ON TORIC ORBITS IN THE AFFINE SIEVE

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Abstract. We give a detailed analysis of a heuristic model for the failure of "saturation" in instances of the Affine Sieve having toral Zariski closure. Based on this model, we formulate precise conjectures on several classical problems of arithmetic interest, and test these against empirical data.

Contents

1. Introduction

The Fundamental Theorem of the Affine Sieve, introduced by Bourgain-Gamburd-Sarnak [BGS10] and proved by Salehi Golsefidy-Sarnak [SGS13] extends the Brun sieve to orbits of affine-linear group actions. The goal of this paper is to study the behavior of prime factors of orbits outside the purview of this theorem.

More precisely, let $\Gamma < \mathrm{GL}_N(\mathbb{Q})$ be a finitely generated group, that is, $\Gamma = \langle A_1, A_2, \cdots, A_k\rangle$, fix a base point $\mathbf{v}_0 \in \mathbb{Q}^N$, and let

$$\mathcal{O} := \Gamma \cdot \mathbf{v}_0 \ \subset \ \mathbb{Z}^N$$

be the orbit of $\mathbf{v}_0$ under Γ, assumed to be integral[1]. Let $\Omega(n)$ denote the number of primes dividing an integer n, counted with multiplicity. Given $R \geq 1$, an integer n with

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[1] One can work more generally with entries in the ring of S-integers $\mathbb{Z}_S$, but we restrict to $\mathbb{Z}$ for ease of exposition. Note that there exist $\Gamma < \mathrm{GL}_N(\mathbb{Q})$ having no non-zero vector giving an integral orbit, e.g., $\Gamma = \langle A\rangle$ with $A = \left(\begin{smallmatrix} \frac12 & 0 \\ 0 & \frac12 \end{smallmatrix}\right)$.

$\Omega(n) \le R$ is called R-almost prime. Fix a polynomial $f(x_1, x_2, ..., x_N) \in \mathbb{Q}[x_1, \dots, x_N]$ taking integer values on $\mathcal{O}$, and let

$$\mathcal{O}_R := \{\mathbf{v} \in \mathcal{O} : \Omega(f(\mathbf{v})) \le R\}$$

be the points in $\mathcal{O}$ taking R-almost prime values under f. The pair $(\mathcal{O}, f)$ is said to *saturate* if there exists some $R < \infty$ so that

$$\mathrm{Zcl}(\mathcal{O}_R) = \mathrm{Zcl}(\mathcal{O}). \tag{1.1}$$

Here Zcl refers to Zariski closure in affine space.[2] The *saturation number* is the least R for which (1.1) holds; this can be determined exactly or at least well-approximated in some special instances, see [Kon14] for more discussion. Let $V(f)$ be the affine $\mathbb{Q}$-variety given by $f = 0$. In general, we assume that f is non-constant on (any irreducible component of) $\mathrm{Zcl}(\mathcal{O})$. This is equivalent to

$$\dim(V(f) \cap \mathrm{Zcl}(\mathcal{O})) < \dim \mathrm{Zcl}(\mathcal{O}), \tag{1.2}$$

viewing the Zariski closure $\mathrm{Zcl}(\mathcal{O})$ inside $\mathbb{C}^N$. Then the aforementioned Fundamental Theorem of Salehi Golsefidy and Sarnak [SGS13, Theorem 1], states the following.

Theorem 1.1 ([SGS13])**.** *Let Γ be a finitely generated subgroup of $GL_N(\mathbb{Q})$ having Zariski closure $\mathbb{G} = \mathrm{Zcl}(\Gamma)$ in $GL_N(\mathbb{C})$. Let $\mathbf{v}_0 \in \mathbb{Q}^N$ and let $\mathcal{O} = \Gamma\mathbf{v}_0 \subset \mathbb{Z}^N$ be the Γ-orbit of $\mathbf{v}_0$. Suppose that $f(x) \in \mathbb{Q}[x_1, \dots, x_N]$ is such that $f(\mathcal{O}) \subset \mathbb{Z}$ and (1.2) is satisfied. Then the pair $(\mathcal{O}, f)$ saturates, as long as no algebraic torus[3] is a homomorphic image of the connected component $\mathbb{G}_0$ of the identity of $\mathbb{G}$.*

In [SGS13, Appendix], Salehi Golsefidy-Sarnak give a heuristic argument, based on the Borel-Cantelli lemma, that the condition of having no tori is necessary in certain cases. Their model considered an algebraic torus (that is, Γ is a free abelian group of rank D with generators $A_1, \dots, A_D \in \mathrm{GL}_N(\mathbb{Z})$, and there is a $g \in \mathrm{GL}_N(\mathbb{C})$ so that for all j, the matrices gA_jg^{-1} are diagonal) and the test polynomial $f(x_{1,1}, ..., x_{N,N}) = \prod_{j=1}^k f_j(\mathbf{x})$, with $f_j(\mathbf{x}) = j + \sum_{m,n=1}^N x_{m,n}^2$. The test polynomial f has (at least) k irreducible factors over $\mathbb{Q}[x_{1,1}, ...x_{N,N}]$, all of the same degree (so they have roughly the same "size" on points of $\mathcal{O}$). Their heuristic was that the prime factorizations of the k elements $f_j(\mathbf{x})$ evaluated at a point $\mathbf{x} \in \mathcal{O}$ ought to be "independent," at least at the level of the number of prime factors, $\Omega(f_j(\mathbf{x}))$, since they are just integer shifts of each other.

In this paper, we refine this heuristic and make precise predictions on the failure of saturation in the toric case, which we then test empirically in a number of natural settings of classical interest.

[2]Recall that this Zariski closure can be thought of as the zero set of all polynomials vanishing on $\mathcal{O}$.
[3]E.g. $(\mathbb{C}^\times)^n$.

1.1. Main Probabilistic Model.

We model the k irreducible factors of f as k randomly and independently chosen integers in an exponentially growing interval, depending on a parameter n. The parameter n is to be viewed as modeling elements of a toral orbit, which grow exponentially.

Theorem 1.2. *Let $k \geq 1$ be a fixed integer. Fix a constant $C > 1$ and for each $n \geq 1$, draw an integer vector*

$$(x_{1,n}, x_{2,n}, \ldots, x_{k,n}) \ \in \ [1, C^n]^k$$

with uniform distribution. Then with probability one,

$$\liminf_{n \geq 1} \frac{\Omega(x_{1,n} \cdot x_{2,n} \cdots x_{k,n})}{\log n} \ = \ \beta_k, \tag{1.3}$$

where β_k denotes the unique solution in $[0, k-1]$ to

$$\beta_k(1 - \log \beta_k + \log k) \ = \ k - 1, \tag{1.4}$$

with $\beta_1 = 0$ and $\beta_k > 0$ for $k \geq 2$.

The constants β_k are absolute, in particular, independent of C. The first few values of β_k are:

$$\beta_2 = 0.373365, \ \beta_3 = 0.913728, \ \beta_4 = 1.52961, \beta_5 = 2.19252, \ldots, \beta_{10} = 5.8754, \ldots$$

Note that the expected size[4] of $\Omega(m)$ for a random integer m is $\log \log m$, and of course

$$\Omega(x_{1,n} \cdot x_{2,n} \cdots x_{k,n}) = \sum_j \Omega(x_{j,n}),$$

whence the expected size of this sum is $k \log \log C^n \sim k \log n$. Thus we may interpret (1.3) as showing that, up to a multiplicative constant k/β_k, one never sees (asymptotically) a deficient number of prime factors.

