MATRICES IN THE HOSOYA TRIANGLE

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Abstract. In this paper we use well-known results from linear algebra as tools to explore some properties of products of Fibonacci numbers. Specifically, we explore the behavior of the eigenvalues, eigenvectors, characteristic polynomials, determinants, and the norm of non-symmetric matrices embedded in the Hosoya triangle. We discovered that most of these objects either embed again in the Hosoya triangle or they give rise to Fibonacci identities.

We also study the nature of these matrices when their entries are taken mod 2. As a result, we found an infinite family of non-connected graphs. Each graph in this family has a complete graph with loops attached to each of its vertices as a component and the other components are isolated vertices. The Hosoya triangle allowed us to show the beauty of both, the algebra and geometry.

1. Introduction

The *Hosoya triangle* [11], is a triangular array where the entries are products of Fibonacci numbers. Our main purpose of this paper is to show some connection between this triangle and several aspects of geometry of the Hosoya triangle, graph theory, and the Fibonacci number sequence, all of them by means of linear algebra techniques. Several authors have been geometrically representing the beauty of some elementary concepts of algebra, number theory, and combinatorics bridging them with Fibonacci numbers using the Hosoya triangle [1, 5, 6, 7, 8, 11, 12, 13].

The recurrence relations of Fibonacci numbers provide an interesting way to study properties of matrices with these entries. For example, in 2002 Lee et. al [14] studied matrices that have squares of Fibonacci numbers in the diagonal and the rest of the entries are generalized Fibonacci numbers. In particular, they studied the Cholesky factorizations and the eigenvalues of these matrices. In 2008 Stanimirović et. al. [18] studied the inverses of generalized Fibonacci and Lucas matrices.

In this paper we study the nature of non-symmetric matrices embedded in the Hosoya triangle, (matrices with product of Fibonacci numbers as entries). We use well-known results in linear algebra as tools to explore different patterns within this triangle. We present three infinite families of matrices where the eigenvalues and eigenvectors satisfy a "closure property" in the set of Fibonacci numbers and partially in the Hosoya triangle. We classify those three families in rank one matrices, skew-triangular matrices, and antidiagonal matrices.

The first family (matrices of rank one) have the feature that the matrices are products of two vectors $\mathbf{u}$ and $\mathbf{v}^T$. The entries of the vectors are consecutive Fibonacci numbers —in fact, the vectors $\mathbf{u}$ and $\mathbf{v}$ are located on the sides of the Hosoya triangle. The matrices of this family have exactly one non-zero eigenvalue which is a combination of Lucas and Fibonacci numbers. We would like to emphasize that these matrices are diagonalizable (with exactly one non-zero eigenvalue) where the entries of the eigenvectors are again points of the Hosoya triangle. For instance, the vector $\mathbf{u}$ is one of those eigenvectors.

We connect the first family of matrices with graph theory by observing the fractal generated by the Hosoya triangle mod 2. Therefore, the matrices of our first family give rise to the adjacency matrices of undirected and non-connected graphs. The graphs consist of a complete graph with loops attached to each of its vertices as one component and some isolated vertices as the other components.

Several of the main results in this paper were found by the first author while working on his undergraduate research project under the guidance of the second and third authors.

The second family satisfies that the non-zero eigenvalues are Fibonacci numbers convolved with themselves (see [3, 15] or [17] at A001629). The matrices of the third family have the feature that the product of their eigenvalues (that are not necessarily integers) is a product of Fibonacci numbers.

1.1. **Hosoya triangle.** In this section we give the classic definition of the Hosoya triangle denoted by $\mathcal{H}$.

The *Hosoya sequence* $\{H(r,k)\}_{r,k>0}$ is defined using the double recursion

$$H(r,k) = H(r-1,k) + H(r-2,k) \text{ and } H(r,k) = H(r-1,k-1) + H(r-2,k-2) \tag{1}$$

with initial conditions $H(1,1) = H(2,1) = H(2,2) = H(3,2) = 1$, where $1 \le k \le r$. For brevity, we write $H_{r,k}$ instead of $H(r,k)$ throughout the paper.

This sequence gives rise to the *Hosoya triangle*, where the entry in position k (taken from left to right) of the rth row is equal to $H_{r,k}$ see Table 1 (and also [17] at A058071). For simplicity, in this paper we use $\mathcal{H}$ to denote the Hosoya triangle. One can also refer to [5, 11, 12, 13] for the definition of the Hosoya triangle. Another way to represent each entry (or point) of the Hosoya triangle is $H_{r,k} = F_k F_{r-k+1}$ for positive integers r and k with $1 \le k \le r$ (see [5, 12]). It is easy to see that an nth *diagonal* (either slash or backslash diagonal) in $\mathcal{H}$ is the collection of all Fibonacci numbers multiplied by F_n.

