $\lambda \in P^+$, the $S(\mathfrak{n}^-)$-module $V^a(\lambda)$ is studied in [15]. A monomial basis of $V^a(\lambda)$, parametrized by the Feigin-Fourier-Littelmann-Vinberg

polytope, is constructed in [1]; and the defining ideal of the $S(\mathfrak{n}^-)$-module $V^a(\lambda)$ is made explicit.

A geometric interpretation of the PBW degeneration is constructed by Feigin in [11]. We briefly recall his construction. Let $\mathfrak{n}^{-,a}$ be the unique abelian Lie algebra such that $U(\mathfrak{n}^{-,a}) = S(\mathfrak{n})$; and

$$N^{-,a} := \exp(\mathfrak{n}^{-,a}) \simeq \mathbb{G}_a^N$$

be the N-fold product of the additive group $\mathbb{G}_a$ where $N = \dim \mathfrak{n}^-$. The degenerate flag variety is defined in [11] as the highest weight orbit

$$\mathcal{F}^a(\lambda) := \overline{N^{-,a} \cdot [v_\lambda^a]} \subseteq \mathbb{P}(V^a(\lambda)).$$

As shown in the same paper, this projective variety is a flat degeneration of the flag variety

$$\mathcal{F}(\lambda) := \exp(\mathfrak{n}^-) \cdot [v_\lambda] = \overline{B \cdot [v_\lambda]} \subseteq \mathbb{P}(V(\lambda)).$$

Moreover, in [8], the authors realized $\mathcal{F}^a(\lambda)$ as a Schubert variety in a partial flag variety of $\mathsf{SL}_{2n}(\mathbb{C})$; furthermore, in [9], the PBW degenerate modules $V^a(\lambda)$ are realized as Demazure modules, and the defining ideal of the $S(\mathfrak{n}^-)$-module $V^a(\lambda)$ is recovered from that of the Demazure module.

2.3. **PBW degeneration of Demazure modules.** The goal of this paper is to study another situation of the PBW degeneration for the Demazure module associated to certain elements $\tau \in W$. In the setting of the last subsection, we consider $\mathfrak{a} = \mathfrak{n}_\tau^-$, and $M = V_\tau^-(\lambda)$ for $\lambda \in \Lambda^+$.

The construction in the last subsection gives an $\mathfrak{n}_\tau^{-,a}$-module $V_\tau^{-,a}(\lambda)$.

Since $\mathfrak{n}_\tau^- \subseteq \mathfrak{n}^-$ and $V_\tau^-(\lambda) \subseteq V(\lambda)$, we first compare the PBW filtration on $V_\tau^-(\lambda)$ itself and the one inherit from $V(\lambda)$.

Proposition 2.1. *For any $\lambda \in \Lambda^+$, $V_\tau^{-,a}(\lambda) \subseteq V^a(\lambda)$.*

Proof. It suffices to show that the following filtrations on $V_\tau^-(\lambda)$ coincide:

(1) the filtration obtained from the PBW filtration on $V(\lambda)$ by intersection;
(2) the filtration arising from the PBW filtration on $U(\mathfrak{n}_\tau^-)$.

We show that for any $k \in \mathbb{N}$,

$$U(\mathfrak{n}^-)_{\leq k} \cap U(\mathfrak{n}_\tau^-) = U(\mathfrak{n}_\tau^-)_{\leq k}.$$

The inclusion $\supseteq$ is clear by definition. To show the other inclusion, we fix a total order on a weight basis $x_1, \cdots, x_N$ of $\mathfrak{n}^-$, which induces a totally ordered basis of $\mathfrak{n}_\tau^-$. Take an element $x \in U(\mathfrak{n}^-)_{\leq k} \cap U(\mathfrak{n}_\tau^-)$ and write it in the fixed basis of $\mathfrak{n}^-$ using the PBW theorem, then every monomial appearing with a non-zero coefficient has PBW degree less or equal to k. This element is contained in $U(\mathfrak{n}_\tau^-)$, which means that one can rewrite x in the induced PBW basis of $\mathfrak{n}_\tau^-$. According to [18, Theorem 6.3] (see also the statement [23, (0.2)]), this rewriting procedure will not increase the PBW degree, showing that $x \in U(\mathfrak{n}_\tau^-)_{\leq k}$. □

Let $N_\tau^{-,a} := \exp(\mathfrak{n}_\tau^{-,a}) \simeq \mathbb{G}_a^M$ where $M = \ell(\tau)$. According to this proposition, we can look the highest weight orbit

$$X_\tau^a(\lambda) := \overline{N_\tau^{-,a} \cdot [v_\lambda^a]} \subseteq \mathbb{P}(V_\tau^{-,a}(\lambda))$$

as a closed subvariety of $\mathcal{F}^a(\lambda)$. As their constructions are similar to the twisted Schubert varieties, we call them *degenerate Schubert varieties* in degenerate flag varieties. One should compare this definition with the one given by the third author in [17].

We recall the definition of triangular elements introduced by the third author in [17]; later, in Proposition 3.13 we characterize triangular elements as the permutations avoiding the two patterns 4231 and 2413.

Definition 2.2. *An element $\tau \in W \simeq \mathfrak{S}_n$ is called* triangular*, if the following property is satisfied: for any $1 \leq i < k \leq j < \ell$ with $\tau(i) > \tau(j)$, $\tau(k) > \tau(\ell)$, one has $\tau(i) > \tau(\ell)$ and $\tau(k) \geq \tau(j)$.*

In [17], for a triangular element $\tau \in W$, a monomial basis of $V_\tau^{-,a}(\lambda)$ is constructed in the spirit of [15], which is parametrised by a suitable face of the polytope described in [15].

The goal of this paper is to study for which triangular element $\tau \in W$, the degenerate Schubert variety is indeed a Schubert variety in a higher rank special linear group.

2.4. **Statement of the main result.** Let $\widetilde{G} = \mathsf{SL}_{2n}$. When an object X (roots, weights, subalgebras, Weyl group elements, etc...) is referred to $\widetilde{G}$, we write it as $\widetilde{X}$.

In order to state the main result of this paper, we give here a direct definition of rectangular elements in the Weyl group W, later in 3.2 we will present a more convenient but equivalent definition and then in Corollary 3.12 we will see a characterization in terms of pattern avoidance. It is clear from the following definition that rectangular elements are triangular.

Definition 2.3. *An element $\tau \in W \simeq \mathfrak{S}_{n+1}$ is called* rectangular*, if for any $1 \leq i < k < j < \ell \leq n+1$, the following conditions are equivalent:*

(1) $\tau(i) > \tau(j)$ and $\tau(k) > \tau(\ell)$;
(2) $\tau(i) > \tau(\ell)$ and $\tau(k) > \tau(j)$.

The main result of the paper is the following:

Theorem 2.4. *Let $\tau \in W$ be a rectangular element and $\lambda \in \Lambda^+$. Then there exists $\widetilde{\lambda} \in \widetilde{\Lambda}^+$ and $\widetilde{\tau} \in \widetilde{W} \simeq \mathfrak{S}_{2n}$ such that*

$$X_\tau^a(\lambda) \text{ is isomorphic to } X_{\widetilde{\tau}}(\widetilde{\lambda})$$

as $\mathbb{G}_a^M$-projective varieties where $M = \ell(\tau)$.

The rest of the paper is devoted to the proof of this theorem. In Section 3 we study the combinatorics of rectangular elements in terms of rectangular subsets. In Section 4 we give the explicit construction of the Weyl group element $\widetilde{\tau}$ and the weight $\widetilde{\lambda}$ appearing in Theorem 2.4, and study the properties of these elements. The proof of Theorem 2.4 will be given in Section 5.

3. Rectangular elements

3.1. **Triangular and rectangular subsets.** The dominance order in Λ results in the following poset structure on Φ^+: for $\alpha_{i,j}$, $\alpha_{k,\ell} \in \Phi^+$,

$$\alpha_{i,j} \geq \alpha_{k,\ell} \text{ precisely when } i \leq k \text{ and } j \geq \ell.$$

Definition 3.1. *Let $\alpha_{i,j}, \alpha_{k,\ell} \in \Phi^+$ be two positive roots.*

(1) Their join *is defined by:*

$$\alpha_{i,j} \vee \alpha_{k,\ell} := \alpha_{\min(i,k),\max(j,\ell)};$$

it is the minimal positive root greater than both $\alpha_{i,j}$ and $\alpha_{k,\ell}$.

(2) If $\mathrm{supp}(\alpha_{i,j}) \cup \mathrm{supp}(\alpha_{k,\ell}) = [i,j] \cup [k,\ell]$ is connected, or equivalently if $\max(i,k) \leq \min(j,\ell)$, then we say the meet *of the two roots exists, and define it by:*

$$\alpha_{i,j} \wedge \alpha_{k,\ell} := \alpha_{\max(i,k),\min(j,\ell)}.$$

If the meet exists it is the maximal positive root smaller than both $\alpha_{i,j}$ and $\alpha_{k,\ell}$; otherwise no positive root is smaller than the two roots.

Definition 3.2. *Let $A \subseteq \Phi^+$ be a subset.*

(1) The set A is called a triangular *subset (see [17]), if*

(R1) *for any $\alpha, \beta \in A$ such that $\mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta)$ is a connected subset of $[n]$, we have $\alpha \vee \beta \in A$; moreover, $\alpha \wedge \beta \in A$ if it exists.*

(2) A triangular subset A is called a rectangular *subset, if the following condition holds:*

(R2) *if $\alpha, \beta \in \Phi^+$, $\alpha \wedge \beta$ exists and $\alpha \vee \beta, \alpha \wedge \beta \in A$, then $\alpha, \beta \in A$.*

(3) A rectangular subset A is said to be irreducible*, if the longest root θ in Φ^+ is contained in A; otherwise it is* reducible.

3.2. **Structure of rectangular subsets.** We start by giving a characterisation of the rectangular subsets. The first result reduces this problem to irreducible ones.

Proposition 3.3. *Let $A \subseteq \Phi^+$ be a rectangular subset. Then there exist uniquely determined connected subsets $I_1, \cdots, I_r$ of $[n]$ such that*

(i) any union of two or more of the subsets $I_1, \cdots, I_r$ is not connected;
(ii) $A_t := A \cap \Phi^+_{I_t}$ is an irreducible rectangular subset of $\Phi^+_{I_t}$;
(iii) $A = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_r$.

Proof. Let $I_1, \cdots, I_r$ be the connected components of the union of $\mathrm{supp}(\alpha)$ with α running through A.

We verify that A_t is an irreducible rectangular subset of $\Phi^+_{I_t}$. We fix $1 \leq t \leq r$ and verify (R1) and (R2).

Let $\alpha, \beta \in A_t$ be such that $\mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta)$ is connected. Clearly $\alpha \vee \beta \in A$ being A rectangular. Moreover, by definition $\mathrm{supp}(\alpha), \mathrm{supp}(\beta) \subseteq I_t$, and $\alpha \vee \beta \in \Phi^+_{I_t}$ follows by $\mathrm{supp}(\alpha \vee \beta) = \mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta)$. So $\alpha \wedge \beta \in A \cap \Phi^+_{I_t} = A_t$.

If $\alpha \wedge \beta$ exists, $\mathrm{supp}(\alpha \wedge \beta) = \mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta) \subseteq I_t$ implies $\alpha \wedge \beta \in \Phi^+_{I_t}$. But we have also $\alpha \wedge \beta \in A$ since A is rectangular, so $\alpha \wedge \beta \in A_t$. We have thus proved that A_t fulfils (R1).

For (R2) let $\alpha, \beta \in \Phi^+$ be such that $\alpha \wedge \beta$ exists and $\alpha \vee \beta, \alpha \wedge \beta \in A_t$. Since $A_t \subseteq A$ and A is rectangular, $\alpha, \beta \in A$. Moreover, $\alpha \vee \beta \in A_t$ implies $\mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta) \subseteq I_t$, which gives $\mathrm{supp}(\alpha), \mathrm{supp}(\beta) \subseteq I_t$, hence $\alpha, \beta \in A_t$. So also (R2) is fulfilled.