To test the validity of this model empirically, it will be useful to understand how large n should be to experimentally observe the behavior (1.3). Naively we may expect from this equation that the largest $\mathfrak{n} = n_{max}$ for which $x_{1,n} \cdots x_{k,n}$ is R-almost prime satisfies:

$$\frac{R}{\log \mathfrak{n}} \approx \beta_k,$$

or

$$\mathfrak{n} \approx \exp(R/\beta_k). \tag{1.5}$$

It turns out that the probabilistic model sometimes makes a different prediction.

Theorem 1.3. *Fix $k \geq 2$, $C > 1$, and for each $n \geq 1$, draw a vector*

$$\mathbf{x}_n = (x_{1,n}, x_{2,n}, \ldots, x_{k,n}) \in [1, C^n]^k$$

[4] e.g. in the normal order sense of the Erdős-Kac theorem.

uniformly. Let $\mathcal{X} = (\mathbf{x}_1, \mathbf{x}_2, \dots)$ be a random variable consisting of a sequence of independent such draws, one for each n. For any fixed $R \geq k$, consider the random variable

$$\mathfrak{n} = \mathfrak{n}(R; \mathcal{X}) := \max\{n : \Omega(x_{1,n} \cdots x_{k,n}) \leq R\},$$

with $\mathfrak{n} = 0$ if there are no such n, and $\mathfrak{n} = \infty$ if the event occurs infinitely often. Then

(1) *with probability one,*

$$\mathfrak{n} < \infty, \tag{1.6}$$

and moreover,

(2) *for all $m \geq k-1$, the m-th moment of $\mathfrak{n}$ diverges,*

$$\mathbb{E}[\mathfrak{n}^m] = \infty. \tag{1.7}$$

Remark 1. In the case $k = 1$ not covered in Theorem 1.3, one has instead that with probability one, $\mathfrak{n} = +\infty$.

Remark 2. In many natural examples treated below, we have $k = 2$, so taking $m = 1$ means that the expected value of $\mathfrak{n}(R)$ is infinite for all $R \geq 2$. Thus we should not expect $\mathfrak{n}(R)$ to behave nicely like $\exp(R/\beta_2)$, as suggested naively by (1.5). One may interpret this as saying that for $k = 2$ there may exist extremely large "sporadic" solutions to $\Omega(x_{1,n}, \dots, x_{k,n}) = R$.

Remark 3. The proofs of Theorems 1.2 and 1.3 apply and give the same result in the more general case of $\mathbf{x}_n$ chosen from non-identically growing intervals, that is $(x_{1,n}, ..., x_{k,n}) \in [1, C_1^n] \times [1, C_2^n] \cdots \times [1, C_k^n]$, for fixed constants $C_1, \dots, C_k > 1$.

1.2. **The Toral Affine Sieve Conjecture.**

The probabilistic model above, motivates a heuristic prediction concerning the number of prime factors of certain sequences, associated to toric orbits, the (rank one) "Toral Affine Sieve Conjecture" stated below. We will derive as consequences of this conjecture other predictions in several settings of classical interest.

Conjecture 1.1 (Toral Affine Sieve Conjecture). *Let $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ be a hyperbolic matrix, that is, one having two distinct real eigenvalues; equivalently*

$$\mathrm{tr}(\gamma)^2 - 4\det(\gamma) > 0.$$

Let $\Gamma = \langle\gamma\rangle^+ := \{\gamma^n : n \geq 0\}$ be the semigroup generated by γ, and suppose that $\mathbf{v}_0 \in \mathbb{Q}^2 \setminus (0,0)$ is a nonzero vector such that the orbit $\mathcal{O} := \Gamma \cdot \mathbf{v}_0 \subset \mathbb{Z}^2$ is integral and infinite. Then

$$\liminf_{(x,y)\in\mathcal{O}} \frac{\Omega(xy)}{\log\log|xy|} \;\geq\; \beta_2 \approx 0.373365. \tag{1.8}$$

Since the Zariski closure of Γ in $GL(2, \mathbb{C})$ is an algebraic torus, and since the orbit $\mathcal{O}$ is assumed to be infinite, it is a one-dimensional torus, so it follows that the Zariski closure of $\mathcal{O}$ in $\mathbb{C}^2$ has $\dim(\mathrm{Zcl}(\mathcal{O})) = 1$ in (1.2). We have taken the test function $f(x, y) = xy$, whence $V(f) \cap \mathrm{Zcl}(\mathcal{O})$ is finite, having dimension 0. The points in $(x_n, y_n) := \gamma^n \mathbf{v}_0 \in \mathcal{O}$ grow exponentially, that is, there are $C > c > 1$ so that

$$c^n \ < \ |x_n y_n| \ = \ |f(\gamma^n \mathbf{v}_0)| \ < \ C^n.$$

In consequence, the factor $\log\log|xy|$ in (1.8) can be replaced by $\log n$, that is, (1.8) is equivalent to

$$\liminf_{n\to\infty} \frac{\Omega(x_n y_n)}{\log n} \ \geq \ \beta_2.$$

The conjecture is based on applying the model of Theorem 1.2 with $k = 2$ having two "independent" factors (x_n, y_n) for $f(\gamma^n \mathbf{v}_0)$. In the "generic" situation, we might have equality in these limits. However there are cases of orbits whose limiting values may involve β_k for larger k, see the examples in §2.

Remark 4. We did not need to assume in Conjecture 1.1 any coprimality condition (e.g. $\gcd(\mathcal{O}) = 1$) on the orbit. Indeed, if all entries of $\mathbf{v} = (x, y) \in \mathcal{O}$ have a common factor, then this factor, divided by $\log\log|xy|$, is irrelevant in the lim inf in (1.8).

1.3. **Consequences.**

The basic Conjecture 1.1 implies other striking predictions, of which we present two below; the first applies to integer points on affine quadrics, and the second applies to the continued fraction convergents of quadratic surds.

Theorem 1.4. *Let $Q(x,y) = Ax^2 + Bxy + Cy^2$ be an indefinite (that is, $D = B^2 - 4AC$ is positive), non-degenerate (D is not a square) binary quadratic form over $\mathbb{Z}$. Fix a square-free $t \in \mathbb{Z}$ so that the set $V(\mathbb{Z})$ of $\mathbb{Z}$-points of the affine quadric $V = V_{Q,t}$ given by*

$$V \ : \ Q(x,y) = t$$

is non-empty. Then, assuming Conjecture 1.1,

$$\liminf_{\substack{(x,y)\in V(\mathbb{Z}) \\ |xy|\to\infty}} \frac{\Omega(xy)}{\log\log|xy|} \ \geq \ \beta_2.$$

Theorem 1.5. *Let α be a real quadratic irrational, and let p_n/q_n denote the n-th convergent of its ordinary continued fraction expansion. Then, assuming Conjecture 1.1,*

$$\liminf_{n} \frac{\Omega(p_n q_n)}{\log n} \ \geq \ \beta_2.$$

These two theorems will not be surprising to experts, but the (conditional) conclusions, particularly the appearance of the precise number $\beta_2 \approx 0.373365$, are unexpected.

1.4. **Organization.**

In §2, we give a number of illustrative examples and numerics which, one may argue, provide support for the heuristic provided by the probabilistic model in the context of Conjecture 1.1. We prove Theorem 1.2 in §3, followed by Theorem 1.3 in §4. In the final §5, we sketch proofs of Theorems 1.4 and 1.5.