$$\begin{array}{ccccccccccccc}
 & & & & & & H_{1,1} & & & & & & \\
 & & & & & H_{2,1} & & H_{2,2} & & & & & \\
 & & & & H_{3,1} & & H_{3,2} & & H_{3,3} & & & & \\
 & & & H_{4,1} & & H_{4,2} & & H_{4,3} & & H_{4,4} & & & \\
 & & H_{5,1} & & H_{5,2} & & H_{5,3} & & H_{5,4} & & H_{5,5} & & \\
 & H_{6,1} & & H_{6,2} & & H_{6,3} & & H_{6,4} & & H_{6,5} & & H_{6,6} & \\
H_{7,1} & & H_{7,2} & & H_{7,3} & & H_{7,4} & & H_{7,5} & & H_{7,6} & & H_{7,7}
\end{array}$$

TABLE 1. Hosoya triangle $\mathcal{H}$.

2. Non-symmetric matrices of rank one in the Hosoya triangle

In this section we study the first family where every matrix is of rank one (matrices that are products of two vectors). We first define matrices using the backslash diagonals of $\mathcal{H}$ and then present one of the main results of this paper. In this result we give a closed formula for the trace of these matrices. We also present a result on the eigenvectors and the eigenvalues of matrices found in the Hosoya triangle.

For m, n, and t, all positive integers with $m, t \le n$, we define in (2) the *backslash matrix* $B(m,n,t)$ (the *slash matrix* is defined similarly). Let $s = (t-1)$ and $r_i = (m+n-1)+i$ for $i = 0, 1, 2, \dots, s$, then

$$B(m,n,t) = \begin{bmatrix} H_{r_0,m} & H_{r_0-1,m} & H_{r_0-2,m} & \cdots & H_{r_0-s,m} \\ H_{r_1,m+1} & H_{r_1-1,m+1} & H_{r_1-2,m+1} & \cdots & H_{r_1-s,m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{r_s,m+s} & H_{r_s-1,m+s} & H_{r_s-2,m+s} & \cdots & H_{r_s-s,m+s} \end{bmatrix}. \tag{2}$$

For example, Figure 1 Part (a) and Part (b) depicts $B(3,7,5)$ and $B(1,7,7)$, respectively. Note that the first entry of $B(m,n,t)$ is the point in the intersection of the m-th backslash diagonal and the n-th slash diagonal. In particular, the entry (point) in position $(1,1)$ of $B(3,7,5)$ (which is represented by $H_{r_0,m}$) can be determined by writing $r_0 = 9$ and $m = 3$ and using that $H_{r,k} = F_k F_{r-k+1}$. Therefore, $H_{9,3} = F_3 F_7 = 26$. This technique may be used to find all entries of the matrix.

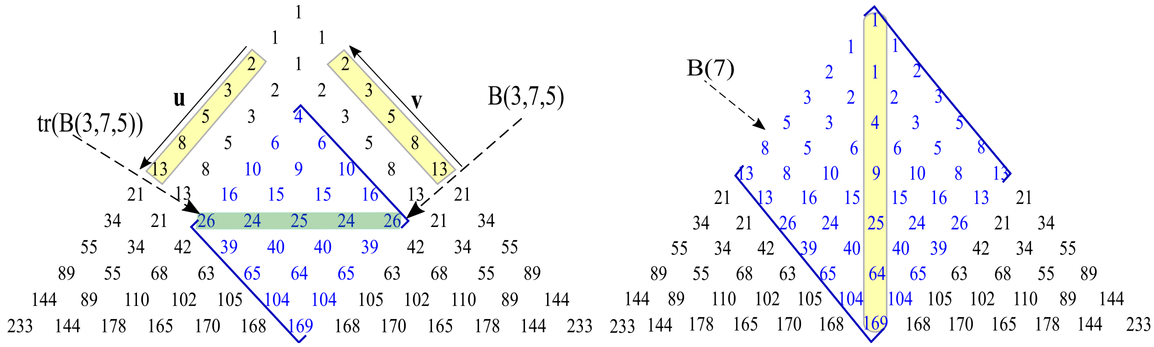


FIGURE 1. (a) Matrix $B(3,7,5)$ in Hosoya triangle. (b) Matrix $B(7)$ in Hosoya triangle.

2.1. **Diagonalizable Matrices in Hosoya triangle.** In this section we describe some properties of diagonalizable matrices in the Hosoya triangle.

We use $\operatorname{tr}(B(m,n,t))$ to denote the trace of $B(m,n,t)$ throughout this section. The formula given in Corollary 2.3 generalizes the convolutions given in [3, 15] and [17, A06733].

