EXOTIC MATRIX MODELS: THE ALBERT ALGEBRA AND THE SPIN FACTOR

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ABSTRACT. The matrix models attached to real symmetric matrices and the complex/quaternionic Hermitian matrices have been studied by many authors. These models correspond to three of the simple formally real Jordan algebras over $\mathbb{R}$. Such algebras were classified by Jordan, von Neumann, and Wigner in the 30s, and apart from these three there are two others: (i) the spin factor $\mathbb{S} = \mathbb{S}_{1,n}$, an algebra built on $\mathbb{R}^{n+1}$, and (ii) the Albert algebra $\mathbb{A}$ of 3×3 Hermitian matrices over the octonions $\mathbb{O}$. In this paper we investigate the matrix models attached to these remaining cases.

1. Introduction

1.1. Let $V = V_{\mathbb{C}}$ be the real vector space of $n \times n$ complex Hermitian matrices equipped with Lebesgue measure. For any polynomial function $f: V \to \mathbb{R}$, define

$$\langle f \rangle_0 = \int_V f(X) \exp(-\operatorname{Tr} X^2/2)\, dX,$$

where $\operatorname{Tr}(X) = \sum_i X_{ii}$ is the sum of diagonal entries, and put

$$\langle f \rangle = \langle f \rangle_0 \,/\, \langle 1 \rangle_0 \,. \tag{1}$$

Let $k \geq 0$ be an integer, and consider (1) evaluated on the polynomial given by taking the trace of the kth power:

$$C_{\mathbb{C}}(n,k) = \left\langle \operatorname{Tr} X^k \right\rangle. \tag{2}$$

For k odd (2) clearly vanishes for all n. On the other hand, for k even and n fixed, it turns out that $C_{\mathbb{C}}(n,k)$ is an integer, and as a function of n is a polynomial of degree $(k+2)/2$ with integral coefficients.

Furthermore, the number $C_{\mathbb{C}}(n,k)$ has the following remarkable combinatorial interpretation. Let Π_k be a polygon with k sides. Any pairing π of the sides of Π_k determines a topological surface $\Sigma(\pi)$ endowed with an embedded graph (the images

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of the edges and vertices of Π_k). Let $N(\pi)$ be the number of vertices in this embedded graph. Then we have

$$C_{\mathbb{C}}(n,k) = \sum_{\pi} n^{N(\pi)}, \tag{3}$$

where the sum is taken over all oriented pairings of the edges of Π_k such that the resulting surface Σ_π is orientable. For example, we have

$$C_{\mathbb{C}}(n,4) = 2n^3 + n, \quad C_{\mathbb{C}}(n,6) = 5n^4 + 10n^2, \quad C_{\mathbb{C}}(n,8) = 14n^5 + 70n^3 + 21n.$$

The pairings yielding $C_{\mathbb{C}}(n,4)$ are shown in Figure 1. For more information, see Harer–Zagier [3], Etingof [2, §4], or Lando–Zvonkin [7].

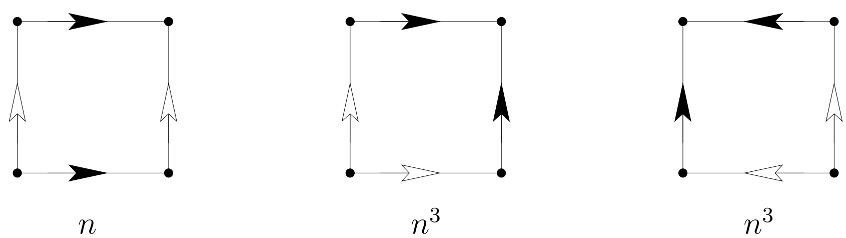

FIGURE 1. Computing $C_{\mathbb{C}}(n,4) = 2n^3 + n$.

1.2. More generally, one can consider integrals over other spaces of matrices, in particular, the space $V_{\mathbb{R}}$ of $n \times n$ real symmetric matrices, and the space $V_{\mathbb{H}}$ of $n \times n$ quaternionic Hermitian matrices. The resulting matrix integrals were investigated by Mulase–Waldron [9], who found explicit combinatorial expressions for the analogues of (2). In the real symmetric case, they found

$$C_{\mathbb{R}}(n,k) = 2^{-k/2} \sum_{\pi} n^{N(\pi)},$$

where the sum is now taken over *all* possible oriented pairings of the edges of Π_k, regardless of whether the resulting surface is orientable or not. The quaternionic case is similar, except that now (3) takes the form

$$C_{\mathbb{H}}(n,k) = 2^{-k/2} \sum_{\pi} \alpha(\pi) n^{N(\pi)},$$

where $\alpha(\pi) \in \{\pm 1\}$ depends on the topology of Σ_π.

1.3. The spaces of matrices $V_{\mathbb{R}}, V_{\mathbb{C}}, V_{\mathbb{H}}$ have another interpretation that is less familiar: they are examples of *simple formally real Jordan algebras over* $\mathbb{R}$ [1, 6, 8]. Briefly, a *Jordan algebra* over a field k is a nonassociative algebra over k whose multiplication satisfies $x \bullet y = y \bullet x$ and $(x \bullet x) \bullet (y \bullet x) = ((x \bullet x) \bullet y) \bullet x$; it is *simple* if it cannot be written as a direct sum ... Although nonassociative, Jordan algebras are power-associative: if one puts $x^n := x \bullet x^{n-1}$ for $n > 1$, then x^n can be computed as $x \bullet \cdots \bullet x$ with any choice of bracketing.

A Jordan algebra A over $\mathbb{R}$ is called *formally real* if $\sum_{i=1}^{n} x_i^2 = 0$ implies each $x_i = 0$. It is known that a real Jordan algebra being formally real is equivalent to it having a *positive definite trace form* $\operatorname{Tr}\colon A \to \mathbb{R}$. This is a linear map satisfying $\operatorname{Tr}(x^2) > 0$ for all $x \in A$, $x \neq 0$, and one has in addition that the trace pairing $\operatorname{Tr}(x \bullet y)$ is a positive definite quadratic form on A.

If the characteristic of k is different from 2, then any associative algebra A over k can be turned into a Jordan algebra by putting $x \bullet y = (xy + yx)/2$, where the multiplication on the right is the usual multiplication in A. This is the Jordan structure on the spaces $V_{\mathbb{R}}$, $V_{\mathbb{C}}$, $V_{\mathbb{H}}$, and the trace form Tr is of course the usual matrix trace.

1.4. Simple formally real Jordan algebras were classified by Jordan, von Neumann, and Wigner in 1934 [5]. Apart from $V_{\mathbb{R}}$, $V_{\mathbb{C}}$, $V_{\mathbb{H}}$, there are two others:

- The *spin factor* $\mathbb{S} = \mathbb{S}_{1,n}$ of pairs $\mathbf{x} = (x_0, x) \in \mathbb{R} \times \mathbb{R}^n$ equipped with the Jordan product $\mathbf{x} \bullet \mathbf{y} = (x_0 y_0 + x \cdot y, x_0 y + y_0 x)$, where $\cdot$ denotes the usual dot product on $\mathbb{R}^n$. The trace form in this case is $\operatorname{Tr}(\mathbf{x}) = x_0$.
- The *Albert algebra* $\mathbb{A}$ of 3×3 Hermitian matrices over the *octonions* $\mathbb{O}$, equipped with the same Jordan product as $V_{\mathbb{R}}$, $V_{\mathbb{C}}$, $V_{\mathbb{H}}$, and with the usual trace as trace form.

1.5. Hence one has the natural problem of investigating the "matrix models" for the Jordan algebras $\mathbb{S}$ and $\mathbb{A}$, and of understanding the underlying combinatorics. In this paper we carry this out. In both cases we give a combinatorial method to compute the expectations $\left\langle \operatorname{Tr} X^k \right\rangle_{\mathbb{J}}$, where $\mathbb{J}$ is one of the algebras $\mathbb{A}, \mathbb{S}$, and where

$$\left\langle \operatorname{Tr} X^k \right\rangle_{\mathbb{J}} = \left\langle \operatorname{Tr} X^k \right\rangle_{0,\mathbb{J}} / \left\langle 1 \right\rangle_{0,\mathbb{J}}, \quad \left\langle f(X) \right\rangle_{0,\mathbb{J}} = \int_{\mathbb{J}} f(X) \exp(-\operatorname{Tr} X^2/2)\, dX. \tag{4}$$

The answers are quite different for the two algebras $\mathbb{A}, \mathbb{S}$, although they do have some similarities with the classical cases. For the Albert algebra, the result (Theorem 3.7) is given in terms of contributions from (orientable and nonorientable) surfaces glued together from polygons, as in the classical matrix algebra cases. For the spin factor, the result is given in terms of colored one-manifolds glued together from closed intervals.

The next results describe how to compute the full perturbation series attached to the trace monomials. Let $t, g_3, g_4, g_5, \ldots$ be formal parameters, and let $\mathbb{Q}[g_3, g_4, \ldots][\![t]\!]$

be the ring of formal power series in t whose coefficients are rational polynomials in the g_k. Then we compute

$$\left\langle \exp\Bigl(\sum_{k\geq 3} \operatorname{Tr} X^k g_k t^k\Bigr) \right\rangle_{\mathbb{J}} \in \mathbb{Q}[g_3, g_4, \dots][\![t]\!] \tag{5}$$

in terms of assembling surfaces (respectively one-manifolds) from polygons (resp. intervals) of various sizes.

The classical matrix algebras $V_{\mathbb{R}}$, $V_{\mathbb{C}}$, $V_{\mathbb{H}}$ depend on a parameter n. This allows one to investigate the expectations/perturbation series as a function of n. This is also true for the spin factor $\mathbb{S}$, and accordingly we are able to incorporate the parameter n into our results. The Albert algebra, on the the other hand, is not part of an infinite family: there is no general matrix model of $n \times n$ Hermitian matrices over $\mathbb{O}$. Indeed, it is exactly the failure of associativity for $\mathbb{O}$ that prevents such matrices for $n > 3$ from having the structure of a Jordan algebra. For $n < 3$, however, the matrix model does make sense: $n = 1$ is just $V = \mathbb{R}$, and $n = 2$ is a special case of the spin factor $\mathbb{S}$, namely $\mathbb{S}_{1,9}$.

Nevertheless, our results show how to express (4) as a *polynomial* $C_{\mathbb{O}}(n, k)$ in n that for $n \leq 3$ agrees with $\langle \operatorname{Tr} X^k \rangle$ evaluated over the appropriate space of matrices. Thus for $n \geq 4$ the combinatorial expansion allows us to define the expectation $\langle \operatorname{Tr} X^k \rangle$, even though the algebraic structure giving rise to it doesn't exist! It would be interesting to find an actual model computing these expectations for $n \geq 4$.

