PERMUTATIONS AVOIDING 312 AND ANOTHER PATTERN, CHEBYSHEV POLYNOMIALS AND LONGEST INCREASING SUBSEQUENCES

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ABSTRACT

We study the longest increasing subsequence problem for random permutations from $S_n(312,\tau)$, the set of all permutations of length n avoiding the pattern 312 and another pattern τ, under the uniform probability distribution. We determine the exact and asymptotic formulas for the average length of the longest increasing subsequences for such permutations specifically when the pattern τ is monotone increasing or decreasing, or any pattern of length four.

1. Introduction

The study of longest increasing subsequences for uniformly random permutations is a wonderful example of a research program which begins with an easy-to-state question whose solution makes surprising connections with different branches of mathematics, and culminates with many astonishing results that have interesting applications in statistics, computer science, physics and biology, see [1,10–12]. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation of $[n] := \{1,\cdots,n\}$. We denote by $L_n(\sigma)$ the length of a *longest increasing subsequence* in σ, that is,

$$L_n(\sigma) = \max\{k \in [n] : \text{there exist } 1 \le i_1 < i_2 < \cdots < i_k \le n \text{ and } \sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_k}\}.$$

Note that, in general, such a subsequence is not unique. Erdös-Szekeres theorem [15] states that every permutation of length $n \ge (r-1)(s-1)+1$ contains either an increasing subsequence of length r or a decreasing subsequence of length s. After this celebrated result, many researchers worked on the problem of determining the asymptotic behavior of L_n on S_n, the set of all permutations of length n, under the *uniform* probability distribution [16, 22, 29, 34, 35]. The problem has been studied by several distinct methods from probability theory, random matrix theory, representation theory and statistical mechanics, see [1, 4, 21, 28] and references therein. It is finally known that $\mathbf{E}(L_n) \sim 2\sqrt{n}$ [22, 30, 35] and $n^{-1/6}(L_n - \mathbf{E}(L_n))$ converges in distribution to the Tracy-Widom distribution as $n \to \infty$ [3, 33]. For a thorough exposition of the subject, see the books [4, 28].

We shall study the longest increasing subsequence problem for some pattern-avoiding permutation classes. First, we shall recall some definitions. For permutations $\tau \in S_k$ and $\sigma \in S_n$, we say that τ appears as a *pattern* in σ if there is a subsequence $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ of length k in σ which has the same relative order of τ, that is, $\sigma_{i_s} < \sigma_{i_t}$ if and only $\tau_s < \tau_t$ for all $1 \le s, t \le k$. For example, the permutation 312 appears as a pattern in 52341 because it

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has the subsequences $523--$, $52-4-$ or $5-34-$. If τ does not appear as a pattern in σ, then σ is called a *τ-avoiding* permutation. We denote by $S_n(\tau)$ the set of all permutations of length n that avoids the pattern τ. More generally, for a set T of patterns, we use the notation $S_n(T) = \cap_{\tau\in T} S_n(\tau)$. Pattern-avoiding permutations have been studied from combinatorics perspectives for many years (for example, see [8]). Recently probabilistic study of random pattern-avoiding permutations has also been initiated, and many interesting results have already appeared [5, 17–20, 23, 24, 26].

The longest increasing subsequence problem for the pattern-avoiding permutations was first studied for the patterns of length tree, that is, for $\tau \in S_3$ on $S_n(\tau)$ with uniform probability distribution in [13]. The case $S_n(\tau^1, \tau^2)$ with $\tau^1, \tau^2 \in S_3$ is studied for all possible cases in [24]. One of the corollaries of our main result, Theorem 2.1, covers the case $S_n(312, \tau)$ with either $\tau \in S_3(312)$ or $\tau \in S_4(312)$ and hence add some new results to this research program.

Note that for any $\sigma \in S_n$, we have

$$L_n(\sigma) = L_n(\sigma^{rc}) = L_n(\sigma^{-1}) \tag{1.1}$$

where the *reverse, complement* and *inverse* of σ are defined as $\sigma_i^r = \sigma_{n+1-i}$, $\sigma_i^c = n+1-\sigma_i$ and $\sigma_i^{-1} = j$ if and only if $\sigma_j = i$, respectively. These symmetries significantly reduces the number of cases needed to be studied.

In a different direction of research, the longest increasing subsequence problem has also been studied on S_n under some *non-uniform* probability distributions such as Mallow distribution [6, 7, 27] and in some other context such as colored permutations [9], i.i.d sequences and random walks [2, 14].