Lemma 2.1. *Let $m, n,$ and t be fixed positive integers. If L_k is the kth Lucas number, then the following hold*

(a) $\displaystyle\sum_{i=0}^{t} F_{m+i}F_{n-i} = \left[(t+1)L_{m+n} - \sum_{i=0}^{t}(-1)^{n-i}L_{m-n+2i}\right]/5$ *and*

(b) $\displaystyle\sum_{i=0}^{t-1}(-1)^{n-i-1}L_{m-n+2i} = (-1)^{n-t}F_{m-n+2t-1} + (-1)^{m-1}F_{n-m+1}.$

Proof. We prove Part (a), the proof of Part (b) follows easily using mathematical induction on t and we omit it. Simplifying the expression $F_{m+i}F_{n-i}$ using the Binet formula for Fibonacci numbers [12] we obtain

$$\sum_{i=0}^{t} F_{m+i}F_{n-i} = \left[(t+1)L_{m+n} - \sum_{i=0}^{t}(-1)^{n-i}L_{m-n+2i}\right]/5.$$

This completes the proof of Part (a). □

Proposition 2.2. *If m, n, t are fixed positive integers with $m, t \le n$ and L_k represents Lucas numbers for $k \ge 0$ then*

$$\operatorname{tr}(B(m,n,t)) = \left[tL_{m+n} + (-1)^{n-t}F_{m-n+2t-1} + (-1)^{m-1}F_{n-m+1}\right]/5.$$

Proof. From (2) we have that $\operatorname{tr}(B(m,n,t)) = \sum_{i=0}^{t-1} H_{r_i-i,m+i}$, where $r_i = (m+n-1)+i$ for $i = 0, 1, 2, \ldots, (t-1)$. Since $H_{r,k} = F_kF_{r-k+1}$, we have $\operatorname{tr}(B(m,n,t)) = \sum_{i=0}^{t-1} F_{m+i}F_{n-i}$. The conclusion follows from Lemma 2.1. □

Corollary 2.3. *If $m, n,$ and t are fixed positive integers, then*

$$\sum_{i=0}^{t-1} F_{m+i}F_{n-i} = \left[tL_{m+n} + (-1)^{n-t}F_{m-n+2t-1} + (-1)^{m-1}F_{n-m+1}\right]/5.$$

Let A be an $n \times n$ matrix. Using linear algebra techniques it is easy to verify the following statement.

A is of rank 1 if and only if $A = \mathbf{u} \cdot \mathbf{v}^T$ for some non-zero column vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Moreover, the characteristic equation of A is given by $x^{n-1}(x - \operatorname{tr}(A)) = 0$ and that $\mathbf{u}$ is the eigenvector associated with $\operatorname{tr}(A)$ (where $\operatorname{tr}(A)$ denotes the trace of the matrix A). Notice that the multiplicity of $x = 0$ is $(n-1)$, therefore there are $(n-1)$ eigenvectors associated with $x = 0$. Finally, $\mathbf{u}$ is not orthogonal to $\mathbf{v}$.

Now we establish certain notations needed to prove Proposition 2.4. Let $W = \{u, v_1, v_2, \dots, v_{t-1}\}$ be the set of eigenvectors of $B(m,n,t)$, then v_i is orthogonal to u for all i. Let $E_j(x)$ indicate the elementary matrix obtained by multiplying the j-th row of the $t \times t$ identity matrix by x.

Proposition 2.4. *Let $B' = B(m,n,t)$ be a backslash matrix embedded in the Hosoya triangle $\mathcal{H}$. Then the following hold*

(1) the eigenvalues of B' are $\lambda = 0$ with algebraic multiplicity $(t-1)$ and $\lambda = \operatorname{tr}(B')$ with algebraic multiplicity one.

(2) The matrix B' is diagonalizable and if $s = t-1$ then the eigenvectors of B' are given by,

$$\mathbf{u} = \begin{bmatrix} F_m \\ F_{m+1} \\ F_{m+2} \\ \vdots \\ F_{m+s} \end{bmatrix}, \mathbf{v_1} = \begin{bmatrix} -F_{n-1} \\ F_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{v_s} = \begin{bmatrix} -F_{n-s} \\ 0 \\ 0 \\ \vdots \\ F_n \end{bmatrix}.$$

Proof. By the definition of B' above, $H_{r,k} = F_k F_{r-k+1}$ using (2) we have

$$B' = \begin{bmatrix} F_m F_n & F_m F_{n-1} & F_m F_{n-2} & \cdots & F_m F_{n-s} \\ F_{m+1} F_n & F_{m+1} F_{n-1} & F_{m+1} F_{n-2} & \cdots & F_{m+1} F_{n-s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{m+s} F_n & F_{m+s} F_{n-1} & F_{m+s} F_{n-2} & \cdots & F_{m+s} F_{n-s} \end{bmatrix}.$$

Now it is easy to verify that $B' = \mathbf{u}\mathbf{v}^T$ where $\mathbf{u}$ is as given in the statement of Part (2) and $\mathbf{v}^T = [F_n, F_{n-1}, F_{n-2}, \dots, F_{n-s}]$. This and the application of the linear algebra techniques described above completes the proof of Part (1).

