Enumerating five families of pattern-avoiding inversion sequences; and introducing the powered Catalan numbers

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Abstract

The first problem addressed by this article is the enumeration of some families of pattern-avoiding inversion sequences. We solve some enumerative conjectures left open by the foundational work on the topics by Corteel et al., some of these being also solved independently by Kim and Lin. The strength of our approach is its robustness: we enumerate four families $F_1 \subset F_2 \subset F_3 \subset F_4$ of pattern-avoiding inversion sequences ordered by inclusion using the same approach. More precisely, we provide a generating tree (with associated succession rule) for each family F_i which generalizes the one for the family F_{i-1}.

The second topic of the paper is the enumeration of a fifth family F_5 of pattern-avoiding inversion sequences (containing F_4). This enumeration is also solved *via* a succession rule, which however does not generalize the one for F_4. The associated enumeration sequence, which we call of *powered Catalan numbers*, is quite intringuing, and further investigated. We provide two different succession rules for it, denoted Ω_{pCat} and Ω_{steady}, and show that they define two types of families enumerated by powered Catalan numbers. Among such families, we introduce the *steady paths*, which are naturally associated with Ω_{steady}. They allow to brigde the gap between the two types of families enumerated by powered Catalan numbers: indeed, we provide a size-preserving bijection between steady paths and valley-marked Dyck paths (which are naturally associated to Ω_{pCat}).

Along the way, we provide several nice connections to families of permutations defined by the avoidance of vincular patterns, and some enumerative conjectures.

1 Introduction and preliminaries

1.1 Context of our work

An *inversion sequence* of length n is any integer sequence $(e_1, \ldots, e_n)$ satisfying $0 \leq e_i < i$, for all $i = 1, \ldots, n$. There is a well-known bijection $\mathtt{T} : S_n \to I_n$ between the set S_n of all permutations of length (or size) n and the set I_n of all inversion sequences of length n, which maps a permutation $\pi \in \mathcal{S}_n$ into its *left inversion table* $(t_1, \ldots, t_n)$, where $t_i = |\{j : j > i \text{ and } \pi_i > \pi_j\}|$. This bijection is actually at the origin of the name inversion sequences.

The study of pattern-containment or pattern-avoidance in inversion sequences was first introduced in [22], and then further investigated in [13]. Namely, in [22], Mansour and Shattuck studied inversion sequences that avoid permutations of length 3, while in [13], Corteel et al. proposed the study of inversion sequences avoiding subwords of length 3. The definition of inversion sequences avoiding words (which may in addition be permutations) is straightforward: for instance, the inversion sequences that avoid the word 110 (resp. the permutation 132) are those with no $i < j < k$ such that $e_i = e_j > e_k$ (resp. $e_i < e_k < e_j$). Pattern-avoidance on special families of inversion sequences has also been studied in the literature, namely Duncan and Steingrímsson on *ascent sequences* – see [15].

The pattern-avoiding inversion sequences of [13] were further generalized in [23], extending the notion of pattern-avoidance to triples of binary relations (ρ_1, ρ_2, ρ_3). More precisely, they denote by $\mathbf{I}_n(\rho_1, \rho_2, \rho_3)$ the set of all inversion sequences in I_n having no three indices $i < j < k$

such that $e_i \rho_1 e_j$, $e_j \rho_2 e_k$, and $e_i \rho_3 e_k$, and by $\mathbf{I}(\rho_1, \rho_2, \rho_3) = \cup_n \mathbf{I}_n(\rho_1, \rho_2, \rho_3)$. For example, the sets $\mathbf{I}_n(=, >, >)$ and $\mathbf{I}_n(110)$ coincide for every n. In [23] all triples of relations in $\{<, >, \leq, \geq, =, \neq, -\}^3$ are considered, where "$-$" stands for any possible relation on a set S, *i.e.* $- = S \times S$. Therefore, all the 343 possible triples of relations are examined and the resulting families of pattern-avoiding inversion sequences are subdivided into 98 equivalence classes. Many enumeration results complementing those in [13, 22] have been found in [23]. In addition, several conjectures have been formulated in [23].

In this paper we study five families of inversion sequences which form a hierarchy for the inclusion order. We enumerate these classes, which result to be counted by well-known enumeration sequences, such as those of the Catalan, the Baxter, and the newly introduced semi-Baxter numbers [9]. These results prove some of the conjectures in [23]. Along our study, we further try to establish bijective correspondences between these families of inversion sequences and other known combinatorial structures. A remarkable feature of our work is that all the families of inversion sequences are presented and studied in a unified way by means of *generating trees*. Before proceeding, let us briefly recall some basics about generating trees. Details can be found for instance in [2, 3, 6, 26].

1.2 Basics on generating trees

Consider a combinatorial class $\mathcal{C}$, that is to say a set of discrete objects equipped with a notion of size such that the number of objects of size n is finite, for any n. We assume also that $\mathcal{C}$ contains exactly one object of size 1. A *generating tree* for $\mathcal{C}$ is an infinite rooted tree whose vertices are the objects of $\mathcal{C}$ each appearing exactly once in the tree, and such that objects of size n are at level n (with the convention that the root is at level 1). The children of some object $c \in \mathcal{C}$ are obtained by adding an *atom* (*i.e.* a piece of object that makes its size increase by 1) to c. Since every object appears only once in the generating tree, not all possible additions are acceptable. We enforce the unique appearance property by considering only additions that follow some prescribed rules and call *growth* of $\mathcal{C}$ the process of adding atoms according to these rules.

To illustrate these definitions, we describe the classical growth for the family of Dyck paths, as given by [3]. Recall that a Dyck path of semi-length n is a lattice path using up $U = (1, 1)$ and down $D = (1, -1)$ unit steps, running from $(0, 0)$ to $(2n, 0)$ and remaining weakly above the x-axis. The atoms we consider are UD factors, a.k.a. *peaks*, which are added to a given Dyck path. To ensure that all Dyck paths appear exactly once in the generating tree, a peak is inserted only in a point of the *last descent*, defined as the longest suffix containing only D letters. More precisely, the children of the Dyck path $w\,UD^k$ are $w\,U\boldsymbol{UD}D^k$, $w\,UD\boldsymbol{UD}D^{k-1}$,..., $w\,UD^{k-1}\boldsymbol{UD}D$, $w\,UD^k\boldsymbol{UD}$.

The first few levels of the generating tree for Dyck paths are shown in Figure 1 (left).

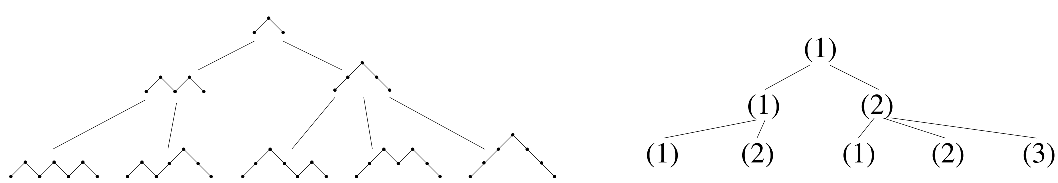


Figure 1: Two ways of looking at the generating tree for Dyck paths: with objects (left) and with labels from the succession rule Ω_{Cat} (right).

When the growth of $\mathcal{C}$ is particularly regular, we encapsulate it in a *succession rule*. This applies more precisely when there exist statistics whose evaluations control the number of objects produced in the generating tree. A succession rule consists of one starting label (*axiom*) corresponding to the value of the statistics on the root object and of a *set of productions* encoding the way in which these evaluations spread in the generating tree – see Figure 1(right). The growth of Dyck paths presented earlier is governed by the statistics "length of the last descent", so that it corresponds to the following succession rule, where each label (k) indicates the number of D steps of the last descent in a Dyck path,

$$\Omega_{Cat} = \begin{cases} (1) \\ (k) \quad \rightsquigarrow (1), (2), \ldots, (k), (k+1). \end{cases}$$

Obviously, as we discuss in [10], the sequence enumerating the class $\mathcal{C}$ can be recovered from the succession rule itself, without reference to the specifics of the objects in $\mathcal{C}$: indeed, the nth term of the sequence is the total number of labels (counted with repetition) that are produced from the root by $n-1$ applications of the set of productions, or equivalently, the number of nodes at level n in the generating tree. For instance, the well-know fact that Dyck paths are counted by Catalan numbers (sequence A000108 in [21]) can be recovered by counting nodes at each level n in the above generating tree.

1.3 Content of the paper

In our study we focus on five different families of pattern-avoiding inversion sequences, which are depicted in Figure 2. As the figure shows, these families are naturally ordered by inclusion, and are enumerated by well-known number sequences.

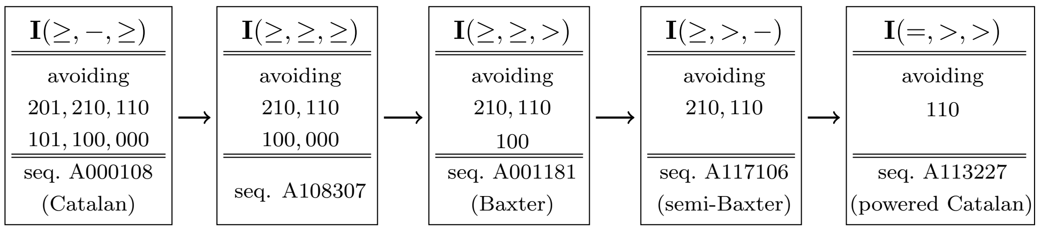


Figure 2: A chain of families of inversion sequences ordered by inclusion, with their characterization in terms of pattern avoidance, and their enumerative sequence.

The objective of our study is twofold. On the one hand we provide (and/or collect) enumerative results about the families of inversion sequences of Figure 2. On the other hand we aim at treating all these families in a unified way. More precisely, in each of the following sections we first provide a simple combinatorial characterization for the corresponding family of inversion sequences, and then we show a recursive growth that yields a succession rule.

The main noticeable property of the succession rules provided in Sections 2, 3, 4, and 5 is that they reveal the hierarchy of Figure 2 at the abstract level of succession rules. Specifically, the recursive construction (and consequently the succession rule) provided for each family is obtained by extending the construction (and thus the succession rule) of the immediately smaller family. Let us spend a few words in commenting the classes of our hierarchy and our results on them.

i) We start in Section 2 with $\mathbf{I}(\geq, -, \geq)$, which we call the family of Catalan inversion sequences. We define two recursive growths for this family, one according to Ω_{Cat} (hence proving that $\mathbf{I}(\geq, -, \geq)$ is enumerated by the Catalan numbers) and a second one that turns out to be a new succession rule for the Catalan numbers. The fact that this family of inversion sequences is enumerated by the Catalan numbers was conjectured in [23] and it has recently been proved independently from us by D. Kim and Z. Lin in [20]. Moreover, we are able to relate the family of Catalan inversion sequences to a family of permutations defined by the avoidance of *vincular patterns*, proving that they are in bijection with a family of pattern-avoiding permutations.

ii) In Section 3 we consider the family $\mathbf{I}(\geq, \geq, \geq)$. By defining a growth for this family, we prove the conjecture (formulated in [23]) that these inversion sequences are counted by sequence

A108307 on [21], which is defined as the enumerative sequence of set partitions of $\{1,\ldots,n\}$ that avoid enhanced 3-crossings [8].

iii) In Section 4 we study inversion sequences in $\mathbf{I}(\geq,\geq,>)$, which we call Baxter inversion sequences. This family of inversion sequences was conjectured in [23] to be counted by Baxter numbers. The proof of this conjecture was provided in [20] by means of a growth for Baxter inversion sequences that neatly generalizes our growth for the family $\mathbf{I}(\geq,\geq,\geq)$.

iv) In Section 5, we deal with the family $\mathbf{I}(\geq,>,-)$, which we call semi-Baxter inversion sequences. Indeed, this family of inversion sequences was conjectured in [23] to be counted by the sequence A117106 [21]; these numbers have been thoroughly studied and named semi-Baxter in the article [9], which among other results proves this conjecture of [23].

v) Finally, in Section 6 we deal with $\mathbf{I}(=,>,>)$, which is the rightmost element of the chain of Figure 2. We call the elements of $\mathbf{I}(=,>,>)$ *powered Catalan inversion sequences*, since the succession rule we provide for them is a "powered version" of the classical Catalan succession rule.

When turning to powered Catalan inversion sequences, the hierarchy of Figure 2 is broken at the level of succession rules. Indeed, although the combinatorial characterization of these objects generalizes naturally that of semi-Baxter inversion sequences, we do not have a growth for powered Catalan inversion sequences that generalizes the one of semi-Baxter inversion sequences. This motivates the second part of the paper, devoted to the study of this "'powered Catalan" enumerative sequence from Section 6 on.

The enumeration of powered Catalan inversion sequences (by A113227, [21]) was already solved in [13]. Our first contribution (in Section 6) is to prove that they grow according to the succession rule Ω_{pCat}, which generalizes the classical rule Ω_{Cat} by introducing powers in it. This motivates the name *powered Catalan numbers* which we have coined for the numbers of sequence A113227.

Quite many combinatorial families are enumerated by powered Catalan numbers. Some are presented in Section 7. These families somehow fall into two categories. Inside each category, the objects seem to be in rather natural bijective correspondence. However, between the two categories, the bijections are much less clear. Our result of Section 7 is to provide a second succession rule for powered Catalan numbers (more precisely, for permutations avoiding the vincular pattern 1-23-4), which should govern the growth of objects in one of these two categories, the other category being naturally associated with the rule Ω_{pCat}.

In Section 8, we describe a new occurrence of the powered Catalan numbers in terms of lattice paths. More precisely, we introduce the family of *steady paths* and prove that they are enumerated by the powered Catalan numbers. This is proved by showing a growth for steady paths that is encoded by (a variant called Ω_{steady} of) the succession rule for permutations avoiding the pattern 1-23-4. We also provide a simple bijection between steady paths and permutations avoiding the vincular pattern 1-34-2, therefore recovering the enumeration of this family, already known [4] to be enumerated by A113227.

Finally, in Section 9 we bridge the gap between the two types of powered Catalan structures, by showing a bijection between steady paths (representing the succession rule Ω_{steady}) and valley-marked Dyck paths (emblematic of the succession rule Ω_{pCat}).

2 Catalan inversion sequences: $\mathbf{I}(\geq,-,\geq)$

The first family of inversion sequences considered is $\mathbf{I}(\geq,-,\geq)$. It was conjectured in [23] to be counted by the sequence of Catalan numbers [21, A000108] (hence the name *Catalan inversion sequences*) whose first terms we recall:

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, \ldots$$

We note that this conjectured enumeration has recently been proved independently from us by D. Kim and Z. Lin in [20]. Their proof does not involve generating trees, but displays a nice Catalan recurrence for the filtration $\mathbf{I}_{n,k}(\geq,-,\geq)$ of $\mathbf{I}_n(\geq,-,\geq)$ where the additional parameter k is the value of the last element of an inversion sequence.

We provide another proof of this conjecture in Proposition 3 by showing that there exists a growth for $\mathbf{I}(\geq,-,\geq)$ according to the well-known Catalan succession rule Ω_{Cat}. Moreover, we show a second growth for $\mathbf{I}(\geq,-,\geq)$, thereby providing a new Catalan succession rule, which is appropriate to be generalized in the next sections. In addition, we show a direct bijection between $\mathbf{I}(\geq,-,\geq)$ and a family of pattern-avoiding permutations, which thus results to be enumerated by Catalan numbers.

2.1 Combinatorial characterization

Let us start by observing that the family of Catalan inversion sequences has a simple characterization in terms of inversion sequences avoiding patterns of length three.

Proposition 1. *An inversion sequence is in* $\mathbf{I}(\geq,-,\geq)$ *if and only if it avoids* 000, 100, 101, 110, 201 *and* 210*.*

Proof. The proof is rather straightforward, since containing e_i, e_j, e_k such that $e_i \geq e_j, e_k$, with $i < j < k$, is equivalent to containing the listed patterns. □

In addition to the above characterization, we remark the following combinatorial description of Catalan inversion sequences, as it will be useful to define a growth according to the Catalan succession rule Ω_{Cat}.