1.5. **Notation.**

We use the following standard notation. We use the symbol $f \sim g$ to mean $f/g \to 1$. The symbols $f \ll g$ and $f = O(g)$ are used interchangeably to mean the existence of an implied constant $C > 0$ so that $f(x) \le Cg(x)$ holds for all $x > C$; moreover $f \asymp g$ means $f \ll g \ll f$. Unless otherwise specified, implied constants depend at most on k, which is treated as fixed. The letter $\varepsilon > 0$ is an arbitrarily small constant, not necessarily the same at each occurrence. The Gamma function is denoted $\Gamma(z)$ and a product $\prod_p$ denotes a product over primes. The floor function, $\lfloor \cdot \rfloor$, returns the largest integer not exceeding its argument.

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2. Examples and Numerics

It should be clear that running decent numerics to test Conjecture 1.1 is a daunting task. Indeed, orbits increase exponentially in size, and hence become ever more difficult to factor. Thankfully, others have already exerted tremendous effort in tabulating prime factorizations for certain sequences of classical interest, in particular, the Fibonacci, Lucas, and Mersenne numbers. We mine their factorization data to test our predictions for Conjecture 1.1 and its consequences. We have made the raw data and Mathematica file used to construct the figures available at: `http://sites.math.rutgers.edu/~alexk/files/AllOmegasData.nb`.

2.1. **Fibonacci and Lucas Numbers Factorization Statistics.**

Let F_n and L_n denote the nth Fibonacci and Lucas numbers, respectively. Recall that both sequences are defined by the same recursive relation, $F_{n+1} = F_n + F_{n-1}$ and $L_{n+1} = L_n + L_{n-1}$, but differ in the initialization, namely, $F_1 = F_2 = 1$, while $L_1 = 1$, $L_2 = 3$. They are related by

$$F_{2n} = F_n L_n. \tag{2.1}$$

Both sequences have been completely factored for $1 \le n \le 1\,000$ and partially factored for n going up to $10\,000$, see the website [Mer].

In the following calculations, when we encounter in the (incomplete) factorization data a composite number having no known prime factors, we treat that number as a product of exactly two primes (which may be an undercount in Ω). We use this data to study orbits giving several different combinations of Fibonacci numbers and Lucas numbers.

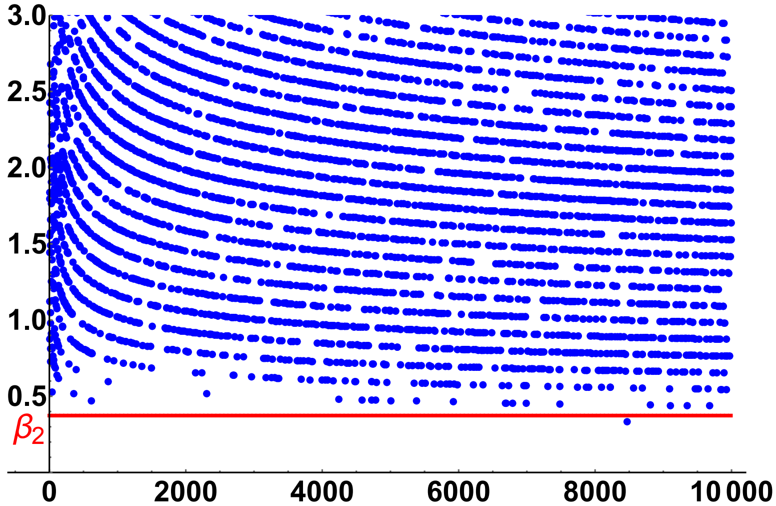

FIGURE 1. A plot of $n < 10\,000$ vs. $\Omega(F_nL_n)/\log\log(F_nL_n)$. Also shown is the horizontal line $y = \beta_2 \approx 0.37$.

Example 2.1. One can easily verify that, if one takes

$$\gamma = \begin{pmatrix} 1/2 & 1/2 \\ 5/2 & 1/2 \end{pmatrix}, \qquad \Gamma = \langle \gamma \rangle^+, \qquad \mathbf{v}_0 = (1,1)^t,$$

then the orbit $\mathcal{O} = \Gamma \cdot \mathbf{v}_0 = \{(F_n, L_n) : n \geq 1\}$. A plot of n versus

$$\frac{\Omega(F_nL_n)}{\log\log(F_nL_n)} \tag{2.2}$$

appears in Figure 1. This plot seems to give rather good evidence for equality in (1.8).

Remarks:

(i) The plot in Figure 1 appears to be a union of curves, and a moment's thought reveals that these are roughly the level sets of $y = R/\log x$ for various integer values of R. Conjecture 1.1 predicts that the number of elements on each curve is finite, since each curve eventually dips below the line $y = \beta_2$.

(ii) From Figure 1, one notices a single value of $n < 10\,000$ for which (2.2) seems to dip below $\beta_2 \approx 0.37$. This occurs at $n = 8\,467$, for which L_n is prime and F_n is composite, with each number spanning 1 770 decimal digits. Since we do not know any factors of F_n, we follow our protocal, declaring that $\Omega(F_nL_n) = 3$. But the true value could perhaps be higher, in which case there may be no values of n up to 10 000 dipping below (2.2). Since Conjecture 1.1 only predicts a lim inf, there may in fact be infinitely many points in the plot dipping below β_2, as long as the amount by which they dip below decreases.

(iii) The data in Figure 1 also provide an instance of (the conditional) Theorem 1.4, since the pair (F_n, L_n) are integer solutions to the Pellian binary quadratic form

$$x^2 - 5y^2 = \pm 4. \tag{2.3}$$

(*iv*) While Figure 1 may seem promising towards Conjecture 1.1, this computation is limited to the humble scale $n = 10\,000$, where $\log n \approx \log\log(F_n L_n) \approx 10$.
With current computing technology it would be difficultto go significantly farther.

One may also object to using the Fibonacci and Lucas sequences to test Conjecture 1.1, as these are "strong divisibility sequences"; i.e., $m \mid n \Longrightarrow a_m \mid a_n$. While it seems likely that this fact could affect some statistics of total number of primes seen in individual draws (see, e.g., [BLMS05]), it appears not to affect the lim inf value in (2.2). Either way, any effect would only increase the limiting value, which Figure 1 suggests is not the case.

Example 2.2. Next we consider the simpler setting of consecutive Fibonacci numbers:

$$\gamma \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma = \langle\gamma\rangle^+, \qquad \mathbf{v}_0 = (1,0)^t, \qquad \mathcal{O} \;=\; \Gamma\cdot\mathbf{v}_0 \;=\; \{(F_{n+1}, F_n)^t\}.$$

Applying Conjecture 1.1, one may surmise that the correct liminf for $\Omega(F_nF_{n+1})/\log\log(F_nF_{n+1})$ is $\beta_2 \approx 0.37$. But a moment's inspection of Figure 2 reveals that the truth seems to be closer to $\beta_3 \approx 0.91$. This is because one of the indices n or $n+1$ is *even*, so that Fibonacci number splits according to (2.1) into a Fibonacci times a Lucas. Thus this sequence F_nF_{n+1} behaves like the product of *three* independent sequences, resulting in the predicted lim-inf of β_3, not β_2.