Proof of Part (2). It is clear that $\mathbf{u}$ is an eigenvector associated with $\lambda = \operatorname{tr}(B')$ (see the discussion above). In order to find the eigenvectors associated with $\lambda = 0$, it is enough to find a basis for the null space of B'. Since

$$E_s\left(\frac{1}{F_{m+s}}\right) E_{s-1}\left(\frac{1}{F_{m+s-1}}\right) \cdots E_2\left(\frac{1}{F_{m+1}}\right) E_1\left(\frac{1}{F_m}\right) B' = \begin{bmatrix} F_n & F_{n-1} & F_{n-2} & \cdots & F_{n-s} \\ F_n & F_{n-1} & F_{n-2} & \cdots & F_{n-s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_n & F_{n-1} & F_{n-2} & \cdots & F_{n-s} \end{bmatrix},$$

from this last matrix it is easy to see that $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_s}\}$ is a basis for the null space of B'. This completes the proof of Part (2). □

Notice, in general, if in the definition given in (1) instead of $H_{1,1} = H_{2,1} = H_{2,2} = H_{3,1} = 1$, we consider $H_{1,1} = a^2; H_{2,1} = ab; H_{2,2} = ab; H_{3,2} = b^2$ with $a, b \in \mathbb{Z}$ (see examples in [17], A284115, A284129, A284126, A284130, A284127, A284131, A284128) then the matrix $B(m,n,t)$ defined over this general recursive sequence satisfies the following properties: first $B(m,n,t)$ has rank 1; second, there are vectors $\mathbf{u}$ and $\mathbf{v}$ from the sides of the general triangle such that $B(m,n,t) = \mathbf{u} \cdot \mathbf{v}^T$ and third, that the only non-zero eigenvalue of the matrix is given by the trace $\operatorname{tr}(B(m,n,t))$ which is equal to $\mathbf{u} \cdot \mathbf{v}^T$ (similar to the vectors $\mathbf{u}$ and $\mathbf{v}$ shown in Figure 1(a)).

A particular case of the matrices $B(m,n,t)$ when $m = 1$ and $t = n$ are persymmetric matrices (matrices that are symmetric with respect to the antidiagonal), we denote these matrices simply by $B(n)$ (or for brevity by B). Therefore, persymmetric matrices are square matrices that are symmetric along the skew-diagonal (see Figure 1(b)).

If $\mathbf{u}$, $\mathbf{v_i}$ for $i = 1, \dots, n-1$ are the eigenvectors of B (as given in Proposition 2.4 Part (2)), then the matrix of eigenvectors of B is

$$Q_n = [\mathbf{u}, \mathbf{v_1}, \dots, \mathbf{v_{n-1}}]. \tag{3}$$

If $\mathrm{tr}(B)$ is the non-zero eigenvalue associated to $\mathbf{u}$ (recall that the eigenvalues associated to $\mathbf{v_i}$ for $i = 1, \dots, n-1$ are all zero), then the diagonal matrix of B is $D_n = [d_{ij}]$, where the only non-zero entry is $d_{11} = \mathrm{tr}(B)$.

Corollary 2.5. *If Q_n and $B(n)$ (or B) are as described above, then for $k > 0$ the following holds*

$$B^k Q_n = Q_n \left(\frac{nL_{n+1} + 2F_n}{5} \right)^k.$$

Proof. From Proposition 2.4 Part (2) we know that B is diagonalizable. This and (3) imply that $B = Q_n D_n Q_n^{-1}$. Therefore, $B^k = Q_n D_n^k Q_n^{-1}$. Therefore, $B^k Q_n = Q_n (\mathrm{tr}(B))^k I_n$, where I_n is the $n \times n$ identity matrix. To complete the proof recall from [3, 15] that $\mathrm{tr}(B) = \sum_{i=1}^{n} F_{n-i} F_i = (nL_{n+1} + 2F_n)/5$. □

2.2. **Normal matrices.** Suppose that $A = B^T B$, where B is the persymmetric matrix $B(n)$. Note that A has rank one and therefore it has exactly one non-zero eigenvalue λ. So, the spectral radius of A is $\rho(A) = \lambda$. From linear algebra we know that $||A||_2 = \sqrt{\rho(A^T A)} = \sqrt{\lambda}$ (see [9, 19]). The identity in Proposition 2.6 Part (4) is well-known. However, this tells us that adding the square of every entry of the matrix gives the sum of all points in the antidiagonal which is again a point of the Hosoya triangle. The identity in Proposition 2.6 Part (5) is also a well-known identity, but this tells us that the sum of all entries in the matrix is the difference of two points in the Hosoya triangle. Another interpretation is given by recalling that the matrix norm of A measures how much a vector X can be extended by applying matrix A on it. Applying these concepts to the matrices in the Hosoya triangle, we obtain that the norm is again a point within the triangle. So, this norm provides a good geometric interpretation in the Hosoya triangle of two well-known Fibonacci identities. One can refer back to Figures 1 (b) and 2 (b) to explore the geometric significance of the value of those norms. The proof of Proposition 2.6 Part (2) gives the geometry of the identity given in Part 4 (it is easy to see that A is a normal). Proposition 2.6 Part 6 shows that the singular value of B (see [9]) is again an entry of the Hosoya triangle.