Proposition 2. *Any inversion sequence* $e = (e_1, \ldots, e_n)$ *is a Catalan inversion sequence if and only if for any* i*, with* $1 \leq i < n$*,*

$$\text{if } e_i \text{ forms a weak descent, i.e. } e_i \geq e_{i+1}, \text{ then } e_i < e_j, \text{ for all } j > i+1.$$

Proof. One direction is clear. The other direction can be proved by contrapositive. More precisely, suppose there are three indices $i < j < k$, such that $e_i \geq e_j, e_k$. Then, if $e_j = e_{i+1}$, e_i forms a weak descent and the fact that $e_i \geq e_k$ concludes the proof. Otherwise, since $e_i \geq e_j$, there must be an index i', with $i \leq i' < j$, such that $e_{i'}$ forms a weak descent and $e_{i'} \geq e_k$. This concludes the proof as well. □

The previous statement means that any of our inversion sequences has a neat decomposition: they are concatenations of shifts of inversion sequences having a single weak descent, at the end. A graphical view of this decomposition is shown in Figure 3.

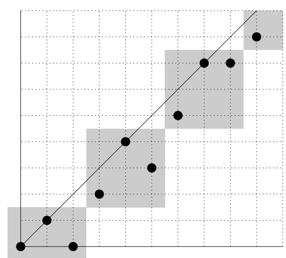


Figure 3: A Catalan inversion sequence and its decomposition.

2.2 Enumerative results

Proposition 3. *Catalan inversion sequences grow according to the succession rule Ω_{Cat},*

$$\Omega_{Cat} = \begin{cases} (1) & \\ (k) & \rightsquigarrow (1), (2), \dots, (k), (k+1). \end{cases}$$

Proof. Given an inversion sequence $e = (e_1, \dots, e_n)$, we define the inversion sequence $e \odot i$ as the sequence $(e_1, \dots, e_{i-1}, i-1, e_i, \dots, e_n)$, where the entry $i-1$ is inserted in position i, for some $1 \leq i \leq n+1$, and the entries $e_i, \dots, e_n$ are shifted rightwards by one. By definition of inversion sequence, $i-1$ is the largest possible value that the ith entry can assume. And moreover, letting $e' := e \odot i$, it holds that $e'_j = e_{j-1} < j-1$, for all $j > i$; namely the index i is the rightmost index such that $e'_k = k-1$. For example, if $i = 4$ and $e = (0, 0, 1, 3, 4, 5)$, then $e \odot i = (0, 0, 1, 3, 3, 4, 5)$.

Then, we note that given a Catalan inversion sequence e of length n, by removing from e the rightmost entry whose value is equal to its position minus one, we obtain a Catalan inversion sequence of length $n-1$. Note that $e_1 = 0$ for every Catalan inversion sequence, thus such an entry always exists.

Therefore, we can describe a growth for Catalan inversion sequences by inserting an entry $i-1$ in position i. By Proposition 2, since the entry $i-1$ forms a weak descent in $e \odot i$, the inversion sequence $e \odot i$ is a Catalan inversion sequence of length $n+1$ if and only if $e_{i+1}, \dots, e_n > i-1$. Then, we call *active positions* all the indices i, with $1 \leq i \leq n+1$, such that $e \odot i$ is a Catalan inversion sequence of length $n+1$. According to this definition, $n+1$ and n are always active positions: indeed, both $e \odot (n+1) = (e_1, \dots, e_n, n)$ and $e \odot n = (e_1, \dots, n-1, e_n)$ are Catalan inversion sequences of length $n+1$.

We label a Catalan inversion sequence e of length n with (k), where k is the number of its active positions decreased by one. Note that the smallest inversion sequence has label (1), which is the axiom of rule Ω_{Cat}.

Now, we show that given a Catalan inversion sequence e of length n with label (k), the labels of $e \odot i$, where i ranges over all the active positions, are precisely the label productions of (k) in Ω_{Cat}.

Let $i_1, \dots, i_{k+1}$ be the active positions of e from left to right. Note that $i_k = n$ and $i_{k+1} = n+1$. We argue that, for any $1 \leq j \leq k+1$, the active positions of the inversion sequence $e \odot i_j = (e_1, \dots, i_j - 1, e_{i_j}, \dots, e_n)$ are $i_1, \dots, i_{j-1}$, $n+1$ and $n+2$. Indeed, on the one hand any position which is non-active in e is still non-active in $e \odot i_j$. On the other hand, by Proposition 2, the index i_j becomes non-active in $e \odot i_j$, since $e_{i_j} < i_j$ by definition. Similarly, any position i_h, with $i_j < i_h < n+1$, which is active in e becomes non-active in $e \odot i_j$. Thus, the active positions of $e \odot i_j$ are $i_1, \dots, i_{j-1}$, $n+1$ and $n+2$. Hence, $e \odot i_j$ has label (j), for any $1 \leq j \leq k+1$. ☐

Furthermore, we can provide a new succession rule for generating Catalan inversion sequences: the growth we provide in the following is remarkable as it allows generalizations in the next sections.

Proposition 4. *Catalan inversion sequences grow according to the following succession rule*

$$\Omega_{Cat_2} = \begin{cases} (1,1) & \\ (h,k) & \rightsquigarrow (0,k+1)^h, \\ & \quad (h+1,k), (h+2,k-1), \dots, (h+k,1). \end{cases}$$

Proof. We consider the growth of Catalan inversion sequences that consists in adding a new rightmost entry, and we prove that this growth defines the succession rule Ω_{Cat_2}. Obviously, this growth is different from the one provided in the proof of Proposition 3.

Let $\max(e)$ be the maximum value among the entries of e. And let $\mathrm{mwd}(e)$ be the maximum value of the set of all entries e_i that form a weak descent of e; if e has no weak descents, then mwd$(e) := -1$. By Proposition 1, since e avoids 100, 201 and 210, the value $\max(e)$ is e_{n-1} or e_n. In particular, if $\max(e) = e_{n-1} \geq e_n$, then $\mathrm{mwd}(e) = \max(e)$.

By Proposition 2, it follows that $f = (e_1, \ldots, e_n, p)$ is a Catalan inversion sequence of length $n+1$ if and only if $\mathrm{mwd}(e) < p \leq n$. Moreover, if $\mathrm{mwd}(e) < p \leq \max(e)$, then e_n forms a new weak descent of f, and $\mathrm{mwd}(f)$ becomes the value e_n; whereas, if $\max(e) < p \leq n$, then $\mathrm{mwd}(f) = \mathrm{mwd}(e)$ since the weak descents of f and e coincide.

Now, we assign to any Catalan inversion sequence e of length n the label (h, k), where $h = \max(e) - \mathrm{mwd}(e)$ and $k = n - \max(e)$. In other words, h (resp. k) marks the number of possible additions smaller than or equal to (resp. greater than) the maximum entry of e.

The sequence $e = (0)$ has no weak descents, thus it has label $(1, 1)$, which is the axiom of Ω_{Cat_2}. Let e be a Catalan inversion sequence of length n with label (h, k). As Figure 4 illustrates, the labels of the inversion sequences of length $n+1$ produced by adding a rightmost entry p to e are

- $(0, k+1)$, for any $p \in \{\mathrm{mwd}(e)+1, \ldots, \max(e)\}$,
- $(h+1, k), (h+2, k-1), \ldots, (h+k, 1)$, when $p = \max(e)+1, \ldots, n$,

which concludes the proof that Catalan inversion sequences grow according to Ω_{Cat_2}. □

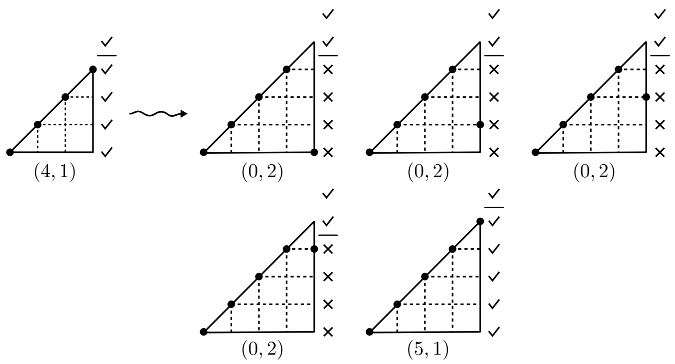


Figure 4: The growth of a Catalan inversion sequence according to Ω_{Cat_2}.

It is well worth noticing that although the above succession rule Ω_{Cat_2} generates the well-known Catalan numbers, we do not have knowledge of this succession rule in the literature.

2.3 One-to-one correspondence with $AV(1\text{-}23, 2\text{-}14\text{-}3)$

In this section we show that Catalan inversion sequences are just *left inversion tables* permutations avoiding the patterns 1-23 and 2-14-3, thereby proving that the family of pattern-avoiding permutations $AV(1\text{-}23, 2\text{-}14\text{-}3)$ forms a new occurrence of the Catalan numbers. We start by recalling some terminology and notation.

A *(Babson-Steingrímsson-)pattern* τ of length k is any permutation of $\mathcal{S}_k$ where two adjacent entries may or may not be separated by a dash – see [1]. Such patterns are also called *generalized* or *vincular*. The absence of a dash between two adjacent entries in the pattern indicates that in any pattern-occurrence the two entries are required to be adjacent: a permutation π of length $n \geq k$ *contains* the vincular pattern τ, if it contains τ as pattern, and moreover, there is an occurrence of the pattern τ where the entries of τ not separated by a dash are consecutive entries of the permutation π; otherwise, π *avoids* the vincular pattern τ. Let $\mathcal{T}$ be a set of patterns. We denote by $AV_n(\mathcal{T})$ the family of permutations of length n that avoid any pattern in $\mathcal{T}$, and by $AV(\mathcal{T}) = \cup_n AV_n(\mathcal{T})$.

Proposition 5. *For any n, Catalan inversion sequences of length n are in bijection with AV_n(1-23, 2-14-3).*

Proof. Let $\mathtt{T}$ be the mapping associating to each $\pi \in \mathcal{S}_n$ its left inversion table $\mathtt{T}(\pi) = (t_1, \ldots, t_n)$. We will use many times the following simple fact: for every $i < j$, if $\pi_i > \pi_j$ (*i.e.* the pair (π_i, π_j) is an *inversion*), then $t_i > t_j$.

Let $\mathtt{R}$ be the reverse operation on arrays. We can prove our statement by using the mapping $\mathtt{R} \circ \mathtt{T}$, which is a bijection between the family $\mathcal{S}_n$ of permutations and integer sequences $(e_1, \ldots, e_n)$ such that $0 \leq e_i < i$. We will simply show that the restriction of the bijection $\mathtt{R} \circ \mathtt{T}$ to the family $AV(1\text{-}23, 2\text{-}14\text{-}3)$ yields a bijection with Catalan inversion sequences. Precisely, we want to prove that for every n, an inversion sequence is in the set $\{(\mathtt{R} \circ \mathtt{T})(\pi) : \pi \in AV_n(1\text{-}23, 2\text{-}14\text{-}3)\}$ if and only if it is a Catalan inversion sequence of length n (*i.e.* belongs to $\mathbf{I}_n(\geq, -, \geq)$).

$\Rightarrow$) We prove the contrapositive: if $e \notin \mathbf{I}_n(\geq, -, \geq)$, then $\pi = (\mathtt{R} \circ \mathtt{T})^{-1}(e)$ contains 1-23 or 2-14-3. Let $t = (t_1, \ldots, t_n) = (e_n, \ldots, e_1)$. Then, t is the left inversion table of a permutation $\pi \in \mathcal{S}_n$, *i.e.* $\mathtt{T}(\pi) = t$. Since $e \notin \mathbf{I}_n(\geq, -, \geq)$, there exist three indices, $i < j < k$, such that $t_i \leq t_k$ and $t_j \leq t_k$.

Without loss of generality, we can suppose that there is no index h, such that $j < h < k$ and $t_i \leq t_h$ and $t_j \leq t_h$. Namely t_k is the leftmost entry of t that at least as large as both t_i and t_j. Then, we have two possibilities:

1. either $j + 1 = k$,
2. or $j + 1 \neq k$, and in this case it holds that $t_j > t_{k-1}$ or $t_i > t_{k-1}$.

First, from $t_i \leq t_k$ and $t_j \leq t_k$ it follows that $\pi_i < \pi_k$ and $\pi_j < \pi_k$.

Now, we prove that both in case 1. and in case 2. above we have $\pi \notin AV_n(1\text{-}23, 2\text{-}14\text{-}3)$.

1. Let us consider the subsequence $\pi_i \pi_j \pi_{j+1}$. We have $\pi_i < \pi_{j+1}$ and $\pi_j < \pi_{j+1}$. If also $\pi_i < \pi_j$, then it forms a 1-23.

   Otherwise, it must hold that $\pi_i > \pi_j$, and thus $t_j < t_i \leq t_{j+1}$. Since the pair (π_i, π_j) is an inversion of π and $t_i \leq t_{j+1}$, there must be a point π_s on the right of π_{j+1} such that (π_{j+1}, π_s) is an inversion and (π_i, π_s) is not. Thus, $\pi_i \pi_j \pi_{j+1} \pi_s$ forms a 2-14-3.
2. First, if $t_j > t_{k-1}$, consider the subsequence $\pi_i \pi_j \pi_{k-1} \pi_k$. It follows that $t_{k-1} < t_k$, since $t_j \leq t_k$, and thus $\pi_{k-1} < \pi_k$. In addition, we know that $\pi_j < \pi_k$. Then, $\pi_j \pi_{k-1} \pi_k$ forms an occurrence of 1-23 if $\pi_j < \pi_{k-1}$. Otherwise, it must hold that $\pi_j > \pi_{k-1}$. As in case 1. the pair (π_j, π_{k-1}) is an inversion, and $t_j \leq t_k$. Therefore, there must be an element π_s on the right of π_k such that (π_k, π_s) is an inversion and (π_j, π_s) is not. Hence $\pi_j \pi_{k-1} \pi_k \pi_s$ forms a 2-14-3.

   Now, suppose $t_j \leq t_{k-1}$, and consider the subsequence $\pi_i \pi_j \pi_{k-1} \pi_k$. According to case 2. it must be that $t_i > t_{k-1}$, and since $t_i \leq t_k$, it holds that $t_{k-1} < t_k$. Since both $\pi_j < \pi_{k-1}$ and $\pi_{k-1} < \pi_k$ hold, $\pi_j \pi_{k-1} \pi_k$ forms an occurrence of 1-23.

$\Leftarrow$) By contrapositive, if a permutation π contains 1-23 or 2-14-3, then $e = (\mathtt{R} \circ \mathtt{T})(\pi)$ is not in $\mathbf{I}(\geq, -, \geq)$.

- If π contains 1-23, there must be two indices i and j, with $i < j$, such that $\pi_i \pi_j \pi_{j+1}$ forms an occurrence of 1-23. We can assume that no points $\pi_{i'}$ between π_i and π_j are such that $\pi_{i'} < \pi_i$. Otherwise we consider $\pi_{i'} \pi_j \pi_{j+1}$ as our occurrence of 1-23.

  Then, two relations hold: $t_i \leq t_{j+1}$ and $t_j \leq t_{j+1}$, and thus $e \notin \mathbf{I}(\geq, -, \geq)$.
- If π contains 2-14-3, and avoids 1-23, there must be three indices i, j and k, with $i < j < j + 1 < k$, such that $\pi_i \pi_j \pi_{j+1} \pi_k$ forms an occurrence of 2-14-3. We can assume that no points $\pi_{i'}$ between π_i and π_j are such that $\pi_{i'} < \pi_i$. Indeed, in case $\pi_{i'} < \pi_j$ held, $\pi_{i'} \pi_j \pi_{j+1}$ would be an occurrence of 1-23; whereas, if $\pi_j < \pi_{i'} < \pi_i$, we could consider $\pi_{i'} \pi_j \pi_{j+1} \pi_k$ as our occurrence of 2-14-3.

