Enumerating Anchored Permutations with Bounded Gaps

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Abstract

Say that a permutation of $1, 2, \ldots, n$ is *k-bounded* if every pair of consecutive entries in the permutation differs by no more than k. Such a permutation is *anchored* if the first entry is 1 and the last entry is n. We give a explicit recursive formulas for the number of anchored k-bounded permutations of n for $k = 2$ and $k = 3$, resolving a conjecture listed on the Online Encyclopedia of Integer Sequences (entry A249665). We also pose the conjecture that the generating function for the enumeration of k-bounded anchored permutations is always rational, mirroring the known result on (non-anchored) k-bounded permutations due to Avgustinovich and Kitaev.

1 Introduction

Suppose one starts on the first stair of a staircase with n steps labeled $1, \ldots, n$ in order, and at each step one either steps forwards or backwards by at most k steps, such that every stair

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is used exactly once and the climb ends on the nth stair. How many distinct such ways are there to climb the stairs?

This question can be stated more precisely as follows. For a positive integer k, define a **k-bounded** permutation of $[n] = \{1, 2, \ldots, n\}$ to be a bijection $\pi : [n] \to [n]$ such that for all $i \in \{1, 2, \ldots, n-1\}$ we have

$$|\pi(i) - \pi(i+1)| \leq k.$$

We say that such a permutation is **anchored** if $\pi(1) = 1$ and $\pi(n) = n$. We are interested in enumerating the k-bounded anchored permutations in terms of k and n.

Example 1. The permutation $1, 4, 2, 3, 6, 5, 7, 8, 9$ is a 3-bounded anchored permutation of $\{1, 2, \ldots, 9\}$, since the first entry is 1, the last entry is 9, and no pair of consecutive entries differs by more than 3.

The question of explicitly enumerating 3-bounded anchored permutations was first posed on the Online Encyclopedia of Integer Sequences, entry A249665 [5]. Our results resolve the stated conjectures in this entry.

Several related questions have been studied previously. Positive stair climbing problems were studied by Goins and Washington [3], extending the well-known fact that the number of ways to climb a staircase of length n using positive steps of $+1$ or $+2$ each time is the nth Fibonacci number.

Avgustinovich and Kitaev [1] studied "k-determined permutations", which they show are equivalent to $(k-1)$-bounded, *non-anchored* permutations, as well as certain Hamiltonian paths in graphs. They resolve a conjecture of Plouffe [7] by providing the generating function for 2-bounded non-anchored permutations, which were originally defined as *key permutations* [6]. Avgustinovich and Kitaev further show that the generating function for any k is always rational using the transfer-matrix method described by Stanley [8, ch. 4].

1.1 Main results

For $k = 1$, there is clearly only one 1-bounded anchored permutation for each n, namely the identity permutation. In this paper, we resolve the cases $k = 2$ and $k = 3$ completely, as well as the $k = 2$ non-anchored setting.

Our main results can be summarized in the following two theorems.

Theorem 2. *Let R_n be the number of 2-bounded anchored permutations of $[n]$. Then the sequence $(R_n)_{n \geq 1}$ is given by the recurrence $R_1 = 1$, $R_2 = 1$, $R_3 = 1$, and*

$$R_n = R_{n-1} + R_{n-3} \tag{1}$$

for all $n \geq 4$. The generating function of the sequence is

$$R(x) = \sum_{n=1}^{\infty} R_n x^n = \frac{x}{1 - x - x^3}.$$

This sequence R_n is also known as *Narayana's cows sequence* [4], and the particular interpretation as 2-bounded anchored permutation is stated without proof (in a slightly different but equivalent form) in Flajolet and Sedgewick [2, p. 373]. We include a proof in this paper for completeness. Note the similarity to the Fibonacci recurrence. It is interesting that for steps of $+1$ and $+2$ only, the recurrence is precisely the Fibonacci sequence, and here, with the added steps of -1 and -2 where every step is reached, it is one index off of the Fibonacci recurrence.