Note that if $C = B \circ B$ is the Hadamard product [10] of the matrix B with itself, then the only non-zero eigenvalue of C is $\lambda = \mathrm{tr}(C)$.

Proposition 2.6. *If $A = B^T B$, then*

(1) A has exactly one non-zero eigenvalue λ.

(2) If λ is the non-zero eigenvalue of A, then λ is the product of the sum of antidiagonal elements of B with the sum of antidiagonal elements of B^T. Thus, $\lambda = \left(\sum_{i=1}^{n} H_{2i+1,i}\right)^2 = (F_n F_{n+1})^2$.

(3) The eigenvalue $\lambda = \mathrm{tr}(A) = \sum_{i,j=1}^{n} (F_i F_j)^2 = (F_n F_{n+1})^2$.

(4) If λ is the non-zero eigenvalue of A, then

$$\sqrt{\lambda} = ||B||_2 = F_n F_{n+1} = \sqrt{\sum_{i,j=1}^{n} (F_i F_j)^2} = \sum_{i=1}^{n} F_i^2.$$

(5) $\sqrt{\sum_{i,j=1}^{n} F_i F_j} = \sum_{i=1}^{n} F_i = F_{n+2} - 1 = ||B||_\infty / F_n$.

(6) $\sqrt{\lambda} = H_{2n,n}$.

Proof. The proofs of Parts (1), (3), and (4)–(6), are straightforward using linear algebra techniques mentioned in the above comments, therefore we omit those details. Since A is a normal matrix the proof of Part (4) follows from the Schur inequality [19] (an alternative proof can be found using parts (2) and (3) or basic algebra).

Proof of Part (2). We know that $A = B^T B = (\mathbf{u}\cdot\mathbf{v}^T)^T(\mathbf{u}\cdot\mathbf{v}^T) = \mathbf{v}\cdot(\mathbf{u}^T\cdot\mathbf{u})\cdot\mathbf{v}^T$. Since, $(\mathbf{u}^T\cdot\mathbf{u})$ is a real number we have $A = (\mathbf{u}^T\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{v}^T$. Therefore, the eigenvalues of A are actually the eigenvalues of $\mathbf{v}\cdot\mathbf{v}^T$ multiplied by $\mathbf{u}^T\cdot\mathbf{u}$. We know that the non-zero eigenvalue of $\mathbf{v}\cdot\mathbf{v}^T$ is given by $\mathrm{tr}(\mathbf{v}\cdot\mathbf{v}^T)$.

Since

$$\mathbf{u}^T\cdot\mathbf{u} = [F_n\, F_{n-1}\, \cdots\, F_2\, F_1]\begin{bmatrix} F_n \\ F_{n-1} \\ F_{n-2} \\ \vdots \\ F_1 \end{bmatrix} = \sum_{i=1}^{n} F_i^2 = F_n F_{n+1} \text{ and } \mathrm{tr}(\mathbf{v}\cdot\mathbf{v}^T) = \sum_{i=1}^{n} F_i^2 = F_n F_{n+1},$$

we have that the eigenvalue λ of A is $(F_n F_{n+1})^2$. This completes the proof of part (2). □

2.3. **Graphs in the Hosoya triangle.** Several authors have been interested in graphs generated by considering the Pascal triangle entries mod 2. The first example is the well-known Sierpiński triangle. Other examples can be found in Koshy [13, Chapter 9]. Here he discussed Pascal graphs in the Pascal Binary Triangle (see also [4, 2, 16]). We use a similar procedure for a family of matrices in $\mathcal{H}$.

We consider the adjacency matrix constructed by taking each entry of the persymmetric matrix $B(n)$ modulo 2 where $n \equiv 2 \bmod 3$. This gives rise to a family of adjacency matrices of undirected and non-connected graphs. The graphs are composed of a complete graph with loops attached to each of its vertices as a component and the other components are some isolated vertices (see Table 2).

Proposition 2.7. *If $k \geq 0$ and $n = 3k+2$, then the graph of the adjacency matrix corresponding to $B(n) \bmod 2$ is a complete graph on $2(k+1)$ vertices with loops at every vertex and k isolated vertices.*

Proof. It is known that the Fibonacci number $F_n \equiv 0 \bmod 2 \iff 3 \mid n$. This and the definition of $B(n)$ imply that every third row and every third column of $B(n)$ are formed by even numbers and that the remaining rows and columns are formed by odd numbers only. Thus, if b_{ij} is an entry of $B(n)$, then $b_{ij} \equiv 0 \bmod 2 \iff i \equiv 0 \mod 3$ or $j \equiv 0 \mod 3$. This and $n = 3k+2$ imply that $B(n) \bmod 2$ contains k columns and k rows with zeros as entries. The remaining 2(k+1) rows and columns have ones as entries. These two features of $B(n) \bmod 2$ give us a complete graph on $2(k+1)$ vertices with loops at every vertex and k isolated vertices. This completes the proof. □