Then, as above $t_j \leq t_{j+1}$, and $t_i + 1 \leq t_{j+1}$ because (π_i, π_j) is an inversion of π. Nevertheless, (π_{j+1}, π_k) is an inversion of π as well, and $\pi_i < \pi_k$. Thus, $t_i \leq t_{j+1}$ and $e \notin \mathbf{I}(\geq, -, \geq)$. □

We mention that although inversion sequences are actually a coding for permutations, it is not easy to characterize the families $\mathbf{I}(\rho_1, \rho_2, \rho_3)$ in terms of families of pattern-avoiding permutations. In fact, the above example is the only one that we provide, and to our knowledge the only one in the literature about these families.

Corollary 6. *The family AV(1-23, 2-14-3) is enumerated by Catalan numbers.*

3 Inversion sequences $\mathbf{I}(\geq, \geq, \geq)$

Following the hierarchy of Figure 2, the next family we turn to is $\mathbf{I}(\geq, \geq, \geq)$. This family was conjectured in [23] to be counted by sequence A108307 on [21], which is defined as the enumerative sequence of set partitions of $\{1, \ldots, n\}$ that avoid enhanced 3-crossings [8]. In [8, Proposition 2] it is proved that the number $E_3(n)$ of these set partitions is given by $E_3(0) = E_3(1) = 1$ and the recursive relation

$$8(n+3)(n+1)E_3(n) + (7n^2 + 53n + 88)E_3(n+1) - (n+8)(n+7)E_3(n+2) = 0\,, \qquad (1)$$

which holds for all $n \geq 0$. Thus, the first terms of sequence A108307 according to recurrence (1) are

$$1, 1, 2, 5, 15, 51, 191, 772, 3320, 15032, 71084, 348889, 1768483, 9220655, 49286863, \ldots$$

This section is aimed at proving that the enumerative sequence of the family $\mathbf{I}(\geq, \geq, \geq)$ is indeed the sequence A108307 [21]. To our knowledge, we provide the first proof of this result, and we succeed in this goal by means of a succession rule that generalizes the one in Proposition 4.

3.1 Combinatorial characterization

To start, we provide a combinatorial description of the family $\mathbf{I}(\geq, \geq, \geq)$, which will be used later to define a growth in the proof of Proposition 9.

As Figure 2 shows, the family $\mathbf{I}(\geq, \geq, \geq)$ properly includes $\mathbf{I}(\geq, -, \geq)$ as subfamily. For instance, the inversion sequence $(0, 0, 1, 1, 4, 2, 6, 5)$ is both in $\mathbf{I}_8(\geq, -, \geq)$ and in $\mathbf{I}_8(\geq, \geq, \geq)$, while $(0, 1, 0, 1, 4, 2, 3, 5)$ is not in $\mathbf{I}_8(\geq, -, \geq)$ despite being in $\mathbf{I}_8(\geq, \geq, \geq)$. The following characterization makes explicit this fact.

Proposition 7. *An inversion sequence belongs to $\mathbf{I}(\geq, \geq, \geq)$ if and only if it avoids* 000*,* 100*,* 110 *and* 210*.*

Proof. The proof is a quick check that containing e_i, e_j, e_k such that $e_i \geq e_j \geq e_k$, with $i < j < k$, is equivalent to containing the above patterns. □

The above result makes clear that every Catalan inversion sequence is in $\mathbf{I}(\geq, \geq, \geq)$. In addition, Proposition 7 proves the following property stated in [23, Observation 7].

Remark 8. Let any inversion sequence $e = (e_1, \ldots, e_n)$ be decomposed in two subsequences e^{LTR}, which is the increasing sequence of left-to-right maxima of e (*i.e.* entries e_i such that $e_i > e_j$, for all $j < i$), and e^{bottom}, which is the (possibly empty) sequence comprised of all the remaining entries of e.

Then, an inversion sequence e is in the set $\mathbf{I}(\geq, \geq, \geq)$ if and only if it e^{LTR} and e^{bottom} are both strictly increasing sequences - see decomposition in Figure 5 where the sequence e^{LTR} is highlighted.

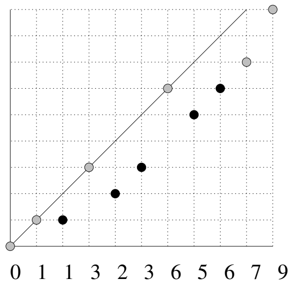


Figure 5: An inversion sequence in $\mathbf{I}(\geq,\geq,\geq)$ and its decomposition according to e^{LTR} and e^{bottom}.

3.2 Enumerative results

Proposition 9. *The family* $\mathbf{I}(\geq,\geq,\geq)$ *grows according to the following succession rule*

$$\Omega_{\mathbf{I}(\geq,\geq,\geq)} = \begin{cases} (1,1) \\ \\ (h,k) \quad \rightsquigarrow (h-1,k+1),(h-2,k+1),\dots,(0,k+1), \\ \qquad\qquad (h+1,k),(h+2,k-1),\dots,(h+k,1). \end{cases}$$

Proof. As in Proposition 4, we let inversion sequences of $\mathbf{I}(\geq,\geq,\geq)$ grow by adding a new rightmost entry, and we show that this growth is encoded by the claimed succession rule.

Let $e \in \mathbf{I}_n(\geq,\geq,\geq)$. Let last$(e)$ be the rightmost entry of e^{bottom}, if there is any, otherwise last$(e) := -1$. By Remark 8, it follows that $f = (e_1,\dots,e_n,p)$ is an inversion sequence of $\mathbf{I}_{n+1}(\geq,\geq,\geq)$ if and only if last$(e) < p \leq n$. Moreover, if last$(e) < p \leq \max(e)$, where $\max(e)$ is the maximum value of e, then p cannot be a left-to-right maximum and last(f) becomes p; whereas, if $\max(e) < p \leq n$, then last$(f) =$ last(e) since p is a left-to-right maximum of f.

Now, we assign to any $e \in \mathbf{I}_n(\geq,\geq,\geq)$ the label (h,k), where $h = \max(e) - \text{last}(e)$ and $k = n - \max(e)$. Note that the label interpretation extends the one in the proof of Proposition 4.

The sequence $e = (0)$ of size one has label $(1,1)$, which is the axiom of $\Omega_{\mathbf{I}(\geq,\geq,\geq)}$. Let e be an inversion sequence of $\mathbf{I}_n(\geq,\geq,\geq)$ with label (h,k). The labels of the inversion sequences of $\mathbf{I}_{n+1}(\geq,\geq,\geq)$ produced by adding a rightmost entry p to e are

- $(h-1,k+1),(h-2,k+1),\dots,(0,k+1)$, when $p = \text{last}(e)+1,\dots,\max(e)$,
- $(h+1,k),(h+2,k-1),\dots,(h+k,1)$, when $p = \max(e)+1,\dots,n$.

Figure 6 depicts an example of the growth according to $\Omega_{\mathbf{I}(\geq,\geq,\geq)}$. □

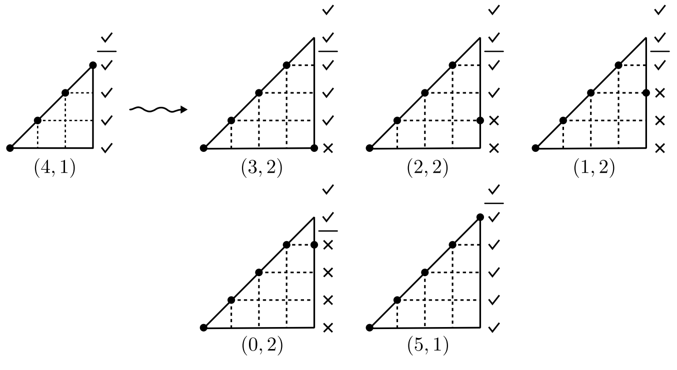


Figure 6: The growth of inversion sequences in $\mathbf{I}(\geq,\geq,\geq)$.

Now, with the aim of enumerating completely the family $\mathbf{I}(\geq,\geq,\geq)$, we solve the functional equation that the succession rule of Proposition 9 yields. The method we use to treat this functional equation is a variant of the so-called kernel method - see [6, 19] and references therein. And, in Proposition 14 we successfully prove that the enumerative sequence of family $\mathbf{I}(\geq,\geq,\geq)$ is A108307 on [21].

For $h,k \geq 0$, let $A_{h,k}(x) \equiv A_{h,k}$ denote the size generating function of inversion sequences of the family $\mathbf{I}(\geq,\geq,\geq)$ having label (h,k). The rule $\Omega_{\mathbf{I}(\geq,\geq,\geq)}$ translates using a stantard technique into a functional equation for the generating function $A(x;y,z) \equiv A(y,z) = \sum_{h,k\geq 0} A_{h,k}\, y^h z^k$.

Proposition 10. *The generating function $A(y,z)$ satisfies the following functional equation*

$$A(y,z) = xyz + \frac{xz}{1-y}\left(A(1,z) - A(y,z)\right) + \frac{xyz}{z-y}\left(A(y,z) - A(y,y)\right). \tag{2}$$

Proof. Starting from the succession rule $\Omega_{\mathbf{I}(\geq,\geq,\geq)}$, we can write

$$\begin{aligned} A(y,z) &= xyz + x\sum_{h,k\geq 0} A_{h,k}\left((1+y+\cdots+y^{h-1})z^{k+1} + (y^{h+k}z + y^{h+k-1}z^2 + \cdots + y^{h+1}z^k)\right) \\ &= xyz + x\sum_{h,k\geq 0} A_{h,k}\left(\frac{1-y^h}{1-y}\,z^{k+1} + \frac{1-\left(\frac{y}{z}\right)^k}{1-\frac{y}{z}}y^{h+1}z^k\right) \\ &= xyz + \frac{xz}{1-y}\left(A(1,z) - A(y,z)\right) + \frac{xyz}{z-y}\left(A(y,z) - A(y,y)\right). \end{aligned}$$

□

Equation (2) is a linear functional equation with two catalytic variables, y and z, in the sense of Zeilberger [28]. Similar functional equations have been solved by using the *obstinate kernel method* [6], which allows to provide the following expression for the generating function of $\mathbf{I}(\geq,\geq,\geq)$.

Theorem 11. *Let $W(x;a) \equiv W$ be the unique formal power series in x such that*

$$W = x\bar{a}(W+1+a)(W+a+a^2).$$

The series solution $A(y,z)$ of Equation (2) satisfies

$$A(1+a,1+a) = \left[\frac{Q(a,W)}{(1+a)^3}\right]^{\geq},$$

where $Q(a,W)$ is a polynomial in W whose coefficients are Laurent polynomials in a defined in Equation (6), and the notation $[Q(a,W)/(1+a)^3]^{\geq}$ stands for the formal power series in x obtained by considering only those terms in the series expansion of $Q(a,W)/(1+a)^3$ that have non-negative powers of a.

Note that W and $Q(a,W)$ are algebraic series in x whose coefficients are Laurent polynomials in a. It follows, as in [6, page 6], that $A(1+a,1+a)$ is D-finite. Hence, the specialization $A(1,1)$, which is the generating function of $\mathbf{I}(\geq,\geq,\geq)$, is D-finite as well.

Proof of Theorem 11. First, set $y = 1+a$ and collect all the terms with $A(y,z)$ of Equation (2) to obtain the kernel form

$$K(a,z)A(1+a,z) = xz(1+a) - \frac{xz}{a}A(1,z) - \frac{xz(1+a)}{z-1-a}A(1+a,1+a), \tag{3}$$

where the kernel is

$$K(a,z) = 1 - \frac{xz}{a} - \frac{xz(1+a)}{z-1-a}.$$

For brevity, we write the right-hand side of Equation (3) as $\mathcal{H}(x, a, z, A(1, z), A(1 + a, 1 + a))$, where

$$\mathcal{H}(x_0, x_1, x_2, w_0, w_1) = x_0 x_2 (1 + x_1) - \frac{x_0 x_2}{x_1} w_0 - \frac{x_0 x_2 (1 + x_1)}{x_2 - 1 - x_1} w_1 .$$

The kernel equation $K(a, z) = 0$ is quadratic in z, and thus it has two roots. We denote $Z_+(a)$ and $Z_-(a)$ the two solutions of $K(a, z) = 0$ with respect to z,

$$\begin{aligned} Z_+(a) &= \frac{1}{2} \frac{a + x - a^2 x - \sqrt{a^2 - 2ax - 2a^3 x + x^2 - 2a^2 x^2 + a^4 x^2 - 4a^2 x}}{x} \\ &= (1 + a) + (1 + a)^2 x + \frac{(1 + a)^4}{a} x^2 + \frac{(a^2 + 3a + 1)(1 + a)^4}{a^2} x^3 + O(x^4), \end{aligned}$$

$$\begin{aligned} Z_-(a) &= \frac{1}{2} \frac{a + x - a^2 x + \sqrt{a^2 - 2ax - 2a^3 x + x^2 - 2a^2 x^2 + a^4 x^2 - 4a^2 x}}{x} \\ &= \frac{a}{x} - (1 + a)a - (1 + a)^2 x - \frac{(1 + a)^4}{a} x^2 - \frac{(a^2 + 3a + 1)(1 + a)^4}{a^2} x^3 + O(x^4) . \end{aligned}$$

Note that the kernel root Z_- is not a power series in x. On the contrary, the other kernel root, Z_+, is a power series in x with Laurent polynomial coefficients in a. Then, the function $A(1 + a, Z_+)$ is a power series in x and the right-hand side of Equation (3) is equal to zero by setting $z = Z_+$, hence

$$\mathcal{H}(x, a, Z_+, A(1, Z_+), A(1 + a, 1 + a)) = 0.$$

We follow the steps of the obstinate variant of the kernel method (see [6, 9]) and exploit the birational transformations that leave $K(a, z)$ unchanged.

Examining the kernel shows that the transformations

$$\Phi : (a, z) \to \left(\frac{z - 1 - a}{1 + a}, z \right) \quad \text{and} \quad \Psi : (a, z) \to \left(a, \frac{z + za - 1 - a + a^2 + a^3}{z - 1 - a} \right)$$

leave the kernel unchanged and generate a group of order 12.

Among all the elements of this group we consider those pairs $(f_1(a, z), f_2(a, z))$ such that $f_1(a, Z_+)$ and $f_2(a, Z_+)$ are power series in x with Laurent polynomial coefficients in a. More precisely, we consider the pairs $(f_1(a, z), f_2(a, z))$ depicted in Figure 7.

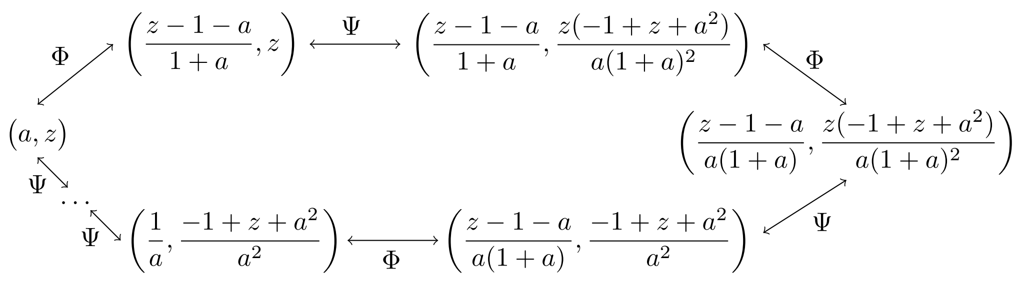


Figure 7: The action of Φ and Ψ on the pair (a, z).