Theorem 3. *Let F_n be the number of 3-bounded anchored permutations of $[n]$. Then the sequence $(F_n)_{n\geq 1}$ is given by the recurrence $F_1 = 1$, $F_2 = 1$, $F_3 = 1$, $F_4 = 2$, $F_5 = 6$, $F_6 = 14$, $F_7 = 28$, $F_8 = 56$, and*

$$F_n = 2F_{n-1} - F_{n-2} + 2F_{n-3} + F_{n-4} + F_{n-5} - F_{n-7} - F_{n-8} \tag{2}$$

for all $n \geq 9$. The generating function of the sequence is

$$F(x) = \frac{x - x^2 - x^4}{1 - 2x + x^2 - 2x^3 - x^4 - x^5 + x^7 + x^8}.$$

In Section 2, we prove Theorem 2, and in Section 3 we prove Theorem 3. Interestingly, we do not know a direct combinatorial proof of the recursion (2), and some open problems in this and other directions are posed in Section 4.

1.2 Notation

We write our permutations $\pi : [n] \to [n]$ in list notation, where the ith entry of the list is $\pi(i)$.

A **gap** of a permutation π is a difference $\pi(i+1) - \pi(i)$ between two consecutive entries. We will always write our gaps with a $+$ or $-$ sign in front to indicate the sign, even if the sign is clear, to distinguish gaps from entries. For instance, we would say that the first gap of the permutation $1, 3, 2, 4$ is $+2$, and the second gap is -1. We sometimes refer to the gaps of a sequence that is not a permutation as well, defined in the same way as consecutive differences between entries.

A sequence whose gaps are all between $-k$ and $+k$ is said to be **blocked** or **stuck** at the end if the last entry a has the property that $a \pm 1, \ldots, a \pm k$ all either occur in the sequence or are less than or equal to 0. For instance, if $k = 3$, the sequence $1, 3, 4, 6, 5, 2$ is blocked at 2; the next possible positive integer that has not been used is 7, which is more than a gap of k away.

The **graph** of a permutation of $\{1, \ldots, n\}$ is the plot of all points $(i, \pi(i))$ in the plane. The **main diagonal** is the line with equation $y = x$. Note that a point in the graph of a permutation is on the main diagonal if and only if it is a fixed point of the permutation.

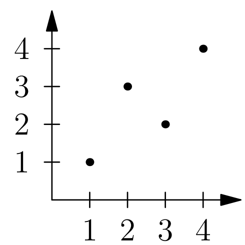

Figure 1: The graph of the 2-bounded permutation $1, 3, 2, 4$.

2 Structure and enumeration for $k = 2$

As in Theorem 2, we define R_n to be the number of 2-bounded anchored permutations of $[n]$. To get a handle on these permutations, we first prove the following lemma. It is worth noting that a weaker version of the lemma suffices to prove recursion (1), but the stronger statement explicitly describes the structure of a 2-bounded permutation.

Lemma 4. *Let π be an anchored 2-bounded permutation of $[n]$. Then there exists a subset $I \subseteq \{2, \ldots, n-2\}$ such that*

1. *Any pair of numbers in I differ by at least three, and*
2. *For all $i \in [n]$,*

$$\pi(i) = \begin{cases} i+1, & \text{if } i \in I; \\ i-1, & \text{if } i-1 \in I; \\ i, & \text{otherwise.} \end{cases}$$

In other words, the graph of the permutation can only deviate from the diagonal $x = y$ in consecutive pairs, with an up-step of 2 and a down-step of 1, before returning to the diagonal with an up-step of 2. (See Figure 2.)

Proof. The lemma is clearly true when $n = 1$. We proceed by strong induction on n. Assume that the lemma holds for all positive integers $n' < n$, and let π be a permutation of $[n]$.

If π is the identity permutation then $I = \emptyset$ and we are done, so we may assume that π is not the identity. Let i be the smallest index for which $\pi(i) \neq i$. Note that $i \in \{2, \ldots, n-2\}$. Then since $\pi(j) = j$ for all $j < i$, the gap from $\pi(i-1)$ to $\pi(i)$ cannot be -1, -2, or $+1$. It therefore must be $+2$, and we have

$$\pi(i) = \pi(i-1) + 2 = i - 1 + 2 = i + 1.$$

Now, the next gap, from $\pi(i)$ to $\pi(i+1)$, can either be -1, $+1$, or $+2$. We claim that it is not $+1$ or $+2$. If the gap were $+1$, then $i+1$ and $i+2$ both occur, before the value i appears in the permutation. So for some $j > i+1$, $\pi(j) = i$. But then the value of $\pi(j+1)$