1.6. Here is a guide to the paper. Part 1 treats the Albert algebra $\mathbb{A}$. Section 2 gives the basic definitions, including background on the octonions, and discusses how the nonassociativity affects the trace computations. Then in §3 we compute the expectation of the traces of the powers in terms of gluings of polygons labelled by octonions (Theorem 3.7). Our approach is a generalization of that of Mulase–Waldron [9], although as one might expect, the nonassociativity of $\mathbb{O}$ causes some new wrinkles to appear. The polynomials obtained by considering the "$n \times n$ Hermitian matrices over $\mathbb{O}$" are given in Table 1. We end this part by explaining in §4 how to compute the perturbation series (Theorem 4.3). Next, part 2 gives a parallel treatment of the spin factor $\mathbb{S}$. Background is recalled in §5, the trace integrals are computed in Theorem 6.4 in §6, and the perturbation series in Theorem 8.1 in §8. One difference between $\mathbb{A}$ and $\mathbb{S}$ is that for the latter we are able to give another model that allows us to incorporate automorphisms, and to give a simple generating function for the connected structures with their automorphisms; this is done in §7 in Theorem 7.3.

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Part 1. The Albert Algebra.

2. Background

2.1. We begin with the octonions $\mathbb{O}$. As a real vector space we have $\mathbb{O} \simeq \mathbb{R}^8$; for a basis we take elements $U = \{e_1, \ldots, e_8\}$ satisfying $e_1 = 1$ and $e_i^2 = -1$ for $i > 1$. For $i \neq j$ and $i, j > 1$, we compute products e_ie_j using the Fano mnemonic (Figure 2): $e_ie_j = \pm e_k$ where k is the third index on the line joining i and j, and where the sign is $+1$ (respectively, -1) if i, j, k are in cyclic order (respectively, out of cyclic order) with respect to the arrow on the line through i, j, k. For example, we have $e_2e_3 = e_4$ and $e_2e_7 = -e_8$. The unit e_1 is called *real*, and $e_2, \ldots, e_8$ are called *imaginary*.

We define the product $\alpha\beta$ of two general octonions $\alpha = \sum_i a_ie_i$, $\beta = \sum_i b_ie_i$ using Figure 2 and linearity. The conjugate $\bar{\alpha}$ of α is defined by $\bar{\alpha} = a_1e_1 - \sum_{i>1} a_ie_i$. We have two maps $\mathrm{Tr}, \mathrm{Norm}$ from $\mathbb{O}$ to $\mathbb{R}$ defined by

$$\mathrm{Tr}\,\alpha = \alpha + \bar{\alpha}, \quad \mathrm{Norm}\,\alpha = \alpha \cdot \bar{\alpha}.$$

These maps satisfy

$$\mathrm{Tr}(\alpha + \beta) = \mathrm{Tr}\,\alpha + \mathrm{Tr}\,\beta, \quad \mathrm{Norm}(\alpha\beta) = \mathrm{Norm}(\alpha)\,\mathrm{Norm}(\beta).$$

We remark that if i, j, k lie on a line, then the $\mathbb{R}$-subalgebra spanned by e_1, e_i, e_j, e_k is isomorphic to $\mathbb{H}$. In this case the triple product $e_ie_je_k$ is associative. If i, j, k do not lie on a line, then $e_ie_je_k$ is not associative, but it is *alternative*: we have $(e_ie_j)e_k = -e_i(e_je_k)$.

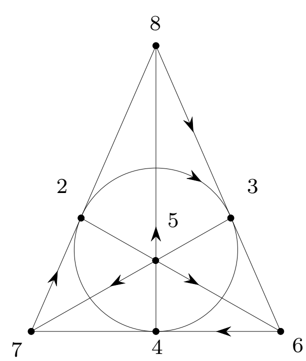


FIGURE 2. The Fano plane mnemonic for multiplication in $\mathbb{O}$.

2.2. Now we define the *Albert algebra* $\mathbb{A}$. As a real vector space $\mathbb{A}$ is the space of 3×3 Hermitian matrices over $\mathbb{O}$:

$$\mathbb{A} = \left\{ \begin{pmatrix} a_1 & \alpha_1 & \alpha_2 \\ \bar{\alpha}_1 & a_2 & \alpha_3 \\ \bar{\alpha}_2 & \bar{\alpha}_3 & a_3 \end{pmatrix} \middle| \, a_i \in \mathbb{R}, \alpha_i \in \mathbb{O} \right\}.$$

As described above, $\mathbb{A}$ becomes a formally real simple Jordan algebra after we put

$$X \bullet Y = (XY + YX)/2, \tag{6}$$

where the product on the right is the usual matrix product. We define powers X^k by

$$X^k = \begin{cases} X & \text{if } k = 1, \\ X \bullet X^{k-1} & \text{if } k > 1. \end{cases} \tag{7}$$

The powers X^k are the first place where the nonassociativity of $\mathbb{O}$ has an effect. As mentioned before, the spaces of Hermitian matrices over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ can be turned into Jordan algebras using (6). For them, the power X^k is exactly $X \cdots X$ (k factors), where the implied product is the usual associative matrix product. But since $\mathbb{O}$ is not associative, the expression $X \cdots X$ is not well-defined. Indeed, this expression must be carefully bracketed using (6) and (7) to guarantee that $\mathbb{A}$ is power-associative (as all Jordan algebras are). Proposition 2.4 below explains how to evaluate X^k using the ordinary matrix product. Before we can state it we need more notation.

2.3. It is well known that any bracketing of a word of length k in a nonassociative algebra can be encoded by a rooted binary tree with k leaves, and that both bracketings and such trees are counted by the Catalan numbers. In our case, not all bracketings arise when computing X^k. Let us call a rooted binary tree *fully nested* if one and only one vertex has two leaves. We also call a bracketing fully nested if the corresponding tree is. In Figure 3, for example, we see the five bracketings of $XXXX$ with their corresponding rooted binary trees. Only the first four are fully nested.

2.4. Proposition.

(i) *Let $k \geq 3$. There are 2^{k-2} fully nested bracketings of a word of length k in a nonassociative algebra.*

(ii) *In $\mathbb{A}$, we have*

$$X^k = \frac{1}{2^{k-2}} \sum_P P(X \cdots X),$$

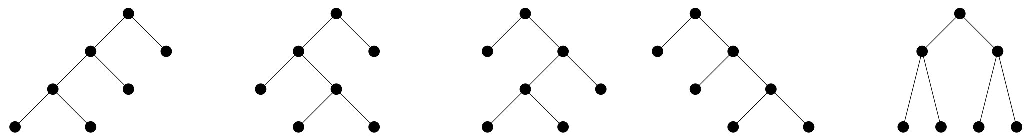
where the sum is taken over all fully nested bracketings P of $X \cdots X$ (k factors).

Proof. Any fully nested rooted binary tree can be encoded by a word of length $k-2$ in the symbols L, R: one reads down from the root and writes down the sequence of children that are non-leaves. This proves (i). The proof of (ii) is a straightforward application of the definition (7) of X^k and the Jordan product (6). $\square$

For later use, we introduce some notation. Given any word w of length k representing a product of elements in an algebra, we define

$$[w]_{\mathrm{fn}} = \frac{1}{2^{k-2}} \sum_P P(w),$$

where the sum is taken over all fully nested bracketings of w.



$((XX)X)X$ $(X(XX))X$ $X((XX)X)$ $X(X(XX))$ $(XX)(XX)$

LL LR RL RR

FIGURE 3. The five bracketings of $XXXX$ and the corresponding rooted binary trees. The first four expressions are fully nested, and the encodings of their trees in terms of words over $\{L, R\}$ are shown below.

2.5. We conclude by stating Wick's theorem, which is the fundamental combinatorial result used in evaluating Gaussian integrals. Let $f(x)$ be a polynomial function on $\mathbb{R}$ and let dx be the usual Lebesgue measure on $\mathbb{R}$. Define

$$\langle f \rangle_{0,\mathbb{R}} = \int_{\mathbb{R}} f(x) \exp(-x^2/2)\, dx, \quad \langle f \rangle_{\mathbb{R}} = \langle f \rangle_{0,\mathbb{R}} \,/\, \langle 1 \rangle_{0,\mathbb{R}}\,. \tag{8}$$

Recall that a *pairing* on a finite set $S = \{1, \dots, n\}$ is a partition of S into nonintersecting subsets of order 2. If n is even, there are $w(n) = (n-1)!! = (n-1)(n-3)\cdots 1$ pairings of S.

2.6. Theorem. *If k is odd, then $\langle x^k \rangle_{\mathbb{R}}$ vanishes. If k is even, then $\langle x^k \rangle_{\mathbb{R}} = w(k)$, the number of pairings of a set of order k.*

The proof of Theorem 2.6 can be found in many places, for example [2, §3.1] and [7]. One learns from the proof that the value $(k-1)!!$ actually arises combinatorially: it should be thought of as counting all pairings of the different x's in the expression $x \cdots x$ (k factors). Thus, for example, we have $\langle x^4 \rangle_{\mathbb{R}} = 3$. If we use subscripts to distinguish the positions and write $xxxx$ as $x_1x_2x_3x_4$, then the 3 arises from the three pairings

$$(x_1x_2)(x_3x_4), \quad (x_1x_3)(x_2x_4), \quad (x_1x_4)(x_2x_3).$$

The numbers $w(k)$ are called the *Wick numbers.*

3. Computation of the basic trace integral

3.1. We begin by recalling some notation from Mulase–Waldron [9], adapted to our case. Any variable $n \times n$ matrix X over $\mathbb{O}$ can be written as

$$X = \sum_{i=1}^{8} A^i e_i, \tag{9}$$

where the A^i are $n \times n$ matrices of real variables. Furthermore, X is Hermitian if and only if A^1 is symmetric ($A^1 = (A^1)^t$) and A^i is antisymmetric for $i > 1$ ($A^i = -(A^i)^t$).