Consequently, these pairs share the property that $A(1 + f_1(a, Z_+), f_2(a, Z_+))$ is a power series in x. Hence, substituting each of these pairs for (a, z) and setting $z = Z_+$ in Equation (3), we obtain a system of six equations whose left-hand sides are all equal to 0,

$$\begin{cases} 0 = \mathcal{H}(x, a, Z_+, A(1, Z_+), A(1+a, 1+a)) \\ 0 = \mathcal{H}\left(x, \frac{Z_+-1-a}{1+a}, Z_+, A(1, Z_+), A\left(1+\frac{Z_+-1-a}{1+a}, 1+\frac{Z_+-1-a}{1+a}\right)\right) \\ 0 = \mathcal{H}\left(x, \frac{Z_+-1-a}{1+a}, \frac{Z_+(-1+Z_++a^2)}{a(1+a)^2}, A\left(1, \frac{Z_+(-1+Z_++a^2)}{a(1+a)^2}\right), A\left(1+\frac{Z_+-1-a}{1+a}, 1+\frac{Z_+-1-a}{1+a}\right)\right) \\ 0 = \mathcal{H}\left(x, \frac{Z_+-1-a}{a(1+a)}, \frac{Z_+(-1+Z_++a^2)}{a(1+a)^2}, A\left(1, \frac{Z_+(-1+Z_++a^2)}{a(1+a)^2}\right), A\left(1+\frac{Z_+-1-a}{a(1+a)}, 1+\frac{Z_+-1-a}{a(1+a)}\right)\right) \\ 0 = \mathcal{H}\left(x, \frac{Z_+-1-a}{a(1+a)}, \frac{-1+Z_++a^2}{a^2}, A\left(1, \frac{-1+Z_++a^2}{a^2}\right), A\left(1+\frac{Z_+-1-a}{a(1+a)}, 1+\frac{Z_+-1-a}{a(1+a)}\right)\right) \\ 0 = \mathcal{H}\left(x, \frac{1}{a}, \frac{-1+Z_++a^2}{a^2}, A\left(1, \frac{-1+Z_++a^2}{a^2}\right), A\left(1+\frac{1}{a}, 1+\frac{1}{a}\right)\right). \end{cases} \tag{4}$$

Now, by eliminating all unknowns except for $A(1+a, 1+a)$ and $A(1, 1+\bar{a})$, where $\bar{a}$ denotes $1/a$, System (4) reduces to the following equation,

$$(1+a)^3 A(1+a, 1+a) - \frac{(1+a)^3}{a^4} A(1, 1+\bar{a}) + P(a, Z_+) = 0, \qquad \text{where} \tag{5}$$

$$P(a,z) = \frac{(-z+1+a)(a-1)}{a^6}\Big(a^8 + 4a^7 + 7a^6 + a^5 z + 8a^5 - z^2a^4 + 2za^4 + 8a^4 + 3za^3 + 7a^3 - a^3z^2 + 6a^2 z + 4a^2 - 3z^2a^2 + a + az^3 - 4z^2 a + 6az - z^2 + 2z\Big).$$

The form of Equation (5) allows us to separate its terms according to the power of a:

- $(1+a)^3 A(1+a, 1+a)$ is a power series in x with polynomial coefficients in a whose lowest power of a is 0,
- $A(1, 1+\bar{a})$ is a power series in x with polynomial coefficients in $\bar{a}$ whose highest power of a is 0; consequently, since $(1+a)^3\bar{a}^4 = a^{-4}+3a^{-3}+3a^{-2}+a^{-1}$, we obtain that $(1+a)^3\,\bar{a}^4 A(1, 1+\bar{a})$ is a power series in x with polynomial coefficients in $\bar{a}$ whose highest power of a is -1.

Then, when we expand the series $-P(a, Z_+)$ as a power series in x, the non-negative powers of a in the coefficients must be equal to those of $(1+a)^3 A(1+a, 1+a)$, while the negative powers of a come from $-(1+a)^3\,\bar{a}^4 A(1, 1+\bar{a})$.

In order to have a better expression for the series $P(a, z)$, we perform a further substitution setting $z = w + 1 + a$. More precisely, let $W \equiv W(x; a)$ be the power series in x defined by $W = Z_+ - (1+a)$. We have the following expression for $Q(a, W) := -P(a, Z_+)$:

$$\begin{aligned} Q(a, W) = -P(a, W+1+a) = {} & \left(-\frac{1}{a^6} - \frac{3}{a^5} - \frac{3}{a^4} - \frac{1}{a^3} + 1 + 3a + 3a^2 + a^3\right) W \\ & + \left(\frac{1}{a^5} + \frac{1}{a^4} - \frac{1}{a} - 1\right) W^2 \\ & + \left(\frac{1}{a^6} - \frac{1}{a^4} + \frac{1}{a^3} - \frac{1}{a}\right) W^3 \\ & + \left(-\frac{1}{a^5} + \frac{1}{a^4}\right) W^4. \end{aligned} \tag{6}$$

Since the kernel annihilates if $z = W + 1 + a$, namely $K(a, W+1+a) = 0$, the function W is recursively defined by

$$W = x\bar{a}(W + 1 + a)(W + a + a^2). \tag{7}$$

Therefore, Equations (6) and (7) conclude the proof. □

From the expression of $A(1+a,1+a)$ in Theorem 11, we can derive an explicit, yet very complicated, expression for the coefficients of the generating function $A(1,1)$, *i.e.* $[x^n]A(1,1) = [x^na^0]A(1+a,1+a)$. In order to do this, we need to calculate first the coefficients $[x^na^i]W^j$, for $j=1,2,3,4$.

Lemma 12. *Let $W(x;a) \equiv W$ be the unique formal power series in x such that Equation (7) holds. Then, for $r \geq 1$*

$$[x^na^s]W^r = \frac{r}{n}\sum_{k=0}^{n-r}\binom{n}{k}\binom{n}{k+r}\binom{n+r}{k-s+2r}. \tag{8}$$

Proof. It follows straightforward from Equation (7) by applying the Lagrange inversion formula. □

Proposition 13. *The number of inversion sequences of the set $\mathbf{I}_n(\geq,\geq,\geq)$, for all $n \geq 1$, is given by $\sum_{k=0}^{n} I(n,k)$, where*

$$\begin{aligned} I(n,k) = \tfrac{1}{n}\tbinom{n}{k}\Big[&\tbinom{n}{k+1}\Big[-\tbinom{n+1}{k-4} - 3\tbinom{n+2}{k-2} - \tbinom{n+1}{k-1} + \tbinom{n+1}{k+2} + 3\tbinom{n+2}{k+4} + \tbinom{n+1}{k+5}\Big] \\ &+2\tbinom{n}{k+2}\Big[\tbinom{n+3}{k} - \tbinom{n+3}{k+4}\Big] + 3\tbinom{n}{k+3}\Big[\tbinom{n+3}{k} - \tbinom{n+3}{k+2} + \tbinom{n+3}{k+3} - \tbinom{n+3}{k+5}\Big] \\ &+4\tbinom{n}{k+4}\Big[-\tbinom{n+4}{k+3} + \tbinom{n+4}{k+4}\Big]\Big]. \end{aligned}$$

Proof. The number of inversion sequences of $\mathbf{I}_n(\geq,\geq,\geq)$ is the coefficient of x^n in $A(1,1)$, which is also the coefficient of x^na^0 in $A(1+a,1+a)$. So by Theorem 11 it is the coefficient of x^na^0 in $Q(a,W)$ as expressed in Equation (6), namely

$$\begin{aligned} [x^na^0]Q(a,W) = &-[x^na^6]W - 3[x^na^5]W - 3[x^na^4]W - [x^na^3]W + [x^na^0]W + 3[x^na^{-1}]W \\ &+ 3[x^na^{-2}]W + [x^na^{-3}]W + [x^na^5]W^2 + [x^na^4]W^2 - [x^na]W^2 - [x^na^0]W^2 \\ &+ [x^na^6]W^3 - [x^na^4]W^3 + [x^na^3]W^3 - [x^na]W^3 - [x^na^5]W^4 + [x^na^4]W^4. \end{aligned}$$

Then, by Lemma 12 we can substitute into the above expression the coefficients of $[x^na^s]W^i$, for each s and $i=1,2,3,4$, proving Proposition 13. □

Although Proposition 13 shows a rather complicated expression for the number of inversion sequences of $\mathbf{I}_n(\geq,\geq,\geq)$, Zeilberger's method of creative telescoping [24, 27] allows us to provide a much simpler recursive formula satisfied by these numbers and to prove that they are indeed the sequence A108307 on [21].

Proposition 14. *Let $a_n = |\mathbf{I}_n(\geq,\geq,\geq)|$. The numbers a_n are recursively defined by $a_0 = a_1 = 1$ and for $n \geq 0$,*

$$8(n+3)(n+1)a_n + (7n^2+53n+88)a_{n+1} - (n+8)(n+7)a_{n+2} = 0\,.$$

Thus, $\{a_n\}_{n\geq 0}$ is sequence A108307 on [21].

Proof. From Proposition 13, we can write $a_n = \sum_{k=0}^{n} I(n,k)$, where the summand $I(n,k)$ is hypergeometric. Then, we prove the announced recurrence using the method of creative telescoping. The Maple package `SumTools` includes the command `Zeilberger`, which implements this approach. On input $I(n,k)$ it shows that

$$\begin{aligned} &-(n+9)(n+8)(n+6)I(n+3,k) + (464n+6n^3+776+92n^2)I(n+2,k) \\ &\quad+(n+2)(15n^2+133n+280)I(n+1,k) + 8(n+3)(n+2)(n+1)I(n,k) \\ &\qquad= G(n,k+1) - G(n,k), \end{aligned} \tag{9}$$

where the certificate function $G(n,k)$ has an expression extremely complicated and is not reported here. Nevertheless, it can be checked that $G(n,0) = G(n,n+9) = 0$.

Thus, to complete the proof it is sufficient to sum both sides of Equation (9) over k, k ranging from 0 to $n+8$. Since the coefficients on the left-hand side of Equation (9) are independent of k, and $I(n,k) = 0$ when $k > n$, summing Equation (9) over k gives

$$\begin{aligned} &-(n+9)(n+8)(n+6)a_{n+3} + (464n + 6n^3 + 776 + 92n^2)a_{n+2} \\ &+(n+2)(15n^2 + 133n + 280)a_{n+1} + 8(n+3)(n+2)(n+1)a_n \quad = \quad 0\,. \end{aligned} \tag{10}$$

This is not exactly the recursive equation for a_n given in Proposition 14. Nevertheless, it is straightforward to check that the sequence defined in Proposition 14 (with initial conditions $a_0 = a_1 = 1$) and the sequence defined by the above P-recursion (with initial conditions $a_0 = a_1 = 1, a_2 = 2$) are the same. Indeed, Equation (10) can be obtained by applying the operator $(n+2)+(n+6)N$ to the recursion of Proposition 14, where N denotes the forward shift operator. The proof is completed by checking that the recursion of Proposition 14 coincides with Equation (1). $\square$

Finally, recall that [21] indicates that the sequence A108307 counts set partitions that avoid enhanced 3-crossings [8]. Our proof that $\mathbf{I}(\geq,\geq,\geq)$ is counted by the same sequence does not explain this enumerative coincidence, and it could be interesting to have a combinatorial (ideally, bijective) explanation of this fact.

4 Baxter inversion sequences: $\mathbf{I}(\geq,\geq,>)$

The next family of inversion sequences according to the hierarchy of Figure 2 is $\mathbf{I}(\geq,\geq,>)$. This family of inversion sequences was conjectured in [23] to be counted by the sequence A001181 [21] of Baxter numbers, whose first terms are

$$1, 2, 6, 22, 92, 422, 2074, 10754, 58202, 326240, 1882960, 11140560, 67329992, \ldots$$

This conjecture has recently been proved in [20, Theorem 4.1]. Accordingly, we call $\mathbf{I}(\geq,\geq,>)$ the family of *Baxter inversion sequences*.

The proof of [20, Theorem 4.1] is analytic. Precisely, [20, Lemma 4.3] provides a succession rule for $\mathbf{I}(\geq,\geq,>)$. It is then shown to generate Baxter numbers using the obstinate kernel method as in our Section 3. This succession rule is however not a classical one associated with Baxter numbers, and no other Baxter family is known to grow according to this new Baxter succession rule. It would be desirable to establish a closer link (either via generating trees, or via bijections) between $\mathbf{I}(\geq,\geq,>)$ and any other known Baxter family.

4.1 Combinatorial characterization

The family of Baxter inversion sequences clearly contains $\mathbf{I}(\geq,\geq,\geq)$, as shown by the following characterization.

Proposition 15. *An inversion sequence is a Baxter inversion sequence if and only if it avoids* 100, 110 *and* 210.

Proof. The statement is readily checked, as in Propositions 1 and 7. $\square$

Another characterization for this family is the following. Recall that for an inversion sequence $e = (e_1,\ldots,e_n)$, we call an entry e_i a LTR maximum (resp. RTL minimum), if $e_i > e_j$, for all $j < i$ (resp. $e_i < e_j$, for all $j > i$).

Proposition 16. *An inversion sequence $e = (e_1,\ldots,e_n)$ is a Baxter inversion sequence if and only if for every i and j, with $i < j$ and $e_i > e_j$, both e_i is a LTR maximum and e_j is a RTL minimum.*

Proof. The proof in both directions is straightforward by considering the characterization of Proposition 15. $\square$

4.2 Enumerative results

We choose to report here a proof of [20, Lemma 4.3] (which is omitted in [20]). This proof is essential in our work, since it displays a growth for Baxter inversion sequences that generalizes the one for the family $\mathbf{I}(\geq,\geq,\geq)$ provided in Proposition 9.

Proposition 17. *Baxter inversion sequences grow according to the following succession rule*

$$\Omega_{Bax} = \begin{cases} (1,1) \\ \\ (h,k) & \rightsquigarrow (h-1,k+1),\ldots,(1,k+1), \\ & \quad (1,k+1), \\ & \quad (h+1,k),\ldots,(h+k,1). \end{cases}$$

Proof. We show that the growth of Baxter inversion sequences by addition of a new rightmost entry (as in the proofs of Propositions 4 and 9) can be encoded by the above succession rule Ω_{Bax}.

As in the proof of Proposition 9, let $\mathrm{last}(e)$ be the value of the rightmost entry of e which is not a LTR maximum, if there is any. Note that $\mathrm{last}(e)$ is also the largest value not being a LTR maximum, since e avoids 210 by Proposition 15. Otherwise, if such an entry does not exists, we set $\mathrm{last}(e)$ equal to the smallest value of e, *i.e.* $\mathrm{last}(e) := 0$.

Moreover, if this rightmost entry of e which is not a LTR maximum exists, it can either form an inversion (*i.e.* there exists an entry e_i on its left such that $e_i > \mathrm{last}(e)$) or not. We need to distinguish two cases in order to define the addition of a new rightmost entry to e:

(a) in case either all the entries of e are LTR maxima, or the rightmost entry of e which is not a LTR maximum does not form an inversion;

(b) in case the rightmost entry of e which is not a LTR maximum exists and does form an inversion.

Then, according to Proposition 16, we have that

(a) The sequence $f = (e_1,\ldots,e_n,p)$ is a Baxter inversion sequence of length $n+1$ if and only if $\mathrm{last}(e) \leq p \leq n$. Moreover, if $\mathrm{last}(e) \leq p < \max(e)$, where as usual $\max(e)$ is the maximum value of e, then $\mathrm{last}(f) = p$ and f falls in case **(b)**. Else if $p = \max(e)$, then again $\mathrm{last}(f) = p$, yet f falls in case **(a)**. While, if $\max(e) < p \leq n$, p is a LTR maximum of f, which thus falls in the same case **(a)** as e, and $\mathrm{last}(f) = \mathrm{last}(e)$.