The paper is organized as follows. We present our main result, Theorem 2.1, in Section 2 and as a first case apply it to $S_n(312, \tau)$ with $\tau \in S_3(312)$ which gives an alternative proof for some cases considered in [24] through generating functions. In Section 3, we consider three specific longer patterns where τ is monotone increasing/decreasing pattern or the pattern $(m-1)m(m-2)(m-3)\cdots 321$. The last section presents the results for the case $S_n(312, \tau)$ with $\tau \in S_4(312)$.

For the rest of the paper, we only deal with random variables defined on sets $S_n(312, \tau)$ under the uniform probability distribution. That is, for any subset $A \subset S_n(312, \tau)$, $\mathbf{P}^\tau(A) = \frac{|A|}{|S_n(312,\tau)|}$. The notation $\mathbf{E}^\tau(X)$ denotes the expected value of a random variable X on $S_n(312, \tau)$ under $\mathbf{P}^\tau$. We denote the coefficient of x^n in a generating function $G(x)$ by $[x^n]G$. For two sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, we write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

2. General Results

Note that if $\tau \notin S_k(312)$, then $S_n(312, \tau) = S_n(312)$ for all $n \geq 1$. For any $\tau \in S_k(312)$ with $k \geq 2$, we define the generating function

$$F_\tau(x, q) = \sum_{n\geq 0} \sum_{\sigma\in S_n(312,\tau)} x^n q^{L_n(\sigma)} \tag{2.1}$$

with $F_1(x, q) \equiv 1$.

For any sequence $w = w_1 w_2 \cdots w_m$ of m-distinct integers, we define the corresponding *reduced form* to be the unique permutation $v = v_1 v_2 \cdots v_m$ where $v_i = \ell$ if the w_i is the ℓ-th smallest term in w. For example, the reduced form of 253 is 132. For any sequence w, we define $F_w(x,q)$ to be $F_v(x,q)$ where v is the *reduced form* of w.

In order to determine $F_\tau(x,q)$ explicitly, we shall introduce some notations. Let w^1, w^2 be two sequences of integers, we write $w^1 < w^2$ or $w^2 > w^1$ if $w^1_i < w^2_j$ for all possible i, j. Recall that for any permutation $\tau = \tau_1 \cdots \tau_k$, τ_i is called a *right-to-left minimum* if $\tau_i < \tau_j$ for all $j > i$. Let $\tau \in S_k(312)$ and let $m_0 = 1 < m_1 < \ldots < m_r$ be the right-to-left minima of τ written from left to right. Then τ can be represented as

$$\tau = \tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(r)} m_r,$$

where $m_0 < \tau^{(0)} < m_1 < \tau^{(1)} < \cdots < m_r < \tau^{(r)}$, and $\tau^{(j)}$ (may possibly be empty) avoids 312 for each $j = 0, 1, \ldots, r$. In this case we call this representation the *normal form* of τ. For instance, if $\tau = 214365$ then the normal form of τ is $\tau^{(0)} 1 \tau^{(1)} 3 \tau^{(2)}$ with $\tau^{(0)} = 2$, $\tau^{(1)} = 4$ and $\tau^{(2)} = 65$.

Assume that $\tau \in S_k(312)$ is given in its normal form, that is, $\tau = \tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(r)} m_r$. We use $\Theta^{(j)}$ to denote the reduced form of $\tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(j)} m_j$, which we call the j^{th} *prefix* of τ. Similarly, $\Theta^{<j>}$ denotes the reduced form of $\tau^{(j)} m_j \tau^{(j+1)} m_{j+1} \cdots \tau^{(r)} m_r$, which we call the j^{th} *suffix* of τ.

We are ready to state our main result which can be considered a q-analog of the main result of [25].

Theorem 2.1. *Let $\tau \in S_k(312)$ be given in its normal form $\tau^{(0)} m_0 \tau^{(1)} m_1 \cdots \tau^{(r)} m_r$. Then,*

$$\begin{aligned} F_\tau(x,q) &= 1 + xq + x(F_{\tau^{(0)}} - \delta_{\tau^{(0)} \neq \emptyset})\delta_{r=0} + x(F_\tau - 1)\delta_{r>0} \\ &\quad + xq(F_{\Theta^{<1>}}(x,q) - 1)\delta_{\tau^{(0)}=\emptyset} + xq(F_\tau(x,q) - 1)\delta_{\tau^{(0)} \neq \emptyset} \\ &\quad + x\sum_{j=1}^{r} (F_{\Theta^{(j)}}(x,q) - F_{\Theta^{(j-1)}}(x,q))(F_{\Theta^{<j>}}(x,q) - 1) \\ &\quad + x(F_{\tau^{(0)}}(x,q) - \delta_{\tau^{(0)} \neq \emptyset})(F_\tau(x,q) - 1), \end{aligned}$$

where we define $F_\emptyset(x,q) = 0$.

The notation δ_χ denotes 1 if the condition χ holds, and 0 otherwise.