To evaluate $\left\langle \operatorname{Tr} X^k \right\rangle_{\mathbb{A}}$, we must first compute the real polynomial $\operatorname{Tr} X^k$. This is accomplished by Proposition 3.2 below. Before stating the result, let $R_k \subset \{1, \dots, 8\}^k$ be the set of all tuples $(i_1, \dots, i_k)$ such that the product of octonion units $e_{i_1} \cdots e_{i_k}$ with respect to some fixed bracketing is *real*. We remark that this property is independent of whatever bracketing of this product is used to evaluate it, although the actual value, which must be ± 1, depends on the bracketing. To see this, consider any product of octonion units $e_{i_1} \cdots e_{i_k}$, whether real-valued or not. After choosing a bracketing and evaluating, one obtains $\pm e_i$ for some $i = 1, \dots, 8$. Any two bracketings are related through a sequence of replacements of the form $e_a(e_b e_c) \mapsto (e_a e_b)e_c$ applied to three consecutive factors. If the indices a, b, c lie on a line or at least one equals 1, then $e_a(e_b e_c) = (e_a e_b)e_c$. Otherwise, alternativity implies $e_a(e_b e_c) = -(e_a e_b)e_c$. This implies that the unbracketed product $e_{i_1} \cdots e_{i_k}$ is well-defined up to sign, and in particular requiring it be real-valued makes sense.

3.2. Proposition. *Let $n \leq 3$ and let X be as in (9). Then we have*

$$\operatorname{Tr} X^k = \sum_{\substack{1 \leq j_1, \dots, j_k \leq n, \\ (i_1, \dots, i_k) \in R_k}} A^{i_1}_{j_1 j_2} A^{i_2}_{j_2 j_3} \cdots A^{i_k}_{j_k j_1} [e_{i_1} \cdots e_{i_k}]_{\mathrm{fn}}. \tag{10}$$

Proof. Let M be any $n \times n$ matrix over an associative algebra. Then it is well known that

$$\operatorname{Tr} M^k = \sum_{1 \leq j_1, \dots, j_k \leq n} M_{j_1 j_2} M_{j_2 j_3} \cdots M_{j_k j_1}. \tag{11}$$

For the convenience of the reader, we recall the proof. Let G be the directed graph on n vertices with directed edges from each vertex to another and with a loop on each vertex. Then M can be interpreted as the weighted adjacency matrix of G, where the edge from vertex i to j is labelled with M_{ij}. The entries of M^k correspond to k-step walks on G: $(M^k)_{ij}$ is the sum over all k-step walks from i to j of the product of the weights along each walk. The result (11) follows from the observation that a walk contributes to $\operatorname{Tr} M^k$ if and only if it starts and stops at the same vertex.

Now we apply (11) to $X = \sum A^i e_i$. If we expand the expression for $\operatorname{Tr} X^k$ using Proposition 2.4, we obtain

$$\operatorname{Tr} X^k = \sum_{1 \le j_1, \ldots, j_k \le n} A^{i_1}_{j_1 j_2} A^{i_2}_{j_2 j_3} \cdots A^{i_k}_{j_k j_1} [e_{i_1} \cdots e_{i_k}]_{\mathrm{fn}}. \tag{12}$$

We claim in (12) we only need to consider tuples $(i_1, \ldots, i_k)$ that are in R_k. Certainly if $(i_1, \ldots, i_k) \in R_k$, then $[e_{i_1} \cdots e_{i_k}]_{\mathrm{fn}}$ is real. On the other hand, if $(i_1, \ldots, i_k) \notin R_k$, then it can happen that $[e_{i_1} \cdots e_{i_k}]_{\mathrm{fn}}$ is real even if the individual terms in the conmputation of $[\ ]_{\mathrm{fn}}$ are not real. However, the discussion immediately before the statement of Proposition 3.2 shows that this is possible only if $[e_{i_1} \cdots e_{i_k}]_{\mathrm{fn}}$ vanishes. This completes the proof. □

3.3. Let Π_k be a polygon with k sides. Suppose for now that k is even. We fix an embedding of Π_k in the plane and distinguish one vertex with a star. We write Π_k^* to indicate that this has been done.

Let $\mathscr{E} = \mathscr{E}(\Pi_k) = \{E_1, \ldots, E_k\}$ be the edges of Π_k^*, numbered counterclockwise around Π_k^* with the first and last edges adjacent to the distinguished vertex. By an *oriented gluing* π of Π_k^* we mean a choice of orientation for each $E \in \mathscr{E}$ together with a pairing on $\mathscr{E}$. An oriented gluing determines a topological surface $\Sigma(\pi)$ by gluing each edge to its pair consistent with the orientations. We say that two edges E, E' are glued *without a twist* if their orientations are *opposite* as we move from the first to the second around the boundary of Π_k^*. Otherwise we say they are glued *with a twist*. For example, in Figure 1 all pairs of edges are glued without a twist, and in Figure 5 the gluing on the right is done with a twist. Let $N(\pi)$ be the number of equivalence classes of the vertices under the gluing.

Suppose Π_k^* has been equipped with an oriented gluing π. We say that a function $f\colon \mathscr{E} \to U$ from the edges to the units $\{e_1, \ldots, e_8\}$ is *compatible* with π if the following hold:

(i) The product $\prod_{E \in \mathscr{E}} f(E)$ is real-valued.
(ii) If E, E' are identified by π, then $f(E) = f(E')$.

3.4. Definition. Let π be an oriented gluing of Π_k^*, with k even, and let $f\colon \mathscr{E} \to U$ be compatible with π. Then we define the *value* $\Omega(\pi, f)$ of the pair (π, f) to be

$$\Omega(\pi, f) = \alpha(\pi, f)[f(E_1) \cdots f(E_k)]_{\mathrm{fn}} \in \mathbb{Q}, \tag{13}$$

where the sign $\alpha(\pi, f) \in \{\pm 1\}$ is defined by the following rules:

(i) $\alpha(\pi, f)$ is computed by taking a product of signs $\alpha(E, E')$ over all edge pairs E, $E' = \pi(E)$.
(ii) If $f(E) = f(E')$ is e_1, then $\alpha(E, E') = 1$.
(iii) If $f(E) = f(E')$ is imaginary, then $\alpha(E, E') = 1$ if E is glued to E' *without* a twist.

(iv) If $f(E) = f(E')$ is imaginary, then $\alpha(E, E') = -1$ if E is glued to E' *with* a twist.

3.5. Example. We give an example of computing $\Omega(\pi, f)$ for $k = 6$. Suppose π is as in Figure 7. Suppose that f satisfies $f(E_1) = f(E_3) = e_4$, $f(E_2) = f(E_4) = e_5$, and $f(E_5) = f(E_6) = e_6$. We have $\alpha(E_1, E_3) = \alpha(E_5, E_6) = -1$ and $\alpha(E_2, E_4) = 1$. There are 16 fully nested bracketings for the expression $e_4e_5e_4e_5e_6e_6$. The contribution $\Omega(\pi, f)$ is $-5/8$. This example also shows that $\Omega(\pi, f)$ can be nonintegral, thanks to the averaging in (13).

3.6. We are now ready to state our first main result:

3.7. Theorem. *Let $k \geq 2$ be even and let* $\mathrm{Tr}\colon \mathbb{A} \to \mathbb{R}$ *be the trace. Let X be a 3×3 Hermitian matrix of variables as in* (9)*. Then we have*

$$\left\langle \mathrm{Tr}\, X^k \right\rangle_{\mathbb{A}} = 2^{-k/2} \sum_{\pi} \sum_{f} \Omega(\pi, f) 3^{N(\pi)}, \tag{14}$$

where the first sum is taken over all oriented gluings of the edges of Π_k^*, and the second sum is taken over all functions $f\colon \mathscr{E} \to U$ compatible with π.

Proof. We use the expression (10) for $\langle \mathrm{Tr}\, X^k \rangle_{\mathbb{A}}$ together with Wick's theorem (Theorem 2.6). A term in (10) can contribute to $\langle \mathrm{Tr}\, X^k \rangle_{\mathbb{A}}$ if and only if there is a complete pairing of the variables $A^i_{jj'}$, and if the product $e_{i_1} \cdots e_{i_k}$ is real. Following [3,9], any such pairing can be visualized by labelling the polygon Π_k^*. We assign the vertices the labels $j_1, \ldots, j_k$, starting with the distinguished vertex and proceeding clockwise. The edge between j_r and j_{r+1} then corresponds to the variable $A^{i_r}_{j_r j_{r+1}}$. If two variables are to be paired in Theorem 2.6, we can represent this by gluing the corresponding edges together in Π_k^*.

Thus fix a monomial in (10), and let A^i_{ab}, $A^{i'}_{cd}$ be two of the variables that are to be paired (Figure 4). For these variables to be equal, we must have $i = i'$, along with some other constraints.

First, if $i = i' = 1$, then A^1 is symmetric. The variables can be equal if either (i) $a = c$ and $b = d$ or (ii) $a = d$ and $b = c$. Case (i) corresponds to gluing with a twist, whereas (ii) corresponds to gluing without a twist. In both cases, the symmetry of A^1 does not affect the sign of this monomial.

On the other hand, if $i = i' > 1$, then A^i is antisymmetric. Again the variables can be equal if either (i) $a = c$ and $b = d$ or (ii) $a = d$ and $b = c$, and again these represent gluing with and without a twist respectively. As before case (ii) introduces no extra sign, but case (i) does. For these pairings, we have the same imaginary unit attached to each edge. These units square to -1, which have the effect of flipping the signs in (i), (ii). Thus these two possiblities exactly correspond to the computation of $\alpha(\pi, f)$ in Definition 3.4.

After a pairing π has been chosen on the perimeter of Π_k^*, we must compute how many different monomials the pairing can contribute to. This is done by (i) summing over all assignments of units to edges compatible with π, and (ii) varying the subscripts $j_1, \ldots, j_k$ over all possibilities. The first is handled by summing over all f compatible with π. For the second, after applying π one finds that certain subscripts j_l must be equal, but apart from that one can allow them to range over any of the three possibilities $1, 2, 3$ (since $\mathbb{A}$ consists of 3×3 matrices over $\mathbb{O}$). The number of different classes of subscripts is the same as $N(\pi)$, which is the factor appearing in (14).