It remains to show the irreducibility of A_t: the longest root $\theta_t \in \Phi^+_{I_t} \cap A = A_t$. Let $I_t = \{r, r+1, \cdots, s\}$ with $r \leq s$. Then for any $r \leq k \leq s$ there exists $\beta_k \in A_t$ such that $k \in \mathrm{supp}(\beta_k) \subseteq I_t$. Since I_t is a connected component, the triangularity of A_t implies that

$$\theta_t = \beta_r \vee \beta_{r+1} \vee \cdots \vee \beta_s \in A_t.$$

The part (iii) is clear. Finally the subsets $I_1, I_2, \ldots, I_r$ are uniquely determined by A since they are the supports of the maximal elements in A. □

Given $A \subseteq \Phi^+$, we define two subsets

$$A^- := A \cap \{\alpha_{1,1}, \alpha_{1,2}, \cdots, \alpha_{1,n}\}, \quad A^+ := A \cap \{\alpha_{1,n}, \alpha_{2,n}, \cdots, \alpha_{n,n}\}.$$

The following lemma is clear.

Lemma 3.4. *The following properties holds:*

(1) for any $\gamma \in \Phi^+$, there exists a unique $\alpha \in \{\alpha_{1,1}, \cdots, \alpha_{1,n}\}$ and a unique $\beta \in \{\alpha_{1,n}, \cdots, \alpha_{n,n}\}$ such that $\gamma = \alpha \wedge \beta$;
(2) if $\alpha, \beta \in A^-$ (resp. A^+), then $\alpha \wedge \beta, \alpha \vee \beta \in A^-$ (resp. A^+);
(3) for $\alpha, \alpha' \in \Phi^+$, if $\alpha \wedge \alpha' \in A^-$ (resp. A^+), then $\alpha, \alpha' \in A^-$ (resp. A^+);
(4) for $\alpha \in A^-$, $\beta \in A^+$, we have $\alpha \vee \beta = \theta$;
(5) if $\gamma = \alpha \wedge \beta$, $\gamma' = \alpha' \wedge \beta'$ with $\alpha, \alpha' \in A^-$ and $\beta, \beta' \in A^+$, then

$$\gamma \vee \gamma' = (\alpha \vee \alpha') \wedge (\beta \vee \beta').$$

The following result characterises the irreducible rectangular subsets by A^- and A^+.

Proposition 3.5. *A subset $A \subseteq \Phi^+$ containing θ is irreducible rectangular if and only if*

$$A = \{\alpha \wedge \beta \mid \alpha \in A^-,\ \beta \in A^+,\ \alpha \wedge \beta \text{ exists}\}.$$

In particular, if A is irreducible rectangular and $\alpha, \beta \in A$, then $\alpha \vee \beta \in A$.

Proof. Assume that A is irreducible rectangular. We first prove that all existing meets $\alpha \wedge \beta$, with $\alpha \in A^-$, $\beta \in A^+$, are in A: indeed, let $\alpha := \alpha_{1,k}$ and $\beta := \alpha_{\ell,n}$ and suppose that $\alpha \wedge \beta$ exists, then $\ell \leq k$ and $\mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta) = [n]$ is connected so that $\alpha \wedge \beta \in A$ by (R1).

On the other hand, take $\gamma := \alpha_{i,j} \in A$; since $\theta = \alpha_{1,n} \in A$, by (R2) we have $\alpha_{1,j} \in A^-$ and $\alpha_{i,n} \in A^+$, hence $\gamma = \alpha_{1,j} \wedge \alpha_{i,n}$ is a meet of an element of A^- and an element of A^+.

Conversely, assume that A consists of all the elements of the form $\alpha \wedge \beta$ for $\alpha \in A^-$ and $\beta \in A^+$. We want to verify (R1) and (R2) for A.

Assume that $\gamma = \alpha \wedge \beta$, $\gamma' = \alpha' \wedge \beta'$ with $\alpha, \alpha' \in A^-$ and $\beta, \beta' \in A^+$. By Lemma 3.4 (5),

$$\gamma \vee \gamma' = (\alpha \vee \alpha') \wedge (\beta \vee \beta') \in A.$$

Moreover, if the meet $\gamma \wedge \gamma'$ exists, it equals $(\alpha \wedge \alpha') \wedge (\beta \wedge \beta')$, which is in A. We have proved (R1).

For (R2), assume that $\gamma, \gamma \in \Phi^+$ are such that $\gamma \wedge \gamma'$ exists and is in A. By Lemma 3.4 (1), we may assume that $\gamma = \alpha \wedge \beta$ and $\gamma' = \alpha' \wedge \beta'$ with $\alpha, \alpha' \in \{\alpha_{1,1}, \cdots, \alpha_{1,n}\}$ and $\beta, \beta' \in \{\alpha_{1,n}, \cdots, \alpha_{n,n}\}$.

Then $\gamma \wedge \gamma' = (\alpha \wedge \alpha') \wedge (\beta \wedge \beta')$. This is an element of A and moreover $\alpha \wedge \alpha' \in \{\alpha_{1,1}, \ldots, \alpha_{1,n}\}$ and $\beta \wedge \beta' \in \{\alpha_{1,n}, \ldots, \alpha_{n,n}\}$, so we find that $\alpha \wedge \alpha'$ is A^- and $\beta \wedge \beta' \in A^+$ by the uniqueness property stated in Lemma 3.4 (1). So, by Lemma 3.4 (3), $\alpha, \alpha' \in A^-$ and $\beta, \beta' \in A^+$. Hence $\gamma = \alpha \wedge \beta$ and $\gamma' = \alpha' \wedge \beta'$ are elements of A.

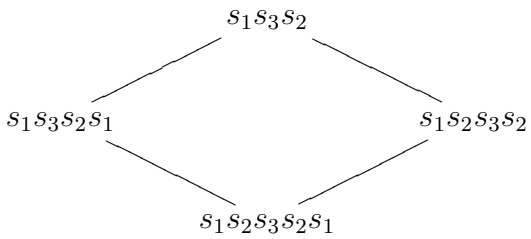
The final statement of the Proposition follows at once by Lemma 3.4 (5) and (2). □

3.3. **Rectangular elements.** We define rectangular elements in a similar way to the definition of triangular elements in [17, Proposition 1].

Definition 3.6. *An element $\tau \in W$ is called rectangular if $N(\tau)$ is a rectangular subset of Φ^+.*

Example 3.7. *Let $n = 3$ and $G = \mathsf{SL}_4(\mathbb{C})$. The elements $s_1s_3s_2$ and $s_1s_2s_3s_2s_1$ are not triangular hence not rectangular. The elements $s_1s_2s_3s_2$ and $s_1s_3s_2s_1$ are triangular but not rectangular. There are 22 triangular elements and 20 rectangular elements in $\mathfrak{S}_4$. Among the rectangular elements, there exists a unique element s_1s_3 which is not irreducible.*

These non-rectangular elements can be arranged into a diamond in the Bruhat order:



Example 3.8. *Let $n = 4$ and $\tau = s_1s_2s_3s_4s_1s_2s_1 \in W$. Then*

$$N(\tau) = \{\alpha_1, \alpha_{1,2}, \alpha_2, \alpha_{1,4}, \alpha_{2,4}, \alpha_{3,4}, \alpha_4\} \text{ and } N(\tau^{-1}) = \{\alpha_1, \alpha_{1,2}, \alpha_2, \alpha_{1,3}, \alpha_{2,3}, \alpha_3, \alpha_{1,4}\}.$$

It is clear that both τ and τ^{-1} are triangular elements in W, but only τ is a rectangular element. Indeed, for $\alpha_{1,3}, \alpha_{2,4} \in \Phi^+$, $\alpha_{1,3} \wedge \alpha_{2,4} = \alpha_{2,3}$ and $\alpha_{1,3} \vee \alpha_{2,4} = \alpha_{1,4}$ are contained in $N(\tau^{-1})$, but $\alpha_{2,4} \notin N(\tau^{-1})$.

Remark 3.9. *The Kempf elements studied in [17], while being triangular, are not necessarily rectangular, for example $s_1s_2s_3s_2$ is a Kempf element but it is not rectangular.*

The following proposition proves that the definition of rectangular elements in 2.4 is equivalent to the above one.

Proposition 3.10. *An element $\tau \in W$ is rectangular if and only if: for any $1 \leq i < k < j < \ell \leq n+1$ the following two conditions are equivalent*

(i) $\tau(i) > \tau(j)$ *and* $\tau(k) > \tau(\ell)$
(ii) $\tau(i) > \tau(\ell)$ *and* $\tau(k) > \tau(j)$.

Proof. The proof is a direct consequence of the following observation:

$$\alpha_{i,j-1} \in N(\tau) \text{ if and only if } \tau(i) > \tau(j).$$

Assume that $\tau \in W$ is a rectangular element, we prove the equivalence of (i) and (ii). For $1 \leq i < k < j < \ell \leq n+1$, we define $\alpha = \alpha_{i,j-1}$, $\beta = \alpha_{k,\ell-1} \in \Phi^+$. Moreover $\alpha \vee \beta = \alpha_{i,\ell-1}$ and $\alpha \wedge \beta = \alpha_{k,j-1}$. In any case $\mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta) = \{i, i+1, \ldots, \ell-1\}$ is connected.

So the condition (i) is equivalent to $\alpha, \beta \in N(\tau)$; condition (ii) is equivalent to: $\alpha \vee \beta, \alpha \wedge \beta \in N(\tau)$. Since $\mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta)$ is connected, (i) implies (ii) by (R1) in the definition of rectangular elements and (ii) implies (i) by (R2).

Now assume that (i) and (ii) are equivalent and we show that τ is rectangular. Let $\alpha = \alpha_{i,j-1}$ and $\beta = \alpha_{k,\ell-1}$ for some $1 \leq i < j \leq n+1$ and $1 \leq k < \ell \leq n+1$; by symmetry we can assume that $i \leq k$. Note that if $i = k$ or $j \geq \ell$ then (R1) and (R2) are obvious, so we may assume that $i < k$ and $j < \ell$.

We prove (R1). So let $\alpha, \beta \in N(\tau)$ with $\mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta)$ connected; this last condition forces $k \leq j$. If $k = j$ then $\alpha \vee \beta = \alpha + \beta \in N(\tau)$ and $\alpha \wedge \beta$ does not exist; if $k < j$ then (i) is fulfilled by $\alpha, \beta \in N(\tau)$, hence (ii) holds, and $\alpha \vee \beta, \alpha \wedge \beta \in N(\tau)$.

Finally we prove (R2). Let $\alpha = \alpha_{i,j-1}, \beta = \alpha_{k,\ell-1} \in \Phi$ with $i \leq k$ be such that $\alpha \wedge \beta$ exists and $\alpha \vee \beta, \alpha \wedge \beta \in N(\tau)$. The existence of $\alpha \wedge \beta$ implies $k < j$, so $1 \leq k < j < \ell$ and (ii) is fulfilled. Then (i) holds and $\alpha, \beta \in N(\tau)$. □

The previous proposition has the following two corollaries.

Corollary 3.11. *If $\tau \in W$ is a non–rectangular permutation and, in particular, $1 \leq i < k < j < \ell \leq n+1$ are such that the condition of Proposition 3.10 is violated by τ, then all permutations in the parabolic cosets $\tau \cdot (\mathfrak{S}_t \times \mathfrak{S}_{n+1-t})$, for all $k \leq t \leq j$, are non-rectangular.*

Proof. Let $a_h = \tau(h)$, for $h = 1, 2, \ldots, n+1$, and let $\mathbf{a} = a_1 a_2 \cdots a_{n+1}$ be the word representation of τ. The word representation $\mathbf{a}'$ of a permutation $\tau' = \tau\eta$ with $\eta \in \mathfrak{S}_t \times \mathfrak{S}_{n+1-t}$ for a fixed $k \leq t \leq j$, is the word $\mathbf{a}$ up to a shuffling of the first t entries and a shuffling of the last $n+1-t$ entries. In particular this may result in a possible swapping of a_i and a_k and a possible swapping of a_j and a_ℓ in $\mathbf{a}'$. But any such swapping just exchange condition (i) and (ii) in Proposition 3.10, so τ' is still non-rectangular. □

Corollary 3.12. *A permutation is rectangular if and only if it avoids the four patterns* 2413, 2431, 4213 *and* 4231. *The number r_n of rectangular elements in $\mathfrak{S}_n$ is given by the recursively defined sequence: $r_1 = 1$, $r_2 = 2$ and $r_n = 4r_{n-1} - 2r_{n-2}$ for any $r \geq 3$ (see entry* `A006012` *in* `OEIS`, [21]*).*

Proof. By Proposition 3.10, being rectangular is a property of the subwords of length 4 of the one-line expression of a permutation; in particular, this property is expressed by some inequalities. Hence it may be expressed as a condition about patterns of length 4. But the four patterns of the claim are the one-line expressions of the four non-rectangular elements of $\mathfrak{S}_4$ as in Example 3.7, so avoiding them is equivalent to being rectangular.