(b) The sequence $f = (e_1,\ldots,e_n,p)$ is a Baxter inversion sequence of length $n+1$ if and only if $\mathrm{last}(e) < p \leq n$. In particular, if $\mathrm{last}(e) < p < \max(e)$, then $\mathrm{last}(f) = p$ and f falls in case **(b)**. Else if $p = \max(e)$, then again $\mathrm{last}(f) = p$ and f falls in case **(a)**. While, if $\max(e) < p \leq n$, as above p is a LTR maximum of f, which thus falls in the same case **(b)** as e, and $\mathrm{last}(f) = \mathrm{last}(e)$.

Now, we assign to any Baxter inversion sequence e of length n a label according to the above distinction: in case **(a)** (resp. **(b)**) we assign the label (h,k), where $h = \max(e) - \mathrm{last}(e) + 1$ (resp. $h = \max(e) - \mathrm{last}(e)$) and $k = n - \max(e)$.

The sequence $e = (0)$ of size one falls in case **(a)**, thus it has label $(1,1)$, which is the axiom of Ω_{Bax}. Now, let e be a Baxter inversion sequence of length n with label (h,k). Following the above distinction, the inversion sequences of length $n+1$ produced by adding a rightmost entry p to e have labels:

(a)
- $(h-1,k+1),\ldots,(1,k+1)$, when $p = \mathrm{last}(e),\ldots,\max(e)-1$,
- $(1,k+1)$, for $p = \max(e)$,
- $(h+1,k),(h+2,k-1),\ldots,(h+k,1)$, when $p = \max(e)+1,\ldots,n$,

(b)
- $(h-1,k+1),\ldots,(1,k+1)$, when $p = \mathrm{last}(e)+1,\ldots,\max(e)-1$,

- $(1, k+1)$, for $p = \max(e)$,
- $(h+1, k), (h+2, k-1), \dots, (h+k, 1)$, when $p = \max(e)+1, \dots, n$,

which concludes the proof that Baxter inversion sequences grow according to Ω_{Bax}. □

The growth of Baxter inversion sequences is depicted in Figure 8.

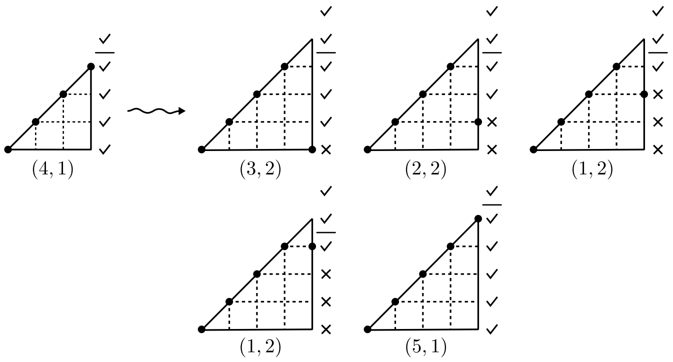


Figure 8: The growth of Baxter inversion sequences.

5 Semi-Baxter inversion sequences: $\mathbf{I}(\geq, >, -)$

The hierarchy of Figure 2 continues with the family $\mathbf{I}(\geq, >, -)$. This family of inversion sequences was conjectured in [23] to be counted by the sequence A117106 [21]. The validity of this conjecture follows from [9] where the numbers of sequence A117106 are named semi-Baxter, hence the name *semi-Baxter inversion sequences*. We recall the first terms of this enumeration sequence

$$1, 2, 6, 23, 104, 530, 2958, 17734, 112657, 750726, 5207910, 37387881, 276467208, \dots$$

5.1 Combinatorial characterization

Proposition 18. *An inversion sequence is in* $\mathbf{I}(\geq, >, -)$ *if and only if it avoids* 110 *and* 210.

The proof of the above statement is elementary, and we omit it. Yet it shows clearly that semi-Baxter inversion sequences avoid only two of the three patterns avoided by the Baxter inversion sequences (see Proposition 15 for a comparison).

Moreover, the following characterization is an extension of that provided in Proposition 16 for the family $\mathbf{I}(\geq, \geq, >)$. Recall that for $e = (e_1, \dots, e_n)$ an inversion sequence, we call an entry e_i a LTR maximum if $e_i > e_j$, for all $j < i$, and we say that e_i and e_j form an inversion if $i < j$ and $e_i > e_j$.

Proposition 19. *An inversion sequence* $e = (e_1, \dots, e_n)$ *is in* $\mathbf{I}(\geq, >, -)$ *if and only if for every* e_i *and* e_j *that form an inversion,* e_i *is a LTR maximum.*

Proof. Similarly to the proof of Proposition 16, the above statement follows immediately by considering that e is an inversion sequence of $\mathbf{I}(\geq, >, -)$ if and only if it avoids 110 and 210. □

5.2 Enumerative results

For the sake of completeness, we choose to report here a direct proof of the fact that the family $\mathbf{I}(\geq, >, -)$ can be generated by the rule Ω_{semi} associated with semi-Baxter numbers. It allows to see that the growth of the family $\mathbf{I}(\geq, \geq, >)$ in the proof of Proposition 17 can be easily generalized to a growth for the family $\mathbf{I}(\geq, >, -)$.

Proposition 20. *The family* $\mathbf{I}(\geq,>,-)$ *grows according to the following succession rule*

$$\Omega_{semi} = \begin{cases} (1,1) \\ \\ (h,k) & \rightsquigarrow (h,k+1),\ldots,(1,k+1), \\ & \quad (h+1,k),\ldots,(h+k,1). \end{cases}$$

Proof. As previously, we define a growth for the family $\mathbf{I}(\geq,>,-)$ according to Ω_{semi} by adding a new rightmost entry.

As in the proof of Proposition 17, let $\text{last}(e)$ be the value of the rightmost entry of e which is not a LTR maximum, if there is any. Otherwise, $\text{last}(e) := 0$. Note that differently from Proposition 17, here we do not need to distinguish cases depending on whether or not the rightmost entry of e not being a LTR maximum forms an inversion.

According to Proposition 19, it follows that $f = (e_1,\ldots,e_n,p)$ is an inversion sequence of $\mathbf{I}_{n+1}(\geq,>,-)$ if and only if $\text{last}(e) \leq p \leq n$. Moreover, if $\text{last}(e) \leq p \leq \max(e)$, where as usual $\max(e)$ is the maximum value of e, then $\text{last}(f) = p$; if $\max(e) < p \leq n$, then $\text{last}(f) = \text{last}(e)$, since p is a LTR maximum.

Now, we assign to any $e \in \mathbf{I}_n(\geq,>,-)$ the label (h,k), where $h = \max(e) - \text{last}(e) + 1$ and $k = n - \max(e)$.

The sequence $e = (0)$ of size one has label $(1,1)$, which is the axiom of Ω_{semi}, since $\text{last}(e) = 0$. Let e be an inversion sequence of $\mathbf{I}_n(\geq,>,-)$ with label (h,k). The labels of the inversion sequences of $\mathbf{I}_{n+1}(\geq,>,-)$ produced adding a rightmost entry p to e are

- $(h,k+1),\ldots,(1,k+1)$, when $p = \text{last}(e),\ldots,\max(e)$,
- $(h+1,k),(h+2,k-1),\ldots,(h+k,1)$, when $p = \max(e)+1,\ldots,n$,

which concludes the proof that $\mathbf{I}(\geq,>,-)$ grows according to Ω_{semi}. □

The growth semi-Baxter inversion sequences is depicted in Figure 9.

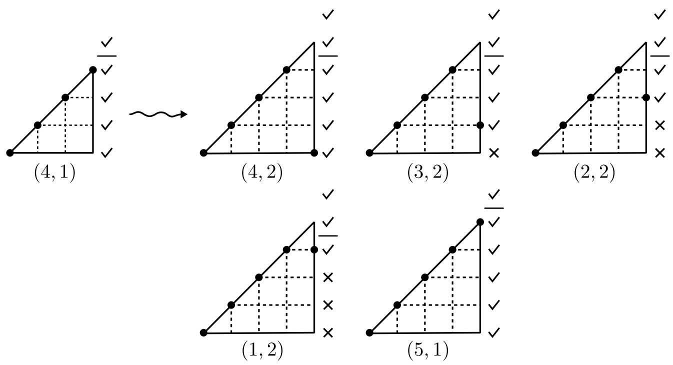


Figure 9: The growth of semi-Baxter inversion sequences.

6 Powered Catalan inversion sequences: $\mathbf{I}(=,>,>)$

The family of inversion sequences $\mathbf{I}(=,>,>)$ is the last element of the chain in Figure 2. These objects that can be characterized by the avoidance of 110 are completely enumerated in [13, Theorem 13], and are called *powered Catalan inversion sequences* in the following. Its enumerative number sequence (of *powered Catalan numbers*) is registered on [21] as sequence A113227, and its first terms are

$$1, 1, 2, 6, 23, 105, 549, 3207, 20577, 143239, 1071704, 8555388, 72442465, 647479819, \ldots$$

D. Callan [11] proves that the nth term of the sequence can be obtained as $p_n = \sum_{k=0}^{n} c_{n,k}$, where $c_{n,k}$ is recursively defined by

$$\begin{cases} c_{0,0} = 1, & \\ c_{n,0} = 0, & \text{for } n \geq 1 \\ c_{n,k} = c_{n-1,k-1} + k \sum_{j=k}^{n-1} c_{n-1,j}, & \text{for } n \geq 1, \text{ and } 1 \leq k \leq n\,. \end{cases} \tag{11}$$

Proposition 21 ([13], Theorem 13)**.** *For $n \geq 1$ and $0 \leq k \leq n$, the number of powered Catalan inversion sequences having k zeros is given by the term $c_{n,k}$ of Equation* (11)*. Thus, the number of powered Catalan inversion sequences of length n is p_n, for every $n \geq 1$.*

Proposition 21 can be rephrased in terms of succession rules, as done below with the rule Ω_{pCat}. More precisely, for $n \geq 1$ and $k \geq 1$, the number of nodes at level n that carry the label (k) in the generating tree associated with Ω_{pCat} is precisely the quantity $c_{n,k}$ given by Equation (11).

Proposition 22. *The family of powered Catalan inversion sequences grows according to the following succession rule*

$$\Omega_{pCat} = \begin{cases} (1) & \\ (k) & \rightsquigarrow (1), (2)^2, (3)^3, \dots, (k)^k, (k+1). \end{cases}$$

We notice that Ω_{pCat} is extremely similar to the Catalan succession rule Ω_{Cat} (see page 3): specifically, the productions of Ω_{pCat} are the same as in Ω_{Cat}, but with multiplicities appearing as "powers". Hence, the name *powered Catalan.*

Proof of Proposition 22. We prove the above statement by showing a growth for the family of powered Catalan inversion sequences. Let $e = (e_1, \dots, e_n)$ be a powered Catalan inversion sequence of length n. Suppose that e has k entries equal to 0, and let $i_1, \dots, i_k$ be their indices. We define a growth that changes the number of 0 entries of e, as follows.

a) First, increase by one all the entries of e that are greater than 0; namely $e' = (e'_1, \dots, e'_n)$, where $e'_i = e_i$, if $i = i_1, \dots, i_k$, otherwise $e'_i = e_i + 1$. Note that $e'_1 = e_1 = 0$, and that e' does not have to be an inversion sequence.

b) Then, insert a new leftmost 0 entry; namely $e'' = (0, e'_1, \dots, e'_n)$. Note that e'' is an inversion sequence of size $n+1$, and moreover it has $k+1$ zeros at positions $1, i_1+1, \dots, i_k+1$.

c) Build the following inversion sequences, starting from e''.

   (1) Replace all the zeros at positions $i_1{+}1, i_2{+}1, \dots, i_k{+}1$ by 1; namely $e^{(1)} = (0, e^*_1, \dots, e^*_n)$, where $e^*_i = e_i + 1$, for all i. Note that $e^{(1)}$ has only one zero.

   (j) For all $1 < j < k+1$, replace all the zeros at positions $i_{j+1}+1, \dots, i_k+1$ by 1, and furthermore replace by 1 only one zero entry among those at indices $i_1+1, \dots, i_j+1$. There are thus j different inversion sequences $e^{(m)} = (0, e^*_1, \dots, e^*_n)$, with $1 \leq m \leq j$, such that $e^*_i = e_i + 1$ except for the indices $i_1+1, \dots, i_{m-1}+1, i_{m+1}+1, \dots, i_j+1$. Note that $e^{(m)}$ has exactly j zeros, for any $1 \leq m \leq j$.

   (k+1) Set $e^{(k+1)} = e''$.

Note that all the inversion sequences of size $n+1$ produced at step c) avoid 110, since the initial inversion sequence e avoids 110. Thus, in each of the above cases we build a powered Catalan inversion sequence of length $n+1$.

Moreover, given any powered Catalan inversion sequence f of length $n+1$, it is easy to retrieve the unique inversion sequence e of length n that produces f according to the operations of c): it is sufficient to replace all the entries equal to 1 by 0, remove the leftmost 0 entry, and finally decrease by one all the entries greater than 0. This procedure is in fact a) - b) - c) backwards.

Finally, we label a powered Catalan inversion sequence e of length n with (k), where k is its number of 0 entries. It is straightforward, and the above itemized list suggests it, that the powered Catalan inversion sequences produced by e following the construction at step c) have labels $(1),(2)^2,(3)^3,\dots,(k)^k,(k+1)$. □

The sequence of powered Catalan numbers proves to be extremely rich of combinatorial interpretations, and quite a few enumerative problems associated with it are open, or beg for a more natural proof. We collect some of them in the remainder of this article, and solve a few by providing bijections between families enumerated by the powered Catalan numbers.

7 Powered Catalan numbers

Recall from the previous section that the sequence of powered Catalan number (p_n) (A113227 on [21]) is defined by $p_n = \sum_{k=0}^{n} c_{n,k}$, where the term $c_{n,k}$ is recursively defined by Equation (11). To our knowledge, there is no information about the ordinary generating function $F_{pCat}(x) = \sum_{n\geq 0} p_n x^n$. On the contrary, the exponential generating function $E_{pCat}(x) = \sum_{n\geq 0} p_n x^n/n!$ has been studied in [16], as well as in [11], where by means of the recurrence (11) a refined version of this exponential generating function is provided.

7.1 Known combinatorial structures enumerated by the powered Catalan numbers

Definition 23. *A* valley-marked Dyck path *(see Figure 10) of semi-length n is a Dyck path P of length $2n$ in which, for each valley (*i.e. *DU factor), one of the lattice points between the valley vertex and the x-axis is marked. In other words, if (i,k) pinpoints any valley of P, then a valley-marked Dyck path associated with P must take a mark in a point (i,j), where $0 \leq j \leq k$. If $j = 0$, we say that the valley has a* trivial mark*.*

In addition, we say that a *return to the mark* of valley-marked Dyck path is any valley whose mark is on the valley itself. Note that returns to the x-axis are a special case of return to the mark. (Here and everywhere, when speaking of return to the x-axis, we do not include the starting nor the ending point of the path.)

We also define the *total mark* of a valley-marked Dyck path as the sum of the heights of the marks.

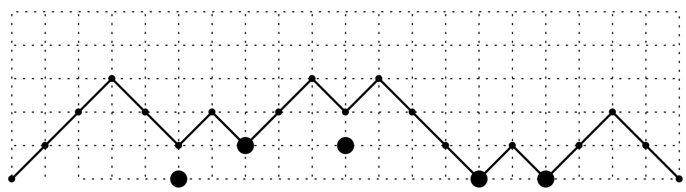


Figure 10: A valley-marked Dyck path.

Valley-marked Dyck paths are enumerated by powered Catalan number according to their semi-length. More precisely, the number of valley-marked Dyck paths of semi-length n having k down steps in the last descent (or symmetrically, k up steps on the main diagonal (of equation $x = y$)) is given by the term $c_{n,k}$ of Equation (11), for every $n \geq 0$ and $0 \leq k \leq n$ ([11, Section 7]).

Increasing ordered trees Another family of objects counted by sequence A113227 is one of labeled ordered trees [11].