Finally we must account for the factor $2^{-k/2}$. This arises because in our correspondence between gluings of Π_k^* and monomials giving a nontrivial pairing we are overcounting. Indeed, the variables A^i_{ab} and $A^{i'}_{cd}$ identified by the gluing π are the same, regardless of whether we glue with a twist or not. This means we see each pair of variables twice, once for gluing with a twist and once for gluing without. Since there are $k/2$ pairs of edges to be glued, we must divide by $2^{k/2}$. This completes the proof. □

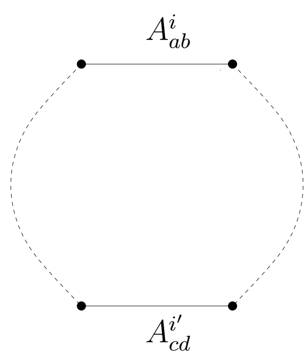


FIGURE 4. Two edges in Π_k^*

3.8. Example. Consider computing $\langle \operatorname{Tr} X^2 \rangle_{\mathbb{A}}$. There are two gluings π_1, π_2 of the bigon Π_2^*, shown in Figure 5. We have $N(\pi_1) = 2$ and $N(\pi_2) = 1$. We have 8 possibilities for labelling the edges E, E' in each. In π_1, the edges are glued without a twist, so all signs $\alpha(E, E')$ are positive. Thus $\Omega(\pi_1, f) = 1$ for any of the 8 possible f, and we obtain $8 \cdot 3^2 = 72$ for this gluing. In π_2, the edges are glued with a twist. This means that when $f(E) = e_1$ is real, we have $\alpha(E, E') = 1$, and when $f(E)$ is imaginary, we have $\alpha(E, E') = -1$. Summing over all f gives -6, and we get a contribution of $-6 \cdot 3 = -18$ from this gluing. The final result is $\langle \operatorname{Tr} X^2 \rangle_{\mathbb{A}} =$

$\frac{1}{2}(8 \cdot 3^2 - 6 \cdot 3) = 27$. This can be checked directly. We have

$$\operatorname{Tr} X^2 = \sum_{j=1}^{3} (A^1_{jj})^2 + \sum_{\substack{1 \le i \le 8 \\ 1 \le j < j' \le 3}} 2(A^i_{jj'})^2. \tag{15}$$

Applying Theorem 2.6, we find $\left\langle (A^1_{jj})^2 \right\rangle_{\mathbb{A}} = 1$ and $\left\langle (A^i_{jj'})^2 \right\rangle_{\mathbb{A}} = 1/2$, which with (15) yields 27.

3.9. Example. Figure 6 shows the 12 oriented gluings of Π_4^* with their contributions. The result is $2^{-2}(128 \cdot 3^3 - 240 \cdot 3^2 + 124 \cdot 3) = 417$. Note that in this example, as in Example 3.8, the products of the e_i are all associative. This follows since there are only at most two different units appearing in any product, so each expression is being computed in a subalgebra isomorphic to $\mathbb{H}$. Thus there is no need to compute the fully nested bracketings.

3.10. Example. The evaluation of $\langle \operatorname{Tr} X^6 \rangle_{\mathbb{A}}$ uses gluings of the hexagon Π_6^*. There are 15 possible ways to pair the edges of Π_6^*, and each pairing has 8 different twisting patterns. For each pairing with twists there are 512 possible assignment of units to the edges, and for each assignment there are 16 fully nested bracketings to compute. After evaluating 983040 terms the final answer is $\langle \operatorname{Tr} X^6 \rangle_{\mathbb{A}} = 7533$.

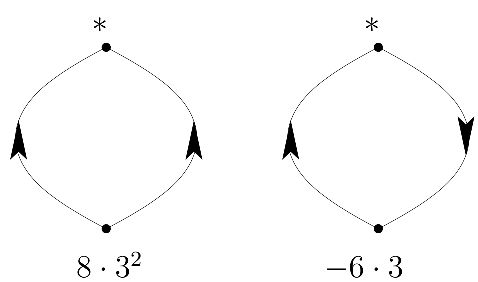


FIGURE 5. The two gluings π_1, π_2 used in computing $\langle \operatorname{Tr} X^2 \rangle_{\mathbb{A}}$ on $\mathbb{A}$.

3.11. Remark. In the computation of $\left\langle \operatorname{Tr} X^k \right\rangle_{\mathbb{A}}$, one can replace $3^{N(\pi)}$ in (14) with $2^{N(\pi)}$. One then finds the result of evaluating $\left\langle \operatorname{Tr} X^k \right\rangle_{\mathbb{A}}$ on the 2×2 Hermitian matrices over $\mathbb{O}$. In fact, one can replace $3^{N(\pi)}$ with $n^{N(\pi)}$ for an indeterminate n, and one obtains a rational polynomial $C_{\mathbb{O}}(n, k)$. By analogy with the associative cases ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$), one can regard this as computing the expectation of $\operatorname{Tr} X^k$ on an "algebra" of $n \times n$ Hermitian matrices over $\mathbb{O}$. Of course this is only an analogy: there is no such algebra, and the computation is purely formal.

Some examples of the polynomials $C_{\mathbb{O}}(n, k)$ are given in Table 1. One sees from the table that these polynomials apparently have surprising properties. For example, they are all *integral* polynomials. Moreover, they are *alternating*. One sees this

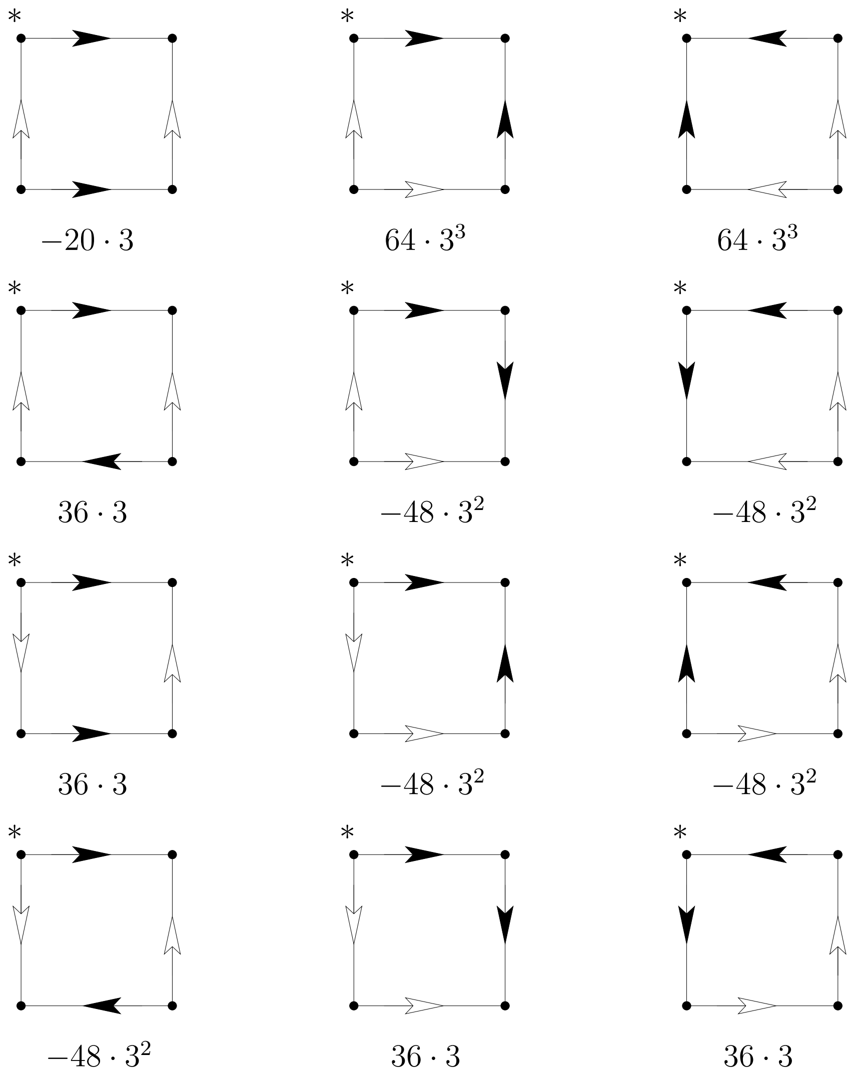


FIGURE 6. Computing $\langle \mathrm{Tr}\, X^4 \rangle_{\mathbb{A}}$ on $\mathbb{A}$.

latter property in [9] for the polynomials over $\mathbb{H}$, which are obtained from those over $\mathbb{R}$ by a changing the parameter n and including an overall sign for contributions from surfaces of odd Euler characteristic, a phenomenon Mulase–Waldron explain via a duality between the Gaussian orthogonal and Gaussian symplectic matrix ensembles.

It would be very interesting to provide an algebraic model with n as a parameter that actually produces these polynomials. Proving that they are integral and alternating would also be interesting, as well as providing a direct combinatorial interpretation of their coefficients.

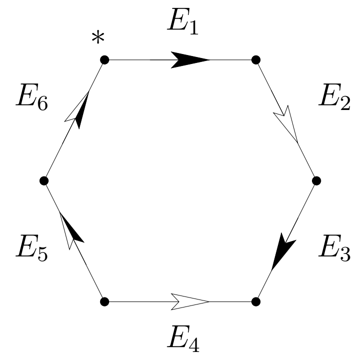


FIGURE 7. One term in evaluating $\langle \operatorname{Tr} X^6 \rangle_{\mathbb{A}}$ on $\mathbb{A}$. The pairs $\{E_1, E_3\}, \{E_5, E_6\}$ have been glued with a twist, the pair $\{E_2, E_4\}$ without. We have $N(\pi) = 1$. This term contributes $-153 \cdot 3 = -459$.

k	$C_{\mathbb{O}}(n,k)$
2	$4n^2 - 3n$
4	$32n^3 - 60n^2 + 31n$
6	$299n^4 - 930n^3 + 1081n^2 - 435n$
8	$5992n^5 - 26577n^4 + 50942n^3 - 46875n^2 + 16728n$

TABLE 1. Octonionic trace polynomials

4. THE PERTURBATION SERIES

4.1. Let $t, g_3, g_4, \dots$ be indeterminates. We regard the g_k as deformation parameters, and package them together into a vector $\mathbf{g} = (g_3, g_4, \dots)$. Let $\mathbf{m} = (m_3, m_4, \dots) \in \prod_{k\geq 3} \mathbb{Z}_{\geq 0}$ be a vector of multiplicities; we assume $m_k = 0$ for all sufficiently large k. Then we write $\mathbf{g}^{\mathbf{m}}$ for the monomial $\prod g_k^{m_k}$. Let $N(\mathbf{m}) = \sum k m_k$.

We consider the pertubation series

$$F(\mathbf{g}, t, X) = \exp\Big(\sum_{k\geq 3} (g_k \operatorname{Tr} X^k) t^k\Big)$$

and the expectation

$$\langle F(\mathbf{g}, t, X) \rangle_{\mathbb{A}} \in \mathbb{Q}[g_3, g_4, \dots][\![t]\!]. \tag{16}$$

Our goal is to compute the coefficient of t^N in (16) in terms of gluings of polygons as in Theorem 3.7. Clearly this coefficient is a homogeneous polynomial of degree N in the monomials $\mathbf{g}^{\mathbf{m}}$, where $N = N(\mathbf{m})$, so it suffices to compute the coefficient of $\mathbf{g}^{\mathbf{m}} t^N$. To state the answer, we need to extend our previous notation.