Now we prove the second statement about the number of rectangular permutations. As is proved in [2], the recursively defined integer sequence as in the statement counts the number of permutations in $\mathfrak{S}_n$ avoiding the four patterns 1324, 1423, 2314 and 2413.

For a permutation τ let $i(\tau)$ be the permutation having as one-line expression the backward reading of the one-line expression of τ^{-1}; the map $\tau \longmapsto i(\tau)$ is an involutive bijection of $\mathfrak{S}_n$. Moreover, if τ avoids the pattern σ then $i(\tau)$ avoids the pattern $i(\sigma)$. This completes the proof since the patterns 2413, 2431, 4213 and 4231 are obtained by the patterns 1324, 1423, 2314 and 2413 by applying i. □

For completeness we prove that also triangular elements are characterized by pattern avoidance.

Proposition 3.13. *A permutation is triangular if and only if it avoids the two patterns* 4231 *and* 2413. *The number of triangular elements in* $\mathfrak{S}_n$ *is counted by* `A032351` *in* `OEIS`, [22].

Proof. The characterization of triangular elements as the permutation avoiding the two patterns 4231 and 2413 is proved as in the previous Corollary 3.12; indeed note that $k \leq j$ is equivalent to $k < j$ in the definition of triangular element since for $k = j$ the condition there is always true.

Now we prove the statement about the number of triangular elements. The permutation avoiding the patterns 4231, 2413 are the same avoiding the patterns 4231 and 3142 by taking the inverse. This second set is in bijection with the set of permutations avoiding 3412 and 4231 by [3]. These are the smooth permutations counted by `A032351` in `OEIS`. □

4. From SL_{n+1} to SL_{2n}

Recall that $\widetilde{G} := \mathsf{SL}_{2n}$, and $\widetilde{\Phi}$ (resp. $\widetilde{\Phi}^+$, $\widetilde{\Phi}^-$, $\widetilde{W}$, $\widetilde{\Lambda}^+$, etc...) is the root system (resp. set of positive roots, set of negative roots, Weyl group, weight lattice, etc...) of $\widetilde{G}$.

4.1. **Special elements in the Weyl group of SL_{2n}.** The following subset of positive roots will play a central role in what follows

$$\nabla_n = \left\{ \widetilde{\alpha}_{i,j} \mid 1 \leq i \leq n \leq j \leq 2n-1,\, j-i \leq n-1 \right\}.$$

Let $\preccurlyeq$ be a total order on $\widetilde{\Phi}^+$ such that for any $\alpha, \beta \in \widetilde{\Phi}^+$, $\mathrm{ht}(\alpha) \leq \mathrm{ht}(\beta)$ implies $\alpha \preccurlyeq \beta$. For a subset $A = \{\beta_r \preccurlyeq \beta_{r-1} \preccurlyeq \cdots \preccurlyeq \beta_1\}$ of $\widetilde{\Phi}^+$, we define

$$\tau_A = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_r} \in \widetilde{W} \simeq \mathfrak{S}_{2n}.$$

Example 4.1. *(1) Let $n = 4$ and $A = \{\widetilde{\alpha}_4, \widetilde{\alpha}_{3,4}, \widetilde{\alpha}_{4,5}, \widetilde{\alpha}_{2,4}, \widetilde{\alpha}_{3,5}, \widetilde{\alpha}_{4,6}, \widetilde{\alpha}_{4,7}\} \subseteq \widetilde{\Phi}^+$. Then $\tau_A = s_4 s_5 s_6 s_7 s_3 s_4 s_2 \in \widetilde{W}$.*

(2) Let $A = \nabla_n$. It is easy to see by induction that

$$\tau_A = (s_n s_{n+1} \cdots s_{2n-1})(s_{n-1} s_{n-2} \cdots s_{2n-3}) \cdots (s_2 s_3) s_1.$$

The following lemma will be repeatedly used in subsequent proofs.

Lemma 4.2. *If $\alpha = \widetilde{\alpha}_{n-k+1,n+h-1}$ and $\beta = \widetilde{\alpha}_{n-v+1,n+u-1}$ are elements of ∇_n then*

$$s_\beta(\alpha) = \begin{cases} \alpha & \text{if } k \neq v \text{ and } h \neq u \\ \alpha - \beta & \text{if } (k = v \text{ and } h \neq u) \text{ or } (k \neq v \text{ and } h = u) \\ -\alpha & \text{if } k = v \text{ and } h = u. \end{cases}$$

Proof. First notice that for any two roots $\gamma, \delta \in \nabla_n$, $\gamma + \delta$ is not a root, since otherwise it would have a coefficient 2 in front of α_n when it is written into simple roots.

Let $a := \langle \alpha, \beta^\vee \rangle$. Since $\widetilde{G}$ is of type A, the possible values of a are $0, \pm 1$ and 2. The above argument shows that $a \neq -1$. The lemma follows from the following observations:

(1) $a = 2$ if and only if $\alpha = \beta$, *i.e.*, $k = v$ and $h = u$;
(2) if $\alpha \neq \beta$, $a = 0$ if and only if $k \neq v$, $h \neq u$.

□

A first consequence of this lemma is the following proposition showing that the order $\preccurlyeq$ is not essential in the definition of τ_A.

Proposition 4.3. *Let $A \subseteq \nabla_n$ be a subset. Then τ_A is independent of the total order $\preccurlyeq$, once it refines the height function.*

Proof. It suffices to show that for $\alpha \neq \beta \in \nabla_n$ with $\mathrm{ht}(\alpha) = \mathrm{ht}(\beta)$, $\langle \alpha, \beta^\vee \rangle = 0$. We take $\alpha = \widetilde{\alpha}_{n-k+1,n+h-1}$ and $\beta = \widetilde{\alpha}_{n-v+1,n+u-1}$, then

$$h + k - 1 = \mathrm{ht}(\alpha) = \mathrm{ht}(\beta) = u + v - 1.$$

Therefore $k \neq v$ is equivalent to $h \neq u$, and then $\langle \alpha, \beta^\vee \rangle = 0$ by Lemma 4.2. □

4.2. ∇_n-ideals.

Definition 4.4. *A subset $A \subseteq \nabla_n$ is called a* ∇_n–ideal, *if $\alpha \in A$, $\beta \in \nabla_n$ and $\beta \leq \alpha$ imply that $\beta \in A$.*

In the following we fix a ∇_n-ideal A. In order to investigate the properties of the element τ_A, we introduce some combinatorial data attached to A.

(1) For $1 \leq k \leq n$, we define $c_k(A)$ either as the maximum h such that $\widetilde{\alpha}_{n-k+1,n+h-1}$ is in A, or as 0 if no such element is in A;

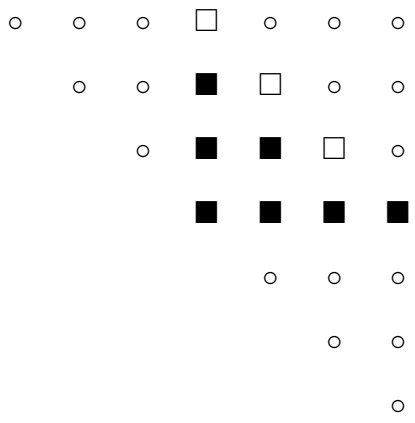
(2) For $1 \leq h \leq n$, we define $r_h(A)$ either as the maximum k such that $\widetilde{\alpha}_{n-k+1,n+h-1}$ is in A, or as 0 if no such element is in A.

Both sequences are non-increasing: $c_1(A) \geq c_2(A) \geq \cdots \geq c_n(A)$ and $r_1(A) \geq r_2(A) \geq \cdots \geq r_n(A)$.

When the set A is clear from the context, we will drop it from the notation.

Example 4.5. *In the following picture we arrange the positive roots as the corresponding root subspaces in a matrix of the special linear group. The elements of $\widetilde{\Phi}^+$ not in ∇_n are illustrated by circles; the elements of ∇_n are labeled by squares. We fix a ∇_n–ideal $A \subseteq \widetilde{\Phi}^+$ as in Example 4.1 (1), whose elements are depicted as black squares.*

Notice that $\alpha \leq \beta$ if and only α is to the left and/or down with respect to β.



In the above example we have: $c_1 = 4$, $c_2 = 2$, $c_3 = 1$, $c_4 = 0$, $r_1 = 3$, $r_2 = 2$, $r_3 = 1$ and $r_4 = 1$; these numbers can be easily deduced by the picture counting the numbers of black squares in A for each row and each column in ∇_n.

Our next aim is to find a formula for τ_A^{-1}, when restricted to the elements of ∇_n.

Proposition 4.6. *Let A be a ∇_n–ideal. We have the following formula for the restriction of τ_A^{-1} to ∇_n:*

$$\nabla_n \ni \alpha = \widetilde{\alpha}_{n-k+1,n+h-1} \longmapsto \begin{cases} -\widetilde{\alpha}_{n+h-r_h,n-k+c_k} & \text{if } \alpha \in A, \\ \widetilde{\alpha}_{n-k+c_k+1,n+h-r_h-1} & \text{if } \alpha \notin A. \end{cases}$$

Proof. We apply induction to $|A|$. If A is empty, then $\tau_A = \mathrm{id}$, $c_1 = c_2 = \cdots = c_n = 0$, $r_1 = r_2 = \cdots = r_n = 0$ and the claimed formula for τ_A^{-1} reads just $\alpha \longmapsto \alpha$ since $\alpha \notin A$ for all $\alpha \in \nabla_n$.

Now suppose $|A| \geq 1$, let $\beta = \widetilde{\alpha}_{n-v+1,n+u-1}$ be a $\preccurlyeq$–maximal element in A and set $A' = A \setminus \{\beta\}$. Then A' is a ∇_n–ideal and the claimed formula holds for A' by induction; note that the $\preccurlyeq$–maximality of β implies that $\tau_A^{-1} = \tau_{A'}^{-1} s_\beta$. We have also $c_v = u$ and $r_u = v$. Moreover for $k \neq v$, $c'_k := c_k(A')$ is equal to c_k, and $c'_v = c_v - 1 = u - 1$; analogously, for $h \neq u$, $r'_h = r_h(A')$ is equal to r_h, and $r'_u = r_u - 1 = v - 1$.

Now let $\alpha = \widetilde{\alpha}_{n-k+1,n+h-1}$ be an element of ∇_n. According to Lemma 4.2, we consider three different cases as follows

$$s_\beta(\alpha) = \begin{cases} \alpha & \text{if } k \neq v,\ h \neq u & \text{(Case 1)} \\ \alpha - \beta & \text{if } (k = v,\ h \neq u) \text{ or } (k \neq v,\ h = u) & \text{(Case 2)} \\ -\alpha & \text{if } k = v,\ h = u & \text{(Case 3)} \end{cases}$$

(Case 1) In this case $\alpha \neq \beta$. The formula holds by noticing that $\tau_A^{-1}(\alpha) = \tau_{A'}^{-1} s_\beta(\alpha) = \tau_A'^{-1}(\beta)$, $c_k = c'_k$ and $r_h = r'_h$.

(Case 2) We suppose $k = v$ and $h \neq u$, the other case is analogous. First assume that $\alpha \in A$, then $\tau_A^{-1}(\alpha) = \tau_{A'}^{-1}(\alpha - \beta) = \tau_{A'}^{-1}(\alpha) - \tau_{A'}^{-1}(\beta)$. We compute these last two terms.

For the first one, note that $\alpha \neq \beta$ implies $\alpha \in A'$. Then we have

$$\tau_{A'}^{-1}(\alpha) = -\widetilde{\alpha}_{n+h-r'_h,n-k+c'_k} = -\widetilde{\alpha}_{n+h-r_h,n-v+u-1}$$

where the first equality holds by induction, and for the second one we have used $r'_h = r_h$ and $c'_k = c'_v = u - 1$.