Definition 24. *An* increasing ordered tree *of size n is a plane tree with $n+1$ labeled vertices, the standard label set being $\{0, 1, 2, \ldots, n\}$, such that each child exceeds its parent. An increasing ordered tree* has increasing leaves *if its leaves, taken in pre-order, are increasing.*

Figure 11 shows two increasing ordered trees, the first has increasing leaves, while the second does not.

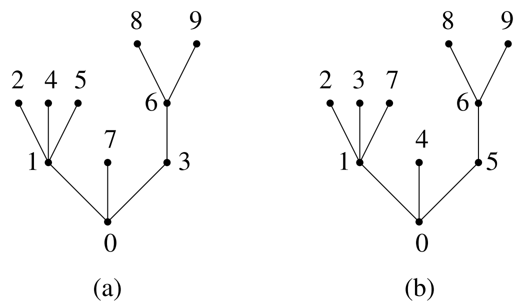


Figure 11: Two increasing ordered trees: (a) with increasing leaves; (b) with non-increasing leaves.

The number of increasing ordered trees of size n is given by the odd double factorial $(2n - 1)!!$ [12]. If we require the additional constraint of having increasing leaves, then the number of these increasing ordered trees of size n results to be the nth powered Catalan number [11]. More precisely, the number of increasing ordered trees with increasing leaves of size n and root degree k is given by the number $c_{n,k}$ of Equation (11).

Remark 25. It is not hard to prove (and this is left to the reader) that valley marked Dyck paths and increasing ordered trees with increasing leaves have a growth according to Ω_{pCat}

Pattern-avoiding permutations Some families of pattern-avoiding permutations are known to be counted by the sequence of powered Catalan numbers. Indeed, the sequence A113227 is actually registered on [21] as the enumerative sequence of permutations avoiding the generalized pattern 1-23-4.

The family AV(1-23-4) has been enumerated by D. Callan in [11], providing a bijection between AV(1-23-4) and increasing ordered trees with increasing leaves.

Furthermore, in [4], some other families of pattern-avoiding permutations are presented as related to the sequence A113227 [21]. In particular, in [4] and subsequent papers, the families AV(1-32-4), and AV(1-34-2), and AV(1-43-2) are proved to be equinumerous to permutations avoiding 1-23-4. It has been conjectured in [5] that also the family AV(23-1-4) is equinumerous to AV(1-23-4). We attempted to prove this conjecture by defining a growth for the family AV(23-1-4) according to Ω_{pCat}. Although our attempts were not successful, they lead us to refine this conjecture as follows.

Conjecture 1. *The number of permutations of AV_n(23-1-4) with k RTL minima is given by $c_{n,k}$ as defined in Equation* (11).

Although we have a little evidence of the above fact (only up to $n = 9$), we suspect that a growth for permutations avoiding the pattern 2-1-3 according to Ω_{Cat}, where the label (k) marks the number of RTL minima, could be generalized as to obtain one for permutations avoiding 23-1-4 according to Ω_{pCat}. We leave open the problem of finding such a growth.

7.2 A second succession rule for powered Catalan numbers

The bijection provided by D. Callan between increasing ordered trees with increasing leaves and AV(1-23-4) in [11] is quite intricate. And the interpretation of the parameter k in $c_{n,k}$ is rather complicated on permutations avoiding 1-23-4. This suggests that the combinatorics of 1-23-4-avoiding permutations is quite different from that of other powered Catalan objects previously presented in our paper. This is also supported by the fact that we can describe a natural growth for $AV(1\text{-}23\text{-}4)$ which is not encoded by Ω_{pCat}. This leads us to presenting a second succession rule associated with powered Catalan numbers (denoted $\Omega_{1\text{-}23\text{-}4}$ below). Our impression is that a powered Catalan family is naturally generated by either Ω_{pCat} or $\Omega_{1\text{-}23\text{-}4}$, but not by both. This is further discussed at the beginning of Section 9.

Following D. Callan [11], we observe that permutations avoiding 1-23-4 have a simple characterization in terms of LTR minima and RTL maxima, as follows.

Proposition 26. *A permutation π of length n belongs to $AV(1\text{-}23\text{-}4)$ if and only if for every index $1 \leq i < n$,*

*if $\pi_i\pi_{i+1}$ is an ascent (*i.e.*, $\pi_i < \pi_{i+1}$), then π_i is a LTR minimum* **or** *π_{i+1} is a RTL maximum.*

Proof. Suppose that there exists an index i, $1 \leq i < n$, such that $\pi_i < \pi_{i+1}$, and neither π_i is a LTR minimum nor π_{i+1} is a RTL maximum. Then, there exists an index $j < i$ such that $\pi_j < \pi_i$, and an index $k > i+1$ such that $\pi_k > \pi_{i+1}$. Thus, $\pi_j\pi_i\pi_{i+1}\pi_k$ forms an occurrence of 1-23-4. Conversely, if π contains an occurrence of 1-23-4, by definition of pattern containment there exists an index i, $1 \leq i < n$, such that $\pi_i < \pi_{i+1}$, and neither π_i is a LTR minimum nor π_{i+1} is a RTL maximum. □

We show now a recursive growth for the family $AV(1\text{-}23\text{-}4)$ that yields a succession rule for powered Catalan numbers whose labels are arrays of length two.

Proposition 27. *Permutations avoiding* 1-23-4 *grows according to the following succession rule*

$$\Omega_{1\text{-}23\text{-}4} = \begin{cases} (1,1) \\ \\ (1,k) & \rightsquigarrow (1,k+1),(2,k),\dots,(1+k,1), \\ (h,k) & \rightsquigarrow (1,h+k),(2,h+k-1),\dots,(h,k+1), \\ & \quad\ (h+1,0),\dots,(h+k,0), \qquad\qquad \textit{if } h \neq 1. \end{cases}$$

Proof. First, observe that removing the rightmost point of a permutation avoiding 1-23-4, we obtain a permutation that still avoids 1-23-4. So, a growth for the permutations avoiding 1-23-4 can be obtained with local expansions on the right. We denote by $\pi \cdot a$, where $a \in \{1,\dots,n+1\}$, the permutation $\pi' = \pi'_1 \dots \pi'_n\pi'_{n+1}$ where $\pi'_{n+1} = a$, and $\pi'_i = \pi_i$, if $\pi_i < a$, $\pi'_i = \pi_i + 1$ otherwise.

For π a permutation in $AV_n(1\text{-}23\text{-}4)$, the active sites of π are by definition the points a (or equivalently the values a) such that $\pi \cdot a$ avoids 1-23-4. The other points a are called non-active sites.

An occurrence of 1-23 in π is a subsequence $\pi_j\pi_i\pi_{i+1}$ (with $j < i$) such that $\pi_j < \pi_i < \pi_{i+1}$. Note that the non-active sites a of π are the values larger than π_{i+1}, for some occurrence $\pi_j\pi_i\pi_{i+1}$ of 1-23. Then, given $\pi \in AV_n(1\text{-}23\text{-}4)$, we denote by $\pi_s\pi_{t-1}\pi_t$ the occurrence of 1-23 (if there is any), in which the point π_t is minimal. Then the active sites of π form a consecutive sequence from the bottommost site to π_t, *i.e.* they are $[1, \pi_t]$. Figure 12 should help understanding which sites are active (represented by diamonds, as usual). If $\pi \in AV_n(1\text{-}23\text{-}4)$ has no occurrence of 1-23, then the active sites of π are $[1, n+1]$.

Now, we assign a label (h,k) to each permutation $\pi \in AV_n(1\text{-}23\text{-}4)$, where h (resp. k) is the number of its active sites smaller than or equal to (resp. greater than) π_n. Remark that $h \geq 1$, since 1 is always an active site. Moreover, $h = \pi_n$: indeed, let $\pi_s\pi_{t-1}\pi_t$ be the occurrence of 1-23 with π_t minimal. There must hold that $\pi_t > \pi_n$, otherwise $\pi_s\pi_{t-1}\pi_t\pi_n$ would form an occurrence of 1-23-4.

The label of the permutation $\pi = 1$ is $(1,1)$, which is the axiom in $\Omega_{1\text{-}23\text{-}4}$. The proof then is concluded by showing that for any $\pi \in AV_n(1\text{-}23\text{-}4)$ of label (h,k), the permutations $\pi \cdot a$ have labels according to the productions of $\Omega_{1\text{-}23\text{-}4}$ when a runs over all active sites of π. To prove this we need to distinguish whether $\pi_n = 1$ or not.

If $\pi_n = 1$, no new occurrence of 1-23 can be generated in the permutation $\pi \cdot a$, for any a active site of π. Thus, the active sites of $\pi \cdot a$ are as many as those of π plus one (since the active site a

of π splits into two actives sites of $\pi \cdot a$). Then, since $\pi_n = 1$, permutation π has label $(1, k)$, for some $k > 0$ (at least one site above 1 in active), and permutations $\pi \cdot a$, for a ranging over all the active sites of π from bottom to top, have labels $(1, k+1), (2, k), \dots, (1+k, 1)$, which is the first production of $\Omega_{1\text{-}23\text{-}4}$.

Otherwise, we have that π has label (h, k), with $h > 1$, and $\pi_n = h$. In this case a new occurrence of 1-23 is generated in the permutation $\pi \cdot a$, for every $a > \pi_n$: indeed, $1\,\pi_n a$ forms an occurrence of 1-23, and moreover is such that a is minimal. Else if $a \leq \pi_n$, no new occurrence of 1-23 can be generated in the permutation $\pi \cdot a$. Thus, permutations $\pi \cdot a$ have labels $(1, h+k), (2, h+k-1), \dots, (h, k+1)$, for any active site $a \leq \pi_n$, and labels $(h+1, 0), (h+2, 0), \dots, (h+k, 0)$, for any active site $a > \pi_n$. Note that this label production coincides with the two lines of the second production of $\Omega_{1\text{-}23\text{-}4}$ concluding the proof. Figure 12 shows an example of the above construction. □

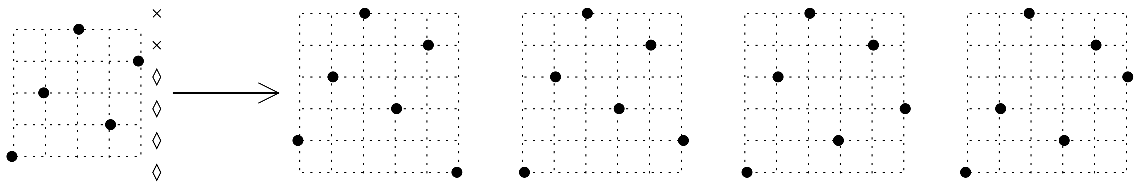


Figure 12: The growth of a permutation avoiding 1-23-4 of label $(4, 0)$.

8 The family of steady paths

In this last part of the paper we provide a further (and new) combinatorial interpretation of powered Catalan numbers in terms of lattice paths.

8.1 Definition and succession rule

Definition 28. *We call* steady path *of size n a lattice path P confined to the cone $\mathfrak{C} = \{(x, y) \in \mathbb{N}^2 : y \leq x\}$, which uses $U = (1, 1)$, $D = (1, -1)$ and $W = (-1, 1)$ as steps, without any factor WD nor DW, starting at $(0, 0)$ and ending at $(2n, 0)$, such that:*

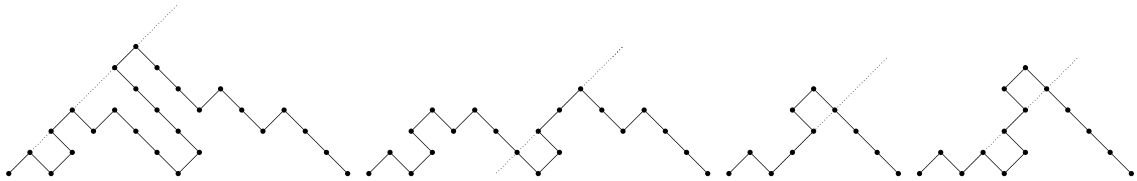
(S1) for any factor UU, the suffix of P following this UU factor lies weakly below the line parallel to $y = x$ passing through the UU factor;

(S2) for any factor WU, the suffix of P following this WU factor lies weakly below the line parallel to $y = x$ passing through the up step of the WU factor.

We call the edge line *of P the line $y = x - t$, with $t \geq 0$ an even integer, which supports the up step of the rightmost occurrence of either UU or WU in P.*

The name "steady" is motivated by the two restrictions (S1) and (S2), which force these paths to remain weakly below a line that moves rightwards and conveys more stability to the mountain range the path would represent. Figure 13 (a) shows an example of a steady path whose edge line coincides with $y = x$, whereas the edge line of the steady path depicted in Figure 13 (b) is $y = x - 6$. Figures 13 (c),(d) show two different examples of paths confined to $\mathfrak{C}$ that are not steady paths. The aware reader may observe that steady paths can be regarded as a subfamily of those "skew Dyck paths" considered in [17] and enumerated according to several parameters.

Remark 29. By Definition 28, the size of a steady path P is equal to the number of its U steps. Moreover, any steady path P of size n is uniquely determined by the set of positions of its up steps $U^{(1)}, \dots, U^{(n)}$ recorded from left to right: precisely, by the set of starting points (i_k, j_k) for



(a) (b) (c) (d)

Figure 13: (a) An example of a steady path P of size 8 with edge line $y = x$; (b) An example of a steady path P of size 8 with edge line $y = x - 6$; (c) a path in $\mathfrak{C}$ that violates (S1); (d) a path in $\mathfrak{C}$ that violates (S2).

any $U^{(k)}$. Indeed, since neither WD nor DW can occur, there is only one way to draw a steady path given the set of positions $\{(0,0) = (i_1, j_1), \ldots, (i_n, j_n)\}$ of its up steps from left to right.

Furthermore, let a set of points $\{(i_1, j_1), \ldots, (i_n, j_n)\}$ in $\mathfrak{C}$ be such that for every index $1 \leq k \leq n$, $j_k = k - 1 - i_k$. This set uniquely defines a steady path of size n provided that for every $1 < k \leq n$, if $i_k \leq i_{k-1} + 1$, then all the points (i_ℓ, j_ℓ), with $\ell > k$, lie weakly below the line parallel to $y = x$ passing through the point (i_k, j_k).

We provide a growth for the family of steady paths that results in the following proposition.

Proposition 30. *The family of steady paths grows according to the following succession rule*

$$\Omega_{steady} = \begin{cases} (0,2) \\ \\ (h,k) \quad \rightsquigarrow (h+k-1, 2), \ldots, (h+1, k), \\ \qquad\qquad\quad (0, k+1), \ldots, (0, h+k+1). \end{cases}$$

Proof. By Remark 29, given a steady path P of size n, we obtain a steady path of size $n-1$ if we remove its rightmost point (i_n, j_n), namely the rightmost up step of P. This allows us to provide a growth for steady paths by addition of a new rightmost up step.

Let P be a steady path of size n, and $(0,0) = (i_1, j_1), \ldots, (i_n, j_n)$ be the positions of its up steps. We describe in which position (i_{n+1}, j_{n+1}) a new rightmost up step can be inserted so that the path obtained is still a steady path. Specifically, according to Definition 28, if the edge line of P is $y = x - 2t$, with t a non-negative integer, then the point (i_{n+1}, j_{n+1}) must remain weakly below this line, that is $j_{n+1} \leq i_{n+1} - 2t$. So, we add a new rightmost up step in any position $(2n, 0), (2n-1, 1), (2n-2, 2), \ldots, (2n-s, s)$, where $s = n - t$. By Remark 29, there exists a unique path of size $n+1$ corresponding to $(0,0) = (i_1, j_1), \ldots, (i_n, j_n), (i_{n+1}, j_{n+1})$, where (i_{n+1}, j_{n+1}) is any point among $(2n, 0), (2n-1, 1), (2n-2, 2), \ldots, (2n-s, s)$, and it is steady by construction.