4.2. Let $\Pi_{\mathbf{m}}$ be the disjoint union of polygons $\coprod \Pi_k^*$, where we take m_k copies of Π_k^*. We write $\Pi \in \Pi_{\mathbf{m}}$ to mean that Π is a connected component of $\Pi_{\mathbf{m}}$. Let $\mathscr{E}$ be the set of all edges of $\Pi_{\mathbf{m}}$, and for each $\Pi \in \Pi_{\mathbf{m}}$ we denote its set of edges by $\mathscr{E}(\Pi)$.

Let π be an oriented gluing of the edges of $\Pi_{\mathbf{m}}$, where as before we allow both twisted and untwisted identifications. Note that π will in general glue together edges in different connected components. We say that a map $f\colon \mathscr{E} \to U$ is compatible with π if it satisfies the extensions of our previous conditions:

(i) For each connected component $\Pi \in \Pi_{\mathbf{m}}$, the product $\prod_{E\in\mathscr{E}(\Pi)} f(E)$ must be real-valued.

(ii) If E, E' are identified by π, then $f(E) = f(E')$.

We define the sign $\alpha(\pi, f)$ exactly as before, and put

$$\Omega(\pi, f) = \alpha(\pi, f) \prod_{\Pi\in\Pi_{\mathbf{m}}} [f(\mathscr{E}(\Pi))]_{\mathrm{fn}},$$

where we write $[f(\mathscr{E}(\Pi))]_{\mathrm{fn}}$ to mean $[f(E_1)\cdots f(E_l)]_{\mathrm{fn}}$, where $E_1, \dots, E_l$ are the edges Π, again arranged clockwise starting from the distinguished vertex.

Finally, we define the group $\operatorname{Aut}\pi$ of automorphisms of π to be the group induced from permuting the connected components. Note that cyclic rotation of the connected components is not allowed, since such symmetries do not preserve the distinguished vertex. We can now state our theorem:

4.3. Theorem. *The coefficient of* $\mathbf{g}^{\mathbf{m}}$ *in the coefficient of* $t^{N(\mathbf{m})}$ *is*

$$2^{-N(\mathbf{m})/2} \sum_{\pi} \sum_{f} \frac{\Omega(\pi, f)}{|\operatorname{Aut}\pi|} 3^{N(\pi)},$$

where the first sum is taken over all oriented gluings of the edges of $\Pi_{\mathbf{m}}$*, and the second sum is taken over all functions* $f\colon \mathscr{E} \to U$ *compatible with* π*.*

Proof. The proof is a simple application of the exponential formula for generating functions together with Theorem 3.7. The only subtlety is the point that one considers all functions f satisfying the condition that $\prod f(E)$ be real-valued on each connected component of $\Pi_{\mathbf{m}}$. But this follows from the formula for the trace given in Proposition 3.2. □

4.4. Example. We give an example to show how to apply Theorem 4.3 and compute the coefficient of g_3^2. Thus $\mathbf{m} = (2, 0, \dots)$ and $\Pi_{\mathbf{m}}$ is two triangles, and the only automorphism is interchanging the two components. Thus the coefficient is $\langle (\operatorname{Tr} X^3)^2 \rangle_{\mathbb{A}}/2$. We will show how to compute the expectation $\langle (\operatorname{Tr} X^3)^2 \rangle_{\mathbb{A}}$.

There are $5 \cdot 3 = 15$ different gluings of the edges of $\Pi_{\mathbf{m}}$. Of these there are three essentially different types (Figure 8); the first occurs 9 times, and the other two 3 times each. For each pairing of the edges, we must choose whether we glue with a twist or not. We can denote the twists by vectors in $(\mathbb{Z}/2\mathbb{Z})^3$. Thus 010 means a and c are glued without a twist, whereas b is glued with a twist.

Consider the gluing of type (I). There are four possibilities for the map f: (i) f is identically 1, (ii) f takes b to 1 and a, c to imaginary units, (iii) f takes a, b to 1 and c to any imaginary unit, and (iv) f takes b, c to 1 and a to any imaginary unit. The contributions of each of these, along with the quantity $N(\pi)$, is summarized in Table 2. One sees that a gluing of type (I) contributes $32 \cdot 3^3 - 32 \cdot 3^2 + 8 \cdot 3$.

Next consider type (II); the contributions are summarized in Table 3. This time there are three different possibilites: (i) f is identically 1, (ii) f takes one of a, b, c to 1 and the other two to imaginary units, and (iii) f takes a, b, c to three different imaginary units. Note that real-valuedness forces that in (iii), the images of f generate a subalgebra isomorphic to $\mathbb{H}$. This means that all products for this type are associative, so there is no need to average over bracketings. Thus the 7 in column (iii) corresponds to the 7 lines in the Fano plane, and the 6 corresponds to the ways to order the three imaginary units in a given line. Note also that type (iii) contributions do not arise in Theorem 3.7, since there we do not have two different sets of edges to label with matching units. The result is that a gluing of type (II) contributes $16 \cdot 3^3 - 24 \cdot 3^2 + 16 \cdot 3$.

Finally consider type (III). We have the same three possibilities for f as in type (II). The contributions are essentially the same as in Table 3, except that each twisting datum should be replaced by its complement. We give the result in Table 4.

To get the final result we sum these contributions and divide by $2^{N(\mathbf{m})/2} = 8$. We obtain $9(32 \cdot 3^3 - 32 \cdot 3^2 + 8 \cdot 3)/8 + (3+3)(16 \cdot 3^3 - 24 \cdot 3^2 + 16 \cdot 3)/8 = 2709$. One can verify that this agrees with a direct evaluation of the integral defining $\langle (\operatorname{Tr} X^3)^2 \rangle_{\mathbb{A}}$.

$N(\pi)$	twisting	(i) all 1	(ii) one 1	(iii) two 1s on left	(iv) two 1s on right
3^3	000	1	7^2	7	7
3^2	100	1	-7^2	7	-7
3^3	010	1	7^2	7	7
3^2	001	1	-7^2	-7	7
3^2	110	1	-7^2	7	-7
3^1	101	1	7^2	-7	-7
3^2	011	1	-7^2	-7	7
3^1	111	1	7^2	-7	-7

TABLE 2. Contributions of gluings of type (I)

4.5. Remark. As in Remark 3.11, one can consider replacing $3^{N(\pi)}$ by $n^{N(\pi)}$ and use our gluing calculus to produce polynomials in n for the mixed moments. For example for $\langle (\operatorname{Tr} X^3)^2 \rangle_{\mathbb{A}}$, one finds the result is $192n^3 - 324n^2 + 147n$. Again, these

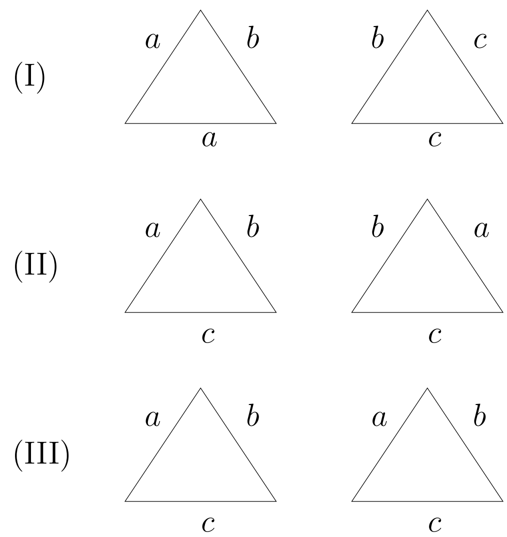


FIGURE 8. Gluings of two triangles

$N(\pi)$	twisting	(i) all 1	(ii) one 1	(iii) 3 different units
3^3	000	1	$3 \cdot 7$	$6 \cdot 7$
3^2	100	1	-7	$-6 \cdot 7$
3^2	010	1	-7	$-6 \cdot 7$
3^2	001	1	-7	$-6 \cdot 7$
3^1	110	1	-7	$6 \cdot 7$
3^1	101	1	-7	$6 \cdot 7$
3^1	011	1	-7	$6 \cdot 7$
3^1	111	1	$3\cdot$	$-6 \cdot 7$

TABLE 3. Contributions of gluings of type (II)

polynomials appear to be integral, if one ignores the denominators coming from the exponential series. They also appear to have alternating coefficients, just as in Table 1. We have no explanation for this fact.

Part 2. The spin factor.

5. Background

5.1. We begin by recalling the definition of the *spin factor* $\mathbb{S} = \mathbb{S}_{1,n}$. As an $\mathbb{R}$-vector space $\mathbb{S}$ is $\mathbb{R} \times \mathbb{R}^n$. Write elements $\mathbf{x} \in \mathbb{S}$ as pairs (x_0, x), where $x_0 \in \mathbb{R}$. Then the Jordan product is defined by $\mathbf{x} \bullet \mathbf{y} = (x_0 y_0 + x \cdot y, x_0 y + y_0 x)$, where $\cdot$ denotes the

$N(\pi)$	twisting	(i) all 1	(ii) one 1	(iii) 3 different units
3^1	000	1	$3 \cdot 7$	$-6 \cdot 7$
3^1	100	1	-7	$6 \cdot 7$
3^1	010	1	-7	$6 \cdot 7$
3^1	001	1	-7	$6 \cdot 7$
3^2	110	1	-7	$-6 \cdot 7$
3^2	101	1	-7	$-6 \cdot 7$
3^2	011	1	-7	$-6 \cdot 7$
3^3	111	1	$3\cdot$	$6 \cdot 7$

TABLE 4. Contributions of gluings of type (III)

usual Euclidean dot product on $\mathbb{R}^n$. The trace map $\mathrm{Tr}\colon \mathbb{S} \to \mathbb{R}$ is defined by $\mathbf{x} \mapsto x_0$. We define powers $\mathbf{x}^k$ as in the case of $\mathbb{A}$, through (7).