3. Antidiagonal matrices and skew-triangular matrices in the Hosoya triangle

In this section we study a family of antidiagonal matrices where the entries of the antidiagonal are points from the "median" of $\mathcal{H}$ (see Figure 2 (a)). Let A be a $n \times n$ in this family. We prove that the eigenvalues of A are again entries of $\mathcal{H}$ as well as the entries of its eigenvectors (except maybe by the sign of those entries). The eigenvectors of A form the rows of a new square matrix E where non-zero entries of E are in the diagonal and antidiagonal. The diagonal of E is formed by all points in a horizontal line of $\mathcal{H}$, while the antidiagonal of E is the same antidiagonal of A seen in Figure 2 (b). Note that every first entry of a row of A is located in the nth backslash diagonal of $\mathcal{H}$, while every first entry of a row of E is located in the first backslash diagonal of $\mathcal{H}$.

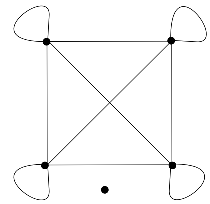
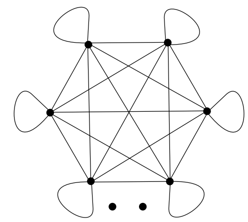
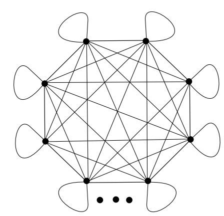
$3k+2$	Matrix	Graph
2	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	
5	$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$	
8	$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$	
11	$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$	

TABLE 2. Adjacency Graphs of $B(n)$ mod 2 in $\mathcal{H}$.

The matrix E can be seen geometrically as a cross in $\mathcal{H}$ where the only non-zero entries of E are the first n entries of the "median" of $\mathcal{H}$ and the entries of the nth row of $\mathcal{H}$. However, some eigenvectors of A have negative entries, but in $\mathcal{H}$ all entries are positive, so our representation is not a perfect geometric representation. Therefore, to have a good geometric representation of the eigenvectors of A we introduce a convention (only for this type of eigenvector). We are going to assume that negative entries of the eigenvector of A are represented by points on the left side of the "median" of $\mathcal{H}$.

3.1. **Eigenvalues of antidiagonal matrices.** First we formally define the matrix A in the following way:

$$A = [a_{ij}]_{1 \le i,j \le n} \quad \text{where} \quad a_{ij} = \begin{cases} F_i^2, & \text{if } j = n - i + 1; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

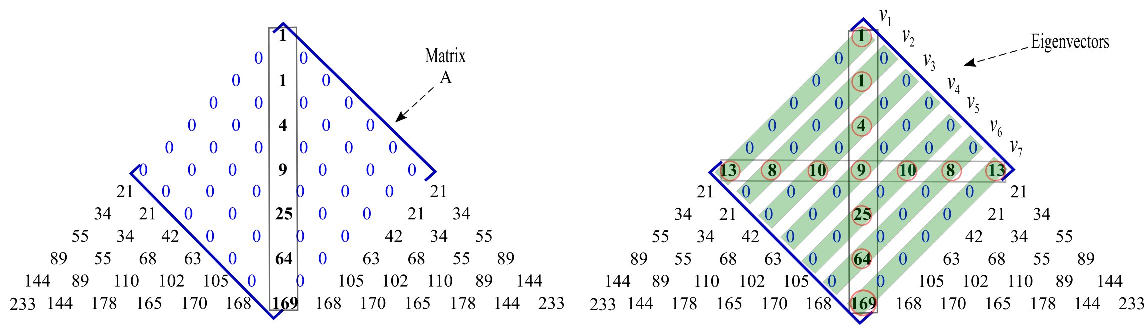


FIGURE 2. (a) Anti-diagonal Matrix A. (b) Eigenvectors of A.

Proposition 3.1. *If A is a matrix as defined in (4), then the following hold*

(1) the rank of A equals n.

(2) If $n = 2k$ and $1 \leq i \leq k$, then the eigenvalues of A are

$$\lambda_j = \begin{cases} F_i F_{n-i+1}, & \text{if } j = i; \\ -F_i F_{n-i+1}, & \text{if } j = k + i. \end{cases}$$

(3) If $n = 2k - 1$, then the eigenvalues of A are $\lambda_k = F_k^2$ and for $1 \leq i < k$

$$\lambda_j = \begin{cases} F_i F_{n-i+1}, & \text{if } j = i; \\ -F_i F_{n-i+1}, & \text{if } j = k + i. \end{cases}$$

(4) If $n = 2k$, then the eigenvectors of A are $x_i = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ for $1 \leq i \leq k$, where

$$\alpha_t = \begin{cases} \pm F_j, & \text{if } t = j \text{ and } j = i; \\ \pm F_{n-j+1}, & \text{if } t = n - j + 1 \text{ and } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