For this reason, Conjecture 1.1 must be stated with an inequality in (1.8); one cannot necessarily determine *a priori* from the data of $\mathcal{O}$ whether there is a "non-obvious" factorization. Indeed, if we keep Γ as is but change $\mathbf{v}_0$ to $\mathbf{v}_0 = (1,2)^t$, then the orbit $\mathcal{O} = \{(L_{n+1}, L_n)^t\}$ becomes consecutive Lucas numbers instead of Fibonaccis. These do not exhibit the extra factorization, so the liminf is restored (though now not very convincingly) to β_2, see Figure 3.

Example 2.3. The previous example suggests the following refinement of Example 2.1. One can easily produce orbits which separately capture the even and odd index Fibonacci/Lucas pairs (F_{2n}, L_{2n}) and (F_{2n+1}, L_{2n+1}). These of course appear simultaneously inside the orbit of Figure 1. Now in Figure 4 we show what happens if the odd values are suppressed: the even values exhibit an increased beta-value, again to β_3.

Example 2.4. We consider pairs (F_{2n}, F_{2n+2}) of consecutive even-indexed Fibonacci numbers. This sequence was already discussed in the initial Bourgain-Gamburd-Sarnak paper on the Affine Sieve, see [BGS10, Section 2.1]. It is obtained by taking $\gamma = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$, which has powers

$$\gamma^n = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{2n+2} & F_{2n} \\ -F_{2n} & -F_{2n-2} \end{pmatrix},$$

and acting on $\mathbf{v}_0 = (1,0)^t$ to give the orbit $\mathcal{O} = \{(F_{2n}, F_{2n+2})^t\}$. Then

$$f(\gamma^n\mathbf{v}_0) = F_{2n}F_{2n-2} = F_nL_nF_{n-1}L_{n-1},$$

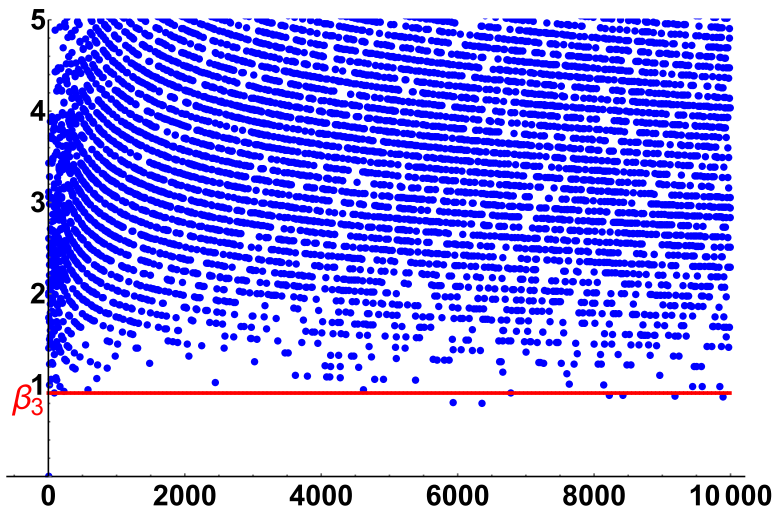

FIGURE 2. A plot of $n < 10\,000$ vs. $\Omega(F_nF_{n+1})/\log\log(F_nF_{n+1})$. Also shown is the horizontal line $y = \beta_3 \approx 0.91$.

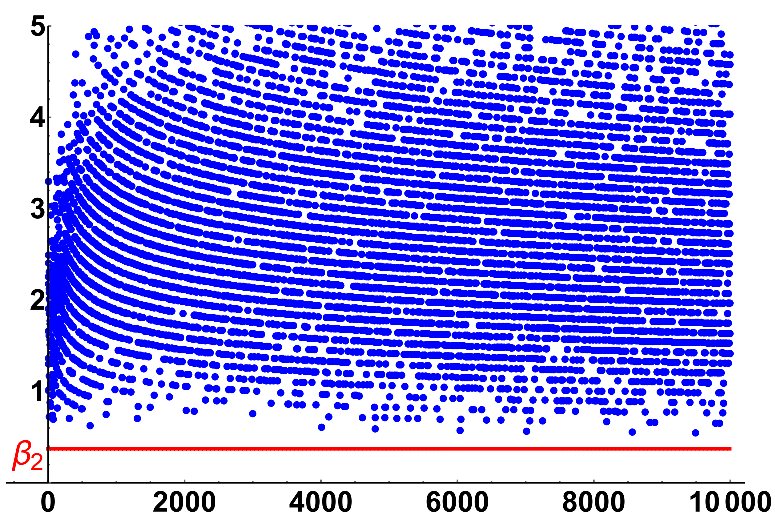

FIGURE 3. A plot of $n < 10\,000$ vs. $\Omega(L_nL_{n+1})/\log\log(L_nL_{n+1})$. Also shown is the horizontal line $y = \beta_2$.

where we have again invoked the Fibonacci identity (2.1). As a consequence we expect four "independent" factors, so the liminf in (1.8) should be no smaller than $\beta_4 \approx 1.52961$. See Figure 5, which confirms the prediction. But on further inspection, it turns out that the lim-inf here should be β_5, not β_4! Indeed, one of the indices n or $n-1$ is even, so one of the factors F_n or F_{n-1} in $f(\gamma^n \mathbf{v}_0)$ should always decompose further into a Fibonacci/Lucas pair. We do not fully understand why the numerics do not agree with this prediction,

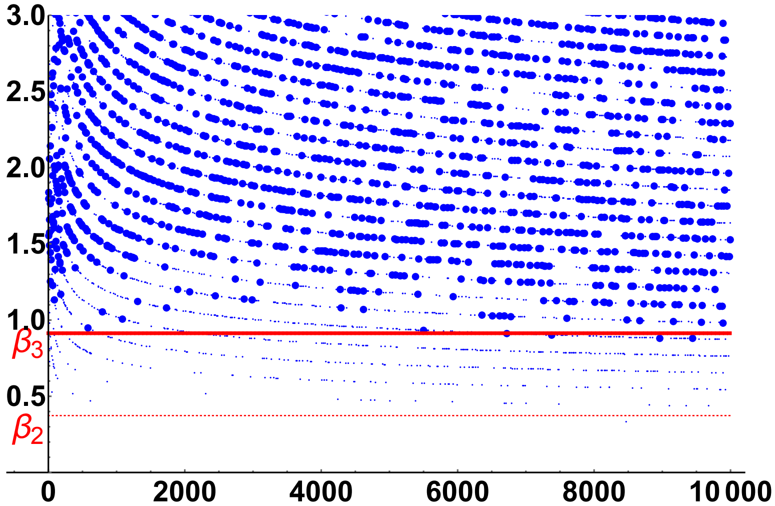

FIGURE 4. A plot of $n < 10\,000$ vs. $\Omega(F_n L_n)/\log\log(F_n L_n)$, with the even index values with large marks and the odd index values with small marks. Also shown are the horizontal lines $y = \beta_2, \beta_3$. Compare to Figure 1.