The algebra $\mathbb{S}$ is called a spin factor because of its connection with Clifford algebras [8, §1.9]. Let W be a real vector space of dimension n with orthonormal basis v_i, $i = 1, \ldots, n$. The Clifford algebra $C(W)$ is the unital associative algebra generated by W modulo the relations $v_i^2 = 1$, and $v_i v_j = -v_j v_i$ for all $i \neq j$. The algebra $C(W)$ has dimension 2^n, with an additive basis given by 1 and all expressions of the form

$$v_{i_1} \cdots v_{i_k}, \quad \text{where } 1 \leq i_1 < \cdots < i_k \leq n \text{ and } k = 1, \ldots, n.$$

The $n+1$-dimensional subspace spanned by 1 and the v_i does not form an associative subalgebra, but it does inherit the structure of a Jordan algebra as in §1.3. It is easy to check that this algebra is exactly $\mathbb{S}_{1,n}$. Since $C(W)$ can be realized as a subalgebra of the $2^n \times 2^n$ symmetric matrices over $\mathbb{R}$, this means that $\mathbb{S}$ can be viewed as a sub-Jordan algebra of $V_{\mathbb{R}}$. For more details we refer to [8].

6. Computation of the basic trace integral

6.1. The goal of this section is to compute combinatorially the expectations $\left\langle \mathrm{Tr}\, \mathbf{x}^k \right\rangle_{\mathbb{S}}$ of the trace monomials. As we shall see, when k is fixed and n is taken as a parameter, we obtain a polynomial $C_{\mathbb{S}}(n,k)$ in n, just as in the classical case of Hermitian matrices. We shall also see that the combinatorial model is one dimensional, as in the case of Feynman diagrams. In fact, we give two closely related models. The first allows one to quickly evaluate $C_{\mathbb{S}}(n,k)$, whereas the second makes it easy to incorporate automorphisms in the model.

A first step is to compute the trace polynomials explicitly. We use the notation of §5, along with the convention that for the vector part x of $\mathbf{x} = (x_0, x)$, the symbol x^k denotes

- the scalar $(x \cdot x)^{k/2}$ if k is even, and
- the vector $(x \cdot x)^{(k-1)/2} x$ if k is odd.

Then we have the following result; we omit the easy proof by induction.

6.2. Proposition. *We have* $\mathbf{x}^k = (z_0, z)$, *where*

$$z_0 = \sum_{\substack{i+j=k\\ j \text{ even}}} \binom{k}{i} x_0^i x^j, \quad z = \sum_{\substack{i+j=k\\ j \text{ odd}}} \binom{k}{i} x_0^i x^j. \tag{17}$$

6.3. Using Proposition 6.2, it is easy to directly compute the expectation $\langle \operatorname{Tr} \mathbf{x}^k \rangle_{\mathbb{S}}$. Our goal is to give a combinatorial description that makes the role of the parameter n more apparent.

Let S be a finite set of points labelled by $\{1, \ldots, k\}$, where we assume k is even. Let $S = S_0 \cup S_b$ be a partition of S into two subsets, each of even order. We take the points in S_b, order them by their labels, and join consecutive ones by an edge. The result is a collection of points and edges that we call a *barbell structure* on S, with the set of edges being called the *barbells*.

Let β be a barbell structure on S. We define a *pairing* $\pi(\beta)$ of β to be a pairing of the elements of S_0 and the elements of S_b. In other words, we can freely join any elements of the two parts together in pairs, but we cannot join an element in one part to an element in the other. Each pairing produces a union of edges (in S_0) and circles (in S_b). Let $N(\pi)$ be the number of connected components of π in S_b. Finally define a *coloring* of π to be an assignment of $\{1, \ldots, n\}$ to each connected component in S_b.

6.4. Theorem. *Let* $\mathbb{S}$ *be the spin factor* $\mathbb{S}_{1,n}$. *For* k *odd we have* $\langle \operatorname{Tr} \mathbf{x}^k \rangle_{\mathbb{S}} = 0$. *For* k *even we have*

$$\langle \operatorname{Tr} \mathbf{x}^k \rangle_{\mathbb{S}} = \sum_{\beta} \sum_{\pi} n^{N(\pi)}, \tag{18}$$

where β *ranges over all barbell structures on* $\{1, \ldots, k\}$ *and* π *ranges over all pairings of* β.

Proof. Proposition 6.2 shows that $\operatorname{Tr} \mathbf{x}^k$ is an odd function of x_0 if k is odd, so certainly $\langle \operatorname{Tr} \mathbf{x}^k \rangle_{\mathbb{S}} = 0$ in that case. So suppose k is even and write $k = i + j$ with i, j even. There are clearly $\binom{k}{i}$ barbell structures on $\{1, \ldots, k\}$ with $|S_0| = i$, and any such one will contribute a factor of $w(k)$ coming from the pairings in S_0. Thus the result will follow if we can show

$$\langle x^j \rangle_{\mathbb{S}} = \Big\langle \big(\sum_{p=1}^{n} x_p^2\big)^{j/2} \Big\rangle_{\mathbb{S}} = \sum_{\pi'} n^{N(\pi')},$$

where π' ranges over the colored pairings of a *fixed* collection of $j/2$ barbells.

This can be seen as follows. Use π' to pair the endpoints of each barbell. Consider labeling the endpoints of each barbell with a variable x_p, $p = 1, \ldots, n$ such that the same variable appears at either end, and such that the variable appearing along each connected component is constant. There are clearly $n^{N(\pi')}$ such assignments of variables. Each one corresponds to a monomial produced by multiplying out $(\sum_{p=1}^{n} x_p^2)^{j/2}$, where the variable x_p from the lth factor is placed on the lth barbell. This completes the proof. □

6.5. Example. In Figure 9 we give the full computation of $\langle \operatorname{Tr} \mathbf{x}^6 \rangle_{\mathbb{S}}$. There are barbell structures with $|S_b| = 0, 2, 4, 6$.

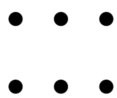
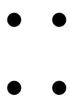
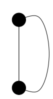
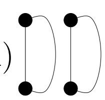
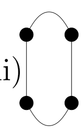
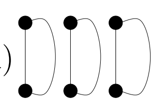
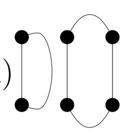
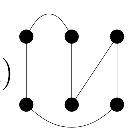
S_0	S_b	Contribution
	$\emptyset$	$w(6) = 15$
		$w(4)\binom{6}{4}n = 45n$
	(i)  (ii) 	(i) $w(2)\binom{6}{2}n^2 = 15n^2$ (ii) $w(2)\binom{6}{2}2n = 30n$
$\emptyset$	(i)  (ii)  (iii) 	(i) n^3 (ii) $2\binom{3}{1}n^2 = 6n^2$ (iii) $8n$

FIGURE 9. Computing $\langle \operatorname{Tr} \mathbf{x}^6 \rangle_{\mathbb{S}}$. The result is $n^3 + 21n^2 + 83n + 15$.

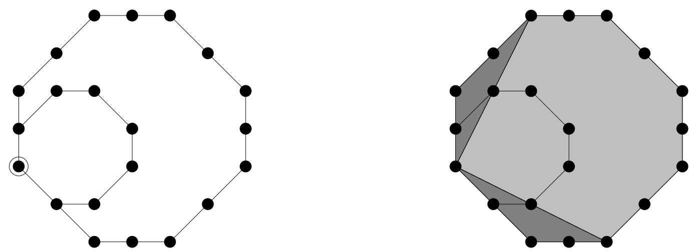
6.6. For k even let $C_{\mathbb{S}}(n, k)$ be the polynomial (18). The polynomials $C_{\mathbb{S}}(n, k)$ can be seen in Table 5. It is clear that they are monic, and the constant terms are the Wick numbers. The coefficients of the codegree one terms

$$1, 8, 21, 40, 65, 96, \ldots$$

are the *octagonal numbers* [10, `A000567`]; these are the analogue of the triangular numbers, in which one arranges dots in an octagon (Figure 10(a)). This can be seen as follows. Write $S = S_0 \cup S_b$ and suppose $|S| = k$ with $k = 2m$ even. There are two ways a paired barbell structure can contribute to this degree. Either $S_0 = \emptyset$ or $|S_0| = 2$. In the former case we have m barbells, and we must choose two of them to pair into a connected component (the other $m-1$ must be paired to themselves). There are two pairings giving one connected component, so there are $\binom{m}{2} \cdot 2$ paired barbell structures of this type. In the latter case we have to pick which two points will go to S_0, so there are $\binom{2m}{2}$ paired barbell structures of this type. Hence altogether this coefficient is

$$\binom{m}{2} \cdot 2 + \binom{2m}{2}. \tag{19}$$

Now in Figure 10(a) we can shave off two triangles to yield a hexagon as in Figure 10(b). Since the hexagonal number is well known to be $\binom{2m}{2}$, this gives (19). Apart from these sequences of coefficients, not much else seems to be known about these polynomials.



(a) The first three octagonal numbers

(b) Shaving the octagon

FIGURE 10.

6.7. We conclude this section by showing how the polynomials $C_{\mathbb{S}}(n, k)$ can be easily computed using standard techniques of generating functions. First, we define the *even Wick numbers* $w_e(k)$ by $w_e(k) = 0$ if k is odd, and $k!! := k(k-2)(k-4)\cdots 2$ if k is even. Given k barbells, there are $w_e(k)$ pairings of the ends that yield one connected component. Then the exponential power series

$$\begin{aligned} A &= \exp\left(\sum_{k\geq 1} w_e(2k) n \frac{x^k}{k!}\right) \\ &= 1 + nx + \left(\frac{n^2}{2} + n\right)x^2 + \left(\frac{n^3}{6} + n^2 + \frac{4n}{3}\right)x^3 + \left(\frac{n^4}{24} + \frac{n^3}{2} + \frac{11n^2}{6} + 2n\right)x^4 + \cdots \end{aligned}$$

gives the generating function of all possible barbell pairings.

To get the polynomials in Table 5, we need to complete the paired barbells into paired barbell structures. As a first step we need to tweak A. Let A_l be the result of applying the series Laplace transform to A,[1] and let A' be the result of replacing x with x^2 in A_l and then convolving with the exponential series. The result is

$$A' = 1 + \frac{n}{2}x^2 + (\frac{n^2}{24} + \frac{n}{12})x^4 + (\frac{n^3}{720} + \frac{n^2}{120} + \frac{n}{90})x^6 + \cdots .$$

This gives the same data as the exponential series A for the barbell pairings, but now the polynomials are placed in the correct (even) degrees.