For the second term, using $\beta \notin A'$, we find

$$\tau_{A'}^{-1}(\beta) = \widetilde{\alpha}_{n-v+c'_v+1,n+u-r'_u-1} = \widetilde{\alpha}_{n-v+u,n+u-v}$$

where the first equality holds by induction, and for the second equality we have used $c'_v = u - 1$ and $r'_u = v - 1$.

Putting them together we conclude that

$$\tau_A^{-1}(\alpha) = -\widetilde{\alpha}_{n+h-r_h,n-v+u-1} - \widetilde{\alpha}_{n-v+u,n+u-v} = \widetilde{\alpha}_{n+h-r_h,n-v+u},$$

and the claimed formula holds since $k = v$ and $c_k = u$.

It remains to consider the case $\alpha \notin A$. Then $\alpha \notin A'$ and, as above, $\tau_A^{-1}(\alpha) = \tau_{A'}^{-1}(\alpha) - \tau_{A'}^{-1}(\beta)$. Moreover, by induction and by $c'_v = u - 1$ and $r'_h = r_h$, we have

$$\tau_{A'}^{-1}(\alpha) = \widetilde{\alpha}_{n-k+c'_k+1,n+h-r'_h-1} = \widetilde{\alpha}_{n-v+u,n+h-r_h-1}.$$

The above proved formula $\tau_{A'}^{-1}(\beta) = \widetilde{\alpha}_{n-v+u,n+u-v}$ is still valid. So we conclude that

$$\tau_A^{-1}(\alpha) = \widetilde{\alpha}_{n-v+u,n+h-r_h-1} - \widetilde{\alpha}_{n-v+u,n+u-v} = \widetilde{\alpha}_{n-v+u+1,n-h-r_h-1}$$

and the claimed formula holds since $k = v$ and $c_v = u$.

(Case 3) In this case we have $\alpha = \beta$, hence $\tau_A^{-1}(\alpha) = -\tau_{A'}^{-1}(\beta) = -\widetilde{\alpha}_{n-v+u,n+u-v}$ where the last equality follows by the computation executed above. Hence the claimed formula holds since $r_h = v$ and $c_k = u$.

□

Example 4.7. *We continue Example 4.5. In this case, we find*

$$\begin{array}{cccc} \widetilde{\alpha}_1 & & & \\ -\widetilde{\alpha}_2 & \widetilde{\alpha}_3 & & \\ -\widetilde{\alpha}_{2,4} & -\widetilde{\alpha}_4 & \widetilde{\alpha}_5 & \\ -\widetilde{\alpha}_{2,7} & -\widetilde{\alpha}_{4,7} & -\widetilde{\alpha}_{6,7} & -\widetilde{\alpha}_7 \end{array}$$

where we are reported the image via τ_A^{-1} of the elements in ∇_n arranged as in the previous picture.

As a first consequence of the previous formula we have the following proposition.

Proposition 4.8. *If A is a ∇_n–ideal then $\widetilde{N}(\tau_A^{-1}) = A$. In particular the length of τ_A is $|A|$.*

Proof. By Proposition 4.6, $A \subseteq \widetilde{N}(\tau_A^{-1})$ and also $\nabla_n \setminus A \subseteq \widetilde{P}(\tau_A^{-1})$. Since $\widetilde{\Phi}^+ = \widetilde{N}(\tau_A^{-1}) \sqcup \widetilde{P}(\tau_A^{-1})$, it suffices to show that for any $\alpha = \widetilde{\alpha}_{i,j} \in \widetilde{\Phi}^+ \setminus \nabla_n$, $\tau_A^{-1}(\alpha)$ is a positive root. There are three cases to consider:

(1) Assume that $j < n$, then $\alpha = \widetilde{\alpha}_{i,n} - \widetilde{\alpha}_{j+1,n}$. We set $k = n - i + 1$ and $k' = n - j$.
- If $\widetilde{\alpha}_{i,n} \in A$, as A is a ∇_n–ideal, $\widetilde{\alpha}_{j,n} \in A$. By Proposition 4.6, we have
$$\tau_A^{-1}(\alpha) = -\widetilde{\alpha}_{n+1-r_1,i-1+c_k} + \widetilde{\alpha}_{n+1-r_1,j+c_{k'}}.$$
Note that $i \leq j$ implies $i < j + 1$, hence $k > k'$ and $c_k \leq c_{k'}$; we conclude $j + c_{k'} \geq i - 1 + c_k$. This proves that $\tau_A^{-1}(\alpha)$ is a positive root.
- If $\widetilde{\alpha}_{i,n} \notin A$ and $\widetilde{\alpha}_{j+1,n} \in A$ then by Proposition 4.6, $\tau_A^{-1}(\widetilde{\alpha}_{i,n}) \in \widetilde{\Phi}^+$ and $\tau_A^{-1}(\widetilde{\alpha}_{j+1,n}) \in \widetilde{\Phi}^-$, so $\tau_A^{-1}(\alpha)$ is positive.
- If $\widetilde{\alpha}_{i,n} \notin A$ and $\widetilde{\alpha}_{j+1,n} \notin A$, by Proposition 4.6,
$$\tau_A^{-1}(\alpha) = \widetilde{\alpha}_{i+c_k,n-r_1} - \widetilde{\alpha}_{j+1+c_{k'},n-r_1}$$
and we conclude by $i + c_k \leq j + 1 + c_{k'}$ as already proved.

(2) When $i > n$, we proceed in a similar way as in (1) by symmetry.

(3) Assume that $j - i \geq n$: in this case $i < n < j$.

If $\widetilde{\alpha}_{i,n} \notin A$ then $\tau_A^{-1}(\widetilde{\alpha}_{i,n}) \in \widetilde{\Phi}^+$ since $\widetilde{\alpha}_{i,n} \in \nabla_n \setminus A$; moreover by the case (2) above, $\tau_A^{-1}(\widetilde{\alpha}_{n+1,j}) \in \widetilde{\Phi}^+$. Writing $\alpha = \widetilde{\alpha}_{i,n} + \widetilde{\alpha}_{n+1,j}$, the above argument shows that $\tau_A^{-1}(\alpha) \in \widetilde{\Phi}^+$. Similarly, if $\widetilde{\alpha}_{n,j} \notin A$ we argue in the same way.

It remains to consider the case $\widetilde{\alpha}_{i,n}, \widetilde{\alpha}_{n,j} \in A$. We write $\alpha = \widetilde{\alpha}_{i,n} - \widetilde{\alpha}_{n,n} + \widetilde{\alpha}_{n,j}$ and note that the image of these three roots are $-\widetilde{\alpha}_{n+1-r_1,i-1+c_k}$, $\widetilde{\alpha}_{n+1-r_1,n-1+c_1}$ and $-\widetilde{\alpha}_{j+1-r_h,n-1+c_1}$ respectively, where $h = j + 1 - n$ and $k = n - i + 1$. By the assumption $j - i \geq n$, it is easy to see $r_h < k$ and $c_k < h$, hence $i - 1 + c_k < j + 1 - r_h$ and this proves that $\tau_A^{-1}(\alpha) \in \widetilde{\Phi}^+$.

□

Another consequence of Proposition 4.6 is the following property related to the Bruhat order of $\widetilde{\Phi}$.

Proposition 4.9. *The map*

$$\nabla_n \supseteq A \ni \alpha \overset{i_A}{\longmapsto} -\tau_A^{-1}(\alpha) \in \widetilde{N}(\tau_A) \subseteq \widetilde{\Phi}^+$$

is a poset anti-isomorphism. Moreover for $\alpha, \beta \in A$ we have:

(i) *if $\alpha \wedge \beta$ exists then it is in A and $i_A(\alpha \wedge \beta) = i_A(\alpha) \vee i_A(\beta)$; on the other hand if $i_A(\alpha) \vee i_A(\beta)$ is in $\widetilde{N}(\tau_A)$ then $\alpha \wedge \beta$ exists;*
(ii) *if $\alpha \vee \beta$ is in A then $i_A(\alpha) \wedge i_A(\beta)$ exists and it is equal to $i_A(\alpha \vee \beta)$; on the other hand if $i_A(\alpha) \wedge i_A(\beta)$ exists then $\alpha \vee \beta$ is in A.*

Proof. By Proposition 4.8, $A = N(\tau_A^{-1})$, hence $i_A(A) \subseteq \widetilde{\Phi}^+$ and i_A maps A to $\widetilde{N}(\tau_A) \subseteq \widetilde{\Phi}^+$.

Let $\alpha = \widetilde{\alpha}_{n-k+1,n+h-1}$ and $\beta = \widetilde{\alpha}_{n-v+1,n+u-1}$ be two elements of A. By Proposition 4.6, $i_A(\alpha) = \widetilde{\alpha}_{n+h-r_h,n-k+c_k}$ and $i_A(\beta) = \alpha_{n+u-r_u,n-v+c_v}$. Note that $\alpha \leq \beta$ if and only if $k \leq v$ and $h \leq u$. Recalling that $t \longmapsto r_t$ and $t \longmapsto c_t$ are both non-increasing maps, we easily see that $k \leq v$ and $h \leq u$ if and only if $n + u - r_u \geq n + h - r_h$ and $n - v + c_v \leq n - k + c_k$; this last condition is clearly equivalent to $i_A(\alpha) \geq i_A(\beta)$. This finishes the proof that i_A is a poset anti-isomorphism.

Now we prove (i). If $\alpha \wedge \beta$ exists then it is an element of A since A is a ∇_n–ideal and $\alpha, \beta \geq \widetilde{\alpha}_{n,n}$. Let $a = \min(k, v)$ and $b = \min(h, u)$. Then $\alpha \wedge \beta = \widetilde{\alpha}_{n-a+1,n+b-1}$. By Proposition 4.8, $i_A(\alpha \wedge \beta) = \widetilde{\alpha}_{n+b-r_b,n-a+c_a}$. Since $t \longmapsto r_t, c_t$ are both non-increasing, we find $\min(n+h-r_h, n+u-r_u) = n+b-r_b$ and $\max(n-k+c_k, n-v+c_v) = n-a+c_a$, this proves that $i_A(\alpha) \vee i_A(\beta) = i_A(\alpha \wedge \beta)$.

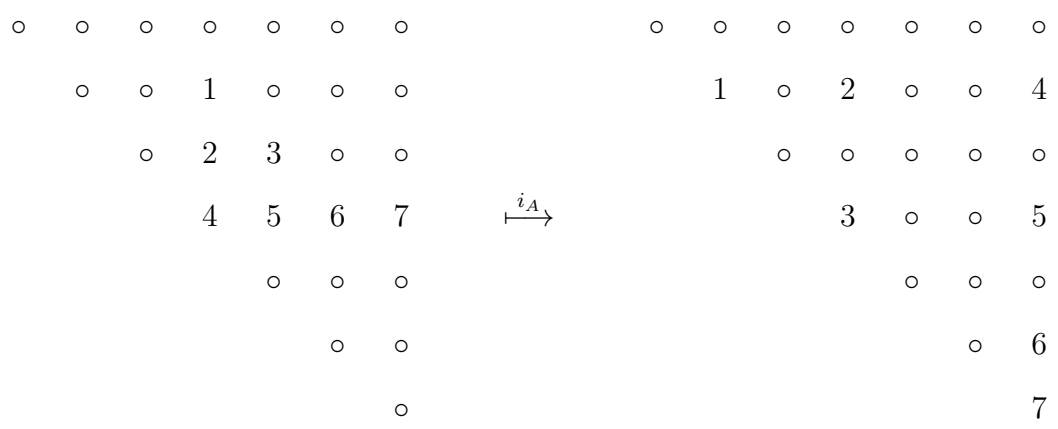
On the other hand note that $\eta = i_A(\alpha) \vee i_A(\beta)$ always exists. If it is an element of $\widetilde{N}(\tau_A)$, then there exists $\gamma \in A$ such that $i_A(\gamma) = \eta$ since $i_A : A \longrightarrow \widetilde{N}(\tau_A)$ is bijective. Since i_A an anti-isomorphism of posets and $\eta \geq i_A(\alpha), i_A(\beta)$, we get $\alpha, \beta \geq \gamma$, and hence $\alpha \wedge \beta$ exists.