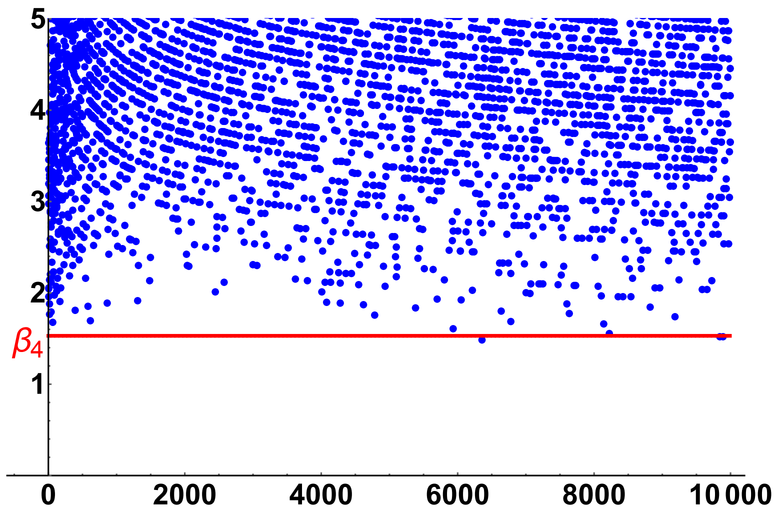

FIGURE 5. A plot of $n < 10\,000$ vs. $\Omega(F_{2n}F_{2n+2})/\log\log(F_{2n}F_{2n+2})$. Also shown is the horizontal line $y = \beta_4$.

though it is plausible that the under-estimation of Ω in inconclusive factorizations may at this point be making a significant contribution.

2.2. **Mersenne Number Factorization Statistics.**

For our last numerical example, we move to Mersenne numbers, $M_n := 2^n - 1$, whose factorizations have also been extensively mined.

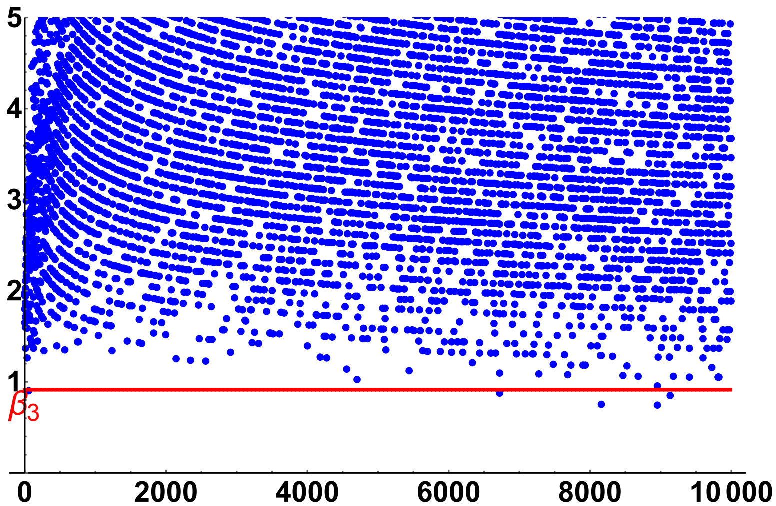

FIGURE 6. A plot of $n < 10\,000$ vs. $\Omega(M_n M_{n+1})/\log n$. Also shown is the horizontal line $y = \beta_3$.

Example 2.5. To produce the orbit $\mathcal{O} = \{(M_{n+1}, M_n)\}$, consider as before $\Gamma = \langle\gamma\rangle^+$ and $\mathcal{O} = \Gamma \cdot \mathbf{v}_0$, where:

$$\gamma = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{v}_0 = (1,0)^t, \qquad \gamma^n \mathbf{v}_0 = (M_{n+1}, M_n)^t.$$

The first 500 values of $\Omega(M_n)$ appear in OEIS (A046051), and the (sometimes partial) factorizations up to 10 000 were kindly provided to us by Sean Irvine using `factordb.com`. These were used to make Figure 6, showing that the liminf of $\Omega(M_n M_{n+1})/\log\log(M_n M_{n+1})$ appears to be tending towards β_3. This is consistent with the fact that one of n or $n+1$ is even, and for the even indices, Mersenne numbers $M_{2\ell}$ factor as $2^{2\ell} - 1 = (2^\ell - 1)(2^\ell + 1)$.

2.3. Extreme Fibonacci and Lucas values with a fixed number of prime factors.

Let us now consider Theorem 1.3 and the (naïve) heuristic (1.5) in the case of the Fibonacci and Lucas sequences, for fixed $R = 2$.

Example 2.6. Define the set

$$\Sigma_{FF} \;:=\; \{n \geq 2 : \Omega(F_n F_{n+2}) = 2\}$$

to be the indices n for which F_n and F_{n+2} are simultaneously prime. Applying (1.5) with $R = k = 2$ would suggest that

$$\max \Sigma_{FF} \;\overset{?}{\approx}\; \exp(2/\beta_2) \approx 212. \tag{2.4}$$

One can now examine the sequence [OEIa] of n for which F_n are prime, to find that

$$\{3, 5, 11, 431, 569\} \;=\; \Sigma_{FF} \cap [1,\ 1\,000\,000]. \tag{2.5}$$

Similarly, consider the set

$$\Sigma_{LL} \ := \ \{n \geq 2 : \Omega(L_n L_{n+2}) = 2\}$$

of indices n for which L_n and L_{n+2} are simultaneously prime; presumably (2.4) should also hold for Σ_{LL}. As before, one can examine the sequence [OEIb] of n for which L_n are prime, to find that

$$\{2, 5, 11, 17\} \ = \ \Sigma_{LL} \cap [1,\ 1\,000\,000]. \tag{2.6}$$

Both these results are compatible, at least to first order, with the naive heuristic (2.4).

Example 2.7. Next define

$$\Sigma_{FL} \ := \ \{n \geq 2 : \Omega(F_n L_n) = 2\}$$

to be the indices n for which the Fibonacci and Lucas sequences are simultaneously prime. As above, the naive heuristic (1.5) predicts $\max \Sigma_{FL} \ \overset{?}{\approx} \ \exp(2/\beta_2) \approx 212$. Using the sequences [OEIa] and [OEIb] of n for which F_n and L_n are primes, respectively, however we find

$$\{4, 5, 7, 11, 13, 17, 47, 148\,091\} \ \overset{*}{=} \ \Sigma_{FL} \cap [1,\ 1\,000\,000]. \tag{2.7}$$

The "$*$" here is to note that for the largest index $\mathfrak{n} := 148\,091$, the corresponding $F_\mathfrak{n}$ and $L_\mathfrak{n}$ (each having around 30 000 decimal digits) have not been certified prime.[5] The pair $(F_\mathfrak{n}, L_\mathfrak{n})$, if indeed both entries are prime, would have

$$\frac{\Omega(F_\mathfrak{n}, L_\mathfrak{n})}{\log\log(F_\mathfrak{n} L_\mathfrak{n})} \ \overset{?}{\approx} \ \frac{2}{\log \mathfrak{n}} \ \approx \ 0.167988,$$

so if we extended Figure 1 to $n < 150\,000$, we would see a huge dip below β_2 at $\mathfrak{n}$. In light of (2.4), this certainly constitutes a massively "sporadic" solution to (2.3). However but the existence of such a solution is *not* shocking, as it is predicted to sometimes occur by the probabilistic model of Theorem 1.3 (see Remark 2). It seems likely to us (though again, this may be naïve) that the left side of (2.7) is actually an equality to Σ_{FL}.[6]

3. Proof of Theorem 1.2

3.1. Analysis of β_k.

Fix an integer $k \geq 1$ let β_k solve (1.4). We first analyze this equation.

[5]The probable primality of $F_\mathfrak{n}$ was found by T. D. Noe while that of $L_\mathfrak{n}$ by de Water; see OEIS for further credits. Both numbers have passed numerous pseudoprimality tests. Assuming GRH, one would need to run about $(30\,000)^4$ trials (that is, $(\log F_\mathfrak{n})^2$ tests at a cost of $(\log F_\mathfrak{n})^2$ each, ignoring epsilons) of the Miller primality test to certify these entries prime. Unconditionally, the exponent 4 would be replaced by a 6, see [LP11]. Or better yet, one could try the elliptic curve primality test, which is also unconditional and in practice runs faster, though a worst-case execution time is currently unknown.