Proof. Note that the contributions of the empty permutation and the permutation of length 1 to the generating function are 1 and xq, respectively. Let $\sigma = \sigma' 1 \sigma'' \in S_n(312)$ with $n \geq 2$. Then $1 < \sigma' < \sigma''$ and each of σ', σ'' avoids 312. If $\sigma = \sigma' 1$ with $\sigma' \neq \emptyset$, then we have the contribution of $x(F_\tau(x,q) - 1)$ if $m_r > 1$, and $x(F_{\tau^{(0)}}(x,q) - \delta_{\tau^{(0)} \neq \emptyset})$ if $m_r = 1$. If $\sigma = 1\sigma''$ with $\sigma'' \neq \emptyset$, then we have a contribution of $xq(F_{\Theta^{<1>}}(x,q) - 1)$ if $\tau^{(0)} = \emptyset$, $xq(F_\tau(x,q) - 1)$ otherwise. As a last case, we need to consider the permutations in the form of $\sigma = \sigma' 1 \sigma''$ with $\sigma', \sigma'' \neq \emptyset$. Observe that σ avoids τ if and only if there exists j, $0 \leq j \leq r$, such that σ' avoids $\Theta^{(j)}$ and contains $\Theta^{(j-1)}$, while σ'' avoids $\Theta^{<j>}$ (see main result of [25] for a similar argument). Thus, the contribution of this case is given by

$$\begin{aligned} &+ x\sum_{j=1}^{r} (F_{\Theta^{(j)}}(x,q) - F_{\Theta^{(j-1)}}(x,q))(F_{\Theta^{<j>}}(x,q) - 1) \\ &+ x(F_{\tau^{(0)}}(x,q) - \delta_{\tau^{(0)} \neq \emptyset})(F_\tau(x,q) - 1), \end{aligned}$$

By summing over all the contributions, we complete the proof. ∎

We can also deduce the rationality of the generating function $F_\tau(x,q)$ for any nonempty pattern τ by using the induction on k with the observations in the proof of Theorem 2.1 and $F_1(x,q)=1$.

Theorem 2.2. *For any $k\geq 1$ and $\tau\in S_k(312)$, the generating function $F_\tau(x,q)$ is a rational function in x and in q.*

Note that $F_1(x,q)=1$ (the only permutation that avoid 1 is the empty permutation). Theorem 2.1 with $\tau=21$ gives

$$F_{21}(x,q)=1+xq+x(F_1(x,q)-1)+xq(F_{21}(x,q)-1)+x(F_1(x,q)-1)(F_{21}(x,q)-1),$$

where $F_1(x,q)=1$. Thus, $F_{21}(x,q)=\frac{1}{1-xq}$.

Theorem 2.1 with $\tau=12$ gives

$$F_{12}(x,q)=1+xq+x(F_{12}(x,q)-1)+xq(F_1(x,q)-1)+x(F_1(x,q)-1)(F_{12}(x,q)-1).$$

Thus, $F_{12}(x,q)=\frac{1+xq-x}{1-x}=1+\frac{xq}{1-x}$.

Corollary 2.3. *For $\tau\in S_2$, the generating functions are given by*

$$F_{21}(x,q)=\frac{1}{1-xq}\ \textit{and}\ F_{12}(x,q)=1+\frac{xq}{1-x}.$$

Now let us focus on patterns of length three, that is, $\tau\in S_3(312)$. In each case we use Theorem 2.1 and Corollary 2.3 with $F_1(x,q)=1$.

Pattern $\tau=$ 123. We have $\Theta^{(0)}=1$, $\Theta^{(1)}=12$, $\Theta^{(2)}=123$, and $\Theta^{<0>}=123$, $\Theta^{<1>}=12$, $\Theta^{<2>}=1$. Thus,

$$F_{123}(x,q)=1+xq+x(F_{123}(x,q)-1)+xq(F_{12}(x,q)-1)+x(F_{12}(x,q)-1)(F_{12}(x,q)-1),$$

which leads to $F_{123}(x,q)=1+xq/(1-x)+\frac{x^2q^2}{(1-x)^3}$.

Pattern $\tau=$ 132. We have $\Theta^{(0)}=1$, $\Theta^{(1)}=132$, and $\Theta^{<0>}=132$, $\Theta^{<1>}=21$. Thus,

$$F_{132}(x,q)=1+xq+x(F_{132}(x,q)-1)+xq(F_{21}(x,q)-1)+x(F_{132}(x,q)-1)(F_{21}(x,q)-1),$$

which gives $F_{132}(x,q)=\frac{1-x}{1-x-xq}$.