Moreover, the positions $(2n, 0), (2n-1, 1), (2n-2, 2), \ldots, (2n-s, s)$ can be divided into two groups: the positions that are ending points of D steps of the last descent of P, and those which are not. This distinction is crucial. Indeed, when we insert a U step in an ending point of a D step of P's last descent no factors WU or UU are generated. On the contrary, denoting $(2n-r, r)$ the topmost point of the last descent of P, when we insert the new rightmost U step at position $(2n-r, r)$, a UU factor is formed, and when we insert it in any point $(2n-i, i)$, with $r < i \leq s$, a WU factor is formed. In both cases, the edge line of the obtained steady path must pass through the point $(2n-r, r)$ (resp. $(2n-i, i)$, for $r < i \leq s$). Thus, the edge line may move rightwards so as to include this point.

Now, we assign the label $(h, k) \equiv (h, r+1)$ to any steady path P of size n and edge line $y = x - 2t$, where $r \geq 1$ is the number of steps in the last descent of P and $h = (n-t) - r$. In other words, the label interpretation is such that h counts the positions in which we insert a new rightmost U step that do not belong to the last descent of P.

The steady path UD of size 1 has edge line $y = x$. Thus its label is $(0, 2)$, which is the axiom of Ω_{steady}. Given a steady path P of size n, edge line $y = x - 2t$, and label $(h, k) \equiv (h, r+1)$, we now prove that the labels of the steady paths obtained by inserting a U step at positions

$(2n,0),\ldots,(2n-s,s)$, with $s=n-t$, are precisely the label productions of Ω_{steady}. Indeed, by inserting the U step at positions $(2n,0),\ldots,(2n-(r-1),r-1)$ the edge line does not change and the paths obtained have labels $(h+k-1,2),\ldots,(h+1,k)$, respectively. Whereas, by inserting the U step at position $(2n-i,i)$, for every $r\le i\le s$, the edge line becomes (or remains) $y=x-2(n-i)$ and the path has label $(0,i+2)$. Thus, we obtain the labels $(0,k+1),\ldots,(0,h+k+1)$, which are the second line of the production of Ω_{steady}, completing the proof. □

Figure 14 depicts the growth of a steady path of size n with edge line $y=x-2$; for any path, the corresponding edge line is drawn.

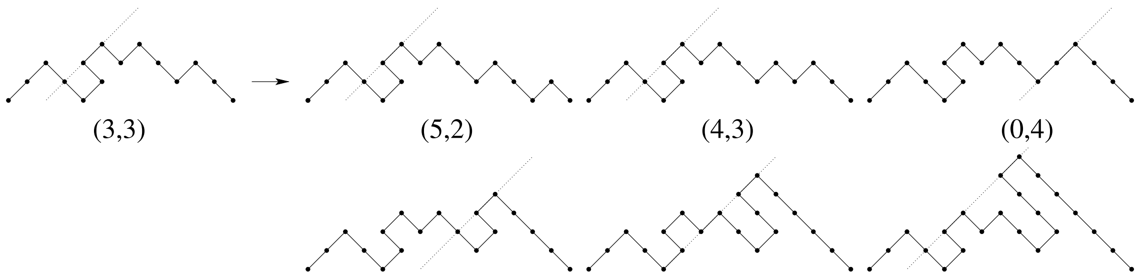


Figure 14: The growth of a steady path according to rule Ω_{steady}.

8.2 Recursive bijection between steady paths and AV(1-23-4)

Although at a first sight the succession rule Ω_{steady} does not resemble the rule $\Omega_{\text{1-23-4}}$ of Proposition 27, the following result follows by the fact that Ω_{steady} and $\Omega_{\text{1-23-4}}$ actually define the same generating tree.

Proposition 31. *The number of steady paths of size n is equal to the number of permutations in AV_n(1-23-4), thus is the n-th powered Catalan number.*

Proof. We prove the above proposition by showing that the succession rule $\Omega_{\text{1-23-4}}$ provided for the family AV(1-23-4) is isomorphic to the rule Ω_{steady}.

First, recall the production of the label $(1,k)$ according to $\Omega_{\text{1-23-4}}$, which appears as

$$(1,k)\rightsquigarrow(1,k+1),(2,k),\ldots,(1+k,1)\,. \tag{12}$$

The same succession rule $\Omega_{\text{1-23-4}}$ yields that the label $(h,0)$ produces according to

$$(h,0)\rightsquigarrow(1,h),(2,h-1),\ldots,(h,1)\,. \tag{13}$$

Now, consider the generating tree defined by $\Omega_{\text{1-23-4}}$ and replace all the labels $(1,k)$ by $(k+1,0)$. According to the production (12) the children of the node with a replaced label are

$$(k+1,0)\rightsquigarrow(k+2,0),(2,k),\ldots,(1+k,1)\,.$$

Setting $h=k+1$, this rewrites as

$$(h,0)\rightsquigarrow(h+1,0),(2,h-1),\ldots,(h,1)\,,$$

which is exactly the production (13) after substituting $(1,h)$ for $(h+1,0)$ in it. Therefore, the substitution of all labels $(1,k)$ by $(k+1,0)$ allows us to rewrite the succession rule $\Omega_{\text{1-23-4}}$ as follows

$$\begin{cases}(2,0)\\ \\ (h,k) \rightsquigarrow (h+k+1,0),\\ \qquad\qquad (2,h+k-1),\ldots,(h,k+1),\\ \qquad\qquad (h+1,0),\ldots,(h+k,0).\end{cases}$$

It is straightforward to check that the growth provided for steady paths in Proposition 30 defines the above succession rule by exchanging the interpretations of the two parameters h and k with respect to Ω_{steady}. □

Having given two generating trees along the same succession rule for steady paths and permutations avoiding 1-23-4, we deduce immediately a bijection between these classes. Namely, this bijection puts in correspondence objects of the two families according to their position in the associated generating tree. Of course, this bijection is not explicit, but recursive (following the way along which each object is built starting from the smallest one). We would of course wish for a nicer bijection between steady paths and permutations avoiding 1-23-4. We did not succeed, but instead were able to provide a bijection between steady paths and permutations avoiding 1-34-2 (which are also known to be enumerated by powered Catalan numbers).

8.3 One-to-one correspondence between steady paths and AV(1-34-2)

Theorem 32. *There exists an explicit bijection between the family of steady paths and AV(1-34-2).*

Proof. By Remark 29, any steady path P of size n is uniquely determined by the positions of its up steps, namely by the points $(0,0) = (i_1, j_1), \ldots, (i_n, j_n)$. These points that encode a unique steady path can in turn be encoded from right to left by a sequence $(t_1, \ldots, t_n)$ of integers that records the Euclidean distance between these points and the main diagonal $y = x$. More precisely, the entry t_k is the distance between the point (i_{n+1-k}, j_{n+1-k}) and the line $y = x$, for any $1 \leq k \leq n$. Note that $t_n = 0$, because the point $(0,0)$ belongs to the main diagonal. Moreover, for any $1 \leq k \leq n$, the entry t_k is in the range $[0, n-k]$, since steady paths are constrained in the cone $\mathfrak{C} = \{(x,y) \in \mathbb{N}^2 : y \leq x\}$. For instance, the steady path depicted in Figure 13 (a) is encoded by the sequence $(5,3,0,4,1,0,1,0)$.

Then, we have that any steady path of size n is defined by a particular sequence $(t_1, \ldots, t_n)$, for which $0 \leq t_k \leq n-k$, for every k. Certainly, the set of all these particular sequences of size n forms a subset of the set $\{\mathtt{T}(\pi) : \pi \in \mathcal{S}_n\}$ of the left inversion tables of permutations of length n. Our aim is to prove that a left inversion table $t = (t_1, \ldots, t_n)$, with $0 \leq t_k \leq n-k$, defines a steady path of size n if and only if $t \in \{\mathtt{T}(\pi) : \pi \in AV_n(\text{1-34-2})\}$.

$\Rightarrow$) We prove the contrapositive. Suppose $t = (t_1, \ldots, t_n)$ is the left inversion table of a permutation $\pi \notin AV_n(\text{1-34-2})$. Then, since π contains 1-34-2, there must be three indices i, j, ℓ, with $i < j < j+1 < \ell$, such that $\pi_i < \pi_\ell < \pi_j < \pi_{j+1}$. Moreover, we can suppose without loss of generality that there are no points π_s between π_i and π_j such that $\pi_s < \pi_i$. Otherwise, we could take $\pi_s \pi_j \pi_{j+1} \pi_\ell$ as our occurrence of 1-34-2.

Then, by definition of the left inversion table $t = \mathtt{T}(\pi)$, since $\pi_j < \pi_{j+1}$ and $\pi_j > \pi_\ell$, we have that $0 < t_j \leq t_{j+1}$. In addition, since there are no points π_s between π_i and π_j such that $\pi_s < \pi_i$, and $\pi_j > \pi_\ell > \pi_i$, it holds that $t_i < t_j$. From this it follows that t cannot encode a steady path P. Indeed, assuming such a path P would exists, t_i (resp. t_j, resp. t_{j+1}) must be the distance between the line $y = x$ and an up step $U^{(i)}$ (resp. $U^{(j)}$, resp. $U^{(j+1)}$), where $U^{(j+1)}$, $U^{(j)}$, and $U^{(i)}$ appear in this order from left to right. Since $t_{j+1} \geq t_j$, the up step $U^{(j)}$ must form either a UU factor or WU factor. Note that the line parallel to the main diagonal passing through $U^{(j)}$ cannot be $y = x$, since $t_j > 0$. Let this line be $y = x - g$, with g even positive number. Then, from $0 \leq t_i < t_j$ it follows that the suffix of P containing the up step $U^{(i)}$ exceeds the line $y = x - g$ passing through $U^{(j)}$.

$\Leftarrow$) Conversely, suppose for the sake of contradiction that there exists a left inversion table $t = (t_1, \ldots, t_n)$ which encodes a non-steady path P of size n.

By definition of steady path, there must be in P an up step $U^{(j)}$ not lying on the main diagonal such that it forms a factor UU or WU, and an up step $U^{(i)}$, which is on the right of $U^{(j)}$, lying above the line parallel to $y = x$ and passing through $U^{(j)}$. This means that $0 < t_j \leq t_{j+1}$, where $U^{(j+1)}$ is the up step which $U^{(j)}$ immediately follows, and $0 \leq t_i < t_j$, with $i < j$. Thus, let $\pi = \mathtt{T}^{-1}(t)$. We have that $\pi_i < \pi_j < \pi_{j+1}$, and from $0 \leq t_i < t_j$, there exists at least a point π_ℓ, with $j < \ell$, such that (π_j, π_ℓ) is an inversion of π and (π_i, π_ℓ) is not. Consequently, $\pi_i\pi_j\pi_{j+1}\pi_\ell$ forms an occurrence of 1-34-2. □

9 Bijection between steady paths and valley-marked Dyck paths

We have exhibited (three but essentially) two succession rules for powered Catalan numbers: Ω_{steady} and Ω_{pCat}. This leads us to classifying powered Catalan structures into two groups:

- those that appear as a rather simple generalization of Catalan structures, for which a growth according to the rule Ω_{pCat} can be found easily;
- those that generalize Catalan structures, but for which a growth according to Ω_{pCat} is not immediate, and the parameter k of Equation (11) is not clearly understood.

Valley-marked Dyck paths are the emblem of the first group; while, steady paths as well as permutations avoiding 1-23-4 rather belong to the second group of structures.

We now take advantage of having a representative in each group which is a family of lattice paths confined to the region $\mathfrak{C}$ to provide a bijective link between the two groups, with Theorem 33 below. Once this theorem will be proved, all powered Catalan structures involved in our study will be related as shown in Figure 15.

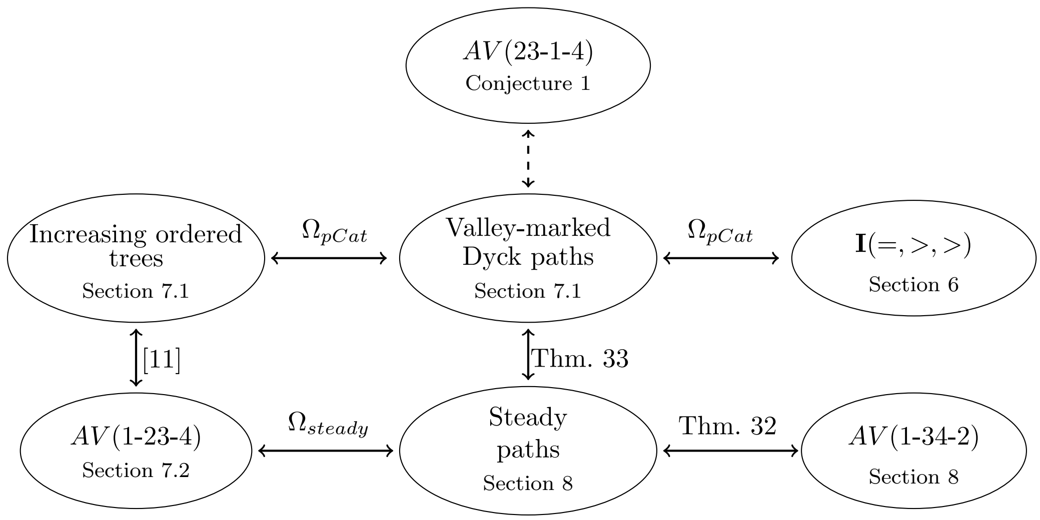


Figure 15: All the structures known or conjectured to be enumerated by the powered Catalan numbers and their relations: a solid-line arrow indicates a bijection (either recursive, or direct), while a dashed-line arrow indicates a conjectured bijection.

Theorem 33. *There is a size-preserving bijection between steady paths and marked-valley Dyck paths, which sends the number of W steps to total mark, preserves the number of steps on the main diagonal, and sends the number of returns to the x-axis to the number of returns to the mark.*

This bijection, hereafter denoted ϕ^*, is explicit and is obtained as follows. Starting from a valley-marked Dyck path, we apply to it a certain transformation ϕ that decreases by one the total mark while introducing one W step. We repeat this operation until all marks have been decreased to 0. To show that ϕ^* is a bijection, we actually describe its inverse. It is denoted θ^* and is also obtained iterating a certain transformation θ (which is the inverse of ϕ).

For this strategy to work, we need to define our transformations on a family of paths having W steps and marks on their valleys. The specific family we consider is the following one.

Definition 34. *A* valley-marked steady path *is a steady path P where each valley receives a mark, according to the following conditions:*

(M1) if (i, k) pinpoints any valley of P, then a valley-marked steady path associated with P must take a mark in a point (i, j), where $0 \le j \le k$;

(M2) all valleys with nontrivial marks are (weakly) below all W steps;

(M3) a valley with a nontrivial mark may be at the same height as a W step only if it appears to the right of the W step.

The definition of total mark and return to the mark are extended to valley-marked steady paths in the obvious way.

Clearly, valley-marked steady paths without W steps are valley-marked Dyck paths, and valley-marked steady paths where all the marks of the valleys are trivial are (in obvious correspondence with) steady paths. Moreover, any of these three families of paths includes as a subfamily the classical Dyck paths (or a family of paths in obvious correspondence with them). More precisely, the set of Dyck paths is the intersection of the families of steady paths and valley-marked Dyck paths (where trivial marks are interpreted as inexistant).

9.1 The transformation ϕ decreasing the number of W steps

Consider a valley-marked steady path P, assumed to contain at least one W step. Consider the rightmost among the bottommost W steps. This W being bottommost, it cannot be preceded by a W. It also cannot be preceded by a D, since DW factors are forbidden. Therefore, it is preceded by a U. This U can only be preceded by a D, otherwise breaking one of the conditions (S1) and (S2) defining steady paths. We have therefore identified that our W is preceded by a valley (drawn in red on the figures). We denote by k the height of this valley, and by $h \le k$ its mark.