[6]Note that in some very special cases, one can sometimes completely determine sets like Σ_{FL}. Indeed, see [BLS09], where all solutions to $x^2 - 3y^2 = 1$ with $\Omega(xy) \leq 3$ are effectively listed.

Lemma 3.1. *For real $k \geq 1$ the function*

$$f_k(t) := t(1 - \log t + \log k) - (k-1)$$

is increasing on $0 < t < k$. It has a unique root $t = \beta_k \in (0, k-1]$.

Proof. The derivative of f is $f_k'(t) = -\log t + \log k$, which is clearly positive on $(0, k)$. For $k = 1$ it has by inspection a root at $\beta_0 = 0 = k - 1$. For $k > 1$, near the origin,

$$\lim_{t \to 0^+} f_k(t) \; = \; -(k-1) \; < \; 0,$$

and at $t = k - 1$, we have

$$f_k(k-1) \; = \; (k-1) \log\left(\tfrac{k}{k-1}\right) \; > \; 0.$$

Hence $f_k(t)$ has a unique root in this interval. □

Remark 5. One can solve for β_k explicitly in terms of the inverse function $g(z)$ to $z \mapsto ze^z$ on the positive real axis. Namely, one finds

$$\beta_k = \frac{1-k}{g\left(\frac{1-k}{ek}\right)},$$

where $e = 2.718....$ We will not need this fact, nor the fact that $\beta_k = k - 1 - O(1/k)$ for k large, which can be shown in a variety of ways.

3.2. **Analysis of the behavior of Ω.**

We next record a uniform asymptotic formula for

$$\mathcal{N}_r(T) \; := \; \#\{x < T : \Omega(x) = r\},$$

that is, the number of positive integers up to T having exactly r prime factors, counted with multiplicity. For fixed r, the formula

$$\mathcal{N}_r(T) \; \sim \; \frac{T}{\log T} \frac{(\log\log T)^{r-1}}{(r-1)!}, \qquad (T \to \infty) \tag{3.1}$$

is well-known, but we shall require an estimate when r is an increasing function of T. Such an estimate can be obtained based on a method of Selberg [Sel54]. A treatment is given in Tenenbaum [Ten95, Chap. II.6, Theorem 5], as stated below.

The result is given in terms of the function

$$\nu(z) \; := \; \frac{1}{\Gamma(z+1)} \prod_p \left(\left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^{z} \right).$$

This infinite product converges on $\mathfrak{Re}(z) > 0$, giving in this region a non-vanishing meromorphic function with simple poles at $z = p$ for all primes p. Note also that $\lim_{z \to 0^+} \nu(z) = 1$; hence for real $z \in [0, 3/2]$, say, $\nu(z)$ is bounded above and below by positive constants.

Proposition 3.1 ([Ten95, eqn. (20), p. 205]). *For $T \geq 3$, we have uniformly in*

$$1 \leq r \leq \frac{3}{2} \log\log T$$

that

$$\mathcal{N}_r(T) \;=\; \frac{T}{\log T} \frac{(\log\log T)^{r-1}}{(r-1)!} \left(\nu\left(\frac{r-1}{\log\log T}\right) + O\left(\frac{r}{(\log\log T)^2}\right)\right), \tag{3.2}$$

with an absolute implied constant.

This asymptotic continues to hold up to $r < (2-\epsilon)\log\log T$, but not beyond this point, as ν has a pole at $z = 2$. A different asymptotic formula takes over at $r > (2+\epsilon)\log\log T$, see [Nic84], but it will not be needed for our purposes.

For our application we derive from (3.2) a simplified estimate.

Lemma 3.2. *Let $r = \gamma \log\log T$ with $\frac{1}{\log\log T} \leq \gamma < \frac{3}{2}$. Then as $T \to \infty$,*

$$\mathbb{P}[\Omega(x) = r] \;:=\; \frac{\mathcal{N}_r(T)}{T} \;\asymp\; (\log T)^{\gamma - \gamma\log\gamma - 1 + o(1)}, \tag{3.3}$$

with absolute implied constants.

Proof. First recall that, on $[0, 3/2]$, the function $\nu(\cdot)$ is bounded above and below by positive constants. Then inserting the Stirling's formula estimate,

$$(r-1)! \;\asymp\; r^{r-1/2}e^{-r}, \qquad\qquad (1 \leq r < \infty)$$

into (3.2) yields

$$\begin{aligned}\mathbb{P}[\Omega(x) = r] \;&\asymp\; \frac{1}{\log T}\left(\frac{\log\log T}{r}\right)^{r-1} r^{-\frac{1}{2}} e^{r} \;=\; \frac{1}{\log T}\,(\gamma)^{-\gamma\log\log T - 1}\,(\gamma\log\log T)^{-\frac{1}{2}}(\log T)^{\gamma} \\ &=\; \gamma^{-\frac{3}{2}}(\log\log T)^{-\frac{1}{2}}(\log T)^{-1-\gamma\log\gamma+\gamma},\end{aligned}$$

from which the estimate (3.3) follows, since $\gamma \geq 1/\log\log T$. ☐

3.3. Estimate for a single draw.

To prove Theorem 1.2, we first obtain upper and lower bounds on the probability density function for a single draw.

Theorem 3.1. *Let $k \geq 1$ be fixed. For any integer $T \geq 2$, draw a vector*

$$(x_1, x_2, \ldots, x_k) \in [1, T]^k$$

uniformly. For any small $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ so that for all $T > T_0(\varepsilon)$,

$$\mathbb{P}[\Omega(x_1 x_2 \cdots x_k) \leq (\beta_k + \varepsilon)\log\log T] \;\gg_\varepsilon\; \frac{1}{(\log T)^{1-\delta}}, \tag{3.4}$$

and, for $k \geq 2$,

$$\mathbb{P}[\Omega(x_1 x_2 \cdots x_k) \leq (\beta_k - \varepsilon)\log\log T] \;\ll_\varepsilon\; \frac{1}{(\log T)^{1+\delta}}, \tag{3.5}$$

there is a $\delta = \delta(\varepsilon) > 0$ so that for all $T > T_0(\varepsilon)$,

3.3.1. *Proof of the lower bound* (3.4).
Suppose $k \geq 1$ and write $k\gamma = \beta_k + \varepsilon$, so that $0 < \gamma < 1$, and let

$$r \ := \ \lfloor \gamma \log\log T \rfloor .$$

Then

$$\mathbb{P}[\Omega(x_1 \dots x_k) \leq k\gamma \log\log T] \ \geq \ \prod_{j=1}^{k} \mathbb{P}[\Omega(x_j) = r].$$

Inserting (3.3) gives

$$\mathbb{P}[\Omega(x_1 \dots x_k) \leq k\gamma \log\log T] \ \ \gg \ \ \left[(\log T)^{\gamma - \gamma\log\gamma - 1 + o(1)}\right]^k .$$

Write $\alpha = k\gamma$; then as $T \to \infty$ the exponent of $\log T$ approaches the limiting value

$$\gamma k - \gamma k \log\gamma - k \ = \ \alpha - \alpha\log\alpha + \alpha\log k - k = f_k(\alpha) - 1.$$

By Lemma 3.1, since $\alpha = k\gamma = \beta_k + \varepsilon > \beta_k$, and $f_k(\beta_k) = 0$, we conclude that as $T \to \infty$ the limiting exponent exceeds -1 by the positive amount $f_k(\alpha) > 0$. Therefore we can pick $\delta(\epsilon) > 0$ and $T_0(\epsilon)$ depending on ε (and k, which is fixed) so that (3.4) holds.