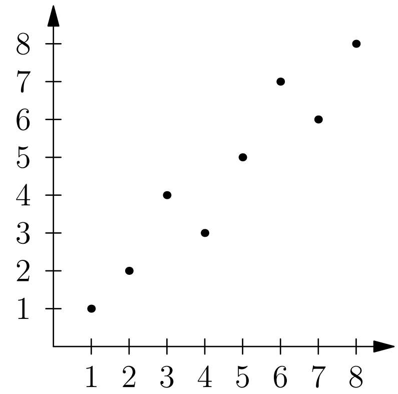

Figure 2: The 2-bounded permutation graphed above, $1, 2, 4, 3, 5, 7, 6, 8$, has subset $I = \{3, 6\}$ as the set of indices i for which $\pi(i) = i + 1$.

must be at least $i+3$ (since all other possible values are already used), and this contradicts 2-boundedness. Otherwise, if the gap between $\pi(i)$ and $\pi(i+1)$ is $+2$, so that $\pi(i+1) = i+3$, then the only way to reach i in the permutation is via a -2 step from $i+2$, and the same argument shows a contradiction.

It follows that the gap at i is -1, so $\pi(i+1) = i$. The only possible value for $\pi(i+2)$ is then $i+2$ (with a $+2$ step from the previous), which is on the diagonal again with all smaller numbers having occurred to the left of it. The remaining entries form a 2-bounded, anchored permutation of $\{i+2, i+3, \ldots, n\}$, which has a corresponding subset $I' \subseteq \{i+3, i+4, \ldots, n-2\}$ that satisfies the conditions above by the inductive hypothesis. Since i is at least 3 less than any element of I', we see that setting $I = \{i\} \cup I'$ gives a valid subset that corresponds to π. $\square$

We now can prove Theorem 2.

Proof. It is easily checked that $R_1 = R_2 = R_3 = 1$. Let $n \geq 4$. Then any anchored 2-bounded permutation π of $[n]$ either starts with $1, 2$ or $1, 3$. In the former case, there are R_{n-1} ways of completing the permutation, since any 2-bounded way of completing it that ends at n is an anchored permutation of $\{2, \ldots, n\}$.

In the latter case, by Lemma 4, the first four entries of the permutation must be $1, 3, 2, 4$, and then the remaining entries starting from 4 form 2-bounded anchored permutation of $\{4, 5, \ldots, n\}$. It follows that there are R_{n-3} possibilities if the permutation starts with $1, 3$.

It follows that $R_n = R_{n-1} + R_{n-3}$.

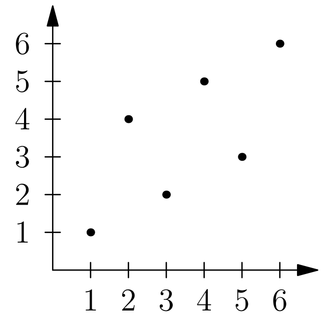

Figure 3: The Joker appears in the above permutation, in its second through sixth entries.

The generating function now follows from a straightforward calculation. We have

$$\begin{aligned} R(x) - xR(x) - x^3R(x) &= \sum_{n=1}^{\infty} R_n x^n - \sum_{n=2}^{\infty} R_{n-1}x^n - \sum_{n=4}^{\infty} R_{n-3}x^n \\ &= x + x^2 + x^3 - (x^2 + x^3) + \sum_{n=4}^{\infty}(R_n - R_{n-1} - R_{n-3})x^n \\ &= x + \sum_{n=4}^{\infty} 0 \cdot x^n \\ &= x, \end{aligned}$$

and it follows that $R(x) = x/(1 - x - x^3)$. □

3 Structure and enumeration for $k = 3$

As in Theorem 3, we define F_n to be the number of 3-bounded anchored permutations of $[n]$. In the 2-bounded case, we saw that there is one possible pattern in which the permutations can veer from the identity, and used that to generate the recursion. Similarly, in the 3-bounded case, we will need to single out a certain special sequence that interferes with an otherwise regular pattern that the permutations must follow.

Definition 5. The **Joker** is the sequence $3, 1, 4, 2, 5$. We say the Joker **appears** in a 3-bounded permutation if for some i, the ith through $(i + 4)$th entries of the permutation are $i + 2, i, i + 3, i + 1, i + 4$.

Aside from the Joker, the 3-bounded permutations turn out to follow a predictable pattern in terms of runs of $+3$ and -3 steps. We will use this structure to devise a three-term recurrence for F_n.