The proof of (ii) is analogous, so omitted. □

Example 4.10. *We continue Example 4.7. Let $\tau_A = s_4 s_5 s_6 s_7 s_3 s_4 s_2$. Then*

$$\widetilde{N}(\tau_A) = \{\widetilde{\alpha}_2, \widetilde{\alpha}_{2,4}, \widetilde{\alpha}_{2,7}, \widetilde{\alpha}_4, \widetilde{\alpha}_{4,7}, \widetilde{\alpha}_{6,7}, \widetilde{\alpha}_7\}.$$

In this example, the map i_A is given by:



Our convention is that the positive root at position $k = 1, 2, \cdots, 7$ is mapped by i_A to the positive root labelled by the same number in the picture on the right hand side.

The following proposition on representable functions on ∇_n–ideals will be used to construct a suitable weight associated to a pair (λ, τ) with τ an element of W and λ a dominant weight for Φ.

Proposition 4.11. *Let A be a ∇_n–ideal and let $f : A \longrightarrow \mathbb{Z}$ be a map such that:*

(i) *for any $\alpha \in A$ we have $f(\alpha) \leq 0$,*
(ii) *if $\alpha, \beta \in A$ and $\alpha \leq \beta$ then $f(\alpha) \leq f(\beta)$,*
(iii) *if $\alpha, \beta, \alpha \vee \beta \in A$ then $f(\alpha \vee \beta) = f(\alpha) + f(\beta) - f(\alpha \wedge \beta)$,*
(iv) *if $\alpha, \beta \in A$ and $f(\alpha) + f(\beta) - f(\alpha \wedge \beta) < 0$ then $\alpha \vee \beta \in A$.*

Then there exists $\mu \in \widetilde{\Lambda}$ such that $\langle \mu, \alpha^\vee \rangle = f(\alpha)$ for all $\alpha \in A$ and $\langle \mu, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \widetilde{\Phi}^+ \setminus A$. In particular $\tau_A^{-1}\mu$ is a dominant weight.

Proof. If $A = \varnothing$ then we may take $\mu = 0$. From now on we assume that A is non-empty. Being a ∇_n–ideal, we have $\widetilde{\alpha}_{n,n} \in A$.

We will define certain integers $c_1, c_2, \ldots, c_{2n-1}$. First we set $c_n := f(\widetilde{\alpha}_{n,n})$. Let h be the minimum such that $\alpha_{h,n} \in A$ and k be the maximum such that $\alpha_{n,k} \in A$. Since the subset A is a ∇_n–ideal, we have:

- $\widetilde{\alpha}_{t,s} \notin A$ for all $1 \leq t \leq h-1$ and all $t \leq s \leq 2n-1$,
- $\widetilde{\alpha}_{t,n} \in A$ for all $h \leq t \leq n$,
- $\widetilde{\alpha}_{t,s} \notin A$ for all $k+1 \leq s \leq 2n-1$ and all $1 \leq t \leq s$,
- $\widetilde{\alpha}_{n,s} \in A$ for all $n \leq s \leq k$.

We set $c_t = f(\widetilde{\alpha}_{t,n}) - f(\widetilde{\alpha}_{t+1,n})$ for all $t = h, h+1, \ldots, n-1$ and $c_t = f(\widetilde{\alpha}_{n,t}) - f(\widetilde{\alpha}_{n,t-1})$ for $t = n+1, n+2, \ldots, k$. Moreover let $c_{h-1} = -f(\widetilde{\alpha}_{h,n})$, $c_{k+1} = -f(\widetilde{\alpha}_{n,k})$ and $c_t = 0$ for all $1 \leq t \leq h-2$ and all $k+2 \leq t \leq 2n-1$.

We will show that

$$\mu := \sum_{t=1}^{2n-1} c_t \widetilde{\varpi}_t \in \widetilde{\Lambda}$$

verifies the properties requested in the statement of the proposition.

We first show that for any $\alpha \in A$, $\langle \mu, \alpha^\vee \rangle = f(\alpha)$. Clearly $\langle \mu, \widetilde{\alpha}_{n,n}^\vee \rangle = c_n = f(\widetilde{\alpha}_{n,n})$. For $h \leq i \leq n-1$ we have

$$\langle \mu, \widetilde{\alpha}_{i,n}^\vee \rangle = c_i + c_{i+1} + \cdots + c_{n-1} + c_n = f(\widetilde{\alpha}_{i,n})$$

and, in the same way, for all $n+1 \leq j \leq k$,

$$\langle \mu, \widetilde{\alpha}_{n,j}^\vee \rangle = c_n + c_{n+1} + \cdots + c_j = f(\widetilde{\alpha}_{n,j}).$$

Now let $\widetilde{\alpha}_{i,j}$ be such that $i \leq n \leq j$ (or, equivalently, such that $\widetilde{\alpha}_{n,n} \leq \widetilde{\alpha}_{i,j}$). Then $\widetilde{\alpha}_{i,n} \vee \widetilde{\alpha}_{n,j} = \widetilde{\alpha}_{i,j}$, $\widetilde{\alpha}_{i,n} \wedge \widetilde{\alpha}_{j,n} = \widetilde{\alpha}_{n,n}$ and $\widetilde{\alpha}_{i,j} = \widetilde{\alpha}_{i,n} + \widetilde{\alpha}_{n,j} - \widetilde{\alpha}_{n,n}$.

Since A is a ∇_n–ideal, $\widetilde{\alpha}_{i,j} \in A$ implies $\widetilde{\alpha}_{i,n}, \widetilde{\alpha}_{n,j} \in A$. By (iii),

$$f(\widetilde{\alpha}_{i,n} \vee \widetilde{\alpha}_{n,j}) = f(\widetilde{\alpha}_{i,n}) + f(\widetilde{\alpha}_{n,j}) - f(\widetilde{\alpha}_{n,n})$$

and, using what we have already proved, we get

$$\langle \mu, \widetilde{\alpha}_{i,j}^\vee \rangle = \langle \mu, \widetilde{\alpha}_{i,n}^\vee \rangle + \langle \mu, \widetilde{\alpha}_{n,j}^\vee \rangle - \langle \mu, \widetilde{\alpha}_{n,n}^\vee \rangle = f(\widetilde{\alpha}_{i,n}) + f(\widetilde{\alpha}_{n,j}) - f(\widetilde{\alpha}_{n,n}) = f(\widetilde{\alpha}_{i,j}).$$

We show that $\langle \mu, \widetilde{\alpha}_{i,j}^\vee \rangle \geq 0$ for all $\widetilde{\alpha}_{i,j} \notin A$. Note that, by (i) and (ii), $c_t \geq 0$ for all $h \leq t \leq k$ but c_n is negative. It is then clear that $\langle \mu, \widetilde{\alpha}_{i,j}^\vee \rangle \geq 0$ if $j < n$ or $n < i$.

We assume that $i \leq n \leq j$. If furthermore $h \leq i$ and $j \leq k$ then, as above,

$$\langle \mu, \widetilde{\alpha}_{i,j}^\vee \rangle = f(\widetilde{\alpha}_{i,n}) + f(\widetilde{\alpha}_{n,j}) - f(\widetilde{\alpha}_{n,n})$$

and this is a non negative integer by (iv) since we are assuming $\widetilde{\alpha}_{i,j} \notin A$. On the other hand, if $i < h$ then

$$\langle \mu, \widetilde{\alpha}_{i,j}^\vee \rangle \geq c_{h-1} + c_h + \cdots + c_n \geq c_{h-1} + c_n = c_{h-1} + f(\widetilde{\alpha}_{h,n}) = 0$$

and, in the same way $\langle \mu, \widetilde{\alpha}_{i,j}^\vee \rangle \geq 0$ if $j > k$. This finishes the proof of the required properties for μ.

It remains to show that $\tau_A^{-1}\mu$ is a dominant weight. This is equivalent to $\langle \tau_A^{-1}\mu, \beta^\vee \rangle \geq 0$ for all $\beta \in \widetilde{\Phi}^+$. Fix such a β and let $\alpha := -\tau_A \beta$. If α is a positive root then $\alpha \in \widetilde{N}(\tau_A^{-1})$, so $\alpha \in A$ by Proposition 4.8. Hence $\langle \tau_A^{-1}\mu, \beta^\vee \rangle = -\langle \mu, \alpha^\vee \rangle = -f(\alpha) \geq 0$. On the other hand, if α is a negative root, then $-\alpha \in \widetilde{P}(\tau_A^{-1})$, so $-\alpha \in \widetilde{\Phi}^+ \setminus N(\tau_A^{-1}) = \widetilde{\Phi}^+ \setminus A$, again by Proposition 4.8; hence $\langle \tau_A^{-1}\mu, \beta^\vee \rangle = \langle \mu, -\alpha^\vee \rangle \geq 0$ by what have already proved for μ. □

Remark 4.12. *The weight μ, such that the associated function f fulfills the properties (i)-(iv) in the proposition, is in general not unique.*

4.3. **From rectangular elements of SL_{n+1} to SL_{2n}.** Let $\tau \in W$ be an irreducible rectangular element. Recall that in Section 3.3 for a subset $A \subseteq \Phi^+$ we have defined $A^- = A \cap \{\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,n}\}$ and $A^+ = A \cap \{\alpha_{1,n}, \alpha_{2,n}, \ldots, \alpha_{n,n}\}$. In particular let $n = j_1 > j_2 > \cdots > j_s \geq 1$ and $1 = i_1 < i_2 < \cdots < i_r \leq n$ be such that $N(\tau)^- = \{\alpha_{1,j_1}, \alpha_{1,j_2}, \ldots, \alpha_{1,j_s}\}$ and $N(\tau)^+ = \{\alpha_{i_1,n}, \alpha_{i_2,n}, \ldots, \alpha_{i_r,n}\}$ (note that, $\alpha_{1,j_1} = \alpha_{i_1,n} = \theta$).

By Proposition 3.5, $N(\tau)$ is the set of all α_{i_h,j_k} with $1 \leq h \leq r$, $1 \leq k \leq s$ and $i_h \leq j_k$. We define a map $\mathcal{D} : N(\tau) \to \nabla_n$ by:

$$N(\tau) \ni \alpha_{i_h,j_k} \overset{\mathcal{D}}{\longmapsto} \widetilde{\alpha}_{n-k+1,n+h-1} \in \nabla_n \subseteq \widetilde{\Phi}^+.$$

Let $A \subseteq \nabla_n$ be the image of $\mathcal{D}$. It is clear that $\mathcal{D}$ is a bijection from $N(\tau)$ to A.

Lemma 4.13. *If τ is an irreducible rectangular element then the image $A = \mathcal{D}(N(\tau))$ is a ∇_n–ideal in $\widetilde{\Phi}^+$.*

Proof. Let $\widetilde{\alpha}_{n-k+1,n+h-1} = \mathcal{D}(\alpha_{i_h,j_k}) \in A$, for some $1 \leq h \leq r$ and $1 \leq k \leq s$, and let $\widetilde{\alpha}_{n-u,n+v}$, for some $0 \leq u, v \leq n-1$, be an element of ∇_n with $\widetilde{\alpha}_{n-u,n+v} \leq \widetilde{\alpha}_{n-k+1,n+h-1}$. Then $u+1 \leq k$, $v+1 \leq h$, hence $i_{v+1} \leq i_h \leq j_k \leq j_{u+1}$, where the second inequality follows by $\alpha_{i_h,j_k} \in N(\tau)$. We have thus proved $i_{v+1} \leq j_{u+1}$, so $\alpha_{i_{v+1},j_{u+1}}$ is an element of $N(\tau)$ and clearly $\mathcal{D}(\alpha_{i_{v+1},j_{u+1}}) = \widetilde{\alpha}_{n-u,n+v}$. We conclude that $\widetilde{\alpha}_{n-u,n+v}$ is an element of A. This finishes the proof that A is a ∇_n–ideal. □

Proposition 4.14. *If τ is an irreducible rectangular element then the map $\mathcal{D} : N(\tau) \longrightarrow A$ is an anti-isomorphism of posets. Moreover for $\alpha, \beta \in N(\tau)$ we have:*