Finally we can incorporate the pairings in S_0. Let B be the exponential generating function of the Wick numbers $w(k)$:

$$\begin{aligned} B &= \sum_{k\geq 0} w(k)\frac{x^k}{k!} \\ &= 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \frac{1}{384}x^8 + \frac{1}{3840}x^{10} + \cdots . \end{aligned}$$

The product $A' \cdot B$ then gives all ways to break up S into $S_0 \cup S_b$ and to pair, along with the data of the number of connected components obtained in S_b. If we take the Laplace transform of $A' \cdot B$, we obtain the ordinary generating function of the polynomials $C_{\mathbb{S}}(n,k)$:

$$(A' \cdot B)_l = 1 + (n+1)\,x^2 + \big(n^2 + 8n + 3\big)\,x^4 + \big(n^3 + 21n^2 + 83n + 15\big)\,x^6 + \cdots \tag{20}$$

7. Automorphisms and connected structures

7.1. Consider the generating function (20) of the polynomials $C_{\mathbb{S}}(n,k)$:

$$1 + (n+1)\,x^2 + \big(n^2 + 8n + 3\big)\,x^4 + \big(n^3 + 21n^2 + 83n + 15\big)\,x^6 + \cdots \tag{21}$$

In this section we modify (21) by taking into account symmetries of barbell diagrams. Let $k = 2m$ be even. Let Ξ_k be the set $\{1, \ldots, k\}$ with barbells drawn between the pairs $\{1,2\}$, $\{3,4\}$, $\ldots$, $\{k-1,k\}$. Let B_m be the wreath product $\mathbb{Z}/2\mathbb{Z} \wr S_m$ of order $2^m m!$ We call B_m the *barbell group*, and think of it as acting on Ξ_m by permuting the barbells and flipping them independently. Thus if we label the points in Ξ_k as above, then we can identify B_m with the subgroup of S_k generated by the transpositions $(1,2)$, $(3,4)$, $\ldots$, $(k-1,k)$ and the products $(1,3)(2,4)$, $(3,5)(4,6)$, $\ldots$, $(k-2,k)(k-3,k-1)$ (see Figure 11).

Our goal is to give a combinatorial meaning to the modified generating function

$$B(x) = 1 + \frac{1}{2}\,(n+1)\,x^2 + \frac{1}{8}\,\big(n^2 + 8n + 3\big)\,x^4 + \frac{1}{48}\,\big(n^3 + 21n^2 + 83n + 15\big)\,x^6 + \cdots , \tag{22}$$

in which each polynomial $C_{\mathbb{S}}(n,k)$ is divided by the order of $B_{k/2}$. In particular, we want to express the coefficient of x^k as a sum over various paired configurations of

[1]This transform takes $\sum a_k x^k/k!$ to $\sum a_k x^k$.

k	$C_{\mathbb{S}}(n,k)$
0	1
2	$n+1$
4	n^2+8n+3
6	$n^3+21n^2+83n+15$
8	$n^4+40n^3+422n^2+1112n+105$
10	$n^5+65n^4+1310n^3+9310n^2+18609n+945$
12	$n^6+96n^5+3145n^4+42720n^3+231259n^2+377664n+10395$
14	$n^7+133n^6+6433n^5+141925n^4+1466059n^3+6476407n^2+9071187n+135135$
16	$n^8+176n^7+11788n^6+383600n^5+6424054n^4+53966864n^3+203378412n^2+252726480n+2027025$
18	$n^9+225n^8+19932n^7+897372n^6+22132614n^5+300621510n^4+2144046428n^3+7109593308n^2+8031454785n+34459425$
20	$n^{10}+280n^9+31695n^8+1885920n^7+64273818n^6+1283152080n^5+14746708430n^4+92004426080n^3+274591498581n^2+287095866840n+654729075$

TABLE 5. The expectations $\left\langle \operatorname{Tr} \mathbf{x}^k \right\rangle_{\mathbb{S}}$, as a function of n.

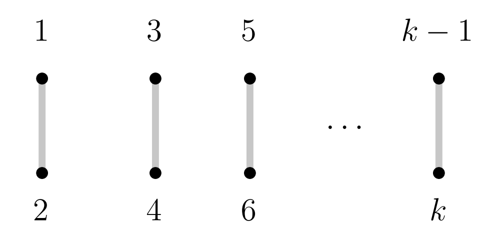


FIGURE 11. The barbell group B_m, where $m = k/2$, permutes the barbells and flips them independently.

k barbells up to isomorphism, where each configuration is weighted by the inverse of the order of its automorphism group. This is analogous to the usual Feynman calculus, which expresses coefficients of certain power series as sums over certain graphs weighted by the inverses of the orders of their automorphism groups. As a simple example, consider the power series (cf. (8))

$$\left\langle \exp(tx^4/4!) \right\rangle_{\mathbb{R}} = 1 + c_1 t + c_2 t^2 + \cdots . \tag{23}$$

Then we have

$$c_j = \sum_{\Gamma \in G_j} \frac{1}{|\operatorname{Aut}\Gamma|},$$

where the sum is taken over all graphs with j vertices of degree 4, and where the automorphisms are induced by permuting vertices and edges (including flips of loops).

For instance, $c_1 = 1/8$ and $c_2 = 35/384$ (cf. Figure 12). For more details, we refer to [2, §3.2] and [4, Ch. 9].

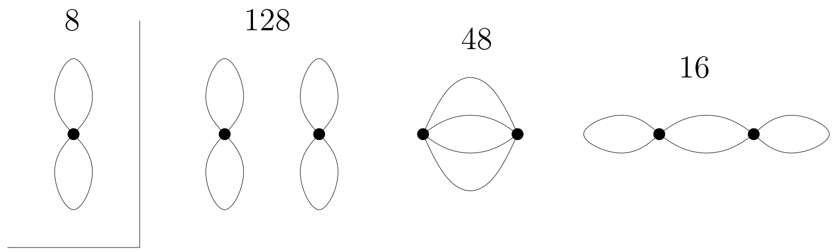


FIGURE 12. Graphs and their numbers of automorphisms used to compute $c_1 = 1/8$ and $c_2 = 35/384$ in (23).

7.2. To carry this out, we give a slightly different model for the terms contributing to the expectation. We define a *barbell graph* to be a graph Γ constructed as follows. Begin with Ξ_k. We partition the vertices into two sets S_{bl} and S_{gr}, where we color the vertices in S_{bl} (respectively S_{gr}) black (respectively green). We do this arbitrarily; in particular the ends of a barbell need not be the same color. Then we make an arbitrary pairing of the ends of the barbells that is compatible with the coloring. This means we only pair black to black and green to green; we never mix colors. We say two barbell graphs Γ, Γ' are equivalent if we can carry Γ to Γ' using the action of B_m. Let $N(\Gamma)$ be the number of connected components of Γ that contain at least one green vertex, and let $\operatorname{Aut}\Gamma \subset B_m$ be the subgroup of automorphisms.

7.3. Theorem. *Let $m = k/2$. Then we have*

$$\frac{1}{|B_m|} C_{\mathbb{S}}(n,k) = \sum_{\Gamma \in G(m)} \frac{n^{N(\Gamma)}}{|\operatorname{Aut}\Gamma|}, \tag{24}$$

where the $G(m)$ is the set of equivalence classes of barbell graphs with m barbells.

Proof. Let $G^*(m)$ be the set of all barbell graphs with m barbells, without modding out by the action of B_m. We will show

$$C_{\mathbb{S}}(n,k) = \sum_{\Gamma \in G^*(m)} n^{N(\Gamma)}, \tag{25}$$

which implies (24). Indeed, by definition the elements of $G(m)$ are the orbits of B_m in $G^*(m)$, and thus (24) follows from the orbit-stabilizer formula. To prove (25), we will show that the total contribution from the barbell graphs $G^*(m)$ agrees with that from the paired barbell structures in §6.3.

Thus let $\beta = S_0 \cup S_b$ be a barbell structure on $\{1, \ldots, k\}$, and consider the fixed collection of barbells Ξ_m as in Figure 11. When we compute the contribution of all pairings π of β, the result has the form $wP(n)$, where w is the Wick number $w(|S_0|)$ and $P(n)$ is a polynomial in n of degree $|S_b|/2$. We claim β tells us how to build a collection of barbell graphs giving the same total contribution, and that by varying β we obtain all graphs in $G^*(m)$.

First, the partition β tells us how to color the vertices in Ξ_m: we color those with labels in S_0 (respectively, S_b) black (resp., green). This determines the sets S_{bl} and S_{gr}. Next, choose an arbitrary pairing of S_{bl}. We claim, once this pairing is fixed, that after adding together the contributions coming from all pairings in S_{gr} one obtains $P(n)$. This proves the result, since there are w possible choices of pairings in S_{bl}.

So let Γ be the union of path graphs formed after pairing the vertices in S_{bl}. The connected components of Γ either have black or green endpoints. The components with black endpoints are irrelevant and can be ignored. Those with green endpoints either have no internal vertices or have all internal vertices black. Since there are $|S_{gr}|/2$ connected components with green vertices, the contribution after all pairings of S_{gr} are formed will be $P(n)$. This shows that either model, barbell structures or barbell graphs, produces the same expectation $\left\langle \operatorname{Tr} \mathbf{x}^k \right\rangle_{\mathbb{S}}$, and completes the proof. □

7.4. The series $B(x)$ counts all the barbell pairings divided by the orders of their automorphism group. Just as in the usual Feynman calculus, one can simplfy the computation by reducing to the connected diagrams, since typically there are far fewer connected than general diagrams. In other words, one considers the generating function

$$B_c(x) := \log B(x).$$

It turns out that this series has a particularly simple form:

7.5. Theorem. *We have*

$$\begin{aligned} B_c(x) &= \frac{1}{2}(n+1)x^2 + \frac{1}{4}(3n+1)x^4 + \frac{1}{6}(7n+1)x^6 + \frac{1}{8}(15n+1)x^8 + \cdots \\ &= \sum_{m\geq 1} \frac{1}{2m}((2^m-1)n+1)x^{2m}. \end{aligned}$$

Proof. First, using the barbell group we can carry any connected barbell graph into a standard form: the pairings all connect the bottom of the ith barbell to the top of the $(i+1)$st barbell for $1 \leq i \leq m$. We take these labels mod m, which means the bottom of the last barbell is connected to the top of the first (cf. Figure 13). For the purposes of this proof, we will say that such a connected barbell graph is *standardly paired.* If there are no green vertices, then the automorphism group of this pairing has order $2m$, and is the group $G \subset B_m$ generated by cyclic permutation of the barbells and simultaneous flipping of all of them about both axes. This implies that the constant term of x^{2m} is $1/2m$.