Definition 6. Define G_n to be the number of 3-bounded permutations π of $\{1, 2, \ldots, n\}$ that start with either $\pi(1) = 1$ or $\pi(1) = 2$ (so they are not necessarily anchored) and end at $\pi(n) = n$.

Definition 7. Define H_n to be the number of 3-bounded permutations π of $\{1, 2, \ldots, n\}$ that start with $\pi(1) = 3$, end with $\pi(n) = n$, and do not start with the Joker as the first five terms.

We claim that for all $n \geq 6$, the sequences F_n, G_n, H_n satisfy the following recurrence relations:

$$\begin{aligned}
F_n &= G_{n-1} + H_{n-1} + F_{n-5}, \\
G_n &= F_n + G_{n-2} + F_{n-3} + G_{n-4} + H_{n-2}, \\
H_n &= F_{n-3} + G_{n-3} + F_{n-4} + G_{n-5} + H_{n-3}.
\end{aligned}$$

To prove these relations, we first prove the following structure lemma.

Lemma 8. *Suppose π is a 3-bounded anchored permutation of $[n]$, and that the first i entries form a 3-bounded anchored permutation of $[i]$, so that $\pi(1) = 1$, $\pi(i) = i$, and the numbers $1, \ldots, i$ comprise the first i entries of the permutation in some order. If the next step is a $+3$, then one of the following two patterns occurs starting at entry i:*

1. *The Joker appears as entries i through $i + 4$.*

2. *There is a positive integer m and a gap $d \in \{\pm 1, \pm 2\}$ such that the sequence of gaps after i is*

$$+3, +3, \ldots, +3, d, -3, -3, \ldots, -3, \overline{d}, +3, +3, \ldots, +3$$

   where the first run of $+3$'s has length m, the run of -3's has length m' where $m' = m-1$ if $d < 0$ and $m' = m$ if $d > 0$, the last run of $+3$'s has length m' as well, and

$$\overline{d} = \begin{cases} +1, & \text{if } d = 1 \text{ or } d = -2; \\ -1, & \text{if } d = 2 \text{ or } d = -1. \end{cases}$$

   *We call such a pattern a **cascading 3-pattern**.*

Proof. First, note that since π restricts to a permutation on $\{1, \ldots, i\}$, we can assume for simplicity that $i = 1$. Now, suppose the next gap is $+3$, so $\pi(2) = 4$.

Let m be the length of the run of consecutive gaps of $+3$ starting from 1 before a gap d not equal to $+3$ occurs. Notice that d cannot be -3 or else the same entry would occur twice in the permutation, and so $d \in \{\pm 1, \pm 2\}$. We will prove that one of the two possibilities above hold by induction on m.

Base Case. Suppose $m = 1$. We consider several subcases based on the value of d.

If $d = -2$, then the first three entries of the sequence are $1, 4, 2$, and the next entry may be 5 or 3. If the next entry is 5 and the fifth entry is larger than 5, then the only way to reach 3 later in the permutation is by a gap of -3 from 6, in which case we would be stuck at 3, having used 1, 2, 4, 5, and 6 already. Thus, if π starts with $1, 4, 2, 5$ then it must continue $1, 4, 2, 5, 3, 6$, which is the Joker. Otherwise, it starts $1, 4, 2, 3$, which is a cascading 3-pattern for $m = 1$ and $d = -2$.

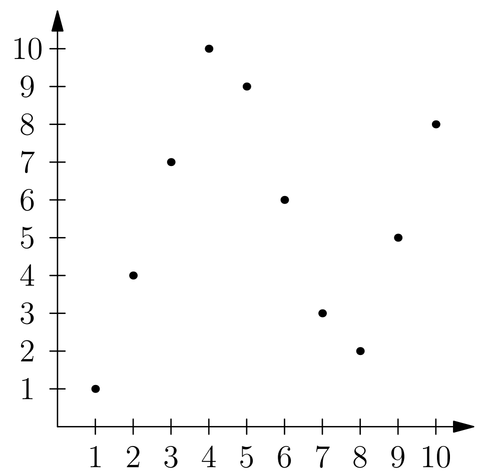

Figure 4: An example of a cascading 3-pattern, with $m = 3$ and $d = -1$.