(5) If $n = 2k - 1$, then the eigenvectors of A are $y_i = [\beta_1, \beta_2, \ldots, \beta_n]$ for $1 \leq i \leq k - 1$ and $y_k = [0, 0, \ldots, 1, 0 \ldots, 0]$, where

$$\beta_t = \begin{cases} \pm F_j, & \text{if } t = j \text{ and } j = i; \\ \pm F_{n-j}, & \text{if } t = n - j \text{ and } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Proof. The proof of Part (1) is straightforward since the rank of the column space of A is n. We now prove Part (2), the proof of Part (3) is similar and it is omitted. Observe that if $n = 2k$ and I_n is the identity matrix, then $\pm F_i F_{n-i+1}$ for $1 \leq i \leq k$ are the solutions of the characteristic equation $\det(A - \lambda I_n) = 0$. Thus, $\lambda_j = F_i F_{n-i+1}$ if $j = i$ and $\lambda_j = -F_i F_{n-i+1}$ if $j = k + i$.

We now prove Part (4), the proof of Part (5) is similar and it is omitted. The eigenvector corresponding to the eigenvalue $\lambda = F_i F_{n-i+1}$ for $1 \leq i \leq k$ can be found by solving the system of equations $(A - F_i F_{n-i+1} I_n)\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ and $\mathbf{0}$ is the zero vector.

Using row operations on $(A - F_i F_{n-i+1} I_n)$, the system of equations given above simplifies to

$$x_i - (F_i/F_{n-i+1})x_{n-i+1} = 0 \text{ and } x_j = 0 \text{ when } i \neq j.$$

Therefore, the eigenvector corresponding to the eigenvalue F_iF_{n-i+1} is given by $[\alpha_1, \alpha_2, \ldots, \alpha_n]$, where

$$\alpha_t = \begin{cases} F_j, & \text{if } t = j \text{ and } i = j; \\ F_{n-j+1}, & \text{if } t = n - j + 1 \text{ and } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

The analysis for the eigenvector associated to $\lambda = -F_iF_{n-i+1}$ is similar. This completes the proof. $\square$

Example. If we have the matrix A seen in Figure 2 (a), then the eigenvalues of A are $\pm F_1F_7$, $\pm F_2F_6$, $\pm F_3F_5$, and F_4^2. The eigenvectors of A are given by [0,0, 0,1,0, 0, 0] and

$$\begin{cases} [F_1^2, 0, 0, 0, 0, 0, F_7F_1] \\ [F_1^2, 0, 0, 0, 0, 0, -F_7F_1] \end{cases}, \begin{cases} [0, F_2^2, 0, 0, 0, F_6F_2, 0] \\ [0, F_2^2, 0, 0, 0, -F_6F_2, 0], \end{cases} \begin{cases} [0, 0, F_3^2, 0, F_5F_3, 0, 0] \\ [0, 0, F_3^2, 0, -F_5F_3, 0, 0]. \end{cases}$$

Comment: The matrix A is diagonalizable. In fact, when $n = 2k$, we have

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & F_1^2 \\ 0 & 0 & \cdots & F_2^2 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ F_n^2 & 0 & \cdots & 0 & 0 \end{bmatrix} P = P \begin{bmatrix} F_1F_{2k} & 0 & 0 & \cdots & 0 & 0 \\ 0 & -F_1F_{2k} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & F_kF_{k+1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & -F_kF_{k+1} \end{bmatrix}$$

where

$$P = \begin{bmatrix} F_1 & F_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & F_2 & F_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & F_k \\ 0 & 0 & 0 & 0 & \cdots & F_{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & F_{2k-1} & -F_{2k-1} & \cdots & 0 \\ F_{2k} & -F_{2k} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Similarly, it is easy to verify that A is diagonalizable when $n = 2k - 1$.

Another interesting property of A is that if

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & F_1^2 \\ 0 & 0 & \cdots & F_2^2 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ F_n^2 & 0 & \cdots & 0 & 0 \end{bmatrix}, \text{ then } A^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{1}{F_n^2} \\ 0 & 0 & \cdots & \frac{1}{F_{n-1}^2} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ \frac{1}{F_1^2} & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The properties we describe for the antidiagonal matrix A are true in general. Thus, if $A' = [a_{i,j}]_{1 \leq i,j \leq n}$ is an antidiagonal matrix with the entries $a_{i,j} \in \mathbb{R}$ —such that the following roots make sense in the set of real numbers— then the eigenvalues of A' are

$$\lambda_i = \begin{cases} \pm\sqrt{a_{i,n}a_{n-i+1,i}}, & \text{if } n = 2k; \\ \pm\sqrt{a_{i,n}a_{n-i,i}}, & \text{if } n = 2k - 1. \end{cases}$$

The eigenvectors have the same behavior as in the previous case. For example, if $n = 2k$, then the eigenvectors of A are $x_i = [\alpha_1, \alpha_2, \ldots, \alpha_{2k}]$, where $\alpha_j = \pm a_{i,n+1-j}$ and $\alpha_{n+1-j} = \pm a_{n+1-j,i}$ if $j = i$ and $\alpha_j = 0$ if $i \neq j$ for $1 \leq i, j \leq k$.