3.3.2. *Proof of the upper bound* (3.5).
The upper bound estimate (3.5) is more subtle and requires $k \geq 2$. Again take a fixed $\epsilon > 0$ and define γ by $k\gamma = \beta_k - \varepsilon$ taking ϵ small enough that $0 < \gamma < 1$, which is possible since $\beta_k > 0$. Since

$$\Omega(x_1 \cdots x_k) \ = \ \Omega(x_1) + \cdots + \Omega(x_k),$$

we have that

$$\mathbb{P}[\Omega(x_1 \dots x_k) \leq k\gamma \log\log T] \ = \sum_{r_1 + \cdots + r_k \leq k\gamma \log\log T} \mathbb{P}[\Omega(x_1) = r_1, \cdots \Omega(x_k) = r_k]$$

We upper bound the total number of summands trivially by

$$\sum_{r_1 + \cdots + r_k \leq k\gamma \log\log T} 1 \ \ll \ (\log\log T)^k \ = \ (\log T)^{o(1)}.$$

It remains to upper bound the contribution of an individual summand

$$\max_{r_1 + \cdots + r_k \leq k\gamma \log\log T} \mathbb{P}[\Omega(x_1) = r_1, \cdots \Omega(x_k) = r_k].$$

Write each r_j as

$$r_j \ = \ \gamma_j \log\log T,$$

so that

$$\gamma_1 + \cdots + \gamma_k \ \leq \ k\gamma \ < \ \beta_k \ < \ k - 1. \tag{3.6}$$

On average these γ_j's are less than one, but individually they could in principle be large, and we can apply (3.3) only when $\gamma_j < 3/2$. Let $\ell \subset \{1, \dots, k\}$ denote the indices j for which $\gamma_j < 3/2$ is "low," and let $h := \{1, \dots, k\} \setminus \ell$ be the "high" indices. Abusing notation, we use the same symbol for their cardinalities, e.g.,

$$\ell + h = k.$$

We have that

$$k\gamma \;\geq\; \sum_{j\in h}\gamma_j \;\geq\; \tfrac{3}{2}h,$$

so

$$\ell \;\geq\; k(1-\tfrac{2}{3}\gamma) \;>\; \tfrac{1}{3}k,$$

and

$$\sum_{j\in\ell}\gamma_j \;=\; \sum_{j}\gamma_j - \sum_{j\in h}\gamma_j \;\leq\; k\gamma - \tfrac{3}{2}h. \tag{3.7}$$

For $j \in h$, we estimate $\mathbb{P}[\Omega(x_j) = r_j] \leq 1$ trivially. This gives a bound

$$\begin{aligned}\mathbb{P}[\Omega(x_1) = r_1, \cdots \Omega(x_k) = r_k] \;&\leq\; \prod_{j\in\ell}\mathbb{P}[\Omega(x_j) = r_j] \\ &\ll\; (\log T)^{o(1)}\prod_{j\in\ell}(\log T)^{\gamma_j-\gamma_j\log\gamma_j-1},\end{aligned}$$

using (3.3). The exponent in this expression, subject to (3.7), is maximized if, for all $j \in \ell$, we set all values equal $\gamma_j = \eta$, in which case,

$$\mathbb{P}[\Omega(x_1) = r_1, \cdots \Omega(x_k) = r_k] \;\ll\; (\log T)^{\ell(\eta-\eta\log\eta-1)+o(1)}. \tag{3.8}$$

Now we have

$$\eta \;=\; \eta(\gamma, k, \ell) \;:=\; \frac{k\gamma}{\ell} - \frac{3h}{2\ell} \;=\; \frac{3}{2} - \frac{k}{\ell}\left(\frac{3}{2} - \gamma\right).$$

We bound the exponent (3.8), varying ℓ. Viewing ℓ as a continuous variable, we

$$\eta' \;:=\; \frac{\partial\eta}{\partial\ell} \;=\; \frac{k}{\ell^2}\left(\frac{3}{2} - \gamma\right) \;=\; \frac{1}{\ell}\left(\frac{3}{2} - \eta\right).$$

The derivative of the exponent of $\log T$ is in the ℓ-variable is then

$$\begin{aligned}\frac{\partial}{\partial\ell}[\ell(\eta - \eta\log\eta - 1)] \;&=\; \eta - \eta\log\eta - 1 - \ell\eta'\log\eta \\ &=\; \eta - \frac{3}{2}\log\eta - 1,\end{aligned}$$

which by inspection is a positive function of $\eta \in (0, 1)$. It follows that the exponent is maximized at the largest allowable value of ℓ, namely the integer $\ell = k$, so $h = 0$. For this value of ℓ, we have $\eta = \gamma$, whence as $T \to \infty$ the exponent of $\log T$ in (3.8) approaches the limiting value

$$k(\gamma - \gamma\log\gamma - 1) = \alpha - \alpha\log\alpha + \alpha\log k - k = f_k(\alpha) - 1.$$

where we have again set $\alpha = k\gamma = \beta_k - \varepsilon$. Again using Lemma 3.1 this limiting exponent is less than -1 since $\alpha < \beta_k$ gives $f_k(\alpha) < 0$. Thus we can choose $\delta(\epsilon)$ and a $T_0(\epsilon)$ so that (3.5) holds. This completes the proof of Theorem 3.1.

3.4. **Proof of Theorem 1.2.**

It is now a simple matter to deduce Theorem 1.2 from Theorem 3.1. Instead of a single draw, here we have a sequence of independent draws, one for each $n = 1, 2, \ldots,$ and with $T = C^n$. By (3.5),

$$\mathbb{P}\left[\frac{\Omega(x_{1,n}x_{2,n}\cdots x_{k,n})}{\log n} \le (\beta_k - \varepsilon)(1 + \log\log C/\log n)\right] \ll_\varepsilon \frac{1}{n^{1+\delta}},$$

and $\sum_{n\ge1} 1/n^{1+\delta} < \infty$. Thus by the Borel-Cantelli Lemma, the probability of these events occurring infinitely often is zero; that is, with probability one, we have

$$\liminf_n \frac{\Omega(x_{1,n}x_{2,n}\cdots x_{k,n})}{\log n} \ge \beta_k - \varepsilon.$$

Similarly, the independent events

$$\left[\frac{\Omega(x_{1,n}x_{2,n}\cdots x_{k,n})}{\log n} \le (\beta_k + \varepsilon)(1 + \log\log C/\log n)\right]$$

occur with probability at least $1/n^{1-\delta}$, the sum of which diverges. By the second Borel-Cantelli Lemma, infinitely many occur with probability one, so

$$\liminf_n \frac{\Omega(x_{1,n}x_{2,n}\cdots x_{k,n})}{\log n} \le \beta_k + \varepsilon.$$

This proves Theorem 1.2.

4. Proof of Theorem 1.3

Let $k \ge 1$, $C > 1$, and $R \ge 1$ be fixed throughout this section (unlike the previous section, where R was growing). In particular, the estimate (3.1) is perfectly valid here and will be used regularly. In this section, we allow implied constants to depend on k, C and R, since they are fixed.