If $d = -1$, suppose for contradiction that the next gap is positive, so that the first four entries are either $1, 4, 3, 5$ or $1, 4, 3, 6$. Then 2 must be reached from a gap of -3 from 5, at which point the permutation is stuck. Thus the next gap must be -1 as well, and the permutation must start $1, 4, 3, 2, 5$, which is a cascading 3-pattern for $m = 1$ and $d = -1$.

If $d = +1$, suppose for contradiction that the next gap is positive, so that the first four entries are either $1, 4, 5, 6$ or $1, 4, 5, 7$ or $1, 4, 5, 8$. Then to reach 2 or 3, there must be a gap of -3 from 6, at which point the permutation is blocked by 4, 5, and 6 and ends at 2 or 3, a contradiction. It follows that the next gap is -2 or -3, and in fact it must be -3 so as to reach the entry 2 without being blocked. Thus, the first five entries are $1, 4, 5, 2, 3, 6$, which is a cascading 3-pattern with $m = 1$ and $d = +1$.

Finally, if $d = +2$, suppose for contradiction that the next gap is positive or -1. Then as in the case above, the permutation becomes blocked once it reaches 2 or 3. So the next gap must be -3 and we have $1, 4, 6, 3$ as the first four entries. We must then have 2 as the fifth entry, or else the sequence would get blocked at 2 later, so the first six entries are $1, 4, 6, 3, 2, 5$, which is a cascading 3-pattern with $m = 1$ and $d = +2$.

Induction step. Suppose $m > 1$ and assume the lemma holds for $m' = m - 1$. Then π starts with $1, 4, 7$. We claim that the entries 2 and 3 must be adjacent in π. Suppose they are not adjacent. If 3 comes first, then the only way to reach 2 is by a -3 gap from 5 (since 1 and 4 are already used) at which point the permutation would be stuck at 2, a contradiction. If 2 comes first, then since 7 comes after 4 we must have reached the 2 using a -3 gap from 5. But then the only possible entry that can follow the 2 is 3, and they are in fact adjacent.

Now, consider the adjacent positions of the 2 and 3. Then the other entry adjacent to 2 must be 5, and 6 must be adjacent to 3 as well, so the 5 and 6 surround the 2 and 3. It follows that if we remove 2, 3, and 4 from the permutation and shift all entries larger than 4 down by 3, we obtain a permutation π' that starts at 1 with a $+3$ gap to 4 (which was the 7 in π). Since the 5 and 6 surrounded the 2 and 3 in π, they become 2 and 3 and are adjacent in π'. All other pairs of adjacent entries in π' still have a difference of at most 3, because they did in π and were both translated down by 3. Thus, π' is a 3-bounded anchored permutation starting with $m-1$ gaps of $+3$, and by the induction hypothesis it must either start with the Joker or a cascading 3-pattern.

Since the 2 and 3 are adjacent in π' it cannot start with the Joker and so it must be of the second form. It follows that π also starts with a cascading 3-pattern, formed by inserting one more $+3$ and -3 and $+3$ into each of the runs of 3's that comprise the gaps of π'. $\square$

We now prove each of the recurrence relations as their own lemma.

Lemma 9. *We have* $F_n = G_{n-1} + H_{n-1} + F_{n-5}$.

Proof. Any 3-bounded anchored permutation either starts with a gap of $+1$, $+2$, or $+3$. If it starts with $+1$ or $+2$, together the number of possibilities are equal to the number of 3-bounded permutations of $\{2, \dots, n\}$ that start with either 2 or 3, which is exactly G_{n-1}.

If it starts with $+3$, then by Lemma 8 it either starts with the Joker sequence or is a cascading 3-pattern. If it starts with the Joker, then $\pi(6) = 6$ and the first six entries are a permutation of $[6]$, so the entries after the fifth form a 3-bounded anchored permutation of $\{6, 7, \dots, n\}$. There are therefore F_{n-5} possibilities in this case. Otherwise, the number of possibilities is equal to the number of 3-bounded permutations of $\{2, \dots, n\}$ that start with 4 and end at n but do not start with the Joker, which is exactly H_{n-1}. The recursion follows. $\square$

Lemma 10. *We have* $G_n = F_n + G_{n-2} + F_{n-3} + G_{n-4} + H_{n-2}$.

Proof. We now wish to enumerate the 3-bounded permutations that start at either 1 or 2 and end at n. The number starting at 1 is F_n, which is the first term in the recurrence.