(i) *if $\alpha \wedge \beta$ exists then it is in $N(\tau)$ and $\mathcal{D}(\alpha \wedge \beta) = \mathcal{D}(\alpha) \vee \mathcal{D}(\beta)$; on the other hand if $\mathcal{D}(\alpha) \vee \mathcal{D}(\beta)$ is in A then $\alpha \wedge \beta$ exists;*
(ii) *$\alpha \vee \beta$ is in $N(\tau)$, $\mathcal{D}(\alpha) \wedge \mathcal{D}(\beta)$ exists and is an element of A, moreover $\mathcal{D}(\alpha \vee \beta) = \mathcal{D}(\alpha) \wedge \mathcal{D}(\beta)$.*

Proof. First notice that the following statements are equivalent:

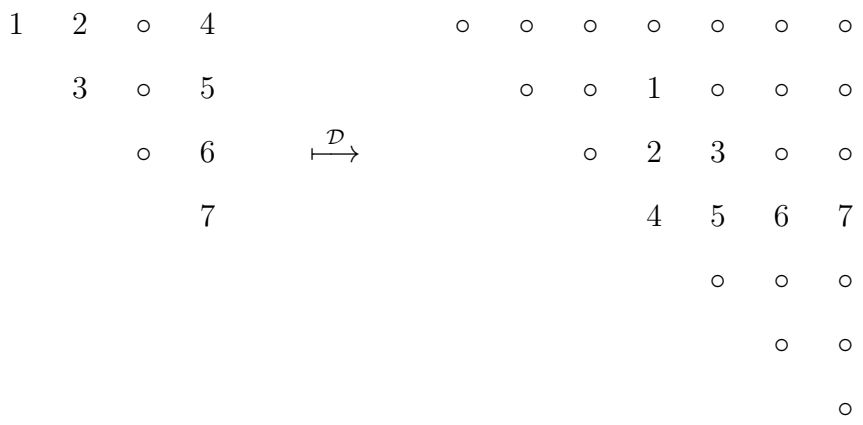
(1) $\alpha = \alpha_{i_h,j_k} \leq \alpha_{i_u,j_v} = \beta$;
(2) $i_u \leq i_h$ and $j_v \geq j_k$;
(3) $u \leq h$ and $v \leq k$;
(4) $\mathcal{D}(\alpha) = \widetilde{\alpha}_{n-k+1,n+h-1} \geq \widetilde{\alpha}_{n-v+1,n+u-1} = \mathcal{D}(\beta)$.

The equivalence between (1) and (4) shows that D is a poset anti-isomorphism.

Now we prove (i). If $\alpha \wedge \beta$ exists in Φ^+ then it is in $N(\tau)$ since τ is (irreducible) rectangular and the formula $\mathcal{D}(\alpha \wedge \beta) = \mathcal{D}(\alpha) \vee \mathcal{D}(\beta)$ follows easily by the definition of $\mathcal{D}$. On the other hand if $\eta = \mathcal{D}(\alpha) \vee \mathcal{D}(\beta)$ is in A, then $\eta = \mathcal{D}(\gamma)$ for certain $\gamma \in N(\tau)$, so $\gamma \leq \alpha, \beta$ by $\eta \geq \mathcal{D}(\alpha), \mathcal{D}(\beta)$. This shows that $\alpha \wedge \beta$ exists.

For (ii) note that $\alpha \vee \beta$ is in $N(\tau)$ by Proposition 3.5 while $\mathcal{D}(\alpha) \wedge \mathcal{D}(\beta)$ exists and is an element of A since A is a ∇_n–ideal by Lemma 4.13. The formula $\mathcal{D}(\alpha \vee \beta) = \mathcal{D}(\alpha) \wedge \mathcal{D}(\beta)$ follows at once by the definition of $\mathcal{D}$. □

Example 4.15. *Let $n = 4$. We consider $\tau = s_1s_2s_3s_4s_1s_2s_1 \in W$ as in Example 3.8. The map $\mathcal{D}$ sending $N(\tau)$ to ∇_n is depicted as follows:*



Here we used the same convention as in Example 4.10.

In Section 4.2 we have associated an element $\widetilde{\tau} := \tau_A$ in the Weyl group $\widetilde{W}$ of $\widetilde{\Phi}$ to the ∇_n–ideal A. Combining the previous proposition with Corollary 4.9 gives:

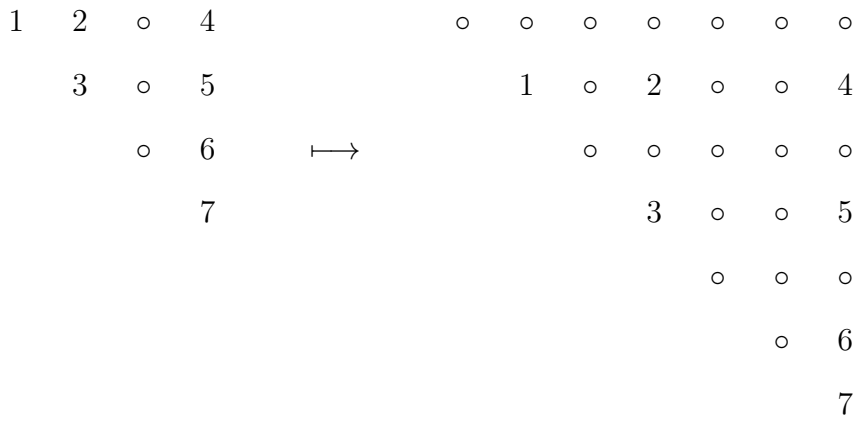
Corollary 4.16. *Let τ be an irreducible rectangular element. The composition*

$$\Phi^+ \supseteq N(\tau) \xrightarrow{\mathcal{D}} A \xrightarrow{i_A} \widetilde{N}(\widetilde{\tau}) \subseteq \widetilde{\Phi}^+$$

is an isomorphism of posets, it preserves the join and, when defined the meet. Moreover $\widetilde{\tau}$ is an irreducible rectangular element of the parabolic subgroup $\widetilde{W}_{\widetilde{I}}$ of $\widetilde{W}$ for a, in general proper, connected subset $\widetilde{I}$ of $\widetilde{\Delta}$.

Proof. By Proposition 4.9 and 4.14, this composition is a poset isomorphism. Being i_A an anti-isomorphism, the union $\widetilde{I}$ of the supports of the roots in $\widetilde{N}(\widetilde{\tau})$ is the support of $i_A(\widetilde{\alpha}_{n,n})$, hence $\widetilde{I}$ is connected. Now it is clear that $\widetilde{N}(\widetilde{\tau})$ is irreducible rectangular, with respect to $\widetilde{I}$, by Proposition 3.5 using the properties of $\mathcal{D}$ and i_A about meet and join stated in Proposition 4.9 and in Proposition 4.14. □

Example 4.17. *Combining Example 4.10 and 4.15 we have the following picture for $i_A \circ \mathcal{D}$:*



Remark 4.18. *As we have seen from the above example, the poset $\widetilde{N}(\widetilde{\tau})$ is isomorphic to $N(\tau)$. But the positive roots in $\widetilde{N}(\widetilde{\tau})$ are commutative, i.e. $\alpha+\beta$ is never a root for $\alpha,\beta\in\widetilde{N}(\widetilde{\tau})$, which is not the case for $N(\tau)$.*

Let $\lambda\in\Lambda^+$ be a dominant weight. We define

$$A\ni\mathcal{D}(\alpha)\overset{f}{\longmapsto}-\langle\lambda,\alpha^\vee\rangle\in\mathbb{Z}.$$

Lemma 4.19. *The subset A and the map $f:A\longrightarrow\mathbb{Z}$ fulfil the hypothesis of Proposition 4.11.*

Proof. By Lemma 4.13 A is a ∇_n–ideal and, being λ dominant, it is clear that $f(\mathcal{D}(\alpha))\leq 0$ for all $\alpha\in N(\tau)$ and (i) in Proposition 4.11 holds.

If $\mathcal{D}(\alpha)\leq\mathcal{D}(\beta)$ are elements of A then by Proposition 4.14 we have $\alpha\geq\beta$. So, being λ dominant, $f(\mathcal{D}(\alpha))\leq f(\mathcal{D}(\beta))$ and also (ii) in Proposition 4.11 holds.

Now we prove that the hypothesis (iii) is fulfilled. Let $\alpha,\beta\in N(\tau)$ and $\widetilde{\alpha}=\mathcal{D}(\alpha)$, $\widetilde{\beta}=\mathcal{D}(\beta)$. If $\widetilde{\alpha}\vee\widetilde{\beta}\in A$ then $\alpha\wedge\beta\in N(\tau)$ by (i) of Proposition 4.14 and $\mathcal{D}(\alpha\wedge\beta)=\widetilde{\alpha}\vee\widetilde{\beta}$. Moreover $\widetilde{\alpha}\wedge\widetilde{\beta}\in A$, being A a ∇_n–ideal, $\alpha\vee\beta\in N(\tau)$, being τ irreducible rectangular, and $\mathcal{D}(\alpha\vee\beta)=\widetilde{\alpha}\wedge\widetilde{\beta}$ by (ii) of Proposition 4.14. Finally since $\alpha\wedge\beta+\alpha\vee\beta=\alpha+\beta$, we find

$$\begin{aligned}
f(\widetilde{\alpha}\vee\widetilde{\beta}) &= -\langle\lambda,(\alpha\wedge\beta)^\vee\rangle\\
&= -\langle\lambda,(\alpha+\beta-\alpha\vee\beta)^\vee\rangle\\
&= -\langle\lambda,\alpha^\vee\rangle-\langle\lambda,\beta^\vee\rangle+\langle\lambda,(\alpha\vee\beta)^\vee\rangle\\
&= f(\widetilde{\alpha})+f(\widetilde{\beta})-f(\widetilde{\alpha}\wedge\widetilde{\beta}),
\end{aligned}$$

and (iii) is proved.

Finally, in order to prove that (iv) holds, note that, using the same notations as above,

$$f(\widetilde{\alpha})+f(\widetilde{\beta})-f(\widetilde{\alpha}\wedge\widetilde{\beta})=-\langle\lambda,(\alpha+\beta-(\alpha\vee\beta))^\vee\rangle.$$

Since the support of $\alpha\vee\beta$ is the smallest convex subset containing $\mathrm{supp}(\alpha)\cup\mathrm{supp}(\beta)$, we see that if this integer is negative then the intersection of $\mathrm{supp}(\alpha)$ and $\mathrm{supp}(\beta)$ is not empty. But τ is rectangular, hence this implies that $\alpha\wedge\beta$ exists in Φ^+; hence it is in $N(\tau)$ by being τ rectangular and so $\widetilde{\alpha}\vee\widetilde{\beta}=\mathcal{D}(\alpha\wedge\beta)$ is an element of A. □

We can now apply Proposition 4.11 and conclude as in the following corollary.

Corollary 4.20. *Let τ be an irreducible rectangular element of W and λ be a dominant weight in Λ^+. Then there exists a dominant weight $\widetilde{\lambda}\in\widetilde{\Lambda}^+$ and an element $\widetilde{\tau}$ of the Weyl group of $\widetilde{\Phi}$ such that: $\langle\widetilde{\tau}\widetilde{\lambda},\widetilde{\alpha}^\vee\rangle=-\langle\lambda,\alpha^\vee\rangle$ for $\widetilde{\alpha}\in A$, with $\widetilde{\alpha}=\mathcal{D}(\alpha)$ (so these scalar products are non-positive), and $\langle\widetilde{\tau}\widetilde{\lambda},\beta^\vee\rangle\geq 0$ if $\beta\in\widetilde{\Phi}^+\setminus A$.*

A particular case will be important in our application to degenerate Schubert varieties.

Proposition 4.21. *Let τ be an irreducible rectangular element of W and let $\lambda=\varpi_r$ be a fundamental weight. Then, with notation as in the previous corollary, we can choose*

$\widetilde{\lambda}$ to be a fundamental weight $\widetilde{\varpi}_{r'}$. In particular $r' = n - v + u$ where $v = |\{\alpha_{1,j} \in N(\tau) \,|\, j \geq r\}|$ and $u = |\{\alpha_{i,n} \in N(\tau) \,|\, i \leq r\}|$.

Proof. We follow the construction of μ as in the proof of Proposition 4.11. First of all $\theta \in N(\tau)$, being τ irreducible rectangular, and $\mathcal{D}(\theta) = \widetilde{\alpha}_{n,n}$ since $\mathcal{D}$ is an anti-isomorphism of posets; in particular $\langle \mu, \widetilde{\alpha}_{n,n}^\vee \rangle = -1$. Moreover $\langle \mu, \widetilde{\alpha}^\vee \rangle = 0, -1$ for all $\widetilde{\alpha} \in A$ by the definition using that λ is fundamental.