Now we claim that the standardly paired connected barbell graphs with $2m$ vertices are in bijection with subsets of $\{1, \ldots, m\}$. Indeed, let $I = \{i_1, \ldots, i_l\}$ be a subset of $\{1,\dots,m\}$. Then we simply take the standardly paired barbell graph with all vertices initially black, and color the bottom of barbell i_j and the top of barbell i_j+1 green for $j=1,\dots,l$. This graph clearly has $N(\Gamma)=1$. Since there are 2^m-1 such graphs with at least one green vertex, the result follows once we divide out by the action of G.

□

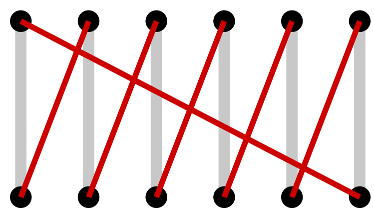
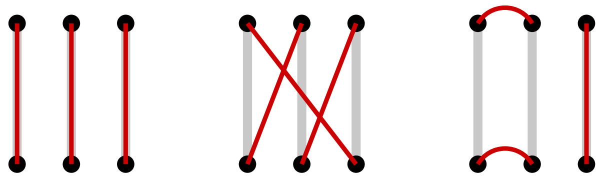


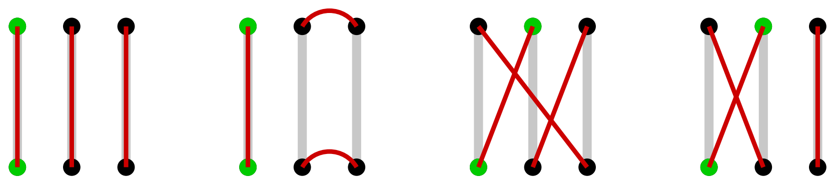
FIGURE 13. A standardly paired connected barbell graph.

7.6. Example. Figures 14–17 show the barbell graphs used in the computation of the coefficient of x^6 in (22), along with their contributions $n^{N(\Gamma)}/|\operatorname{Aut}\Gamma|$. Thus the graph Γ in Figure 14(a) has $|\operatorname{Aut}\Gamma| = 48$ and contributes to the constant term. The connected graphs, which contribute to the coefficient of x^6 in Theorem 7.5, are indicated with a star $\star$.



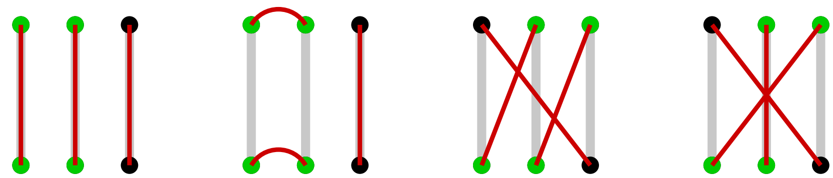
(a) $1/48$ (b) $1/6\star$ (c) $1/8$

FIGURE 14.



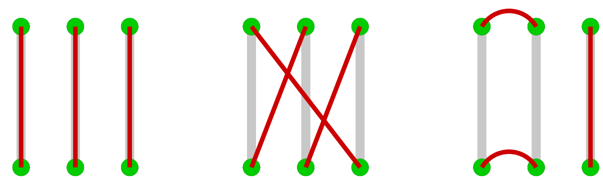
(a) $n/16$ (b) $n/8$ (c) $n/2\star$ (d) $n/4$

FIGURE 15.



(a) $n^2/16$ (b) $n/8$ (c) $n/2\star$ (d) $n^2/4$

FIGURE 16.



(a) $n^3/48$ (b) $n/6\star$ (c) $n^2/8$

FIGURE 17.

8. THE PERTURBATION SERIES

We conclude by computing the pertubation series for the spin factor. We use notation from §4. In particular, let t be an indeterminate and let $g_3, g_4, \dots$ be formal deformation parameters, packaged together into a vector $\mathbf{g} = (g_3, g_4, \dots)$. Let $\mathbf{m} = (m_3, m_4, \dots) \in \prod_{k\geq 3} \mathbb{Z}_{\geq 0}$ be a vector of multiplicities; we assume $m_k = 0$ for all sufficiently large k. We write $\mathbf{g}^{\mathbf{m}}$ for the monomial $\prod g_k^{m_k}$ and set $N(\mathbf{m}) = \sum k m_k$. We will also need to consider Ξ_k and the barbell group for odd k. Thus for any k define

$$m = m(k) = \begin{cases} k/2 & k \text{ even}, \\ (k-1)/2 & k \text{ odd}. \end{cases}$$

Let $M(k) = 2^m m!$ and let B_m be the barbell group of order $M(k)$. For k even we think of B_m acting on a set of k fixed barbells as described in §7.1. For k odd we let Ξ_k be the set $\{1, \dots, k\}$ with $(k-1)/2$ fixed barbells as in Figure 18, and let B_m act by fixing the isolated point. Eventually when building barbell graphs using copies of Ξ_k, we will assume that the vertices are always colored with $|S_{gr}|$ *even*.

Define the pertubation series

$$F(\mathbf{g}, t, \mathbf{x}) = \exp\Big(\sum_{k\geq 3} (g_k \operatorname{Tr} \mathbf{x}^k) t^k / M(k)\Big). \tag{26}$$

Our goal is to compute the coefficient of t^N in

$$\langle F(\mathbf{g}, t, X)\rangle_{\mathbb{S}} \in \mathbb{Q}[g_3, g_4, \dots][\![t]\!].$$

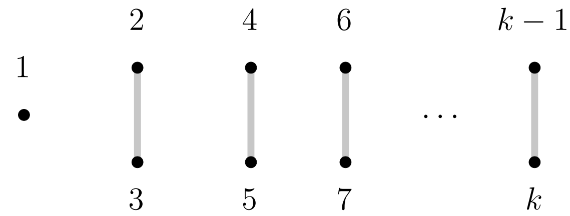


FIGURE 18.

(16) in terms of barbell graphs. As in §4, this coefficient is a homogeneous polynomial of degree N in the monomials $\mathbf{g}^{\mathbf{m}}$, where $N = N(\mathbf{m})$, so we compute the coefficient of each $\mathbf{g}^{\mathbf{m}} t^N$. Let $\Xi_{\mathbf{m}}$ be the union $\coprod \Xi_k$, where we take m_k copies of Ξ_k. Let $G(\mathbf{m})$ be the collection of barbell graphs built from $\Xi_{\mathbf{m}}$, where in every block of barbells Ξ_k the number of green vertices is even.

8.1. Theorem. *In* (26)*, the coefficient of* $\mathbf{g}^{\mathbf{m}}$ *in the coefficient of* $t^{N(\mathbf{m})}$ *is*

$$\sum_{\Gamma \in G(\mathbf{m})} \frac{n^{N(\Gamma)}}{|\operatorname{Aut}\Gamma|},$$

where $\operatorname{Aut}\Gamma$ *consists of automorphisms induced by acting by the barbell groups in the blocks and permutations of the blocks with the same number of vertices.*

Proof. Just like Theorem 4.3, the proof follows from the exponential formula for generating functions together with Theorem 7.5. The only subtleties are that (i) one must have an even number of green vertices in each subcollection of barbells, and (ii) the connected components need not be closed 1-manifolds, but can be 1-manifolds with boundary. The first follows from the trace formula in Proposition 6.2, and the second follows since we now allow odd numbers of vertices in each block of barbells. □

8.2. Example. We give an example to show how to apply Theorem 8.1. We compute the coefficient of g_3^2, which corresponds to $\langle(\operatorname{Tr}\mathbf{x}^3)^2\rangle_{\mathbb{S}}/8$. Thus $\mathbf{m} = (2, 0, \dots)$ and we are building barbell graphs from two blocks, each a copy of Ξ_3. The denominator 8 comes from the product $2 \cdot 2 \cdot 2 = M(3) \cdot M(3) \cdot 2!$. In particular, we have two actions of the barbell group B_1 in each Ξ_3, and we also have the involution that exchanges the blocks. The barbell graphs are shown in Figures 19–21, along with the quantities $n^{N(\Gamma)}/|\operatorname{Aut}\Gamma|$. The result is

$$\frac{\langle(\operatorname{Tr}\mathbf{x}^3)^2\rangle_{\mathbb{S}}}{2 \cdot 2 \cdot 2!} = \frac{9}{8}n^2 + \frac{7}{2}n + \frac{15}{8}.$$

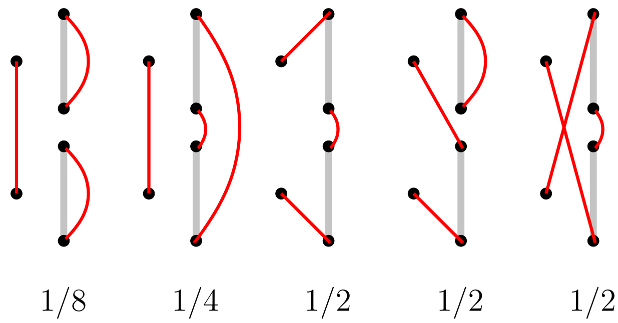


FIGURE 19.

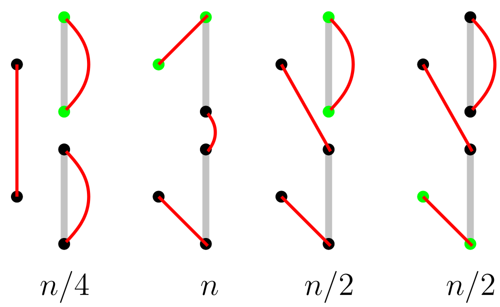


FIGURE 20.

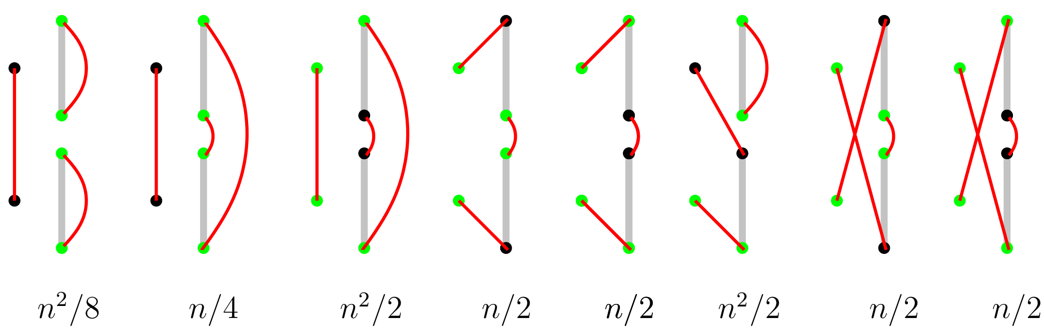


FIGURE 21.

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