Note that if A is an antidiagonal matrix (see definition (4)) then the characteristic polynomial of A is given by

$$p(x) = \begin{cases} \displaystyle\prod_{i=1}^{n} \left(x^2 - F_i^2 F_{n-i+1}^2\right), & \text{when } n \text{ is even };\\ \displaystyle\prod_{i=1}^{(n-1)/2} \left(x^2 - F_i^2 F_{n-i+1}^2\right)\left(x - F_{(n+1)/2}^2\right), & \text{when } n \text{ is odd }. \end{cases}$$

In fact, using Vieta's formula for polynomials we can say that the characteristic polynomial for A is a polynomial of degree n where the coefficient of x^{n-1} is $\operatorname{tr}(A)$, the coefficient of x^0 is $\det(A)$ (determinant of A), and the coefficient of x^{n-k} is given by

$$(-1)^k \sum_{1\le i_1\le \ldots \le i_k \le n} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k} \text{ where } \lambda_{i_j} \text{ represent the eigenvalues of } A.$$

3.2. **Determinants of skew-triangular matrices in the Hosoya triangle.** In this section we define skew-triangular matrices in $\mathcal{H}$. This family of matrices does not necessarily have integers as eigenvalues. So, we analyze their determinants to obtain some information about their eigenvalues. We also discuss some properties of the determinants (see the determinant in (6) on page 10). The determinant of an antidiagonal matrix is well known in linear algebra. Here we use this tool to show that the determinant of a member of the subfamily of matrices with entries in $\mathcal{H}$ is a product of points of this triangle. The geometry of the triangle helps us see these properties very clearly. We now define the family $T(n,k)$ of skew-triangular matrices $\mathcal{H}$. If $k = 2, 3, \ldots, n+1$, then

$$T(n,k) = [a_{ij}]_{1\le i,j\le n} \quad \text{where} \quad a_{ij} = \begin{cases} F_i F_{n-j+1}, & \text{if } k \le i+j \le n+1;\\ 0, & \text{otherwise.}\end{cases} \tag{5}$$

From linear algebra we know that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of a matrix A, then the determinant of A is given by, $\det(A) = \prod_{i=1}^{n} \lambda_i$ (see [19]). It is easy to verify that for a fixed n and $i \neq j \in \{2, \ldots, n+1\}$, the matrices $T(n,i)$ and $T(n,j)$ do not necessarily have the same eigenvalues (in most of the cases the eigenvalues are not integers). Proposition 3.2 shows that if n is fixed, the product of the eigenvalues of $T(n,k)$ is equal to the product of eigenvalues of $T(n,n+1)$ for $1 < k \le n$. This is the product of points located in the "median" of $\mathcal{H}$.

Proposition 3.2. *If $2 \le k \le n+1$, then for every $n \ge 2$ this holds*

$$\det(T(n,k)) = \begin{cases} \prod_{i=1}^{n} F_i^2, & \text{if } n \equiv 0 \text{ or } 1 \bmod 4;\\ -\prod_{i=1}^{n} F_i^2, & \text{if } n \equiv 2 \text{ or } 3 \bmod 4.\end{cases}$$

Proof. This is straightforward using cofactors or using Leibnitz's formula of the sum over all permutations of the numbers $1, 2, \ldots, n$. □

Note that $\det(T(n,k)) = \det(A)$ where A is the $n \times n$ antidiagonal matrix defined in Section 3 on page 6. This result can be extended to matrices which have entries $a_{ij} = F_i F_{n-j+1}$ such that $n-k \le i+j \le n+1$ and k is either 1 or 2 or any positive integer less than $n-2$. For example, if $n = 4$ and $\det(A)$ is denoted by $|A|$ then it holds that,

$$\begin{vmatrix} F_1F_4 & F_1F_3 & F_1F_2 & F_1^2\\ F_2F_4 & F_2F_3 & F_2^2 & \\ F_3F_4 & F_3^2 & & \\ F_4^2 & & 0 & \end{vmatrix} = \begin{vmatrix} 0 & F_1F_3 & F_1F_2 & F_1^2\\ F_2F_4 & F_2F_3 & F_2^2 & \\ F_3F_4 & F_3^2 & & \\ F_4^2 & & 0 & \end{vmatrix} = \cdots = \begin{vmatrix} 0 & 0 & 0 & F_1^2\\ 0 & 0 & F_2^2 & \\ 0 & F_3^2 & & \\ F_4^2 & & 0 & \end{vmatrix} = \prod_{i=1}^{4} F_i^2. \tag{6}$$

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