For those starting at 2, if the next entry is 1 then the third entry can either be 3 or 4. We now wish to count 3-bounded permutations of $\{3, \dots, n\}$ that start at either 3 or 4 and end at n, which is exactly G_{n-2}.

If the first two entries are $2, 3$, then if the next gap is positive it follows that the 1 can only be reached by a gap of -3 from 4, at which point the permutation is stuck. It follows that the next gap is negative, and it must be a gap of -2. So the first four entries are $2, 3, 1, 4$, and the remaining entries starting from 4 form a 3-bounded anchored permutation of $\{4, \dots, n\}$. Thus, there are F_{n-3} possibilities in this case.

If the first two entries are $2, 4$, then 1 can either be reached from a gap of -3 from 4, or later from a gap of -2 from 3. But the latter option becomes stuck at 1, and so there must be a gap of -3 from 4 to 1. It follows that the permutation starts $2, 4, 1, 3$ and then continues with a 3-bounded permutation of $\{5, \dots, n\}$ that starts at either 5 or 6. There are therefore G_{n-4} such possibilities.

Finally, if the first two entries are $2, 5$, then the 1 must occur at some point in π and must be surrounded by 3 and 4. If we remove the 1, then, we obtain a 3-bounded permutation of $\{2, \dots, n\}$ starting at 2 and with a starting gap of $+3$, with the 3 and 4 adjacent. By Lemma 8, the 3 and 4 will always be adjacent in such a permutation with a starting gap of

+3 unless it starts with the Joker pattern, and so, removing the 1 and the 2, we see that there are exactly H_{n-2} possibilities in this case. □

Notice that the final step in the above proof was analogous to the final step of the proof of Lemma 9. Deleting the 1 from the permutation resulted in the H_{n-1} term in the F_n recurrence, just as deleting the 1 and the 2 from the permutation resulted in the H_{n-2} term in the G_n recurrence. We will use this trick once more below, deleting the 1, 2, and 3, resulting in a H_{n-3} term in the H_n recurrence.

Lemma 11. *We have $H_n = F_{n-3} + G_{n-3} + F_{n-4} + G_{n-5} + H_{n-3}$.*

Proof. We wish to enumerate the 3-bounded permutations that start at 3 and end at n but do not start with the Joker sequence $3, 1, 4, 2, 5$. The second entry can either be 1, 2, 4, 5, or 6.

Notice that if we add a 0 to the front of the permutation, we will get a 3-bounded anchored permutation of $\{0, \ldots, n\}$ that starts with a gap of +3. By Lemma 8, since the permutation does not start with the Joker, it must start with a cascading 3-pattern.

Thus, if the first gap after the 3 is not +3, then d is determined and the 3-pattern is determined as well. In particular, if the first two entries are $3, 1$ then the permutation must start with $3, 1, 2$, and so the entries after the third form a 3-bounded permutation of $\{4, \ldots, n\}$ that starts at either 4 or 5 and ends at n. There are exactly G_{n-3} such entries in this case.

If the first two entries are $3, 2$ then since the start is a cascading 3-pattern, the first four entries are $3, 2, 1, 4$. The entries starting at 4 form a 3-bounded permutation of $\{4, \ldots, n\}$ starting at 4 and ending at n, giving us F_{n-3} more possibilities.

If the first two entries are $3, 4$, then by the cascading 3-pattern the first five entries are $3, 4, 1, 2, 5$. The entries starting at 5 form a 3-bounded permutation of $\{5, \ldots, n\}$ starting at 5 and ending at n, giving us F_{n-4} more possibilities.

If the first two entries are $3, 5$, the cascading 3-pattern tells us that the first five entries are $3, 5, 2, 1, 4$, with the next entry either 6 or 7. The entries starting after the fifth form a 3-bounded permutation of $\{6, \ldots, n\}$ starting at either 6 or 7 and ending at n, giving us G_{n-5} more possibilities.