We decompose our valley-marked steady path P around this factor DUW. The D step has a matching step to its left, which has to be a U step, since W has been chosen bottommost. The U and the W have matching D steps to their right. Our path P is therefore decomposed as

$$Pr \cdot U \cdot A \cdot DUW \cdot B \cdot D \cdot C \cdot D \cdot S, \text{ see Figure 16(a),}$$

where Pr (resp. S) is a prefix (resp. suffix) of the path, and A, B and C are factors of this path never going below their starting ordinate. Additionally, B must be non-empty (since WD factors are forbidden). Moreover, all valleys in A or B have a trivial mark, by conditions (M2) and (M3). And similarly, the only valleys in C with nontrivial marks (if any) are at the "ground level" of C (*i.e.*, at height $k+1$).

We define the image by ϕ of this path depending on whether A is empty or not.

- If $A = \emptyset$, its image is

  $$Pr \cdot U \cdot B \cdot UD \cdot C \cdot D \cdot S, \text{ see Figure 16(b).}$$

  Note that in this case, a valley between B and the successive U has been created, replacing the red valley.

- If $A \neq \emptyset$, its image is

  $$Pr \cdot U \cdot A \cdot U \cdot B \cdot D \cdot C \cdot D \cdot S, \text{ see Figure 16(c).}$$

  Note that in this case, a valley between A and the successive U has been created, replacing the red valley.

Performing this transformation, the red valley has been moved to a new valley at height $k+1$, whose mark is set to $h+1$. Every other valley (even if it is moved by ϕ) keeps its mark unchanged.

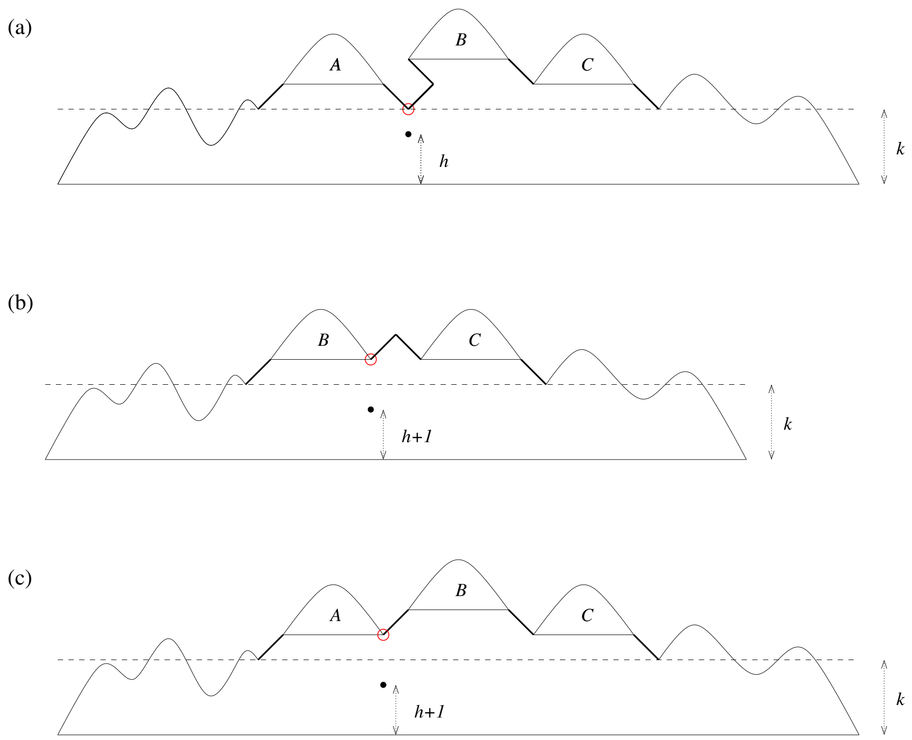


Figure 16: (a) A valley-marked steady path assumed to contain at least one W step; (b) its image by ϕ in the case A is empty; (c) its image by ϕ in the case A is not empty.

Lemma 35. *Let P be a valley-marked steady path with at least one W step, with total mark m, k W steps, r returns to the mark, and d steps on the main diagonal. It holds that*

i) *$\phi(P)$ is a valley-marked steady path.*

ii) *The total mark of $\phi(P)$ is $m+1$.*

iii) *The number of W steps in $\phi(P)$ is $k-1$.*

iv) *$\phi(P)$ has d steps on the main diagonal.*

v) *$\phi(P)$ has r returns to the mark.*

vi) *The valley that has been moved w.r.t. P is the leftmost among the topmost valleys of $\phi(P)$ carrying a nontrivial mark.*

Proof. Items ii) and iii) are clear.

Item v) is also very easy, since the only valley whose height is modified (by +1) has its mark modified accordingly (also by +1).

In the case where A is not empty, the only modification between P and $\phi(P)$ is that the considered DUW factor is replaced by U. Neither of these two U steps may be on the main diagonal (since A is not empty), so P and $\phi(P)$ have the same number of steps on the main diagonal. If A is empty, B is moved one step down parallelly to the main diagonal, the rest of the path being unchanged. Moreover, the U step in the considered DUW factor of P may not be on the main diagonal, and neither the U step following B in $\phi(P)$ (since B is not empty). This proves iv).

Items i) and vi) require more care. Consider first i). We need to check that $\phi(P)$ satisfies conditions (S1), (S2), (M1), (M2) and (M3).

The case where A is not empty is easier. Indeed, in this case, the relative positions of Pr, A, B, C and S, as well as the lines supported by the UU and WU factors are unchanged. So, $\phi(P)$ satisfies conditions (S1) and (S2). For (M1), notice that all valleys except the red one are not moved, and their mark are not changed, so we are just left with checking that the new red valley satisfies condition (M1), which is immediate since both the height and the mark of the red valley are increased by one. Because P satisfies condition (M2), we know that any valley of P with a nontrivial mark is at height at most $k+1$. The same stays true in $\phi(P)$ (the red valley having height $k+1$ exactly), ensuring that $\phi(P)$ also satisfies condition (M2). Finally, the rightmost among the bottommost W steps (if any) of $\phi(P)$ either goes from height $k+j+1$ to $k+j$ with $j \geq 2$ or goes from height $k+2$ to $k+1$ and is in the prefix $Pr \cdot U \cdot A$ of $\phi(P)$. In the first case, all valleys with a nontrivial mark being at height at most $k+1$, condition (M3) clearly holds. In the second case, going from P to $\phi(P)$, the rightmost among the bottommost W steps is at the same height but further to the left, and so the fact that (M3) is satisfied by P ensures that (M3) is also satisfied by $\phi(P)$.

We now consider the case where A is empty. Here, the path is modified more substantially. To check that conditions (S1) and (S2) are satisfied, we first note that the U steps supporting a line parallel to $y = x$ not to be crossed are the same in P and $\phi(P)$. Next, we examine how B and C are moved. First, B is moved one cell to the left and one cell down w.r.t. the prefix $Pr \cdot U$ of P, unchanged in $\phi(P)$. This makes sure that conditions (S1) and (S2) are not violated by steps of B. Second, C is moved one cell to the right and one cell up w.r.t. B. Noticing that at least one up step in B supports a line parallel to $y = x$ imposing a condition to the suffix of the path, this makes sure that no step of C (nor of the suffix following C) violates condition (S1) nor (S2). Third, the small peak that has been added between B and C clearly satisfies conditions (S1) and (S2). That (M1) is satisfied is clear, since again the only valley which is changed is the red valley, for which both the height and the mark are increased by one. To see that (M2) and (M3) are satisfied, observe first that the red valley does not violate them, being either above the rightmost among the bottommost W steps of $\phi(P)$, or at the same height but to its right. For all other valleys, it is enough to notice that all valleys in B have trivial marks (since P satisfies (M2)) and that the rightmost among the bottommost W steps of $\phi(P)$ is either higher than that of P or at the same height but to its left.

We now turn to the proof of vi). First, observe that the valley that has been moved w.r.t. P (the "red" valley) has mark $h+1$ so is nontrivial. Note that its height is $k+1$. Consider next a valley with a nontrivial mark in $\phi(P)$, and assume it is not the red valley. It means that it corresponds to a valley with a nontrivial mark in P. Since P satisfies (M2) and (M3), such a valley may either be a height at most k, and hence lower than the red valley, or it may be at height exactly $k+1$ but to the right of the considered W step of P. In the case where A is not empty, it follows immediately that the red valley is the leftmost among the topmost valleys of $\phi(P)$ with a nontrivial mark. If A is empty, we have to use in addition that B contains no valley with a nontrivial mark (which follows from (M2) on P). In both cases, we obtain that $\phi(P)$ satisfies vi). □

9.2 The transformation θ decreasing the total mark

Consider a valley-marked steady path P, whose total mark is assumed to be non-zero. Among the valleys of P having a nontrivial mark, choose the leftmost among the topmost ones. Denote by $k > 0$ the height of this valley, and by $h > 0$ its mark.

Decompose P around this marked valley DU as follows. Let A be the longest factor of P ending with this D step and which stays (weakly) above height k. (Necessarily, A is not empty.) Let Pr be the prefix of P before A. Note that the last step of Pr is a U or W step going from height $k-1$ to height k, but because of condition (M2), it has to be a U. Let B be the factor of P between the U step of the valley we considered and its matching D step. Let C be the longest factor of P following this D which stays (weakly) above height k. Note that C is followed by a D step. Call S the suffix of P after this D step.

It results that P is decomposed as

$$Pr \cdot A \cdot U \cdot B \cdot D \cdot C \cdot D \cdot S, \text{ see Figure 17(a).}$$

We define the image of P by θ depending on whether B is empty or not.

- If $B = \emptyset$, $\theta(P)$ is the path

$$Pr \cdot D \cdot U \cdot W \cdot A \cdot D \cdot C \cdot D \cdot S, \text{ see Figure 17(b).}$$

- If $B \neq \emptyset$, $\theta(P)$ is the path

$$Pr \cdot A \cdot D \cdot U \cdot W \cdot B \cdot D \cdot C \cdot D \cdot S, \text{ see Figure 17(c).}$$

In both cases, the considered valley of P has been replaced in $\theta(P)$ by the one inside the created DUW factor, which is at height $k-1$. We set its mark to be $h-1$. All other valleys keep their marks unchanged.

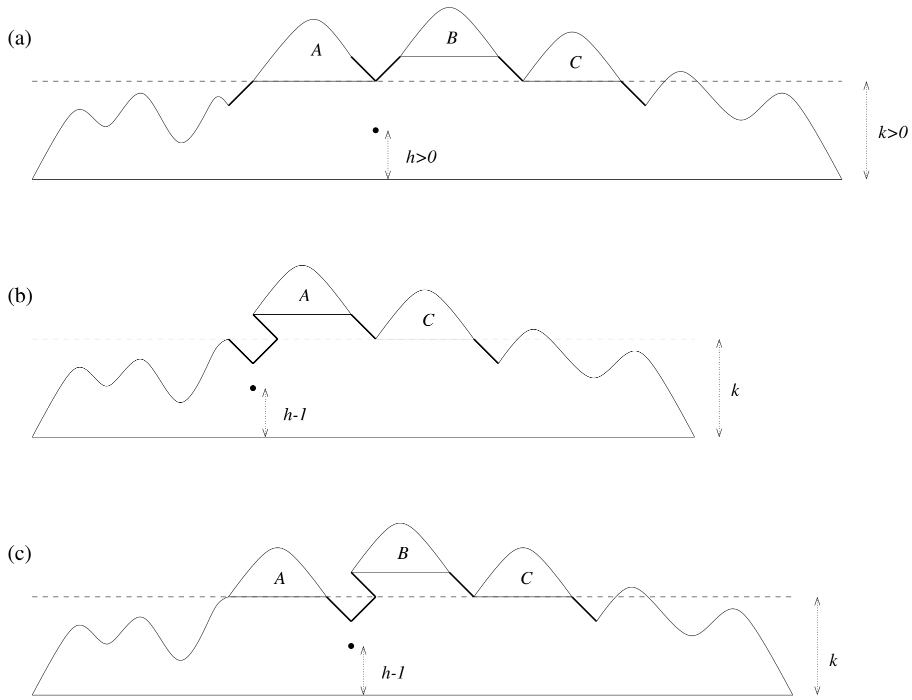


Figure 17: (a) A valley-marked steady path with non-zero total mark; (b) its image by θ in the case B is empty; (c) its image by θ in the case B is not empty.

Lemma 36. *Let P be a valley-marked steady path of total mark at least one, with total mark m, k W steps, r returns to the mark, and d steps on the main diagonal. It holds that*

i) *$\theta(P)$ is a valley-marked steady path.*

ii) *The total mark of $\theta(P)$ is $m-1$.*

iii) *The number of W steps in $\theta(P)$ is $k+1$.*

iv) *$\theta(P)$ has d steps on the main diagonal.*

v) *$\theta(P)$ has r returns to the mark.*

vi) The W step that has been added to P is the rightmost among the bottommost W steps of $\theta(P)$.

Proof. Items ii), iii) and v) are clear. Item iv) is also rather easy. In the case where B is not empty, it follows because the U step of a valley may never lie on the main diagonal (and because all steps of P and $\theta(P)$ are at the exact same place, except around the modified valley). In the case where B is empty, A is moved up parallelly to the main diagonal, and C, Pr and S are not moved, making sure that P and $\theta(P)$ have the same number of steps on the main diagonal.

As in the proof of Lemma 35, the main part of the proof is to show i) and vi).

In the case where B is not empty, conditions (S1) and (S2) are clearly preserved (since the U steps supporting the lines not to be crossed are the same, and Pr, A, B, C and S all stay at the same place). In the case where B is empty, the lines not to be crossed also remain the same. Moreover, Pr, C and S stay at the same place while A is moved, but parallelly to the main diagonal. So, (S1) and (S2) stay satisfied.

Condition (M1) obviously stays satisfied, since θ modifies the valleys only by moving one of them one level down, together with its mark. Since the chosen valley of P is leftmost among the topmost valleys with nontrivial marks, it holds that the valleys of A and B (if any) all have trivial marks, and that the only valleys of C with nontrivial marks (if any) are at "ground level" for C, *i.e.*, at height k. This ensures that $\theta(P)$ satisfies (M2) and (M3) (in both cases $B = \emptyset$ and $B \neq \emptyset$).

We are just left with the proof of vi). Recall that the chosen valley of P is at height k and has a nontrivial mark. By conditions (M2) and (M3), this implies that all W steps of P go from height $k+1$ to k or are higher. Moreover, if P has a W step from height $k+1$ to k, it has to be in Pr or A. This easily ensures vi). □

9.3 Proof of Theorem 33

Let us denote by ϕ^* (resp. θ^*) the transformation that takes a steady path (resp. marked-valley Dyck path) and iteratively applies ϕ (resp. θ) to it as long as the path has some W step (resp. positive total mark). To complete the proof of Theorem 33, we just have to show the following:

Theorem 37. *ϕ^* is a size-preserving bijection between steady paths and marked-valley Dyck paths, whose inverse is θ^*. Moreover, ϕ^* sends the number of W steps to total mark, preserves the number of steps on the main diagonal, and sends the number of returns to the x-axis to the number of returns to the mark.*

Proof. That ϕ^* applied to a steady path produces a marked-valley Dyck path follows immediately from the remark that a marked-valley Dyck path is just a marked-valley steady path with no W step. Similarly, θ^* applied to a marked-valley Dyck path produces a steady path.

To prove that ϕ^* is a bijection, we simply note that θ^* is its inverse. This is an immediate consequence of the fact that ϕ and θ are inverse of each other. This last claim follows by construction and items vi) of Lemmas 35 and 36.

From Lemma 35, ϕ (and hence ϕ^*) preserves the statistics "total mark + number of W steps". This implies that ϕ^* sends the number of W steps to total mark. It follows similarly from Lemma 35 that ϕ^* preserves the number of steps on the main diagonal, and sends the number of returns to the x-axis to the number of returns to the mark. □

A noticeable and nice property of the bijection ϕ^* is that it is the identity on the set of Dyck paths (interpreted either as valley-marked Dyck paths with only trivial marks, or as steady paths with no W step).

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