Pattern $\tau=$ 213. We have $\Theta^{(0)}=21$, $\Theta^{(1)}=213$, and $\Theta^{<0>}=213$, $\Theta^{<1>}=1$. Thus,

$$F_{213}(x,q)=1+xq+x(F_{132}(x,q)-1)+xq(F_{132}(x,q)-1)+x(F_{132}(x,q)-F_{21}(x,q))(F_1(x,q)-1),$$

which gives $F_{213}(x,q)=\frac{1-x}{1-x-xq}$.

Pattern $\tau=$ 231. We have $\Theta^{(0)}=\Theta^{<0>}=231$. Thus,

$$F_{231}(x,q)=1+xq+x(F_{12}(x,q)-1)+xq(F_{231}(x,q)-1)+x(F_{12}(x,q)-1)(F_{231}(x,q)-1),$$

which gives $F_{231}(x,q)=\frac{1-x}{1-x-xq}$.

Pattern $\tau=$ 321. We have $\Theta^{(0)}=\Theta^{<0>}=321$. Thus,

$$F_{321}(x,q)=1+xq+x(F_{21}(x,q)-1)+xq(F_{321}(x,q)-1)+x(F_{21}(x,q)-1)(F_{321}(x,q)-1),$$

which gives $F_{321}(x,q)=\frac{1-xq}{(1-xq)^2-x^2q}$.

Hence, we can state the following result.

Corollary 2.4. *For $\tau \in S_3(312)$, the generating functions are given by*

$$\begin{aligned}
F_{123}(x,q) &= 1 + \frac{xq}{1-x} + \frac{x^2q^2}{(1-x)^3},\\
F_{132}(x,q) &= F_{213}(x,q) = F_{231}(x,q) = \frac{1-x}{1-x-xq},\\
F_{321}(x,q) &= \frac{1-xq}{(1-xq)^2 - x^2q}.
\end{aligned}$$

The results in Corollary 2.4 indeed extend the relevant results of Simion Schmidt [31] for the permutations avoiding two patterns of length 3. Here we find the generating functions for the number of permutations σ in $S_n(312,\tau)$ with $\tau \in S_3(132)$ according to the length of the longest increasing subsequence in σ.

Our next result considers a specific type of pattern in which the last entry is 1.

Corollary 2.5. *Assume $\tau = \rho 1 \in S_k(312)$. Then $F_\tau(x,q) = \frac{1}{1-xF_\rho(x,1)}$. Moreover,*

$$\frac{\partial}{\partial q} F_\tau(x,q)\Big|_{q=1} = xF_\tau^2(x,1)\left(1 + \frac{\partial}{\partial q} F_\rho(x,q)\Big|_{q=1}\right).$$

Proof. By Theorem 2.1, we have $F_\tau(x,q) = \frac{1}{1-xq-x(F_\rho(x,q)-1)}$ (for case $q = 1$, see [25]). Moreover, Theorem 2.1 gives

$$\frac{\partial}{\partial q} F_\tau(x,q)\Big|_{q=1} = \frac{x\left(1 + \frac{\partial}{\partial q} F_\rho(x,q)\Big|_{q=1}\right)}{(1-xF_\rho(x,1))^2},$$

which completes the proof. ∎

Proposition 2.6. *Assume $\tau \in S_k(312)$. Consider $S_n(312,\tau)$ with uniform probability distribution. Then*

$$\mathbf{E}^\tau(L_n) = \frac{1}{s_n}[x^n]\frac{\partial}{\partial q} F_\tau(x,q)\Big|_{q=1} \text{ and } \mathbf{E}^\tau(L_n^2) = \frac{1}{s_n}[x^n]\left(\frac{\partial^2}{\partial q^2} F_\tau(x,q)\Big|_{q=1} + \frac{\partial}{\partial q} F_\tau(x,q)\Big|_{q=1}\right)$$

where $s_n = |S_n(312,\tau)|$.

Note that $s_n = [x^n]F_\tau(x,1)$. By Corollary 2.4 and Proposition 2.6, we recover some of the relevant results in [24].

Corollary 2.7. *For all $n \geq 1$, we have*

$$\begin{aligned}
&\mathbf{E}^{123}(L_n) = \frac{2(n^2-n+1)}{n^2-n+2}, && \mathbf{E}^{123}(L_n^2) = \frac{2(2n^2-2n+1)}{n^2-n+2},\\
&\mathbf{E}^{132}(L_n) = \mathbf{E}^{213}(L_n) = \mathbf{E}^{231}(L_n) = \frac{n+1}{2}, && \mathbf{E}^{132}(L_n^2) = \mathbf{E}^{213}(L_n^2) = \mathbf{E}^{231}(L_n^2) = \frac{n(n+3)}{4},\\
&\mathbf{E}^{321}(L_n) = \frac{3n}{4}, && \mathbf{E}^{321}(L_n^2) = \frac{n(9n+1)}{16}.
\end{aligned}$$

3. Special cases of longer patterns

Our main result, Theorem 2.1, can be used to obtain general results for several longer patterns. In this subsection, as an example, we apply it to the following three specific patterns $12\cdots m$, $m(m-1)\cdots 21$ and $(m-1)m(m-2)\cdots 21$.