Now let $\alpha_{i,j} \in A$. By $\widetilde{\alpha}_{i,j} + \widetilde{\alpha}_{n,n} = \widetilde{\alpha}_{i,n} + \widetilde{\alpha}_{n,j}$ we find: $\langle \mu, \widetilde{\alpha}_{i,j}^\vee \rangle = -1$ if and only if $\langle \mu, \alpha_{i,n}^\vee \rangle = \langle \mu, \alpha_{n,j}^\vee \rangle = -1$. It is then clear that $\mu = \widetilde{\varpi}_{v-1} - \widetilde{\varpi}_n + \widetilde{\varpi}_{u+1}$, with u and v as in the Proposition.

Hence $\langle \mu, \widetilde{\alpha}^\vee \rangle \geq -1$ for all $\widetilde{\alpha} \in \widetilde{\Phi}^+$ and we conclude that $\widetilde{\lambda}$ is fundamental. Since $\widetilde{\lambda}$ is the unique dominant weight in the $\widetilde{W}$–orbit of μ it is easy to check that $\widetilde{\lambda} = \widetilde{\varpi}_{n-v+u}$. □

5. Proof of Theorem 2.4

The proof of the theorem is executed in the following steps. We fix an irreducible rectangular element $\tau \in W$.

5.1. **Dimension estimation.** The first step of the proof is to show that the corresponding Demazure modules have the same dimension. The proof makes fully use the basis constructed in [15, 17] and its parametrisation by the Feigin-Fourier-Littelmann-Vinberg (FFLV) polytope, as well as all results we have developed for rectangular elements.

Proposition 5.1. *If $\tau \in W$ is an irreducible triangular element, then*

$$\dim V_\tau(\lambda) = \dim V_{\widetilde{\tau}}(\widetilde{\lambda}).$$

Proof. By Corollary 4.16, $\widetilde{\tau}$ is a rectangular element hence a triangular element. By [17, Proposition 2, Corollary 2], there exists a monomial basis of $V_\tau(\lambda)$ (resp. $V_{\widetilde{\tau}}(\widetilde{\lambda})$) parametrised by the lattice points in the marked chain polytope [1, 15] associated to the poset $N(\tau)$ (resp. $\widetilde{N}(\widetilde{\tau})$). The markings in the marked chain polytopes are given by the inner products between λ (resp. $\widetilde{\lambda}$) and some roots in $N(\tau)$ (resp. $\widetilde{N}(\widetilde{\tau})$), which are preserved by Corollary 4.16.

As conclusion, the two marked chain polytopes have the same number of lattice points, which proves the equality on the dimensions. □

5.2. **The case of fundamental weights.** Notice that the negative roots contained in $\mathfrak{n}_\tau^-$ are $-N(\tau)$.

Since both Lie algebras $\mathfrak{n}_\tau^{-,a}$ and $\widetilde{\mathfrak{n}}_{\widetilde{\tau}}^+$ are abelian, the map $\mathcal{D}$ defined in the beginning of Section 4.3 gives an isomorphism of Lie algebras

$$\Psi_\tau : \mathfrak{n}_\tau^{-,a} \xrightarrow{\sim} \widetilde{\mathfrak{n}}_{\widetilde{\tau}}^+$$

sending the root space of weight α to that of weight $-\mathcal{D}(\alpha)$.

Then Ψ_τ induces an isomorphism of algebra

$$\Psi_\tau : S(\mathfrak{n}_\tau^{-,a}) \xrightarrow{\sim} S(\widetilde{\mathfrak{n}}_{\widetilde{\tau}}^+).$$

Let ϖ_r be a fundamental weight of G. Recall that we have defined the cyclic module

$$V_\tau^{-,a}(\varpi_r) := S(\mathfrak{n}_\tau^{-,a}) \cdot v_{\varpi_r}^a \text{ (resp. } V^a(\varpi_r) := S(\mathfrak{n}^{-,a}) \cdot v_{\varpi_r}^a).$$

Let $I_\tau^a(\varpi_r)$ (resp. $I^a(\varpi_r)$) denote its defining ideal (see Section 2.2) in $S(\mathfrak{n}_\tau^{-,a})$ (resp. $S(\mathfrak{n}^{-,a})$).

Let Φ_r^+ denote the nilpotent radical consisting of all roots in Φ^+ supported at r. We first study the defining ideal $I^a(\varpi_r)$.

Lemma 5.2. *The defining ideal $I^a(\varpi_r)$ is generated by the following elements:*

(G1) *f_α for $\alpha \in \Phi^+ \setminus \Phi_r^+$;*

(G2) *$f_\alpha f_\beta + f_{\alpha\vee\beta} f_{\alpha\wedge\beta}$ for $\alpha, \beta \in \Phi_r^+$ such that both $\alpha \vee \beta$ and $\alpha \wedge \beta$ exist.*

Proof. By [15, Theorem 4], the defining ideal $I^a(\varpi_r)$ is given by:

$$I^a(\varpi_r) = S(\mathfrak{n}^{-,a}) \cdot \operatorname{span}\left\{ U(\mathfrak{n}^+) \circ f_\alpha^{\langle \varpi_r, \alpha^\vee \rangle + 1} \mid \alpha \in \Phi^+ \right\},$$

where elements in $\mathfrak{n}^+$ act on $\mathfrak{n}^-$ as derivations by: for $\alpha, \beta \in \Phi^+$,

$$e_\alpha \circ f_\beta = \begin{cases} f_{\beta-\alpha}, & \text{if } \beta - \alpha \in \Phi^+, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that this action preserves the PBW degree. We compute $U(\mathfrak{n}^+) \circ f_\alpha^{\langle \varpi_r, \alpha^\vee \rangle + 1}$:

(1) Let $\alpha \in \Phi^+ \setminus \Phi_r^+$. Since the action of $U(\mathfrak{n}^+)$ preserves the PBW degree, $\{U(\mathfrak{n}^+) \circ f_\alpha\}$ is a linear span of elements of PBW degree one. This shows that

$$\{U(\mathfrak{n}^+) \circ f_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_r^+\} = \operatorname{span}\{f_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_r^+\}.$$

(2) We compute $U(\mathfrak{n}^+) \circ f_\alpha^2$ for $\alpha \in \Phi_r^+$.

Let $\alpha = \alpha_{p,q}$ for $p \le r \le q$. We consider $e_{s,t} \circ f_{p,q}^2$: it is non-zero if and only if either $s = p$, $t < q$ or $p < s$, $t = q$. By symmetry we consider the case $s = p$ and $t < q$, then $e_{p,t} \circ f_{p,q}^2 = f_{t+1,q} f_{p,q}$. If $r \le t$ then it is in the ideal generated by the elements in (G1). We assume that $t < r$, which means that $f_{t+1,q} f_{p,q}$ is of the form in (G2).

We consider $e_{i,j} \circ f_{t+1,q} f_{p,q}$: similar to the above argument, the only case we need to consider is $j = q$ and $i \ge r + 1$. In this case,

$$e_{i,q} \circ f_{t+1,q} f_{p,q} = f_{t+1,i-1} f_{p,q} + f_{t+1,q} f_{p,i-1},$$

the right hand side has the form in (G2).

It remains to show that for $\alpha, \beta \in \Phi_r^+$, $\gamma \in \Phi^+$,

$$e_\gamma \circ (f_\alpha f_\beta + f_{\alpha\wedge\beta} f_{\alpha\vee\beta}) \tag{5.1}$$

is in the ideal generated by (G1) and (G2). According to the above arguments, it suffices to consider the case where $\alpha = \alpha_{i,j}$, $\beta = \alpha_{k,\ell} \in \Phi_r^+$ with $1 \le i < k \le j < \ell \le n$, then $\alpha \vee \beta = \alpha_{i,\ell}$ and $\alpha \wedge \beta = \alpha_{k,j}$.

Consider the case where $\gamma = \alpha_{i,t}$.

- If $t \ge r$ then the element in (5.1) is in the ideal generated by (G1).

- If $t < r$ then the element in (5.1) reads $f_{t+1,j}f_{k,\ell} + f_{t+1,\ell}f_{k,j}$, which is in (G2).

In the similar way one studies the cases $\alpha = \alpha_{k,t}$, $\alpha_{t,j}$ and $\alpha_{t,\ell}$.

□

Corollary 5.3. *The defining ideal $I^a_\tau(\varpi_r)$ is generated by the following elements:*

(G1') *f_α for $\alpha \in N(\tau) \setminus \Phi^+_r$;*
(G2') *$f_\alpha f_\beta + f_{\alpha\vee\beta}f_{\alpha\wedge\beta}$ for $\alpha, \beta \in N(\tau) \cap \Phi^+_r$ such that both $\alpha \vee \beta$ and $\alpha \wedge \beta$ exist.*

Proof. Since $N(\tau)$ is a rectangular subset, it suffices to observe the obvious relation

$$I^a_\tau(\varpi_r) = I^a(\varpi_r) \cap S(\mathfrak{n}^{-,a}_\tau),$$

then apply Lemma 5.2 and the PBW theorem. □

Let r' be the index of the fundamental module for $\widetilde{G}$ as in Proposition 4.21 by taking $\lambda = \varpi_r$. The Demazure module $V_{\widetilde{\tau}}(\widetilde{\varpi}_{r'})$ is a cyclic $S(\widetilde{\mathfrak{n}}^+_{\widetilde{\tau}})$-module with a cyclic vector $v_{\widetilde{\tau}(\widetilde{\varpi}_{r'})}$. Let $\widetilde{I}_{\widetilde{\tau}}(\widetilde{\varpi}_{r'})$ denote its defining ideal.

We consider the composition

$$\varphi_r : S(\mathfrak{n}^{-,a}_\tau) \xrightarrow{\sim} S(\widetilde{\mathfrak{n}}^+_{\widetilde{\tau}}) \longrightarrow S(\widetilde{\mathfrak{n}}^+_{\widetilde{\tau}})/\widetilde{I}_{\widetilde{\tau}}(\widetilde{\varpi}_{r'}) \simeq V_{\widetilde{\tau}}(\widetilde{\varpi}_{r'}).$$

Lemma 5.4. *The map φ_r passes through $I^a_\tau(\varpi_r)$.*

Proof. We show that the generators in Corollary 5.3 are annihilated by φ_r.

Let $\alpha \in N(\tau) \setminus \Phi^+_r$. Then we have

$$\varphi_r(f_\alpha) = \Psi_r(f_\alpha) \cdot v_{\widetilde{\varpi}_{r'}} = \widetilde{e}_{\mathcal{D}(\alpha)} \cdot v_{\widetilde{\tau}(\widetilde{\varpi}_{r'})}.$$

By Corollary 4.20,

$$\langle \widetilde{\tau}(\widetilde{\varpi}_{r'}), \mathcal{D}(\alpha)^\vee \rangle = \langle \tau(\varpi_r), \alpha^\vee \rangle = -\langle \varpi_r, \alpha^\vee \rangle = 0,$$

implying that $f_\alpha \in \ker \varphi_r$.

Let $\alpha, \beta \in N(\tau) \cap \Phi^+_r$. By Corollary 4.16, we have

$$\begin{aligned} \varphi_r(f_\alpha f_\beta + f_{\alpha\vee\beta}f_{\alpha\wedge\beta}) &= (\widetilde{e}_{\mathcal{D}(\alpha)}\widetilde{e}_{\mathcal{D}(\beta)} + \widetilde{e}_{\mathcal{D}(\alpha\vee\beta)}\widetilde{e}_{\mathcal{D}(\alpha\wedge\beta)}) \cdot v_{\widetilde{\tau}(\widetilde{\varpi}_{r'})} \\ &= \widetilde{\tau}((\widetilde{f}_{\mathcal{D}_A(\alpha)}\widetilde{f}_{\mathcal{D}_A(\beta)} + \widetilde{f}_{\mathcal{D}_A(\alpha)\vee\mathcal{D}_A(\beta)}\widetilde{f}_{\mathcal{D}_A(\alpha)\wedge\mathcal{D}_A(\beta)}) \cdot v_{\widetilde{\varpi}_{r'}}) \end{aligned}$$

where $\mathcal{D}_A := i_A \circ \mathcal{D}$. Since the fundamental representation $\widetilde{V}(\varpi_{r'})$ is minuscule, all weight spaces have dimension one, and the above formula gives 0. □

Proposition 5.5. *The map φ_r is an isomorphism of $\mathfrak{n}^{-,a}_\tau$-modules*

$$\varphi_r : V^{-,a}_\tau(\varpi_r) \xrightarrow{\sim} V_{\widetilde{\tau}}(\widetilde{\varpi}_{r'}).$$

Proof. Combine Lemma 5.4 and Proposition 5.1. □

5.3. **General case.** We are now ready for the general case. The idea of the proof is similar to [9, Theorem 4.1], but simpler for the following two reasons:

(1) One of the goal of [9] is to recover the defining ideal of the PBW-degenerate modules, in our case we assume this knowledge.
(2) The dimension estimation (5.1) is established.