Finally, if the first two entries are $3, 6$, then since it is a cascading 3-pattern the 1 and 2 must be adjacent in π. Removing the 1, 2, and 3 then gives a 3-bounded permutation of $\{4, \ldots, n\}$ that starts at 6 and ends at n but avoids the Joker. There are H_{n-3} such possibilities, and the proof is complete. □

We can now eliminate H_n from these recurrences to form a two-term recurrence. Putting $n-1$ in the recurrence for G_n, we have $G_{n-1} = F_{n-1} + G_{n-3} + F_{n-4} + G_{n-5} + H_{n-3}$, which nearly matches the recurrence for H_n. From this we conclude $H_n = F_{n-3} + G_{n-1} - F_{n-1}$. We can now substitute for the H terms in the F and G recurrences to obtain the following relationships:

$$F_n = G_{n-1} + F_{n-4} + G_{n-2} - F_{n-2} + F_{n-5}, \tag{3}$$

$$G_n = F_n + G_{n-2} + G_{n-3} + G_{n-4} + F_{n-5}. \tag{4}$$

Notice that our proofs above actually show that these recursions hold for all n, even $n \le 5$, where we set $F_j = G_j = 0$ for any $j \le 0$. Thus, we can unwind the recursions to find the first few values of F_n and G_n, as follows.

n	1	2	3	4	5	6	7	8
F_n	1	1	1	2	6	14	28	56
G_n	1	1	2	4	10	22	45	93

We now have the tools to prove Theorem 3.

Proof. We first find the generating function for $\{F_n\}$, and use this to find the single-term recurrence for the sequence.

Let $F(x) = \sum_{n=1}^{\infty} F_n x^n$ and $G(x) = \sum_{n=1}^{\infty} G_n x^n$. Then we have

$$\begin{aligned}
F(x) &= x + x^2 + x^3 + 2x^4 + 6x^5 + \sum_{n=6}^{\infty} F_n x^n \\
x^2F(x) &= x^3 + x^4 + x^5 + \sum_{n=6}^{\infty} F_{n-2} x^n \\
x^4F(x) &= x^5 + \sum_{n=6}^{\infty} F_{n-4} x^n \\
x^5F(x) &= \sum_{n=6}^{\infty} F_{n-5} x^n,
\end{aligned}$$

and

$$\begin{aligned}
G(x) &= x + x^2 + 2x^3 + 4x^4 + 10x^5 + \sum_{n=6}^{\infty} G_n x^n \\
xG(x) &= x^2 + x^3 + 2x^4 + 4x^5 + \sum_{n=6}^{\infty} G_{n-1} x^n \\
x^2G(x) &= x^3 + x^4 + 2x^5 + \sum_{n=6}^{\infty} G_{n-2} x^n \\
x^3G(x) &= x^4 + x^5 + \sum_{n=6}^{\infty} G_{n-3} x^n \\
x^4G(x) &= x^5 + \sum_{n=6}^{\infty} G_{n-4} x^n.
\end{aligned}$$

We can now utilize the recursions (3) and (4) to make the infinite summations cancel and keep track of the smaller terms, obtaining the following two equations:

$$F(x) - xG(x) - x^2G(x) + x^2F(x) - x^4F(x) - x^5F(x) = x,$$
$$G(x) - F(x) - x^2G(x) - x^3G(x) - x^4G(x) - x^5F(x) = 0.$$

Solving these two equations for $F(x)$ and $G(x)$ gives us that

$$F(x) = \frac{x - x^2 - x^4}{1 - 2x + x^2 - 2x^3 - x^4 - x^5 + x^7 + x^8}.$$

Finally, we can multiply both sides of the above relation by the denominator of the fraction, and we find that for $n \geq 8$, F_n satisfies the recursion

$$F_n = 2F_{n-1} - F_{n-2} + 2F_{n-3} + F_{n-4} + F_{n-5} - F_{n-7} - F_{n-8},$$

as desired. □

4 Conjectures and open problems

As future work, a direct combinatorial proof of the eight-term recurrence for F_n, without relying on algebraic methods for simplification, may lend new insights into the structure of these permutations. In particular, it would be interesting if there was an intrinsic reason for why the recursion has depth 8.

Along similar lines, for $k \geq 4$, one can ask whether there is always a linear recurrence relation of some depth for the number of k-bounded anchored permutations. Given Avgustinovich and Kitaev's work [1] on non-anchored k-bounded permutations, and given the complicated recurrence that exists for $k = 3$, it seems plausible that there would always exist such a recurrence. This conjecture can be stated in terms of generating functions as follows.

Conjecture 12. Let $F_{k,n}$ be the number of k-bounded anchored permutations of length n. Then the generating function

$$\sum_{n=1}^{\infty} F_{k,n}x^n$$

is a rational function of x for any $k \geq 1$.

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