Recall that the Chebyshev polynomials of the second kind are defined by $U_j(\cos\theta)=\frac{\sin((j+1)\theta)}{\sin\theta}$. It is well known that these polynomials satisfy

$$U_0(t)=1, U_1(t)=2t, \text{ and } U_m(t)=2tU_{m-1}(t)-U_{m-2}(t) \text{ for all integers } m, \tag{3.1}$$

and

$$U_n(t)=2^n\prod_{j=1}^{n}\left(t-\cos\left(\frac{j\pi}{n+1}\right)\right). \tag{3.2}$$

3.1. Monotone increasing pattern $\tau=12\cdots m$.

In this subsection, we study the pattern $\tau=12\cdots m$. By Corollaries 2.3 and 2.4, we see that $F_1(x,q)=1$, $F_{12}(x,q)=1+\frac{xq}{1-x}$ and $F_{123}(x,q)=1+\frac{xq}{1-x}+\frac{x^2q^2}{(1-x)^3}$. By Theorem 2.1 with $\tau=12\cdots m$, we have

$$\begin{aligned}F_{12\cdots m}(x,q)&=1+xq+x(F_{12\cdots m}(x,q)-1)+xq(F_{12\cdots(m-1)}(x,q)-1)\\&\quad+x\sum_{j=2}^{m}(F_{12\cdots j}(x,q)-F_{12\cdots(j-1)}(x,q))(F_{12\cdots(m-j+1)}(x,q)-1).\end{aligned}$$

Define $G(x,q,v)=\sum_{m\geq1}F_{12\cdots m}(x,q)v^m$. Then, by multiplying the above recurrence by v^m and summing over $m\geq2$, we obtain

$$\frac{v}{1-v}+(1+x-qxv)G(x,q,v)-\frac{x(1-v)}{v}G^2(x,q,v)=0.$$

By solving the above equation for $G(x,q,v)$ we obtain

$$G(x,q,v)=\frac{(1+x-qxv-\sqrt{(1+x-qxv)^2-4x})v}{2x(1-v)}.$$

Let $v'=v(1-x)^2/(qx)$, then

$$\frac{G(x,q,v')\frac{1-v'}{v'}-1}{(1-x)v}-1=\frac{1-v(1-x)-\sqrt{1-2v(1+x)+v^2(1-x)^2}}{2xv},$$

which, by definition of Narayana numbers (see Sequence A001263 in [32]), leads to

$$\frac{G(x,q,v')\frac{1-v'}{v'}-1}{(1-x)v}-1=\sum_{n\geq1}\sum_{k=1}^{n}\frac{1}{n}\binom{n}{k}\binom{n}{k-1}(v')^nx^{k-1}.$$

Therefore,

$$\frac{G(x,q,v)\frac{1-v}{v}-1}{qxv/(1-x)}=1+\sum_{n\geq1}\sum_{k=1}^{n}\frac{1}{n}\binom{n}{k}\binom{n}{k-1}\frac{q^nx^{n+k-1}v^n}{(1-x)^{2n}},$$

which implies

$$G(x,q,v)=\frac{v}{1-v}\left(1+\frac{qxv}{1-x}+\sum_{n\geq1}\sum_{k=1}^{n}\frac{1}{n}\binom{n}{k}\binom{n}{k-1}\frac{q^{n+1}x^{n+k}v^{n+1}}{(1-x)^{2n+1}}\right).$$

By finding the coefficient of v^m, we obtain the following result.

Theorem 3.1. *For all* $m \geq 2$,

$$F_{12\cdots m}(x,q) = 1 + \frac{qx}{1-x} + \sum_{j=2}^{m-1}\left(\frac{q^j x^j}{(1-x)^{2j-1}}\sum_{k=1}^{j-1}\frac{1}{j-1}\binom{j-1}{k}\binom{j-1}{k-1}x^{k-1}\right).$$

3.2. **Monotone decreasing pattern $\tau = m(m-1)\cdots 21$.** In this subsection, we study the pattern $\mathbf{m} = m(m-1)\cdots 21$.

Corollary 3.2. *Let* $\mathbf{m} = m(m-1)\cdots 21$. *Then*

$$F_{\mathbf{m}}(x,q) = \frac{U_{m-2}(t) - \sqrt{x}U_{m-3}(t)}{\sqrt{x}(U_{m-1}(t) - \sqrt{x}U_{m-2}(t))},$$

where $t = \frac{1+x-xq}{2\sqrt{x}}$.