Proposition 5.6. *Let $\lambda \in \Lambda^+$ and $\tau \in W$ be an irreducible rectangular element. There exists an isomorphism*

$$\varphi_\tau : V_\tau^{-,a}(\lambda) \xrightarrow{\sim} V_{\widetilde{\tau}}(\widetilde{\lambda}).$$

of $\mathfrak{n}_\tau^{-,a}$-modules and hence of $N_\tau^{-,a}$-modules.

Proof. Let $\lambda = a_1\varpi_1 + \cdots + a_n\varpi_n$ for $a_1, \cdots, a_n \geq 0$. Then in Proposition 4.11 we can choose $\widetilde{\lambda} = a_1\widetilde{\varpi}_{1'} + \cdots + a_n\widetilde{\varpi}_{n'}$, where for $k = 1, \cdots, n$, the index k' is obtained from Proposition 4.21 by taking $r = k$.

We consider the following diagram:

$$\begin{array}{ccc} V_{\widetilde{\tau}}(\widetilde{\lambda}) & \xrightarrow{\sim} & V_{\widetilde{\tau}}(\widetilde{\varpi}_{1'})^{\otimes a_1} \otimes \cdots \otimes V_{\widetilde{\tau}}(\widetilde{\varpi}_{n'})^{\otimes a_n} \\ {\scriptstyle \varphi_\tau}\uparrow & & \uparrow{\scriptstyle \sim} \\ V_\tau^{-,a}(\lambda) & \xrightarrow{\pi} & V_\tau^{-,a}(\varpi_1)^{\otimes a_1} \otimes \cdots \otimes V_\tau^{-,a}(\varpi_n)^{\otimes a_n}, \end{array}$$

where

- the upper horizontal arrow is the isomorphism onto the Cartan component of the tensor product, the existence of such an isomorphism is a consequence of the standard monomial theory for the multicone over Schubert varieties, see [20], [19] and [10];
- the right vertical arrow is the tensor product of various φ_r's in Proposition 5.5;
- the map π is the projection onto the Cartan component, its existence follows by the compatibility between the PBW-filtration and the tensor product; it is a surjection by the Minkowski property [17, Lemma 3];
- the left vertical map φ_τ is defined as the composition of the other three maps, making the diagram commutative. By construction it is a surjective $\mathfrak{n}_\tau^{-,a}$-module morphism.

Finally note that, by Proposition 5.1, $V_\tau^{-,a}(\lambda)$ and $V_{\widetilde{\tau}}(\widetilde{\lambda})$ have the same dimension, hence the surjective map φ_τ is an isomorphism. □

5.4. **Proof of Theorem 2.4.** We first consider the case where $\tau \in W$ is an irreducible rectangular element.

By Proposition 5.6, the isomorphism φ_τ induces

$$X_\tau^a(\lambda) \subseteq \mathbb{P}(V_\tau^{-,a}(\lambda)) \simeq \mathbb{P}(V_{\widetilde{\tau}}(\widetilde{\lambda})) \supseteq X_{\widetilde{\tau}}(\widetilde{\lambda}).$$

We look at the open dense parts $X_{\widetilde{\tau}}(\widetilde{\lambda})^\circ := \widetilde{B} \cdot v_{\widetilde{\tau}(\widetilde{\lambda})}$ and $X_\tau^a(\lambda)^\circ := N_\tau^{-,a} \cdot v_\lambda^a$ and show that they are isomorphic.

Again by Proposition 5.6, we have the following isomorphisms:

$$\widetilde{B}\cdot v_{\widetilde{\tau}(\widetilde{\lambda})}\simeq\prod_{\gamma\in\widetilde{N}(\widetilde{\tau}^{-1})}\widetilde{U}_{\gamma}\cdot v_{\widetilde{\tau}(\widetilde{\lambda})}\simeq\prod_{\beta\in N(\tau)}U_{-\beta}\cdot v_{\lambda}^{a}\simeq N_{\tau}^{-,a}\cdot v_{\lambda}^{a}.$$

The $\mathbb{G}^M$-equivariancy comes from the fact that both Lie algebras $\mathfrak{n}_{\tau}^{-,a}$ and $\widetilde{\mathfrak{n}}_{\widetilde{\tau}}^{+}$ are abelian.

Let $\tau\in W$ be a rectangular element. According to Proposition 3.3, there exists pairwise disjoint connected subsets $I_1,\cdots,I_t\subseteq[n]$ and $A_k=N(\tau)\cap\Phi_{I_k}^{+}$ such that A_k is a rectangular subset in $\Phi_{I_k}^{+}$ and $N(\tau)=A_1\sqcup\cdots\sqcup A_t$. Then there exist $\tau_1,\cdots,\tau_t$ such that $\tau_k\in W_{I_k}$ is an irreducible rectangular element and $\tau=\tau_1\cdots\tau_t$ where W_{I_k} is the Weyl group of $\Phi_{I_k}^{+}$. To terminate the proof of Theorem 2.4, it suffices to apply the above irreducible rectangular case to each $\tau_1,\cdots,\tau_t$ and the sub-root systems separately.

6. Consequences and examples

Corollary 6.1. *Let $\tau\in W$ be a rectangular element and $\lambda\in\Lambda^{+}$. The degenerate Schubert variety $X_{\tau}^{a}(\lambda)$ is projectively normal, has rational singularities and is Frobenius split.*

Example 6.2. *Let $\tau=w_0\in W$ be the longest element in the Weyl group, which is an irreducible rectangular element. It is easy to verify that the Weyl group element $\widetilde{\tau}\in\widetilde{W}$ coincides with the permutation σ given in* [8]. *In this case, our Theorem 2.4 coincides with* [8, Theorem 1.2].

Remark 6.3. *We consider $\mathfrak{g}=\mathfrak{sl}_4$ and $s_1s_3s_2\in\mathfrak{S}_4$. Our construction provides the element $s_5s_4s_3s_5s_4\in\mathfrak{S}_6$. It is easy to check, that for the first (and the third) fundamental weight of $\mathfrak{sl}_4$, there is no dominant integral weight $\widetilde{\lambda}$ for $\mathfrak{sl}_6$ such that $V_{s_1s_3s_2}^{-,a}(\varpi_1)$ is isomorphic to the Demazure module $V_{s_5s_4s_3s_5s_4}(\widetilde{\lambda})$.*

Remark 6.4. *Our approach is different to the one in* [4]. *There the authors study which "classical" Schubert variety in a flag variety of type* A *stays irreducible and/or is still a Schubert variety when degenerated following a certain degeneration of the flag variety to the degenerate flag variety for type* A. *But this kind of degeneration of Schubert varieties is different from the degenerate Schubert varieties considered in this paper, for example our degenerate Schubert varieties are all irreducible by definition.*

References

[1] F. Ardila, T. Bliem, and D. Salazar, *Gelfand-Tsetlin polytopes and Feigin-Fourier-Littelmann-Vinberg polytopes as marked poset polytopes.* J. Combin. Theory Ser. A 118 (2011), no. 8, 2454–2462.

[2] Y. Biers-Ariel, *The Number of Permutations Avoiding a Set of Generalized Permutation Patterns*, Journal of Integer Sequences, Vol. 20 (2017), Article 17.8.3.

[3] M. Bóna, *The permutation classes equinumerous to the smooth class*, The Electronic Journal of Combinatorics, 5, (1998), #R31.

[4] L. Bossinger, M. Lanini, *Following Schubert varieties under Feigin's degeneration of the flag variety*, arXiv:1802.04320.

[5] N. Bourbaki, Groupes et algèbres de Lie, Ch. 4–6, Éléments de Mathématique, Fasc. XXXIV, Hermann, Paris, 1968; Masson, Paris, 1981.
[6] G. Cerulli Irelli, X. Fang, E. Feigin, G. Fourier, M. Reineke, *Linear degenerations of flag varieties*. Mathematische Zeitschrift, (2017) 287:615–654.
[7] G. Cerulli Irelli, E. Feigin, M. Reineke, *Quiver Grassmannians and degenerate flag varieties*, Algebra and Number Theory 6-1 (2012), 165–194.
[8] G. Cerulli Irelli, M. Lanini, *Degenerate flag varieties of type A and C are Schubert varieties*. Internat. Math. Res. Notices (2014).
[9] G. Cerulli Irelli, M. Lanini, P. Littelmann, *Degenerate flag varieties and Schubert varieties: a characteristic free approach*. Pac. J. Math. 284-2 (2016), 283–308.
[10] R. Chirivì, *Deformation and Cohen-Macaulayness of the multicone over the flag variety*, Comment. Math. Helv. vol.76 (2001), 436–466.
[11] E. Feigin, $\mathbb{G}_a^M$ *degeneration of flag varieties*. Selecta Math. (N.S.), 18 (2012), no. 3, 513–537.
[12] E. Feigin, *Degenerate flag varieties and the median Genocchi numbers*, Mathematical Research Letters, 18 (2011), no. 6, pp. 1–16.
[13] E. Feigin, M. Finkelberg, *Degenerate flag varieties of type A: Frobenius splitting and BW theorem*, Mathematische Zeitschrift October 2013, Volume 275, Issue 1–2, pp 55–77.
[14] E. Feigin, M. Finkelberg, P. Littelmann, *Symplectic degenerate flag varieties*. Canad. J. Math. 66 (2014), no. 6, 1250–1286.
[15] E. Feigin, G. Fourier, P. Littelmann, *PBW filtration and bases for irreducible modules in type* A_n, Transform. Groups 16 (2011), no. 1, 71–89.
[16] E. Feigin, G. Fourier, P. Littelmann, *PBW filtration and bases for symplectic Lie algebras*. Int. Math. Res. Not. IMRN 2011, no. 24, 5760–5784.
[17] G. Fourier, *PBW-degenerated Demazure modules and Schubert varieties for triangular elements*, Journal of Combinatorial Theory, Series A, Volume 139, 2016, Pages 132–152.
[18] I. M. Gelfand, M. I. Graev and A. Postnikov, *Combinatorics of hypergeometric functions associated with positive roots*, in Arnold–Gelfand Mathematics Seminars, Geometry and Singularity Theory (V. I. Arnold, I. M. Gelfand, M. Smirnov and V. S. Retakh, Eds.), Birkhäuser, Boston, 1997, pp. 205–221.
[19] G.R. Kempf, A. Ramanathan, *Multi-cones over Schubert varieties*, Invent. math., 87, 353–363 (1987).
[20] V. Lakshmibai and K. N. Raghavan. *Standard Monomial Theory. Invariant Theoretic Approach*. Encyclopaedia of Mathematical Sciences 137. Invariant Theory and Algebraic Transformation Groups 8. Berlin: Springer, 2008.
[21] OEIS Foundation Inc. (2018), *The On-Line Encyclopedia of Integer Sequences*, `http://oeis.org/A006012`.
[22] OEIS Foundation Inc. (2018), *The On-Line Encyclopedia of Integer Sequences*, `http://oeis.org/A032351`.
[23] H. Ohsugi, T. Hibi, *Quadratic initial ideals of root systems*, Proc. Amer. Math. Soc. 130 (2002), no. 7, 1913–1922.

Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Lecce, Italy
E-mail address: rocco.chirivi@unisalento.it

Mathematisches Institut, Universität zu Köln, 50931, Cologne, Germany
E-mail address: xinfang.math@gmail.com

Lehrstuhl B für Mathematik, RWTH Aachen University, 52056, Aachen, Germany
E-mail address: fourier@mathb.rwth-aachen.de