For each $n \ge 1$, we choose uniformly a vector $\mathbf{x}_n = (x_{1,n}, \ldots, x_{k,n}) \in [1, C^n]^k$, and let

$$\mathfrak{n} \;=\; \mathfrak{n}(R) \;=\; \max\{n \ge 1 : \Omega(x_{1,n}\cdots x_{k,n}) \le R\},$$

with $\mathfrak{n} = 0$ if this set is empty and $\mathfrak{n} = \infty$ if it is unbounded.

First note that (1.6) follows immediately from Theorem 1.2. Indeed, if $\mathfrak{n}(R) = \infty$, then $\Omega(x_{1,n}\cdots x_{k,n}) = R$ occurs for infinitely many n's. But then

$$\liminf_{n\ge1} \frac{\Omega(x_{1,n}\cdots x_{k,n})}{\log n} \;=\; 0,$$

contradicting (1.3). Hence this event has probability zero.

To prepare for the proof of (1.7), we record the following computations. Recall that implied constants in this section may depend on k, C, and R.

Lemma 4.1. *Let $k \geq 1$ and $R \geq 1$ be fixed. Then for $t \geq 1$,*

$$\mathbb{P}[\Omega(x_{1,t} \cdots x_{k,t}) \leq R] \ll \frac{(\log t)^{k(R-1)}}{t^k}. \tag{4.1}$$

Assuming further that $R \geq k$, we have that

$$\mathbb{P}[\Omega(x_{1,t} \cdots x_{k,t}) \leq R] \gg \frac{(\log t)^{R-k}}{t^k}. \tag{4.2}$$

Proof. The event $\Omega(x_{1,t} \cdots x_{k,t}) \leq R$ is contained inside the intersection of the events $\Omega(x_{j,t}) \leq R$, for all $j = 1, 2, \ldots, k$. Thus using (3.1) gives

$$\mathbb{P}[\Omega(x_{1,t} \cdots x_{k,t}) \leq R] \leq \prod_{j=1}^{k} \mathbb{P}[\Omega(x_{j,t}) \leq R] \ll \left[\frac{1}{\log C^t} \frac{(\log\log C^t)^{R-1}}{(R-1)!}\right]^k,$$

from which (4.1) follows immediately.

Now assume that $R/k \geq 1$. Then the event $\Omega(x_{1,t} \cdots x_{k,t}) \leq R$ contains the intersection over all $j = 1, 2, \ldots, k$ of the non-empty events $\Omega(x_{j,t}) \leq R/k$. So

$$\mathbb{P}[\Omega(x_{1,t} \cdots x_{k,t}) \leq R] \geq \prod_{j=1}^{k} \mathbb{P}[\Omega(x_{j,t}) \leq \tfrac{R}{k}] \gg \left[\frac{1}{\log C^t} \frac{(\log\log C^t)^{\frac{R}{k}-1}}{(\frac{R}{k}-1)!}\right]^k,$$

which implies (4.2). ∎

Lemma 4.2. *If $R \geq k \geq 1$ are fixed, then for all sufficiently large t,*

$$\mathbb{P}[\mathfrak{n}(R) = t] \gg \frac{(\log t)^{R-k}}{t^k}.$$

Proof. Consider the event $\mathfrak{n}(R) = t$. This occurs if and only if $\Omega(x_{1,t} \cdots x_{k,t}) \leq R$ and, for all larger integers $s > t$, we have that $\Omega(x_{1,s} \cdots x_{k,s}) > R$. That is,

$$\begin{aligned}\mathbb{P}[\mathfrak{n}(R) = t] &= \mathbb{P}[\Omega(x_{1,t} \cdots x_{k,t}) \leq R] \cdot \prod_{s>t} \Big(1 - \mathbb{P}[\Omega(x_{1,s} \cdots x_{k,s}) \leq R]\Big) \\ &\gg \frac{(\log t)^{R-k}}{t^k} \cdot \prod_{s>t} \left(1 - K\frac{(\log s)^{k(R-1)}}{s^k}\right),\end{aligned}$$

where we used (4.2) and (4.1). (Here $K > 0$ is a constant depending at most on k, C, and R.) Since $s \geq 2$, the infinite product converges absolutely. It bounds the result below by a uniform positive constant for all sufficiently large t that avoid possible nonpositive terms for small s in the infinite product. ∎

Proof of Theorem 1.3. Assume that $R \geq k \geq 1$ and let $m \geq k-1$. Consider the m-th moment of $\mathfrak{n}$, namely,

$$\mathbb{E}[\mathfrak{n}^m] = \sum_{t \geq 0} t^m \, \mathbb{P}[\mathfrak{n}(R) = t] \gg \sum_{t \geq 0} t^m \frac{(\log t)^{R-k}}{t^k},$$

where we used Lemma 4.2. Since $m-k \geq -1$, this sum diverges.

Note the case $R = k = 1$ gives divergence of the $m = 0$-th moment; that is, if $k = 1$ then $\mathfrak{n} = \infty$ with probability 1.) □

5. Proofs of Theorems 1.4 and 1.5

Assume Conjecture 1.1 in this section.

Proof of Theorem 1.4. Let $V : Q = t$ have $V(\mathbb{Z}) \neq \emptyset$. As is well-known and in this case essentially goes back to Gauss, $V(\mathbb{Z})$ decomposes into a finite number of Γ-orbits,

$$V(\mathbb{Z}) \ = \ \bigsqcup_{j=1}^{m} \Gamma \cdot \mathbf{v}_j,$$

where $\Gamma = O_Q(\mathbb{Z})$ is the orthogonal group fixing Q (see, e.g., [Cas78] or [Kon16, §2]). Since Q is indefinite, the Zariski closure of Γ is a torus,

$$\mathbb{G} = \mathrm{Zcl}(\Gamma) = O(1,1).$$

Thus, up to finite index, $\Gamma = \langle \gamma \rangle$ for some hyperbolic matrix γ. By Conjecture 1.1 each orbit $\mathcal{O}_j = \Gamma \cdot \mathbf{v}_j$ has

$$\liminf_{(x,y)\in\mathcal{O}_j} \frac{\Omega(xy)}{\log\log|xy|} \ \geq \ \beta_2,$$

and hence the same holds for all of $V(\mathbb{Z})$. □

Proof of Theorem 1.5. Let α be a quadratic surd having ordinary continued fraction expansion $\alpha = [a_0, a_1, a_2, ...]$ with partial quotients p_n/q_n, given in matrix form by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p_n \\ q_n \end{pmatrix}.$$

Now α has an eventually periodic continued fraction expansion

$$\alpha \ = \ [a_0; a_1, \dots a_k, \overline{a_{k+1}, \dots, a_{k+\ell}}].$$

After the first few terms, the sequence $(p_n, q_n)^t$ decomposes into finitely many Γ-orbits, where

$$\Gamma = \langle \gamma \rangle, \qquad \gamma = M \begin{pmatrix} 0 & 1 \\ 1 & a_{k+1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{k+\ell} \end{pmatrix} M^{-1},$$

with

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_k \end{pmatrix},$$

for the orbits given by

$$\mathbf{v}_j := M \begin{pmatrix} 0 & 1 \\ 1 & a_{k+1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{k+j} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad 0 \leq j \leq \ell - 1.$$

We may apply Conjecture 1.1 to each orbit, since they are infinite, and using the asymptotic $\log\log p_n q_n \sim \log n$ establishes the result. □

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