Proof. The proof is given by induction on m. Clearly, $F_1(x,q) = 1$ and $F_{21}(x,q) = \frac{1}{1-xq}$, so the claim holds for $m = 1, 2$. Assume that the claim holds for $1, 2, \cdots, m$ and let's us prove it for $m+1$. Since $\mathbf{m+1} = (\mathbf{m}+1)1$, then Theorem 2.1 gives $F_{\mathbf{m+1}}(x,q) = \frac{1}{1-xq-x(F_{\mathbf{m}}(x,q)-1)}$. Thus by induction assumption, we obtain

$$\begin{aligned}
F_{\mathbf{m+1}}(x,q) &= \frac{1}{1-xq-x(F_{\mathbf{m}}(x,q)-1)} \\
&= \frac{\sqrt{x}(U_{m-1}(t) - \sqrt{x}U_{m-2}(t))}{x(2tU_{m-1}(t) - U_{m-2}(t)) - x\sqrt{x}(2tU_{m-2}(t) - U_{m-3}(t))} \\
&= \frac{\sqrt{x}(U_{m-1}(t) - \sqrt{x}U_{m-2}(t))}{x(U_m(t) - \sqrt{x}U_{m-1}(t))} \\
&= \frac{U_{m-1}(t) - \sqrt{x}U_{m-2}(t)}{\sqrt{x}(U_m(t) - \sqrt{x}U_{m-1}(t))}
\end{aligned}$$

where we used the fact (3.1) and $2t\sqrt{x} = 1 + x - xq$. ∎

By Corollary 3.2 with $q = 1$ and (3.1), we have $F_{\mathbf{m}}(x,1) = \frac{U_{m-1}(\frac{1}{2\sqrt{x}})}{\sqrt{x}U_m(\frac{1}{2\sqrt{x}})}$, as shown in [25]. Moreover, by Corollary 2.5, we have

$$\frac{\partial}{\partial q}F_{\mathbf{m}}(x,q)\Big|_{q=1} = xF^2_{\mathbf{m}}(x,1)\left(1 + \frac{\partial}{\partial q}F_{\mathbf{m-1}}(x,q)\Big|_{q=1}\right)$$

with $\frac{\partial}{\partial q}F_{\mathbf{1}}(x,q)\Big|_{q=1} = 0$. Thus, by induction on m, we can state the following result.

Corollary 3.3. *Let* $\mathbf{m} = m(m-1)\cdots 21$. *Then*

$$\frac{\partial}{\partial q}F_{\mathbf{m}}(x,q)\Big|_{q=1} = \frac{1}{U^2_m(\frac{1}{2\sqrt{x}})}\sum_{j=1}^{m-1}U^2_j(\frac{1}{2\sqrt{x}}).$$

Since the smallest pole of $1/U_n(x)$ is $\cos\left(\frac{\pi}{n+1}\right)$, it follows, by Corollary 3.3, that the coefficient of x^n in the generating function $\frac{\partial}{\partial q}F_{\mathbf{m}}(x,q)\Big|_{q=1}$ is given by,

$$[x^n]\frac{\partial}{\partial q}F_{\mathbf{m}}(x,q)\Big|_{q=1} \sim \alpha_m n\left(4\cos^2\left(\frac{\pi}{m+1}\right)\right)^n \text{ as } n\to\infty.$$

Let $v_0 = \frac{1}{4\cos^2\left(\frac{\pi}{m+1}\right)}$. The constant α_m can be computed explicitly as

$$\begin{aligned}\alpha_m &= \lim_{x\to v_0}\frac{1-4x\cos^2\left(\frac{\pi}{m+1}\right)}{U_m^2(\frac{1}{2\sqrt{x}})}\sum_{j=1}^{m-1}U_j^2(\frac{1}{2\sqrt{x}})\\ &= \frac{\sum_{j=1}^{m-1}U_j^2\left(\cos\left(\frac{\pi}{m+1}\right)\right)}{\left(4\cos^2\left(\frac{\pi}{m+1}\right)\right)^m\prod_{j=2}^{m-1}\left(1-\frac{\cos\left(\frac{j\pi}{m+1}\right)}{\cos\left(\frac{\pi}{m+1}\right)}\right)^2}\\ &= \frac{\sum_{j=1}^{m-1}U_j^2\left(\cos\left(\frac{\pi}{m+1}\right)\right)}{4^m\cos^4\left(\frac{\pi}{m+1}\right)\prod_{j=2}^{m-1}\left(\cos\left(\frac{\pi}{m+1}\right)-\cos\left(\frac{j\pi}{m+1}\right)\right)^2}.\end{aligned}$$

Moreover, the coefficient of x^n in the generating function $F_{\mathbf{m}}(x,1) = \frac{U_{m-1}(\frac{1}{2\sqrt{x}})}{\sqrt{x}U_m(\frac{1}{2\sqrt{x}})}$ is given by

$$[x^n]F_{\mathbf{m}}(x,1) \sim \tilde{\alpha}_m\left(4\cos^2\left(\frac{\pi}{m+1}\right)\right)^n \text{ as } n\to\infty,$$

where

$$\begin{aligned}\tilde{\alpha}_m &= \lim_{x\to v_0}\frac{\left(1-4x\cos^2\left(\frac{\pi}{m+1}\right)\right)U_{m-1}(\frac{1}{2\sqrt{x}})}{\sqrt{x}U_m(\frac{1}{2\sqrt{x}})}\\ &= \frac{U_{m-1}(\cos(\frac{\pi}{m+1}))}{2^{m-1}\cos^{m-1}(\frac{\pi}{m+1})\prod_{j=2}^{m-1}\left(1-\frac{\cos(\frac{j\pi}{m+1})}{\cos(\frac{\pi}{m+1})}\right)}\\ &= \frac{U_{m-1}(\cos(\frac{\pi}{m+1}))}{2^{m-1}\cos\left(\frac{\pi}{m+1}\right)\prod_{j=2}^{m-1}\left(\cos\left(\frac{\pi}{m+1}\right)-\cos\left(\frac{j\pi}{m+1}\right)\right)}.\end{aligned}$$

Hence, by Proposition 2.6, we have $\mathbf{E}^{\mathbf{m}}(L_n) \sim \frac{\alpha_m}{\tilde{\alpha}_m}n$. By substituting expressions of α_m and $\tilde{\alpha}_m$, it leads to the following result.

Theorem 3.4. *Let $m \geq 1$. When $n\to\infty$, we have*

$$\mathbf{E}^{\mathbf{m}}(L_n) \sim \frac{\sum_{j=1}^{m-1}U_j^2\left(\cos\left(\frac{\pi}{m+1}\right)\right)}{2^{m+1}\cos^3\left(\frac{\pi}{m+1}\right)U_{m-1}(\cos(\frac{\pi}{m+1}))\prod_{j=2}^{m-1}\left(\cos\left(\frac{\pi}{m+1}\right)-\cos\left(\frac{j\pi}{m+1}\right)\right)}n.$$

For example, Theorem 3.4 for $m = 3$ gives that $\mathbf{E}^{321}(L_n) \sim \frac{3n}{4}$ as shown in [24], and for $m = 4$, we have $\mathbf{E}^{4321}(L_n) \sim \left(2-\frac{3}{\sqrt{5}}\right)n$.

3.3. Pattern $\tau = (m-1)m(m-2)(m-3)\cdots 321$.

In this subsection, we study the pattern $\hat{\mathbf{m}} = (m-1)m(m-2)(m-3)\cdots 321$.

Corollary 3.5. *Let* $\hat{\mathbf{m}} = (m-1)m(m-2)(m-3)\cdots 321$ *with* $m \geq 4$. *Then*

$$F_{\hat{\mathbf{m}}}(x,q) = \frac{(1-x)U_{m-3}(t) - \sqrt{x}(1-x+xq)U_{m-4}(t)}{\sqrt{x}(1-x)(U_{m-2}(t) - x(1-x+xq)U_{m-3}(t))},$$

where $t = \frac{1+x-xq}{2\sqrt{x}}$.

By Corollary 3.5 with $q=1$ and (3.1), we have

$$F_{\hat{\mathbf{m}}}(x,1) = \frac{U_{m-1}(\frac{1}{2\sqrt{x}})}{\sqrt{x}U_m(\frac{1}{2\sqrt{x}})}.$$

By induction on m, we can state the following result.

Corollary 3.6. *Let* $\hat{\mathbf{m}} = (m-1)m(m-2)(m-3)\cdots 321$ *with* $m \geq 4$. *Then*

$$\frac{\partial}{\partial q}F_{\hat{\mathbf{m}}}(x,q)\Big|_{q=1} = \frac{1}{U_m^2(\frac{1}{2\sqrt{x}})}\left(U_2(\frac{1}{2\sqrt{x}}) + \sum_{j=2}^{m-1} U_j^2(\frac{1}{2\sqrt{x}})\right).$$

By similar arguments as in the proof of Theorem 3.4, we obtain the following result.

Theorem 3.7. *Let* $m \geq 4$. *As* $n \to \infty$, *we have*

$$\mathbf{E}^{\hat{\mathbf{m}}}(L_n) \sim \frac{U_2\left(\cos\left(\frac{\pi}{m+1}\right)\right) + \sum_{j=1}^{m-1} U_j^2\left(\cos\left(\frac{\pi}{m+1}\right)\right)}{2^{m+1}\cos^3\left(\frac{\pi}{m+1}\right)U_{m-1}(\cos(\frac{\pi}{m+1}))\prod_{j=2}^{m-1}\left(\cos\left(\frac{\pi}{m+1}\right) - \cos\left(\frac{j\pi}{m+1}\right)\right)} n.$$

4. THE CASE $S_n(312, \tau)$ WHERE $\tau \in S_4(312)$

In this section, we present the results for random permutations from $S_n(312, \tau)$ where $\tau \in S_4(312)$. A summary of the results for all $\tau \in S_4(312)$ is given in Table 1. We present the details only for the two patterns, $\tau = 1234$ and $\tau = 1243$. Since the computations for other cases are very similar, the details are omitted.

Example 4.1. *By Theorem 2.1 with* $\tau = 1234$, *we have*

$$\begin{aligned} F_{1234}(x,q) &= 1 + xqF_{123}(x,q) + x(F_{12}(x,q) - F_1(x,q))F_{123}(x,q) \\ &\quad + x(F_{123}(x,q) - F_{12}(x,q))F_{12}(x,q) + x(F_{1234}(x,q) - F_{123}(x,q))F_1(x,q). \end{aligned}$$

By Corollaries 2.3 and 2.4, we have

$$F_{1234}(x,q) = 1 + \frac{xq}{1-x} + \frac{x^2q^2}{(1-x)^3} + \frac{x^3(1+x)q^3}{(1-x)^5},$$

which agrees with Theorem 3.1 with $m = 4$. *Thus, by Proposition 2.6, we have*

$$\mathbf{E}^{1234}(L_n) = \frac{3(n^4 - 4n^3 + 9n^2 - 6n + 4)}{n^4 - 4n^3 + 11n^2 - 8n + 12}$$

and

$$\mathbf{E}^{1234}(L_n^2) = \frac{3(3n^4 - 12n^3 + 23n^2 - 14n + 4)}{n^4 - 4n^3 + 11n^2 - 8n + 12}.$$

τ	$F_\tau(x,q)$	$\mathbf{E}^\tau(L_n)$	Reference
1234	$1+\frac{xq}{1-x}+\frac{x^2q^2}{(1-x)^3}+\frac{x^3(1+x)q^3}{(1-x)^5}$	$\frac{3(n^4-4n^3+9n^2-6n+4)}{n^4-4n^3+11n^2-8n+12}\to 3$	Example 4.1
1243,1324	$1+\frac{xq(qx(2x-1)+(1-x)^2)}{(1-x-qx)^2(1-x)}$	$\frac{2^{n-3}(n^2-n+4)}{(n-1)2^{n-2}+1}\sim\frac{n}{2}$	Example 4.2
2134,2314 1342	$1+\frac{xq(1-x)}{(1-x)^2-qx}$	$\sim\frac{n}{\sqrt{5}}$	Theorem 2.1
2143,3214 2431,3241 3421,1432	$\frac{1-x-qx}{(1-qx)^2-x}$	$\sim\frac{n}{\sqrt{5}}$	Theorem 2.1
2341,4321	$\frac{(1-x)^3}{(1-x)^3-xq(1-x)^2-x^3q^2}$	$\sim\frac{(-5a^2+22a-9)n}{31}$ $a\approx 2.46577\cdots,\ a^3-4a^2+5a-3=0$	Theorem 2.1

Table 1. A summary of the results for $S_n(312,\tau)$ with $\tau\in S_4(312)$.

Example 4.2. *By Theorem 2.1 with* $\tau = 1243$*, we have*

$$\begin{aligned}F_{1243}(x,q) &= 1 + xqF_{132}(x,q) + x(F_{1243}(x,q)-1)\\ &\quad + x(F_{12}(x,q)-1)(F_{132}(x,q)-1) + x(F_{1243}(x,q)-F_{12}(x,q))(F_{21}(x,q)-1),\end{aligned}$$

which, by Corollaries 2.3 and 2.4, leads to

$$F_{1243}(x,q) = 1 + \frac{xq(qx(2x-1)+(1-x)^2)}{(1-x-qx)^2(1-x)}.$$

Thus, by Proposition 2.6, we have

$$\mathbf{E}^{1243}(L_n) = \frac{2^{n-3}(n^2-n+4)}{(n-1)2^{n-2}+1}$$

and

$$\mathbf{E}^{1243}(L_n^2) = \frac{2^{n-4}(n^3+5n+2)}{(n-1)2^{n-2}+1}.$$

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